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Probability Density Functions based Weights for Ordered Weighted Averaging (OWA) Operators: An Example of Water Quality Indices

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ABSTRACT

This paper explores the application of ordered weighted averaging (OWA) operators to develop water quality index, which incorporates an attitudinal dimension in the aggregation process. The major thrust behind selecting the OWA operator for aggregation of multi-criteria decision-making is its capability to encompass a range of operators bounded between *minimum* and *maximum*. A new approach for generating OWA weight distributions using probability density functions (PDFs) is proposed in this paper. The basic parameters (mean and standard deviation) of the probability density functions can be determined using the number of criteria (e.g. water quality indicators) in the aggregation process.

The proposed approach is demonstrated using data provided in a study by Swamee and Tyagi (2000) for establishing water quality indices. The Normal distribution and its inverse form were found suitable for compromising or normative decisions, whereas the Exponential and its inverse form were found suitable for pro-risk and risk-averse decisions, respectively. The proposed OWA weight distributions are also compared with the commonly used regular increasing monotone (RIM) functions for generating OWA weights. Sensitivity analyses are carried out to highlight the utility of the proposed approach for multi-criteria decision-making and establishing water quality indices.

Keywords: OWA operators, fuzzy, probability density function, water quality index, degree of orness, and dispersion.

LIST OF NOTATION

$a_1, a_2, ..., a_n$	Multi-criteria vector
b_j	j^{th} largest element in the vector ($a_1, a_2, ..., a_n$)
BOD_5	Biochemical oxygen demand
$Disp(w)$	Dispersion
DO	Dissolved oxygen
E	Exponential distribution
FN-IOWA	Fuzzy number IOWA
GIOWA	Generalized IOWA operator
GOWA	Generalized OWA operator
i,j	Counters
I	Index (water quality)
IE	Inverse Exponential distribution
IN	Inverse Normal distribution
IOWA	Induced OWA
LOWA	Linguistic OWA
MCDM	Multi criteria decision-making
ME-OWA	*Maximizing* entropy OWA
n	Number of criteria or attributes
N	Normal distribution
OWA	Ordered weighted averaging
PDF	Probability density function
P-OWA	Probabilistic OWA
$Q(r)$	*Linguistic quantifier* as a fuzzy subset
$Q_*(r)$	Linguistic quantifiers "for all"
$Q^*(r)$	Linguistic quantifiers "there exists"
RDM	Regular decreasing monotone
RIM	Regular increasing monotone
RUM	Regular unimodal
s_i	Sub-indices
SOWA	"orlike" S-OWA-OR operator / "andlike" S-OWA-AND operator

UDS	Uniform deceasing sub-indices
UOWA	Uncertain OWA operator
US	Unimodal sub-indices
$\mathrm{w} = (w_1, w_2, \cdots, w_n)^T$	OWA weights vectors
w_j	OWA weights
w_i^N	Normalized Normal OWA weight distribution
$w_i'^N$	Un normalized Normal OWA weight distribution
w_i^{IN}	Normalized Inverse Normal OWA weight distribution
$w_i'^{IN}$	Un normalized Inverse Normal OWA weight distribution
w_i^E	Normalized Exponential OWA weight distribution
$w_i'^E$	Un normalized Exponential OWA weight distribution
w_i^{IE}	Normalized Inverse Exponential OWA weight distribution
$w_i'^{IE}$	Un normalized Inverse Exponential OWA weight distribution
WQI	Water quality index (indices)
WOWA	Weighted OWA
x	Continuous random variable
α	Degree of *orness*
β	Degree of a polynomial function for the RIM functions
δ	Change
λ	Fractiles or quantiles
μ	Mean
μ_n	Mean of the collection $1,2,...,n$
σ	Standard deviation
σ_n	Standard deviation of the collection $1,2,...,n$

1. INTRODUCTION

Environmental indices for water, air and sediments are common communication tools used by regulatory agencies to describe the overall quality of the environmental system. The environmental indices are evaluated from the individual estimates of environmental indicators or criteria. Water quality index (WQI) is evaluated using data collected through routine sampling, which are mandatory in a regulatory framework. The WQI is useful in establishing background levels of water quality for a given aquatic system for implementing regulatory policies and evaluating decision actions planned for the improvement and the rehabilitation of an aquatic system (Silvert 2000). In the interpretation of WQI, an attitudinal tolerance may vary with respect to the intended use of water like drinking, swimming, and fishing etc. Therefore, water that is "excellent" for swimming might be of "poor" quality for drinking.

Water quality index is a risk communication tool used to describe the status of water by translating a large amount of non-commensurate data into a single value (Ott 1978). A significant amount of literature is available on the evaluation and management of water systems using WQI. For this purpose, physical, chemical, and biological water quality indicators (sub-indices) are aggregated in a 'meaningful' way using various statistical and mathematical techniques (Ott 1978). These aggregation approaches generally include logical operators (e.g., minimum, maximum), averaging operators (e.g., arithmetic average, weighted average, geometric mean, weighted product), and many others operators (e.g., simple addition, root sum power, root sum-square, and multiplicative forms) (Somlikova and Wachowiak 2001;Silvert 2000; Sinha *et al.* 1994; Ott 1978).

Swamee and Tyagi (2000) have discussed advantages and shortcomings of different aggregation techniques available for the evaluation of WQI. In the aggregation process, recognition of two potential pitfalls, namely *exaggeration* and *eclipsing*, is important. Exaggeration occurs when all water quality indicators individually posses lower value (meaning in acceptable range), yet the WQI comes out *unacceptably* high. Eclipsing is the reverse phenomenon, where one or more of the water quality indicators are of relatively high value (meaning in an unacceptable range), yet the estimated WQI comes out as *unacceptably* low. These phenomena are typically affected by the method of aggregation, therefore the challenge is to determine the best aggregation method that will simultaneously reduce both *exaggeration* and *eclipsing*.

From a regulatory compliance viewpoint, threshold level of contaminant (or a water quality indicator) concentration in the drinking water is established in the context of possible adverse human (ecological) health impacts. For this reason it is extremely useful to relate WQI to some sort of 'acceptability' measure for drinking water, which can be interpreted as the membership of a fuzzy set. Silvert (2000) argued that "the concept of 'acceptability' is itself seen by some as fuzzy, in the colloquial rather than mathematical sense, but this reflects the reality that we can measure environmental effects far more accurately than we can evaluate their significance." A lack of consensus may exist on the definition of 'acceptability' and its related objectivity; therefore it may be more realistic to convert water quality indicators into a fuzzy membership, which represents the degree of acceptability of those water quality indicators in the set of acceptable conditions. Recently, a large number of applications in developing environmental indices has been reported in the literature where advanced assessment methods such as fuzzy synthetic evaluation are being employed. Simple fuzzy classification, fuzzy similarity method and fuzzy comprehensive assessment, are all subsets of fuzzy synthetic evaluation, which have been recently used in various environmental applications (Sadiq et al. 2006; Sadiq and Rodriguez 2004; Lu and Lo 2002;Chang *et al.* 2001; Lu *et al.* 1999; Tao and Xinmiao 1998).

The aggregation of fuzzy sets requires operations by which several fuzzy numