The Stochastic Goodwill Problem

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Abstract

Utility maximization problems related to optimal advertising under uncertainty are considered. In particular, we determine the optimal strategies for the problem of maximizing the utility of goodwill at launch time and minimizing the disutility of a stream of advertising costs that extends until the launch time. We also consider some generalizations such as problems with constrained budget, optimization under partial information, and discretionary launching.

Key Words: advertising, linear quadratic regulator, new product introduction, stochastic control, utility maximization.

1 Introduction

We consider the optimization problem faced by a firm that, while advertising a product prior to its introduction into the market, wants to determine the optimal advertising policy for the maximization of the product image (also called *goodwill*), and the minimization of the total discounted cost. We shall also consider the problem of optimizing the launching time, thus allowing the firm to decide at its discretion to stop the advertising campaign and start selling the product.

This type of problems can be traced back at least to Nerlove and Arrow [20], who proposed to model the stock of advertising goodwill $X(t)$ at time $t \geq 0$ as

$$\dot{X}(t) = u(t) - \rho X(t),$$

where $u(t)$ is the rate of advertising expenditure, $\rho > 0$ is a factor of deterioration of product image in absence of advertisement, and $\dot{X} := \frac{dX}{dt}$. Then the optimization problem for the firm could be formulated as

$$\sup_u \left(e^{-\beta T} U_1(X(T)) - \int_0^T e^{-\beta t} U_2(u(t))\, dt \right),$$

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where $T > 0$ is the planned launching time, $\beta > 0$ a discount factor, U_1 represents the utility from the product image at launch time, and U_2 accounts for the regret from advertisement spending until product launching (U_2 is of course assumed to be such that $e^{-\beta t}U_2(u(t))$ is integrable over $[0, T]$). This deterministic optimal control formulation has been extended by many authors, to account for delay effects, non-linearity in the response to advertisement, and many other factors. For a very recent work on the subject, which also contains a list of related references, we refer to Buratto and Viscolani [6].

On the other hand, less work has been devoted to the case of stochastic evolution of goodwill level (for a few examples of works in this direction, we refer to the review article by Feichtinger, Hartl and Sethi [12] and references therein. See also the monographs by Tapiero [24] and by Sethi and Thompson [23]). The emergence of randomness in the dynamics of goodwill is quite natural for several reasons: one may think, for example, that random fluctuations in the goodwill level are the effect of external factors beyond the control of the firm. It is also natural to assume that noise enters through the control, since the effect of advertisement may be partly uncertain.

In this work we concentrate on some simple but hopefully still interesting cases of optimal control of the goodwill in the stochastic setting. We do not aim at maximum generality, instead we focus on models whose special structure allows us to obtain explicit solutions.

The paper is organized as follows: first we formulate the problem of determining the optimal advertising policy as a stochastic control problem in continuous time, we fix assumptions and notation, and recall the tools of the general theory of stochastic control problems for linear diffusions with quadratic costs. Then we consider several optimization problems under specific assumptions on the form of the stochastic differential equation modeling the dynamics of the goodwill, and on the utility functions. In some cases the solution is completely explicit, in others it is explicit modulo the solution of first order nonlinear ordinary differential equations. In the case of quadratic utilities we can also explicitly solve problems of optimal control under partial observation and optimal stopping of the controlled diffusion. We conclude suggesting some problems that we did not address here.

2 Problem formulation

Let $X(t)$ be the level of product image at time t, $0 \leq t \leq T$, with $X(0) = x \geq 0$ given. We postulate a dynamics for X of the type

$$dX(t) = (-\rho X(t) + u(t))\,dt + b(X(t), u(t))\,dW(t), \tag{1}$$

where W is a standard Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$, with T a fixed time for introduction of the advertised product into the market. We shall assume $b(x,u) = \sigma$, or $b(x,u) = \sigma x$, or $b(x,u) = \sigma x - \delta u$, with σ and δ positive constants, depending on the problem at hand. The control process u models the rate of advertisement spending by the firm, and is assumed to be measurable, adapted, and non-negative. We will denote by $\mathcal{U}$ the set of controls satisfying these properties.

We are interested in the following types of problems:

1. The maximization of an objective function that weights the utility from goodwill at launching time T and the total discounted disutility from advertisement until time T:
$$\sup_{u\in\mathcal{U}} \mathbb{E}\Big[U_1(X(T)) - \int_0^T e^{-\beta t} U_2(u(t))\,dt\Big].$$

2. The minimization of an objective function obtained by summing the disutility from not reaching a target level of goodwill $k > 0$ at time T and the total discounted disutility from advertisement until time T:
$$\inf_{u\in\mathcal{U}} \mathbb{E}\Big[U_1(X(T) - k) + \int_0^T e^{-\beta t} U_2(u(t))\,dt\Big].$$

3. The maximization of the utility from goodwill at launching time T with constrained advertisement policies:
$$\sup_{u\in\mathcal{M}} \mathbb{E}[U_1(X(T))],$$
where $\mathcal{M}$ is a subset of $\mathcal{U}$ composed by controls u such that
$$\mathbb{E}\Big[\int_0^T e^{-\beta t} U_2(u(t))\,dt\Big] \le M$$
for a fixed positive M.

4. The mixed problem of optimal advertisement policy to meet a goal and optimal launching time
$$\sup_{u\in\mathcal{U},\tau\in\mathcal{S}} \mathbb{E}\Big[U_1(X_\tau - k) - \int_0^\tau e^{-\beta t} U_2(u(t))\,dt\Big],$$
where $\mathcal{S}$ is the set of all stopping times with respect to the filtration $\mathbb{F}$.

Remark 1 Problem 1 in the above list can be interpreted as the single-objective optimization problem corresponding to a multi-objective program of the type
$$\sup_u \left(\mathbb{E}[U_1(X(T))], -\mathbb{E}\Big[\int_0^T e^{-\beta t} U_2(u(t))\,dt\Big]\right),$$

where the objective is a weighted average (modulo rescaling by constants of U_1 or U_2) of the two original criteria. The same could be said, mutatis mutandis, for problems 2–4. For more details on multi-objective optimization theory, see Zeleny [26].

3 Preliminaries

3.1 Notation

Given a function $\phi : [0,T] \times \mathbb{R}$, we shall adopt the following notation for partial derivatives:

$$\begin{aligned} \phi_t(t,x) &:= \frac{\partial \phi}{\partial t}(t,x), \\ \phi_x(t,x) &:= \frac{\partial \phi}{\partial x}(t,x), \\ \phi_{xx}(t,x) &:= \frac{\partial^2 \phi}{\partial x^2}(t,x). \end{aligned}$$

We denote by $C^{p,q}([0,T],\mathbb{R})$ the space of functions $(t,x) \mapsto \phi(t,x)$ whose partial derivatives with respect to t of order up to p, and with respect to x of order up to q, are continuous. Moreover, C_0^∞ will denote the space of infinitely differentiable functions with compact support.

We indicate by L^u the generator of the controlled diffusion (1), i.e. L^u is the differential operator defined by

$$L^u : f(x) \mapsto (-\rho x + u) f'(x) + \frac{1}{2} b^2(x,u) f''(x), \quad f \in C_0^\infty(\mathbb{R}).$$

Given a stochastic process X and a function f, we shall use the following notation for conditional expectations

$$\mathbb{E}^{s,y}[f(X(t))] := \mathbb{E}[f(X(t)) | X(s) = y].$$

3.2 General linear-quadratic regulator problems

We shall extensively use the results of Ait Rami, Moore, and Zhou [1] (see also Yong and Zhou [25]) on the connection between stochastic linear quadratic (LQ) control problems and a certain class of generalized Riccati equations (GRE). Here we simply recall their setup and results.

Consider the following LQ problem in a finite horizon T: minimize the objective functional

$$\begin{aligned} J(s,x;u) := \mathbb{E}^{s,x}\bigg[\int_s^T & [\langle Q(t)X(t), X(t)\rangle + \langle R(t)u(t), u(t)\rangle]\, dt \\ & + \langle HX(T), X(T)\rangle \bigg] \end{aligned}$$

over all $u \in L^2([s,T];\mathbb{R}^n)$ adapted to the filtration generated by W, subject to

$$\begin{aligned} dX(t) &= \Big(A(t)X(t) + B(t)u(t)\Big)\,dt + \Big(C(t)X(t) + D(t)u(t)\Big)\,dW(t), \\ X(s) &= x \in \mathbb{R}^n, \quad 0 \le s \le t \le T, \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^n$, the matrices A, B, C, D, Q, and R belong to $L^\infty([s,T];\mathcal{L}(\mathbb{R}^n))$, and Q, R and H are symmetric. All matrices may depend on t, except for H. Here $\mathcal{L}(\mathbb{R}^n)$ denotes the space of linear mappings of $\mathbb{R}^n$ into itself, and L^∞ is the space of essentially bounded functions.

Define the generalized Riccati equation associated to this LQ problem as the constrained differential equation

$$\begin{aligned} &\dot{P} + PA + A'P + C'PC - (PB + C'PD)(R + D'PD)^{\#}(B'P + D'PC) \\ &\qquad + Q = 0, && (2) \\ &P(T) = H, && (3) \\ &(R + D'PD)(R + D'PD)^{\#}(B'P + D'PC) - (B'P + D'PC) = 0, && (4) \\ &R + D'PD \ge 0, && (5) \end{aligned}$$

where $\cdot'$ and $\cdot^{\#}$ denote, respectively, the transpose and the Moore pseudoinverse of a matrix.

Then, if the GRE admits a solution P, the corresponding LQ problem is well posed, and the value function $V(s,x) := \inf_{u \in \mathcal{L}_2} J(s,x;u)$ is uniquely determined by

$$V(s,x) = \langle P(s)x, x \rangle.$$

Moreover, if $R(t) + D(t)'P(t)D(t) > 0$ for all $t \in [s,T]$, then the optimal control is uniquely determined by the linear feedback law

$$u_*(t) = -\Big(R(t) + D(t)'P(t)D(t)\Big)^{-1}\Big(B(t)'P(t) + D(t)'P(t)C(t)\Big)X(t).$$

A converse result also holds, namely, assuming that the LQ problem is well posed, if P exists such that (2)–(4) are satisfied for all $t \in [s,T]$, then P must satisfy

$$R + D'PD \ge 0 \quad \forall t \in [s,T].$$

A stronger result actually holds, i.e. the solvability of the GRE is equivalent to the existence of an optimal feedback control for the LQ problem.

Remark 2 It would be interesting to allow some of the coefficients in the LQ problem (in particular the decay factor ρ) to be random. Then the corresponding Riccati equation becomes a backward stochastic differential equation (BSDE). The problem of characterizing the finiteness and solvability of a stochastic LQ problem with random coefficients in terms of the solution of an associated Riccati equation is, in its full generality, still open. However, for partial results in this direction see Chen and Yong [7], and Bismut [5] for the first work on this issue.

4 Linear utilities

In this section we assume that the goodwill level X follows a stochastic differential equation (SDE) of the type (1) with $b(x,u) = \sigma \in \mathbb{R}_+$, and that $U_1(x) = \gamma x$, $U_2(x) = x$, with γ a positive constant. Then the problem becomes

$$\sup_{u\in\mathcal{U}} \mathbb{E}\Big[\gamma X(T) - \int_0^T e^{-\beta t} u(t)\,dt\Big], \tag{6}$$

and it is easy to see that one has to restrict the control space to allow only bounded controls, to avoid meaningless situations like infinite spending. For simplicity we assume $u \in [0,m]$, $m \in \mathbb{R}_+$.

The Hamilton-Jacobi-Bellman (HJB) equation associated to this problem is given by

$$\psi_t + \sup_{u\in[0,m]} (L^u\psi - e^{-\beta t}u) = 0, \quad \psi(T,x) = \gamma x. \tag{7}$$

Note that one has

$$\sup_{u\in[0,m]} (L^u\psi - e^{-\beta t}u) = \begin{cases} -\rho x\psi_x + \frac{1}{2}\sigma^2\psi_{xx}, & \psi_x \le e^{-\beta t} \\ -\rho x\psi_x + m(\psi_x - e^{-\beta t}) + \frac{1}{2}\sigma^2\psi_{xx}, & \psi_x > e^{-\beta t}. \end{cases}$$

Let us consider first the case $\psi_x > e^{-\beta t}$. The HJB equation can be written as

$$\psi_t - (\rho x - m)\psi_x + \frac{1}{2}\sigma^2\psi_{xx} - me^{-\beta t} = 0, \quad \psi(T,x) = \gamma x.$$

We guess a solution of the form $\psi(t,x) = \gamma(t)x + b_1(t)$, obtaining

$$x\gamma'(t) + b_1'(t) - (\rho x - m)\gamma(t) - me^{-\beta t} = 0,$$

with terminal conditions $\gamma(T) = \gamma$, $b_1(T) = 0$. Then this equation splits into

$$\gamma'(t) - \rho\gamma(t) = 0, \quad \gamma(T) = \gamma,$$

with solution $\gamma(t) = \gamma e^{-\rho(T-t)}$, and

$$b_1'(t) = -m\gamma(t) + me^{-\beta t}, \quad b_1(T) = 0,$$

with solution

$$b_1(t) = -\frac{m\gamma}{\rho}(1 - e^{-\rho(T-t)}) + \frac{m}{\beta}(e^{-\beta T} - e^{-\beta t}).$$

The case $\psi_x \leq e^{-\beta t}$ is completely similar: Let t_* be the solution of the equation $\gamma(t) = e^{-\beta t}$, i.e. $t_* = \frac{\rho T - \log \gamma}{\rho + \beta}$. Let us now solve the equation

$$\psi_t - \rho x \psi_x + \frac{1}{2}\sigma^2 \psi_{xx} = 0, \quad \psi(t_*, x) = \gamma(t_*)x + b_1(t_*),$$

where the terminal condition is such that a global solution of the HJB equation is at least continuous. It is immediate that the solution of this equation is $\psi(t,x) = \gamma(t)x + b_1(t_*)$, so that the global solution of the HJB is $\psi(t,x) = \gamma(t)x + b(t)$, where $b(t) = b_1(t_*)$ for $\gamma(t) \leq e^{-\beta t}$, and $b(t) = b_1(t)$ for $\gamma(t) > e^{-\beta t}$. It is also easy to see that b is continuously differentiable on $(0,T)$. In fact, one only needs to check whether there is smooth fit at t_*. But since $b_1'(t) = -m\gamma(t) + me^{-\beta t}$, by definition of t_* it immediately follows $b_1'(t_*) = 0$. This also proves that $\psi \in C^{1,2}([0,T], \mathbb{R})$, hence the solution of the HJB equation is the value function of the corresponding control problem (as follows by standard verification theorems, see e.g. Yong and Zhou [25]), and we can conclude that the optimal control is given by the following bang-bang policy:

$$u_*(t) = \begin{cases} 0 & t \leq t_*, \\ m & t > t_*. \end{cases} \tag{8}$$

That is, it is optimal to do nothing until a certain point in time t_*, after which it becomes optimal to advertise at the maximum rate. Note that, depending on γ, it could well be that $t_* > T$, i.e. it would never be optimal to advertise. This situation arises if the reward for improving the image of a product is small compared to the value of resources spent on advertisement.

We collect the findings of this section in the following proposition.

Proposition 3 *The optimal control problem (6) is solved by a control of the type (8), with* $t_* = \frac{\rho T - \log \gamma}{\rho + \beta}$, *and the corresponding value function is given by* $V(t,x) = \gamma(t)x + b(t)$, *where*

$$b(t) = \begin{cases} b_1(t_*) & t \leq t_*, \\ b_1(t) & t > t_*. \end{cases}$$

Remark 4 It is worth noting that the same result holds for quite general diffusion coefficients $b(x,u)$. In fact, guessing a solution ψ of (7) linear in x, so that $\psi_{xx} = 0$, we can carry out exactly the same calculations. An alternative explanation of this fact can be given as follows: assume

$$dX(t) = (-\rho X(t) + u(t))\,dt + b(X(t), u(t))\,dW(t),$$

with u and b such that the integrals below are well defined. Then a simple calculation yields

$$X(T) = e^{-\rho T} X_0 + \int_0^T e^{-\rho(T-t)} u(t)\,dt + \int_0^T e^{-\rho(T-t)} b(X(t), u(t))\,dW(t),$$

and if $b(\cdot,\cdot)$ is such that the stochastic integral is a martingale (e.g. b is bounded), then taking expectations on both sides we get

$$\mathbb{E}[X(T)] = e^{-\rho T}x + \int_0^T e^{-\rho(T-t)}\mathbb{E}[u(t)]\,dt,$$

where the interchange of the order of integration follows by Fubini's theorem using the assumption $u \geq 0$. It is now clear that the functional form of b will not influence the optimal control.

5 Linear utilities with constrained budget

In the same framework as in the previous section, let us now consider the constrained stochastic control problem

$$\sup_{u\in\mathcal{M}} \mathbb{E}[X(T)], \tag{9}$$

where $\mathcal{M}$ is the set of admissible controls $u(\cdot) \in [0,m]$ satisfying the integral constraint

$$\mathbb{E}\left[\int_0^T e^{-\beta t}u(t)\,dt\right] \leq M,$$

for a given constant $M \geq 0$. In order for the constrained problem to be non-trivial, it is also necessary to assume that $M \leq m\int_0^T e^{-\beta t}\,dt$. We actually only need to consider controls u for which the constraint is binding, i.e. advertising policies that use the whole budget M. In fact, denoting by $X^u(T)$ the controlled goodwill at time T, it is clear that $u_1 \geq u_2$ implies $\mathbb{E}[X^{u_1}(T)] \geq \mathbb{E}[X^{u_2}(T)]$, so it is never optimal to leave resources unused.

Let us introduce a Lagrange multiplier $\lambda > 0$, and consider the (unconstrained) problem

$$\sup_{u\in[0,m]} \mathbb{E}\left[X(T) - \lambda\left(\int_0^T e^{-\beta t}u(t)\,dt - M\right)\right]. \tag{10}$$

Then one has

$$\begin{aligned}
\sup_{u\in\mathcal{M}} \mathbb{E}[X(T)] &= \sup_{u\in\mathcal{M}} \mathbb{E}\left[X(T) - \lambda\left(\int_0^T e^{-\beta t}u(t)\,dt - M\right)\right] \\
&\leq \sup_{u\in[0,m]} \mathbb{E}\left[X(T) - \lambda\left(\int_0^T e^{-\beta t}u(t)\,dt - M\right)\right].
\end{aligned}$$

If the unconstrained problem (10) admits a solution u_λ for all $\lambda > 0$, and a λ_* exists such that $\mathbb{E}\int_0^T e^{-\beta t}u_{\lambda_*}(t)\,dt - M = 0$, then $u_* := u_{\lambda_*}$ is an optimal control for the constrained problem (for a proof of this simple fact, in a slightly different setting, see Øksendal [21], chap. 11). So we proceed to solve

$$\sup_{u\in[0,m]} \mathbb{E}\left[\frac{1}{\lambda}X(T) - \int_0^T e^{-\beta t}u(t)\,dt\right],$$

whose solution is

$$\begin{aligned} \gamma(t) \le e^{-\beta t} \quad &\Rightarrow \quad u_*(t) = 0, \\ \gamma(t) > e^{-\beta t} \quad &\Rightarrow \quad u_*(t) = m, \end{aligned}$$

with $\gamma(t) = \dfrac{1}{\lambda}e^{-\rho(T-t)}$.

The starting point for advertisement t_* is given by the solution of the equation $\gamma(t) = e^{-\beta t}$, so that

$$t_* = \frac{\rho T + \log\lambda}{\rho+\beta}. \tag{11}$$

We now need to show that $\lambda > 0$ exists such that

$$\int_{t_*}^T m e^{-\beta t}\,dt = M. \tag{12}$$

The solution of such an equation is given by

$$\lambda_* = e^{\rho T}\left(\beta\frac{M}{m} + e^{-\beta T}\right)^{-\frac{\rho+\beta}{\beta}},$$

which is clearly positive. It is now clear how to associate to such a λ_* the optimal solution for the constrained problem. Namely, given λ_* we obtain the optimal switching time t_* by (11), and hence the optimal control as $u_*(t) = m\mathbb{I}_{\{t>t_*\}}$, where $\mathbb{I}$ is the indicator function.

We have then proved the following result.

Proposition 5 *The optimal advertising policy for the constrained maximization of goodwill (9) is given by*

$$u_*(t) = \begin{cases} 0 & t \le t_*, \\ m & t > t_*, \end{cases}$$

with

$$t_* = \frac{2\rho}{\rho+\beta}T - \frac{1}{\beta}(e^{-\beta T} + \beta M/m).$$

Remark 6 It follows from (12) that the time to start advertising is given by

$$e^{-\beta t_*} - e^{-\beta T} = \beta\frac{M}{m},$$

and therefore we cannot simply consider unbounded controls with cumulative discounted cost less or equal than M, otherwise the optimal policy would be to "do infinite advertising at time T".

6 Linear utility of goodwill and CRRA disutility of investment

Let X be described by the controlled diffusion

$$dX(t) = (-\rho X(t) + u(t))\,dt + \sigma\,dW(t), \quad X(0) = x > 0.$$

Consider $U_1(x) = \gamma x$, $\gamma > 0$, and $U_2(x) = x^\alpha/\alpha$, $\alpha \in]0,1[$. Then the problem we are considering is

$$\sup_{u\in\mathcal{U}} \mathbb{E}\Big[\gamma X(T) - \int_0^T e^{-\beta t}\frac{u(t)^\alpha}{\alpha}\,dt\Big], \tag{13}$$

and its associated HJB equation can be written as

$$\psi_t + \sup_{u\geq 0}(L^u\psi - e^{-\beta t}\frac{u^\alpha}{\alpha}) = 0, \quad \psi(T,x) = \gamma x. \tag{14}$$

Assuming $\psi_x > 0$, the supremum is attained by

$$u_* = \Big(\psi_x e^{\beta t}\Big)^{\frac{1}{\alpha-1}}. \tag{15}$$

Substituting into (14) we obtain

$$\psi_t - \rho x\psi_x + \frac{\alpha-1}{\alpha}e^{\frac{\beta t}{\alpha-1}}\psi_x^{\frac{\alpha}{\alpha-1}} + \frac{1}{2}\sigma^2\psi_{xx} = 0.$$

If we conjecture a solution of the type

$$\psi(t,x) = \gamma(t)x + C(t),$$

then we would have $\psi_x > 0$, and the HJB equation separates into the following two ODEs:

$$\gamma'(t) = \rho\gamma(t), \quad \gamma(T) = \gamma,$$

and

$$C'(t) + \frac{\alpha-1}{\alpha}e^{\frac{\beta t}{\alpha-1}}\gamma(t)^{\frac{\alpha}{\alpha-1}}, \quad C(T) = 0.$$

The solution of the first one is simply $\gamma(t) = \gamma e^{-\rho(T-t)}$, and the second one follows explicitly as well by integration. The solution ψ of the HJB equation is of class $C^{1,2}([0,T],\mathbb{R})$, hence by standard verification theorems it coincides with the value function of the associated control problem, and by (15) we obtain the optimal policy in closed form as

$$u_*(t) = \Big(\gamma e^{-\rho(T-t)}e^{\beta t}\Big)^{\frac{1}{\alpha-1}}.$$

It is worth noting that because of the special structure of the utility functions, the optimal control is an open-loop policy, i.e. it does not depend on the state X.

Summarizing, the following holds.

Proposition 7 *The optimal advertising policy for the problem (13) is the open-loop control*

$$u_*(t) = \left(\gamma e^{-\rho(T-t)} e^{\beta t}\right)^{\frac{1}{\alpha-1}}.$$

7 Quadratic utilities

In this section we assume that both U_1 and U_2 are quadratic. In particular, we assume $U_1(x) = \gamma x^2$, with $\gamma > 0$, and $U_2(x) = x^2$. Then we consider the problem

$$\sup_{u \in \mathcal{U}} \mathbb{E}\Big[\gamma X(T)^2 - \int_0^T e^{-\beta t} u(t)^2\, dt\Big],$$

or, equivalently,

$$\inf_{u \in \mathcal{U}} \mathbb{E}\Big[\int_0^T e^{-\beta t} u(t)^2\, dt - \gamma X(T)^2\Big], \tag{16}$$

where X follows the controlled dynamics (1), under special choices of the diffusion coefficient b, and $\mathcal{U}$ is the set of adapted, square integrable controls. The problem at hand is a linear quadratic regulator problem with indefinite costs, which can be solved by the methods of Ait-Rami, Moore and Zhou [1] recalled earlier.
We shall study the following three cases:

- The noise is additive, i.e. $b(x, u) = \sigma \in \mathbb{R}_+$.
- The noise is multiplicative, i.e. $b(x, u) = \sigma x$, with $\sigma \in \mathbb{R}_+$.
- The control affects the (instantaneous) variance of the noise in a linear fashion, i.e. $b(x, u) = \sigma x - \delta u$, with σ, $\delta \in \mathbb{R}_+$.

More generally, one could replace the constants σ and δ with deterministic functions $\sigma(t)$ and $\delta(t)$, and the LQ technique would still hold. However, explicit solutions to the corresponding Riccati equations, and hence explicit feedback formulae, will be most likely lost.

Remark 8 We have removed the positivity assumption on the set of admissible controls in order to obtain explicit results. In some cases we shall be able to prove that the optimal control is also positive, and in others that it is positive on some "rich enough" set.

Before we begin with the stochastic case, let us briefly recall what the solution would be in the deterministic case. More precisely, if the goodwill evolves according to the controlled linear system

$$\begin{cases} \dot{X}(t) &= -\rho X(t) + u(t) \\ X(0) &= x, \end{cases}$$

then the optimal control for the deterministic optimal control problem

$$\inf_{u \in L^2([0,T];\mathbb{R})} \left[\int_0^T e^{-\beta t} u(t)^2 \, dt - \gamma X(T)^2 \right]$$

is given by

$$u_*(t) = -e^{\beta t} P(t) X(t),$$

where P solves the Riccati equation

$$\dot{P}(t) = 2\rho P + e^{\beta t} P^2(t), \quad P(T) = -\gamma. \tag{17}$$

Moreover, the value function is given by

$$V(x) = P(0)x^2.$$

The Riccati equation for this problem is identical to the Riccati equation for the problem with additive noise. We refer to the following subsection for details about global solvability and explicit solution of (17).

7.1 Additive noise

In this subsection we study the stochastic optimal control problem (16) subject to the dynamics (1) with $b(x,u) = \sigma$, $\sigma > 0$.

The Riccati equation associated to the problem is given by

$$\dot{P}(t) = 2\rho P + e^{\beta t} P^2(t), \quad P(T) = -\gamma,$$

which admits the explicit solution

$$P(t) = \frac{\gamma(\beta + 2\rho)e^{2\rho t}}{\gamma(e^{(\beta+2\rho)T} - e^{(\beta+2\rho)t}) - (\beta + 2\rho)e^{2\rho T}}. \tag{18}$$

By theorem 3.1 in [1] it follows that the LQ problem is well posed if and only if P is finite. By (18) it follows that the problem is well posed on the whole interval $[0,T]$ if and only if

$$e^{\beta T} - \frac{\beta + 2\rho}{\gamma} < e^{-2\rho T}.$$

Under this assumption, since $e^{-\beta t} > 0$, corollary 3.2 in [1] implies that the optimal control is unique and is given by

$$u_*(t) = -e^{\beta t} P(t) X(t). \tag{19}$$

Moreover, the value function can be written as

$$V(s,x) = P(s)x^2 + \sigma^2 \int_s^T P(t) \, dt, \tag{20}$$

where the second term on the right hand side accounts for the effect of the additive noise (in fact P does not depend on σ).

From (19), the optimal trajectory solves the SDE

$$dX(t) = -(\rho + e^{\beta t}P(t))X(t)\,dt + \sigma\,dW(t)$$

with $X(0) = x$, that is, by the variation of constants formula,

$$X(t) = e^{-\int_0^t r(s)\,ds}x + \int_0^t e^{-\int_\xi^t r(s)\,ds}\sigma\,dW(\xi), \tag{21}$$

where $a(s) := \rho + e^{\beta t}P(t)$. Note that, at any time $t \leq T$, the optimally controlled goodwill level is normally distributed with mean μ_t and variance η_t given by

$$\begin{aligned}\mu_t &= e^{-\int_0^t a(s)\,ds}x,\\ \eta_t &= \int_0^t e^{-2\int_\xi^t a(s)\,ds}\sigma^2\,d\xi.\end{aligned}$$

From the expression for the value function V it is immediate to derive its sensitivity (at time zero) with respect to the "size" of the noise. In fact, let us consider the SDE

$$dX^\varepsilon(t) = (-\rho X^\varepsilon(t) + u(t))\,dt + \sqrt{\varepsilon}\,\sigma\,dW(t),$$

with $\varepsilon \in [0, 1]$. It is clear that for $\varepsilon = 0$ we recover the deterministic dynamics. Let us call V^ε the value function for the problem with scaled noise, and u_*^ε the corresponding optimal feedback control. Then from (20) we can write

$$V^\varepsilon(s, x) = P(s)x^2 + \varepsilon\sigma^2\int_s^T P(t)\,dt,$$

where P is again given by (18). Therefore it is clear that

$$\lim_{\varepsilon\to 0} V^\varepsilon(s, x) = V^0(s, x) = P(s)x^2,$$

where $V^0(s, x)$ is the value function of the deterministic problem. One can also write the sensitivity of $V^\varepsilon(x)$ with respect to ε as

$$\frac{\partial V^\varepsilon(x)}{\partial\varepsilon} = \sigma^2\int_0^T P(t)\,dt,$$

i.e. the value function is linear with respect to "intensity" of the noise. Moreover, the optimal control u_*^ε is completely independent of the noise components, since the Riccati equation associated to the stochastic problem is the same as the Riccati equation in the deterministic setting, hence $u_*^\varepsilon \equiv u_*$. One can say that the optimal control policy is robust with respect to

additive noise.
One can also obtain immediately the sensitivity of the value function with respect to the initial value of goodwill as

$$\frac{\partial V(s,x)}{\partial x} = 2P(s)x.$$

Moreover, from (21), it is easy to see that there is continuous and differentiable dependence of the optimal trajectory with respect to the initial condition, with

$$\frac{\partial X(t;x)}{\partial x} = e^{-\int_0^t a(s)\,ds},$$

and in particular $\partial_x \mathbb{E}[X(T;x)] = e^{-\int_0^T a(s)\,ds}$, which gives the sensitivity of the average level of goodwill at launch time with respect to the level of goodwill at the beginning of the advertising period.

Remark 9 In fact Ait Rami, Moore and Zhou [1] consider only the case of $b(x,u)$ linear in x and u, so their methods are not directly applicable to our situation. However, by repeating their arguments with $b(x,u) \equiv \sigma$, one sees that their results hold in this case as well.

Remark 10 Assuming that the problem is well posed at $s = 0$, then $P(t) < 0$ for all $t \in [0,T]$, therefore the sign of the control agrees with the sign of the state X. This does not guarantee that u is always non-negative, but at least assures that while the goodwill does not reach zero, the control will be positive. The solution is still not satisfactory from a mathematical point of view, but should be acceptable in applications. Moreover, if we had to insist on enforcing the non-negativity requirement, we would certainly loose closed-form solutions. The same considerations apply to some of the cases we shall treat in the sequel.

7.2 Multiplicative noise

The solution in the case of multiplicative noise $b(x,u) = \sigma x$ is quite similar, at least as far as the calculations goes, to the case of additive noise. In fact, the GRE can be written as

$$\dot{P}(t) = (2\rho - \sigma^2)P + e^{\beta t}P^2(t), \quad P(T) = -\gamma, \tag{22}$$

and its solution is given by

$$P(t) = \frac{\gamma(\beta + 2\rho - \sigma^2)e^{(2\rho-\sigma^2)t}}{\gamma(e^{(\beta+2\rho-\sigma^2)T} - e^{(\beta+2\rho-\sigma^2)t}) - (\beta + 2\rho - \sigma^2)e^{(2\rho-\sigma^2)T}}. \tag{23}$$

In analogy to the previous case, the optimal control is given by

$$u_*(t) = -e^{\beta t}P(t)X(t),$$

on the subinterval of $[0,T]$ where the problem is well posed. Since the weight on the control is always positive, it is enough to determine the interval where P does not explode. If $\beta + 2\rho - \sigma^2 \neq 0$, then the problem is well posed in the interval $[s,T]$, with

$$s = \frac{(2\rho - \sigma^2)T + \log(e^{\beta T} - (\beta + 2\rho - \sigma^2)/\gamma)}{\beta + 2\rho - \sigma^2},$$

provided the inequality

$$e^{\beta T} > \frac{\beta + 2\rho - \sigma^2}{\gamma}$$

is satisfied, otherwise the problem is nowhere well posed.

We can now state the following proposition, whose proof is an immediate consequence of the above discussion and of general results on stochastic LQ control.

Proposition 11 *Assume that* $2\rho + \beta \neq \sigma^2$ *and*

$$(2\rho - \sigma^2)T + \log(e^{\beta T} - (\beta + 2\rho - \sigma^2)/\gamma) \leq 0.$$

Then the problem (16) with $b(x,u) = \sigma x$ *is well posed at* $s = 0$ *and admits the optimal control in feedback form*

$$u_*(t) = -e^{\beta t} P(t) X(t),$$

where P *is the unique (negative) solution of the Riccati equation (22). Moreover, the corresponding value function is given by*

$$V(s,x) = P(s)x^2.$$

Since the optimal control is linear with respect to X, then one can see that X^{u_*} is always positive. In fact, the closed-loop equation can be written as

$$dX(t) = -(\rho + e^{\beta t} P(t))X(t) + \sigma X(t)\, dW(t),$$

whose explicit solution is given by

$$X(t) = x \exp\Big(-(\rho + \sigma^2/2)t - \int_0^t e^{\beta s} P(s)\, ds + \sigma W(t) \Big).$$

In particular, since P is negative on the whole interval $[0,T]$, this also proves that the optimal control u_* is positive on the same interval, hence eliminating a shortcoming of the model with additive noise, where optimal control policies, in general, are not guaranteed to be positive.

Let us now study the system in the small noise limit, that is, we study the optimal control problem on a dynamics of the type

$$dX^\varepsilon(t) = (-\rho X^\varepsilon(t) + u(t))\, dt + \sqrt{\varepsilon}\, \sigma X^\varepsilon(t)\, dW(t),$$

where $\varepsilon \in [0,1]$. For $\varepsilon = 0$ the problem reduces again to deterministic control. The value function V^ε for this case can be written as

$$V^\varepsilon(x) = P^\varepsilon(0)x^2,$$

with

$$P^\varepsilon(0) = \frac{\gamma(\beta + 2\rho - \varepsilon\sigma^2)}{\gamma(e^{(\beta+2\rho-\varepsilon\sigma^2)T} - 1) - (\beta + 2\rho - \varepsilon\sigma^2)e^{(2\rho-\varepsilon\sigma^2)T}}. \tag{24}$$

It clearly holds $P^\varepsilon(0) \to P^0(0)$ as $\varepsilon \to 0$, where P^0 is the solution of the Riccati equation for the deterministic case, hence

$$\lim_{\varepsilon\to 0} V^\varepsilon(x) = V^0(x).$$

The sensitivity of the value function with respect to the noise intensity can again be computed explicitly as

$$\frac{\partial V^\varepsilon(x)}{\partial\varepsilon} = \frac{\partial P^\varepsilon(0)}{\partial\varepsilon}x^2,$$

which is nonlinear with respect to ε, in contrast to the case of additive noise. An explicit expression for $\partial_\varepsilon P^\varepsilon(0)$ is easily found from (24), but it is rather cumbersome. Here we limit ourselves to writing the expression for the sensitivity of $P^\varepsilon(0)$ at $\varepsilon = 0$, i.e. to

$$\left.\frac{\partial P^\varepsilon(0)}{\partial\varepsilon}\right|_{\varepsilon=0} = \frac{\sigma^2\gamma\Big(\gamma - (2\rho+\beta)^2 T e^{2\rho T} + \gamma((2\rho+\beta)T - 1)e^{(2\rho+\beta)T}\Big)}{(\gamma + (2\rho + \beta - \gamma e^{\beta T})e^{2\rho T})^2}.$$

The optimal feedback policy is clearly sensitive to the noise, as it follows from

$$u_*^\varepsilon(t) = -e^{\beta t}P^\varepsilon(t)X^\varepsilon(t).$$

Since $P^\varepsilon \to P^0(0)$ as $\varepsilon \to 0$, we can still say that the functional specification of the optimal control in the small noise limit converges to the corresponding functional specification for u_*^0 in the deterministic limit, i.e.

$$u_*^\varepsilon(t,x) = e^{\beta t}P^\varepsilon(t)x \longrightarrow e^{\beta t}P^0(t)x = u_*^0(t,x)$$

as $\varepsilon \to 0$.

In complete analogy to the case of additive noise, the sensitivity of the value function with respect to the initial goodwill level is given explicitly by

$$\frac{\partial V(s,x)}{x} = 2P(s)x.$$

Moreover, the optimal trajectory X depends regularly on the initial condition. In particular, we can write

$$\frac{\partial X(t;x)}{x} = \exp\Big(-(\rho + \sigma^2/2)t - \int_0^t e^{\beta s}P(s) + \sigma W(t)\Big),$$

and

$$\frac{\partial \mathbb{E}[X(T;x)]}{\partial x} = \exp\Big(-\rho t - \int_0^t e^{\beta s}P(s)\Big).$$

7.3 Control-dependent diffusion coefficient

Suppose now that the control affects the diffusion term as well, namely that advertisement spending reduces variability in a linear way, i.e. $b(x,u) = \sigma x - \delta u$. Then the GRE for this problem becomes

$$\begin{cases} \dot{P} = (2\rho - \sigma^2)P + (1-\sigma\delta)^2 \dfrac{P^2}{e^{-\beta t} + \delta^2 P}, \\ e^{-\beta t} + \delta^2 P > 0 \\ P(T) = -\gamma. \end{cases} \tag{25}$$

Using again corollary 3.2 in [1], if $e^{-\beta t} + \delta^2 P > 0$ and the problem is well posed, then the optimal control is unique and is given by

$$u^*(t) = -\frac{(1-\sigma\delta)P(t)}{e^{-\beta t} + \delta^2 P(t)} X(t),$$

with associated value function

$$V(s,x) = P(s)x^2.$$

The optimal trajectory is given by the closed-loop equation

$$dX(t) = a(t)X(t)\,dt + c(t)X(t)\,dW(t),$$

with

$$\begin{aligned} a(t) &:= -\rho - \frac{(1-\sigma\delta)P(t)}{e^{-\beta t} + \delta^2 P(t)}, \\ c(t) &:= \sigma + \delta \frac{(1-\sigma\delta)P(t)}{e^{-\beta t} + \delta^2 P(t)}, \end{aligned}$$

which admits the explicit solution

$$X(t) = x \exp\Big(\int_0^t (a(s) - \frac{1}{2} c(s)^2)\,ds + \int_0^t c(s)\,dW(s) \Big). \tag{26}$$

In analogy to the case of multiplicative noise, the optimal trajectory is always positive.

Unfortunately, the Riccati equation for this case does not seem to admit an explicit solution, hence one has to resort on numerical methods to solve it. However, this does not present great difficulties, as it is only a first-order nonlinear equation with terminal condition. Under the assumption $e^{-\beta T} - \delta^2\gamma > 0$, the problem is locally well posed, i.e. there exists $t_0 < T$ such that the Riccati equation (25) admits a solution in $[t_0, T]$. In practice, given a certain set of numerical values for the parameters, one would try to solve (25) numerically, and correspondingly determine in what time interval

P is finite and $e^{-\beta t} + \delta^2 P > 0$, which coincides with the region where the original problem is well posed.

If we assume that the problem is well posed at s, and hence that the Riccati equation (25) has a solution P defined on the whole interval $[s, T]$ (and in particular $e^{-\beta t} + \delta^2 P(t) > 0$ for all $t \in [s, T]$), then $P(t)$ is negative for all $t \in [s, T]$, as demonstrated in the following lemma.

Lemma 12 *If P solves the Riccati equation (25) on $[s, T]$, then $P(t) < 0$ for all $t \in [s, T]$.*

Proof. Assume, by contradiction, that there exists $t_0 \in [s, T[$ such that $P(t_0) = 0$. Then $P(t) \equiv 0$ is a continuous solution of (25) for $t > t_0$. By Proposition 7.1 in Yong and Zhou [25] it follows that this is also the only solution. But this contradicts the terminal condition $P(T) = -\gamma \neq 0$. $\square$

Since we have proved that the optimal trajectory is positive, then, under the additional condition $\sigma\delta < 1$, the optimal control u_* is positive as well. Note that, because of the interpretation of σ and δ, the hypothesis $\sigma\delta < 1$ is not very restrictive.

Let us now consider the small noise limit, i.e. the control problem on the dynamics

$$dX^\varepsilon(t) = (-\rho X^\varepsilon(t) + u(t))\, dt + \sqrt{\varepsilon}\, (\sigma X^\varepsilon(t) - \delta u(t))\, dW(t),$$

where $\varepsilon \in [0, 1]$. The Riccati equation for this problem can be written as

$$\dot{P} = (2\rho - \varepsilon\sigma^2)P + (1 - \varepsilon\sigma\delta)^2 \frac{P^2}{e^{-\beta t} + \varepsilon\delta^2 P}, \quad P(T) = -\gamma. \tag{27}$$

It is clear that local solvability at $t = T$ for the problem with $\varepsilon = 1$ implies local solvability at $t = T$ for all problems with $\varepsilon \in [0, 1[$. Therefore, let us fix an s such that the Riccati equation (27) is solvable in $[s, T]$ for all $\varepsilon \in [0, 1]$, and let us call its unique solution P^ε. We shall now prove that P^ε is continuous with respect to ε. This implies immediately that V^ε converges to V^0 as $\varepsilon \to 0$.

Proposition 13 *There exists a neighborhood J_T of T and a neighborhood H_0 of 0 such that for all $\varepsilon \in H_0$, there exists a unique solution of (27) in J_T with the properties that $P^\varepsilon(T) = -\gamma$ and $(t, \varepsilon) \mapsto P(t, \varepsilon)$ is of class $C^1(J_T \times H_0)$.*

Proof. Let us define

$$f(t, x; \varepsilon) = (2\rho - \varepsilon\sigma^2)x + (1 - \varepsilon\sigma\delta)^2 \frac{x^2}{e^{-\beta t} + \varepsilon\delta^2 x}.$$

Then for any ε the mapping $(t, x) \mapsto f(t, x; \varepsilon)$ is of class C^1. Moreover, f and $\partial_x f$ are continuous. Therefore §10.7.1 of Dieudonné [9] applies, and the result follows. □

One could also apply results on differential dependence of the solution of ODEs with respect to parameters to obtain an equation for $\partial_\varepsilon P^\varepsilon$, but this would be expressed in terms of the solution of equation (27) itself. In particular, $\partial_\varepsilon P^\varepsilon$ solves the linear ODE

$$U' = A(t;\varepsilon)U + B(t;\varepsilon),$$

where

$$\begin{aligned} A(t;\varepsilon) &= \left.\frac{\partial f(t,x;\varepsilon)}{\partial P}\right|_{x=P(t;\varepsilon)}, \\ B(t;\varepsilon) &= \left.\frac{\partial f(t,x;\varepsilon)}{\partial \varepsilon}\right|_{x=P(t;\varepsilon)}. \end{aligned}$$

Once again, the sensitivity of the value function V with respect to x is given by

$$\frac{\partial V(s,x)}{\partial x} = 2P(s)x,$$

and the optimal trajectory X depends regularly on x as follows from (26). In particular, the dependence of the average level of goodwill at launch time with respect to x is given by

$$\frac{\partial \mathbb{E}[X(T;x)]}{\partial x} = \exp\Big(\int_0^t a(s)\,ds\Big).$$

8 Quadratic hedging

We consider in this section a class of problems that can again be solved explicitly using the linear quadratic regulator, while giving much less difficulty in terms of well posedness.

Let k a level of goodwill we want to reach by time T. Then we are interested in minimizing the distance from the objective, while keeping costs low, i.e.

$$\inf_{u\in\mathcal{U}} \mathbb{E}\Big[U_1(X(T)-k) + \int_0^T e^{-\beta t} U_2(u(t))\,dt\Big],$$

where X follows the controlled dynamics (1), and $\mathcal{U}$ is the space of square integrable control processes adapted to the filtration generated by W.
If we assume that both the disutility U_1 from not meeting the target and the disutility of advertisement spending U_2 are quadratic, then we can use the approach of the linear-quadratic regulator to obtain an explicit solution. In particular, let us define the process $Y(t)$ as the distance from the target, i.e. $Y(t) = k - X(t)$. Then one has

$$\begin{cases} dY(t) &= (-\rho Y(t) - u(t) + \rho k)\,dt - b(k - Y(t), u(t))\,dW(t) \\ Y(0) &= y := k - x > 0, \end{cases}$$

and we can consider the following stochastic control problem

$$\inf_{u\in\mathcal{U}} \mathbb{E}^y \left[\int_0^T e^{-\beta t} u(t)^2 \, dt + \gamma Y(T)^2 \right], \tag{28}$$

As in the previous section, we will study three cases, corresponding to $b(x,u) = \sigma$, $b(x,u) = \sigma x$, and $b(x,u) = \sigma x - \delta u$, where σ and δ are positive constants. It is clear that this is the standard linear-quadratic regulator problem with non-negative costs. The main difference with respect to the case with indefinite costs is that, under mild regularity assumptions on the coefficients and the weights, the Riccati equation associated to the problem always admits a solution on the whole interval $[0,T]$ (see, e.g., Theorem 2.1 in Yong and Zhou [25]).

In analogy with the previous section, a deterministic problem is still meaningful, and its solution (except for the value function) coincides with the solution of the case with additive noise. In the following subsection we provide more details.

8.1 Additive noise

In this case the distance from the target Y evolves according to the SDE

$$dY(t) = (-\rho Y(t) - u(t) + \rho k)\, dt - \sigma\, dW(t),$$

hence the Riccati equations associated to (28) read as

$$\begin{aligned} \dot{P} &= 2\rho P + e^{\beta t} P^2, & P(T) &= \gamma, \\ \dot{\phi} &= (\rho + e^{\beta t} P)\phi - \rho k P, & \phi(T) &= 0. \end{aligned}$$

The solution is given by

$$P(t) = \frac{\gamma(2\rho+\beta)e^{2\rho t}}{(2\rho+\beta)e^{2\rho T} + \gamma(e^{(2\rho+\beta)T} - e^{(2\rho+\beta)t})} \tag{29}$$

$$\phi(t) = \frac{\gamma(2\rho+\beta)k(e^{\rho(T-t)} - 1)}{(2\rho+\beta)e^{2\rho T} + \gamma(e^{(2\rho+\beta)T} - e^{(2\rho+\beta)t})} \tag{30}$$

Note that P is well defined on the whole interval $[0,T]$, hence the problem is well defined, and the optimal control is uniquely determined by

$$u_*(t) = e^{\beta t}(P(t)Y(t) + \phi(t)) = e^{\beta t}(P(t)(k - X(t)) + \phi(t)),$$

and the value function can be written as

$$\begin{aligned} V(s,y) &= P(s)y^2 + 2\phi(s)y + \rho k \int_s^T \phi(t)\, dt \\ &\quad + \sigma^2 \int_s^T P(t)\, dt - \int_s^T e^{\beta t}\phi(t)^2\, dt, \end{aligned} \tag{31}$$

where the term $\sigma^2 \int_s^T P(t)\,dt$ accounts for the effect of the noise in the system.

The optimal trajectory can be found by solving the closed loop equation

$$dX(t) = (-a(t)X(t) + c(t))\,dt + \sigma\,dW(t),$$

where

$$\begin{aligned} a(t) &:= \rho + e^{\beta t} P(t), \\ c(t) &:= e^{\beta t}(kP(t) + \phi(t)). \end{aligned}$$

By an application of the variation of constant formula, the solution can be written as

$$X(t) = e^{-\int_0^t a(s)\,ds} x + \int_0^t e^{-\int_\xi^t a(s)\,ds} c(\xi)\,d\xi + \int_0^t e^{-\int_\xi^t a(s)\,ds} \sigma\,dW(\xi), \quad (32)$$

hence $X(t)$ is normally distributed for each t with mean μ_t and variance η_t given by

$$\begin{aligned} \mu_t &= e^{-\int_0^t a(s)\,ds} x + \int_0^t e^{-\int_\xi^t a(s)\,ds} c(\xi)\,d\xi \\ \eta_t &= \sigma^2 \int_0^t e^{-2\int_\xi^t a(s)\,ds}\,d\xi. \end{aligned}$$

Note that, in analogy to the case of subsection 7.1, the control is always positive at least where $Y > 0$, i.e. when the target is not yet met, because clearly $P > 0$, $\phi > 0$ for all $t < T$. The analogy continues with the sensitivity of the value function with respect to the noise, which is given by

$$\frac{\partial V^\varepsilon(y)}{\partial \varepsilon} = \sigma^2 \int_0^T P(t)\,dt.$$

Moreover, the optimal control strategy does not depend on the noise component, i.e. coincides with the optimal control for the corresponding deterministic problem.

The sensitivity of the value function with respect to the initial distance from the target level of goodwill is given by

$$\frac{\partial V(s,y)}{\partial y} = 2P(s)y + 2\phi(s),$$

as follows from (31).

From (32) we also obtain the sensitivities of the optimally controlled goodwill level:

$$\frac{\partial X(t;x)}{\partial x} = \frac{\partial \mathbb{E}[X(t;x)]}{\partial x} = e^{-\int_0^t a(s)\,ds}.$$

8.2 Multiplicative noise

If $b(x,u) = \sigma x$, then the equation for $Y(t)$ becomes

$$dY(t) = (-\rho Y(t) - u(t) + \rho k)\,dt + (\sigma Y(t) - \sigma k)\,dW(t).$$

The associated Riccati equations are then:

$$\begin{aligned} \dot{P} &= (2\rho - \sigma^2)P + e^{\beta t}P^2, & P(T) = \gamma \\ \dot{\phi} &= (\rho + e^{\beta t}P)\phi + (\sigma^2 - \rho)kP, & \phi(T) = 0, \end{aligned}$$

and their solutions can be written as

$$P(t) = \frac{\gamma(\beta + 2\rho - \sigma^2)e^{(2\rho-\sigma^2)t}}{(\beta + 2\rho - \sigma^2)e^{(2\rho-\sigma^2)T} + \gamma(e^{(\beta+2\rho-\sigma^2)T} - e^{(\beta+2\rho-\sigma^2)t})} \tag{33}$$

$$\phi(t) = \frac{(\beta + 2\rho - \sigma^2)\gamma k e^{\rho t}(e^{(\rho-\sigma^2)T} - e^{(\rho-\sigma^2)t})}{(\beta + 2\rho - \sigma^2)e^{(2\rho-\sigma^2)T} + \gamma(e^{(\beta+2\rho-\sigma^2)T} - e^{(\beta+2\rho-\sigma^2)t})} \tag{34}$$

Since P is well defined for all values of $t \in [0,T]$, the unique optimal control policy is given by

$$u_*(t) = e^{\beta t}(P(t)Y(t) + \phi(t)) = e^{\beta t}(P(t)(k - X(t)) + \phi(t)),$$

with corresponding value function

$$\begin{aligned} V(s,y) &= P(s)y^2 + 2\phi(s)y + 2\rho k \int_s^T \phi(t)\,dt \\ &\quad + k^2\sigma^2 \int_s^T P(t)\,dt - \int_s^T e^{\beta t}\phi^2(t)\,dt. \end{aligned} \tag{35}$$

Let us write the SDE for the optimal trajectory: defining $a(\cdot)$ and $c(\cdot)$ as in the previous subsection, we can write

$$dX(t) = (-a(t)X(t) + c(t))\,dt + \sigma X(t)\,dW(t). \tag{36}$$

As in the case of additive noise, and in contrast to the case of subsection 7.2, we can guarantee positivity of the optimal strategy only for those t where $X(t) \le k$, as it follows from P and ϕ being positive on $[0,T]$.

Let us now consider the behavior of the value function and of the optimal control in the small noise limit, i.e. let us write, for $\varepsilon \in [0,1]$,

$$dY^\varepsilon(t) = (-\rho Y^\varepsilon(t) - u(t) + \rho k)\,dt + \sqrt{\varepsilon}\,(\sigma Y^\varepsilon(t) - \sigma k)\,dW(t),$$

and denote by V^ε and u_*^ε, as usual, the value function and the optimal control for the problem (28). Then one can easily prove that $V^\varepsilon \to V^0$ as $\varepsilon \to 0$. In fact, by (33) and (34) follows that $P^\varepsilon \to P^0$ and $\phi^\varepsilon \to \phi^0$ as $\varepsilon \to 0$, and one can exchange integration and limit by dominated convergence, since

$P^\varepsilon(t) \le \gamma$ and $\phi^\varepsilon(t) \le \gamma k$ for all $t \in [0,T]$ and all $\varepsilon \in [0,1]$. The same argument also shows that the optimal control, as a function of t and x, converges to the deterministic control law in the small noise limit, that is $u^\varepsilon(t,x) \to u^0(t,x)$ as $\varepsilon \to 0$. Moreover, we can again write explicitly the sensitivity of the value function with respect to ε as follows:

$$\begin{aligned}\frac{\partial V^\varepsilon(y)}{\partial \varepsilon} &= \frac{\partial P^\varepsilon(0)}{\partial \varepsilon} y^2 + 2\frac{\partial \phi^\varepsilon(0)}{\partial \varepsilon} y + 2\rho k \int_0^T \frac{\partial \phi^\varepsilon(t)}{\partial \varepsilon}\, dt \\ &\quad + \sigma^2 k^2 \int_0^T P^\varepsilon(t)\, dt + \varepsilon \sigma^2 k^2 \int_0^T \frac{\partial P^\varepsilon(t)}{\partial \varepsilon}\, dt \\ &\quad -2 \int_0^T e^{\beta t} \frac{\partial \phi^\varepsilon(t)}{\partial \varepsilon}\, dt,\end{aligned}$$

where one can differentiate under the integral sign by the uniform boundedness of the integrands.

In contrast to the case of indefinite costs and multiplicative noise of subsection 7.2, the sensitivity of the value function with respect to noise intensity depends on P and ϕ on the whole interval $[0,T]$ and not only at time $t = 0$. An explicit expression for $\partial_\varepsilon V^\varepsilon$ at $\varepsilon = 0$ could again be given, although in a very cumbersome form. We omit the tedious but straightforward details.

The sensitivity of the value function with respect to the initial distance from the target is easily obtained by differentiating (35):

$$\frac{\partial V(s,y)}{\partial y} = 2P(s)y + 2\phi(s).$$

Using results of regular dependence of the solution of SDEs on the initial condition (see, e.g., Gihman and Skorohod [13], [14]), we can obtain an explicit representation of $X_x(t;x) := \partial_x X(t;x)$. In fact, X_x solves the SDE

$$\begin{cases} dX_x(t) &= -a(t)X_x(t)\, dt + \sigma X_x(t)\, dW(t) \\ X_x(0) &= 1, \end{cases}$$

then

$$X_x(t) = \exp\Big(- \int_0^t a(s)\, ds - \frac{1}{2}\sigma^2 t + \sigma W(t)\Big).$$

From this expression one can immediately obtain also $\mathbb{E}[X_x(t)]$. However, it is not clear that one can differentiate under the expectation sign. The following argument shows that in fact $\partial_x \mathbb{E}[X(t;x)] = \mathbb{E}[X_x(t;x)]$. Let us write the SDE for the optimal trajectory in integral form:

$$X(t) = x - \int_0^t a(s)X(s)\, ds + \int_0^t \sigma X(s)\, dW(s) + \int_0^t c(s)\, ds,$$

and take expectation on both sides, getting

$$\mathbb{E}[X(t)] = x - \int_0^t a(s)\mathbb{E}[X(s)]\,ds + \int_0^t c(s)\,ds,$$

where the exchange of integration and expectation is justified by Fubini's theorem, and the stochastic integral has expectation zero because X is square integrable. Define now $m(t) = \mathbb{E}[X(t)]$ and take derivatives with respect to t. Then $m(t)$ solves the Cauchy problem

$$\dot{m}(t) = -a(t)m(t) + c(t), \quad m(0) = x,$$

that is

$$\mathbb{E}[X(t)] = xe^{-\int_0^t a(s)\,ds} + \int_0^t e^{\int_0^\xi a(s)\,ds} c(\xi)\,d\xi,$$

and

$$\frac{\partial \mathbb{E}[X(t;x)]}{\partial x} = e^{-\int_0^t a(s)\,ds}.$$

8.3 Control-dependent diffusion coefficient

If $b(x,u) = \sigma x - \delta u$, the equation for the dynamics of Y is

$$dY(t) = (-\rho Y(t) - u(t) + \rho k)\,dt + (\sigma Y(t) + \delta u(t) - \sigma k)\,dW(t). \tag{37}$$

The Riccati equations for this case are

$$\begin{cases} \dot{P} = (2\rho - \sigma^2)P + \dfrac{(1-\sigma\delta)^2 P^2}{e^{-\beta t} + \delta^2 P} \\ e^{-\beta t} + \delta^2 P > 0 \\ P(T) = \gamma, \end{cases} \tag{38}$$

and

$$\begin{cases} \dot{\phi} &= \Big(\rho + (1-\sigma\delta)\frac{P}{e^{-\beta t}+\delta^2 P}\Big)\phi \\ & \quad + \sigma k\Big(\sigma + \delta(1-\sigma\delta)\dfrac{P}{e^{-\beta t} + \delta^2 P} - \dfrac{\rho}{\sigma}\Big)P \\ \phi(T) &= 0. \end{cases} \tag{39}$$

In analogy to the case with indefinite costs, there does not seem to exist an explicit solution for these equations. However, Theorem 7.2 in Yong and Zhou [25] ensures the existence of a solution $P \in C([0,T];\mathbb{R}_+)$ (and hence uniqueness, by Proposition 7.1 of ibid.) As a consequence, the linear equation with bounded coefficients (39) admits a unique continuous solution on $[0,T]$. Then the associated LQ problem is solvable at any $s \in [0,T]$, with optimal control and value function given respectively by

$$u_*(t) = \Big(e^{-\beta t} + \delta^2 P(t)\Big)^{-1}\Big((1-\sigma\delta)P(t)Y(t) + \sigma\delta k P(t) + \phi(t)\Big),$$

and

$$\begin{aligned} V(s,y) &= P(s)y^2 + 2\phi(s)y + 2\rho k \int_s^T \phi(t)\,dt \\ &\quad + \sigma^2 \int_s^T P(t)\,dt - \int_s^T \frac{(\phi(t) + \sigma\delta k P(t))^2}{e^{-\beta t} + \delta^2 P(t)}\,dt. \end{aligned} \tag{40}$$

as follows by Theorem 6.1 in Yong and Zhou [25].

Unfortunately we cannot prove that the optimal control u_* is nonnegative on $\{X(t) \le k\}$. We can prove instead the following weaker result.

Proposition 14 *If $2\rho > \sigma^2$, then the solution P of the Riccati equation (38) is such that $0 < P(t) \le \gamma$ for all $t \in [0,T]$.*

Proof. Suppose, by contradiction, that $t_0 \in [s,T[$ exists such that $P(t_0) = 0$. Then $P(t) = 0$ for all $t \in [s,T]$ is a solution of the Riccati equation on the interval $[t_0,T]$. By uniqueness of solution, we obtain $P(T) = 0 \neq \gamma > 0$. Then it must be $P(s) > 0$. Suppose now, again by contradiction, that there exists $t_0 \in [s,T[$ such that $P(t_0) > \gamma$. Then $\dot{P}(t_0) > 0$, and $P(t)$ is strictly increasing for all $t > t_0$. Then $P(T) > \gamma$. □

The sign of ϕ seems more difficult to determine. In fact, at $t = T$ one has

$$\dot{\phi}(T) = \sigma k \Big(\sigma + \delta(1 - \sigma\delta) \frac{\gamma}{e^{-\beta T} + \delta^2\gamma} - \frac{\rho}{\sigma} \Big)\gamma. \tag{41}$$

Its sign depends on term between parentheses. In particular, if it is positive, then $\phi(t) \le 0$ in a neighborhood of T, and vice-versa. In general, however, it seems not possible to determine a priori the sign of the right hand side in (41) for all $t \in [0,T]$.

Let us now consider the small noise limit, i.e. the control problem on the dynamics

$$\begin{cases} dY^\varepsilon(t) &= (-\rho Y^\varepsilon(t) - u(t) + \rho k)\,dt + \sqrt{\varepsilon}\,(\sigma Y^\varepsilon(t) + \delta u(t) - \sigma k)\,dW(t) \\ Y^\varepsilon(0) &= k - x, \end{cases}$$

where $\varepsilon \in [0,1]$. The Riccati equations for this problem can be written as

$$\begin{cases} \dot{P} = (2\rho - \varepsilon\sigma^2)P + (1 - \varepsilon\sigma\delta)^2 \dfrac{P^2}{e^{-\beta t} + \varepsilon\delta^2 P} \\ e^{-\beta t} + \varepsilon\delta^2 P > 0 \\ P(T) = \gamma, \end{cases} \tag{42}$$

and

$$\begin{cases} \dot{\phi} &= \Big(\rho + (1 - \varepsilon\sigma\delta)\dfrac{P}{e^{-\beta t} + \varepsilon\delta^2 P}\Big)\phi \\ &\quad + \Big(\varepsilon\sigma^2 k + \varepsilon\sigma\delta k(1 - \varepsilon\sigma\delta)\dfrac{P}{e^{-\beta t} + \varepsilon\delta^2 P} - \rho k\Big)P \\ \phi(T) &= 0. \end{cases}$$

As already mentioned, by general results on standard stochastic LQ problems, we obtain well posedness, as well as existence and uniqueness of solutions for the associated Riccati equations, for all $\varepsilon \in [0,1]$.

Let us show that $V^\varepsilon \to V^0$ as $\varepsilon \to 0$. Appealing again to regular dependence of solutions of ordinary differential equation with respect to parameters, we obtain, as in subsection 7.3, that P^ε is continuous with respect to ε, so that $P^\varepsilon \to P^0$ as $\varepsilon \to 0$. Similarly, $\phi^\varepsilon \to \phi^0$ as $\varepsilon \to 0$. In the sequel we shall assume that $2\rho > \sigma^2$, so that $2\rho - \varepsilon\sigma^2 > 0$ for all $\varepsilon \in [0,1]$. The same argument used in the above proposition shows that $|P^\varepsilon(\cdot)| \leq \gamma$. Let us write, with obvious meaning of the symbols,

$$\dot{\eta} = -q^\varepsilon(t)\eta + r^\varepsilon(t)P,$$

where $\eta(t) = P(T-t)$, $\eta(0) = 0$. Then we have

$$\eta(t) = e^{q^\varepsilon(t)} \int_0^t e^{-q^\varepsilon(s)} r^\varepsilon(s)\, ds,$$

where

$$|q^\varepsilon(t)| \leq \rho + \max_{\varepsilon\in[0,1]} |1-\varepsilon\sigma\delta| \frac{\gamma}{e^{-\beta T}} := C_1,$$

and

$$|r^\varepsilon(t)| \leq \gamma\Big(\sigma^2 k + \rho k + \sigma\delta k \max_{\varepsilon\in[0,1]} |1-\varepsilon\sigma\delta| \frac{\gamma}{e^{-\beta T}}\Big) := C_2.$$

Then

$$|\phi(\cdot)| \leq C_2 T,$$

and one can take the limit under the integral sign by the dominated convergence theorem in V^ε as $\varepsilon \to 0$, obtaining the desired result.

An analysis of the sensitivity of the value function with respect to ε could be carried out by "intersecting" results of subsections 7.3 and 8.2.

The sensitivity of the value function with respect to the initial distance from the target level of goodwill is easily obtained by (40):

$$\frac{\partial V(s,y)}{\partial y} = 2P(s)y + 2\phi(s).$$

One can also repeat the same arguments of the previous subsection to show that the sensitivity of X with respect to the initial level of goodwill is given by

$$X_x(t) = \exp\Big(-\int_0^t (a(s) + \frac{1}{2}c(s)^2)\, ds + \int_0^t c(s)\, dW(s)\Big),$$

and, on average,

$$\frac{\partial \mathbb{E}[X(t;x)]}{\partial x} = e^{-\int_0^t a(s)\, ds}.$$

9 Minimum energy control

We study the problem of determining what is the minimum expected quadratic expenditure to meet a goal "on average" at time T, i.e. we solve the constrained LQ problem

$$\inf_{u\in\mathcal{K}} \mathbb{E}\Big[\int_0^T e^{\beta t} u(t)^2\Big],$$

where $\mathcal{K} \subset \mathcal{U}$ is the class of controls such that $\mathbb{E}[(X(T)-k)^2] = 0$, and X follows the controlled dynamics

$$\begin{cases} dX(t) &= (-\rho X(t) + u(t))\,dt + \sigma\,dW(t) \\ X(0) &= x, \end{cases}$$

where we shall assume $x = 0$ for simplicity.

It is intuitively clear that an admissible control that realizes the quadratic hedge always exists. This is proved by the following simple argument. Consider a feedback control of the form

$$u(t) = mX(t) + a,$$

with m, $a \in \mathbb{R}_+$, and let us write the corresponding closed-loop equation

$$\begin{cases} dX(t) &= -rX(t)\,dt + a\,dt + \sigma\,dW(t) \\ X(0) &= 0, \end{cases}$$

where $r := m - \rho$. Then we have

$$X(T) = e^{-rT} aT + e^{-rT}\sigma \int_0^T e^{rt}\,dW(t),$$

hence

$$\mathbb{E}[X(T)] = e^{-rT} aT$$

and

$$\mathbb{E}[X(T)^2] = e^{-2rT} a^2 T^2 + \frac{1}{2r}(1 - e^{-2rT}).$$

Condition $\mathbb{E}[(X^u(T) - k)^2] = 0$ can therefore be written as

$$(e^{-2rT}T^2)a^2 - (2ke^{-rT}T)a + \frac{1}{2r}(1 - e^{-2rT}) + k^2 = 0.$$

Solving this quadratic equation for a we get the two solutions

$$a_\pm = \frac{ke^{-rT}T \pm \sqrt{k^2 e^{-2rT}T^2 - (e^{-2rT}T^2)(\frac{1}{2r}(1-e^{-2rT}) + k^2)}}{e^{-2rT}T^2}.$$

Choosing any $m \geq \rho$, both a_- and a_+ are real, so that the desired control is given by $u(t) = mX(t) + a_+$.

In order to find the control with minimum discounted energy that realizes the hedge, we first transform the constrained stochastic control problem into an unconstrained one by the introduction of a Lagrange multiplier. In particular, let us introduce the distance from the target $Y(t) = k - X(t)$ and a Lagrange multiplier λ. Then we look for the solution of the unconstrained problem

$$\inf_{u\in\mathcal{U}} \mathbb{E}\Big[\lambda Y(T)^2 + \int_0^T e^{\beta t} u(t)^2\, dt\Big],$$

subject to

$$\begin{cases} dY(t) &= (\mu - \rho Y(t) - u(t))\, dt + \sigma\, dW(t) \\ Y(0) &= k. \end{cases}$$

Using the results of the previous section we know that a unique optimal control u_*^λ exists for any λ and is given by

$$u_*^\lambda(t) = e^{\beta t} P^\lambda(t) Y(t) + e^{\beta t}\phi^\lambda(t),$$

where P^λ and ϕ^λ are solutions of the associated Riccati equations. The optimal trajectory solves the following SDE

$$dY(t) = (\mu^\lambda(t) - c^\lambda(t)Y(t))\, dt + \sigma\, dW(t),$$

with

$$\begin{aligned} \mu^\lambda(t) &:= \mu - e^{\beta t}\phi^\lambda(t), \\ c^\lambda(t) &:= \rho + e^{\beta t} P^\lambda(t). \end{aligned}$$

We look now for a λ such that

$$\mathbb{E}[Y^2(T)] = 0.$$

Let us write then

$$e^{\int_0^t c^\lambda(s)\, ds}\Big(dY(t) + c^\lambda(t)Y(t)\, dt\Big) = e^{\int_0^t c^\lambda(s)\, ds}\Big(\mu^\lambda(t)\, dt + \sigma\, dW(t)\Big),$$

hence

$$e^{\int_0^T c^\lambda(s)\, ds} Y(T) = Y(0) + \int_0^T e^{\int_0^t c^\lambda(s)\, ds}\mu^\lambda(t)\, dt + \sigma \int_0^T e^{\int_0^t c^\lambda(s)\, ds}\, dW(t).$$

Squaring both sides, taking expectation, using Itô's isometry and recalling that stochastic integrals are martingales, hence have zero expectation, we obtain that $\mathbb{E}[Y(T)^2] = 0$ is equivalent to the following equation in λ:

$$\begin{aligned} k^2 + \Big[\int_0^T e^{\int_0^t c^\lambda(s)\, ds}\mu^\lambda(t)\, dt\Big]^2 + \sigma^2 \int_0^T e^{2\int_0^t c^\lambda(s)\, ds}\, dt \\ + 2k \int_0^T e^{\int_0^t c^\lambda(s)\, ds}\mu^\lambda(t)\, dt = 0. \end{aligned}$$

This equation is highly nonlinear, and one cannot expect to find an explicit solution for λ. However, it is clear that a solution exists since controls of the form $u(t) = mX(t) + a$ are admissible, and among such controls one can find at least one which realizes the hedge. The issue of finding such λ by numerical techniques will be addressed elsewhere.

10 Quadratic utilities and partial information

Assume that the goodwill X evolves according to the controlled Ornstein-Uhlenbeck dynamics

$$\begin{cases} dX(t) &= (-\rho X(t) + u(t))\,dt + \sigma\,dW(t) \\ X(0) &= x, \end{cases}$$

and that we cannot observe X directly, but that instead we have only a noisy observation of it, i.e. we can observe a process Z defined by

$$\begin{cases} dZ(t) &= hX(t)\,dt + g\,dW_0(t) \\ Z(0) &= 0, \end{cases} \tag{43}$$

with h and g constants, and W_0 a Brownian motion independent of W. This assumption is quite realistic, as it is difficult to "measure" the image of a product, while a firm may have access to marketing data that are good proxies for the goodwill. Equation (43) describes the relationship between the unobserved goodwill X and its noisy proxy Z.
Then we shall be interested in the following stochastic control problem with partial observation: to minimize the functional

$$J(x;u) = \mathbb{E}^x\left[-\gamma X(T)^2 + \int_0^T e^{-\beta t} u(t)^2\,dt\right]$$

among all square-integrable controls u that are adapted to the filtration generated by the observation process Z. Given the special structure of the problem, we can appeal to the so called separation principle (see, e.g. Davis [8] or Bensoussan [3]), according to which the LQ problem with noisy observation reduces to linear filtering and deterministic control on the filtered dynamics. In particular, the optimal control is given by

$$u_*(t) = -e^{\beta t} P(t)\hat{X}(t), \tag{44}$$

where P is again a solution of the Riccati equation

$$\dot{P} = 2\rho P + e^{\beta t} P^2, \quad P(T) = -\gamma,$$

which admits the explicit solution (18), and $\hat{X}$ is given by the Kalman filter

$$\begin{cases} d\hat{X}(t) &= \Big(-\rho - S(t)\frac{h^2}{g^2}\Big)\hat{X}(t)\,dt + u_*(t)\,dt + S(t)\frac{h}{g^2}\,dZ(t) \\ \hat{X}(0) &= \mathbb{E}[X(0)]. \end{cases} \tag{45}$$

Recall that it holds $\hat{X}(t) = \mathbb{E}[X(t)|\mathcal{F}_t^Z]$, where $\mathcal{F}_t^Z$ is the σ-algebra generated by $(Z(s))_{s\in[0,t]}$. Here S is the solution of the (filtering) Riccati equation

$$\dot{S} = -2\rho S - \frac{h^2}{g^2}S^2 + \sigma^2, \quad S(0) = \mathrm{Var}(X(0)),$$

which admits the explicit solution

$$S(t) = \frac{\alpha_1 - C\alpha_2 \exp((\alpha_2 - \alpha_1)g^{-2}h^2t)}{1 - C\exp((\alpha_2 - \alpha_1)g^{-2}h^2t)}, \tag{46}$$

where

$$\begin{aligned}
\alpha_1 &= \frac{-\rho g^2 - g(\rho^2 g^2 + h^2\sigma^2)^{1/2}}{h^2}, \\
\alpha_2 &= \frac{-\rho g^2 + g(\rho^2 g^2 + h^2\sigma^2)^{1/2}}{h^2}, \\
C &= \frac{\mathrm{Var}(X(0)) - \alpha_1}{\mathrm{Var}(X(0)) - \alpha_2},
\end{aligned}$$

and $\mathrm{Var}(A)$ denotes the variance of the random variable A. Substituting the expression for the optimal control (44) into the SDE for the Kalman filter (45), and defining

$$H(t) = -\rho - S(t)\frac{h^2}{g^2} - e^{\beta t}P(t),$$

we obtain the following expression for the optimal goodwill estimate

$$\hat{X}(t) = e^{\int_0^t H(s)\,ds}\hat{X}(0) + \frac{h}{g^2}\int_0^t e^{\int_s^t H(r)\,dr}S(s)dZ(s).$$

Similarly we can solve the hedging type of problem in the case of partial observation. Let Y the distance from the target k, as before, following the SDE

$$dY(t) = (-\rho Y(t) - u(t) + \rho k)\,dt + \sigma\,dW(t).$$

Then we suppose we can observe a process Z which is related to the goodwill X through

$$dZ(t) = hX(t)\,dt + g\,dW_0(t),$$

or equivalently

$$dZ(t) = (-hY(t) + hk)\,dt + g\,dW_0(t).$$

So we can write

$$u_*(t) = e^{\beta t}(P(t)\hat{Y}(t) + \phi(t)),$$

where P and ϕ are solutions of the Riccati equations

$$\begin{aligned}
\dot{P}(t) &= 2\rho P(t) + e^{\beta t} P^2(t), \quad P(T) = \gamma, \\
\dot{\phi} &= (\rho + e^{\beta t} P)\phi - \rho k P, \quad \phi(T) = 0,
\end{aligned}$$

given explicitly by (29)-(30), and $\hat{Y}$ is solution of the SDE (see Liptser and Shiryaev [18])

$$d\hat{Y}(t) = (-\rho - \frac{h^2}{g^2} S(t))\hat{Y}(t)\,dt - u_*(t)\,dt + (\rho - h)k\,dt - \frac{h}{g^2} S(t)\,dZ(t),$$

where S solves the Riccati equation

$$\dot{S} = -2\rho S - \frac{h^2}{g^2} S^2 + \sigma^2, \quad S(0) = \mathrm{Var}(X(0)),$$

whose solution is given by (46).

11 Quadratic hedging with optimal stopping

Problems of mixed optimal stopping and control have recently attracted attention in works of applied probability, see for instance Karatzas and Wang [16] for applications to portfolio optimization, Duckworth and Zervos [10], [11] and Zervos [27] for problems of investment decisions with strategic entry and exit, and Karatzas, Ocone, Wang and Zervos [15] for a singular control problem with finite fuel. For the theory, see, e.g., Krylov [17], Bensoussan and Lions [4], Øksendal and Sulem [22], and Morimoto [19]. One of the first works addressing the issue of finding explicit results was Beneš [2].

In this section we find an explicit representation for the optimal control and the optimal stopping strategy for the case of minimizing an objective function that is the sum of the quadratic distance of the goodwill from a target at the (discretionary) launch time τ and of the cumulative quadratic cost until τ, assuming that the goodwill dynamics is of Ornstein-Uhlenbeck type. For simplicity we also assume $\beta = 0$, i.e. we consider the case without discounting. In order to discourage long waiting before launching the product, we also introduce an extra term in the objective function depending on the time of launching.

So let Y be the distance from a desired target k. We shall find the solution to the problem

$$\inf_{u\in\mathcal{U},\tau\in\mathcal{S}} \mathbb{E}^x\left[Y^2(\tau) + \gamma_1 \int_0^\tau u^2(t)\,dt + \gamma_2\tau\right] =: V(x), \tag{47}$$

where Y is such that

$$dY(t) = (\mu - \rho Y(t) + u(t))\,dt + dW(t),$$

with $\mu := \rho k$ and we have assumed, without loss of generality (in the setting of constant b), $\sigma = 1$. Let $\mathbb{F}$ be the filtration generated by W. Then $\mathcal{U}$ is the space of $\mathbb{F}$-adapted square integrable control processes, and $\mathcal{S}$ is the set of all $\mathbb{F}$-stopping times.

The quasi-variational inequality associated to the mixed problem of optimal control and optimal stopping (47) can be written as

$$\min_x \Big(x^2 - V(x), \min_u (L^u V + \gamma_1 u^2 + \gamma_2) \Big) = 0,$$

where L^u is the generator of the controlled diffusion Y, i.e. L^u is the differential operator defined by

$$L^u f(x) = \frac{1}{2} f''(x) + (\mu - \rho x - u) f'(x).$$

We guess a continuation region D of the type $D = \{x : x \geq x_0\}$, where one must have

$$\min_u (L^u V + \gamma_1 u^2 + \gamma_2) = 0.$$

We have

$$\min_u (L^u V + \gamma_1 u^2 + \gamma_2) = AV - \frac{1}{4\gamma_1} V_x^2 + \gamma_2,$$

where A is the generator of the uncontrolled diffusion, i.e.

$$Af(x) = \frac{1}{2} f''(x) + (\mu - \rho x) f'(x).$$

Then we get

$$AV - \frac{1}{4\gamma_1} V_x^2 + \gamma_2 = 0, \quad x \geq x_0$$

In order to linearize this ODE, we apply the Hopf-Cole transformation $U(x) = e^{\frac{1}{2\gamma_1} V(x)}$, obtaining

$$\frac{1}{2} U_{xx} + (\mu - \rho x) U_x - \frac{\gamma_2}{2\gamma_1} U = 0, \quad x \geq x_0$$

In order to obtain solutions that are ordinary functions, we restrict ourselves to the special case $\frac{\gamma_2}{2\gamma_1} = \rho$. However, one can solve the linear equation for U without this assumption, obtaining solutions that can be expressed in terms of special functions. Two linearly independent solutions are

$$U_1(x) = e^{-2\mu x + \rho x^2}, \quad U_2(x) = e^{\rho(x - \mu/\rho)^2} \int_x^\infty e^{-\rho(s - \mu/\rho)^2}\, ds$$

Then the general solution can be written as $U = \alpha_1 U_1 + \alpha_2 U_2$, with α_1 and α_2 arbitrary real constants.

Guided by the observation that U_1 is unbounded in the continuation region, we set $\alpha_1 = 0$, and impose C^1 fit of $U(x) = \alpha_2 U_2(x)$ to the Hopf-Cole transformation of x^2 at the point x_0, that is

$$\begin{aligned} U(x_0) &= e^{-\frac{1}{2\gamma_1}x_0^2} \\ U'(x_0) &= -\frac{x_0}{\gamma_1}e^{-\frac{1}{2\gamma_1}x_0^2}. \end{aligned}$$

We solve now the following system of equations for the unknowns α_2 and x_0:

$$\begin{aligned} \alpha_2 e^{\rho(x_0-\mu/\rho)^2}\int_x^\infty e^{-\rho(s-\mu/\rho)^2}\,ds &= e^{-\frac{1}{2\gamma_1}x_0^2} \\ \alpha_2\Big[2\rho(x_0-\frac{\mu}{\rho})e^{\rho(x_0-\mu/\rho)^2}\int_{x_0}^\infty e^{-\rho(s-\mu/\rho)^2}\,ds - 1\Big] &= -\frac{x_0}{\gamma_1}e^{-\frac{1}{2\gamma_1}x_0^2}. \end{aligned}$$

Therefore x_0 is given by the solution of the following equation

$$((2\rho+\frac{1}{\gamma_1})x - 2\mu)e^{\rho(x-\mu/\rho)^2}\int_x^\infty e^{-\rho(s-\mu/\rho)^2}\,ds = 1.$$

In fact the solution, if it exists, is unique, because the left hand side, as a function of x, is increasing. Now one can also find α_2 in terms of x_0, so that we have a candidate continuation region (or equivalently an optimal stopping region), and a candidate value function.

We need to show that $AV(x) - \frac{1}{4\gamma_1}(V'(x))^2 + \gamma_2 \geq 0$ in the region $x \leq x_0$, for $V(x) = x^2$. That is, we want to show that

$$(2\rho+\frac{1}{\gamma_1})x^2 - 2\mu x - (1+\gamma_2) \leq 0. \tag{48}$$

The expression on the left hand side takes its maximum either at $x = 0$ or at $x = x_0$. Therefore, if $x_{max} = 0$, condition (48) is trivially verified. Otherwise, if $x_{max} = x_0$, we have

$$\frac{1}{(2\rho+\gamma_1^{-1})x_0 - 2\mu} = U_2(x_0) \geq \frac{1}{\rho y_0 + \sqrt{\rho^2 y_0^2 + 2\rho}},$$

where $y_0 := x_0 - \frac{\mu}{\rho}$, and the inequality follows by standard estimates on the error function. Therefore we get

$$(2\rho+\gamma_1^{-1})x_0 - 2\mu \leq \rho y_0 + \sqrt{\rho^2 y_0^2 + 2\rho}.$$

After some algebraic manipulations, one finds

$$(2\rho+\frac{1}{\gamma_1})x_0^2 - 2\mu\frac{1}{\gamma_1}x_0 \leq 2\rho\gamma_1 = \gamma_2.$$

Therefore, (48) is verified if, e.g., $\gamma_1 > 1$. We shall assume that in the following, but note that this condition can be weakened.

Finally, we need to prove that $V(x) \geq x^2$ in the continuation region $x \geq x_0$, or equivalently that $U(x) \geq e^{-\frac{1}{2\gamma_1}x^2}$ for $x \geq x_0$. In order to prove this, let us consider their ratio $f(x) = U(x)e^{\frac{1}{2\gamma_1}x^2}$ and prove that it is increasing. One has

$$f'(x) = \alpha_2 e^{\frac{1}{2\gamma_1}x^2}\Big(((2\rho + \frac{1}{\gamma_1})x - 2\mu)U_2(x) - 1\Big),$$

and since we have that $((2\rho + \frac{1}{\gamma_1})x - 2\mu)U_2(x)$ is increasing in x and $((2\rho + \frac{1}{\gamma_1})x_0 - 2\mu)U_2(x) = 1$, it follows $((2\rho + \frac{1}{\gamma_1})x - 2\mu)U_2(x) > 1$ for $x \geq x_0$. Therefore we have verified all conditions for optimality, and we summarize our findings in the following proposition.

Proposition 15 *The optimal control policy u_* and optimal stopping time τ_* for the problem (47), with $\gamma_1 > 1$ and $\frac{\gamma_2}{2\gamma_1} = \rho$, are given by*

$$u_*(Y(t)) = \arg\min_u (L^u V(Y(t)) + \gamma_1 u^2 + \gamma_2) = \frac{V'(Y(t))}{2\gamma_1}$$

and

$$\tau_* = \inf\{t \geq 0 : \, Y(t) \leq x_0\}.$$

12 Further problems

In the same spirit of the hedging type of problem addressed above, it would be interesting to compute the strategy that maximizes the "probability of perfect hedge", i.e. to solve the problem

$$\sup_{u \in \mathcal{U}} \mathbb{P}(X^u(T) \geq k). \tag{49}$$

In particular, one could consider $\mathcal{U}$ as the set of controls satisfying an integral constraint of the type $\int_0^T e^{-\beta t} u(t)\, dt \leq M$. Obviously problem (49) is equivalent, $\mathbb{I}$ being the indicator function, to

$$\sup_u \mathbb{E}\Big[\mathbb{I}(X(T) > k)\Big],$$

which suggests that we could also consider other performance criteria of the type

$$\sup_u \mathbb{E}[\ell(X(T) - k)],$$

with ℓ a general loss function. In the special case of a quadratic loss function, a possible approach would be to use a constrained version of the linear quadratic regulator, in analogy with the Lagrange multiplier method that we adopted in section 5.

The optimal control problems studied in this paper are limited to the case of "smooth" disturbances, that is, the driving noise process has continuous paths. It is meaningful to relax this assumption and consider also jump components in the noise, to take into account possible shocks to the image of the advertised product, due, for instance, to bad news on the product itself or similar ones, or to the introduction of superior technologies.

One could also try to study different type of controls, namely impulse controls, or even combinations of classical and impulse controls. This is particularly meaningful for our problems, since impulse controls correspond to the so-called "pulsing advertising" policies that have been studied in the management and marketing literature (see [12] and references therein).

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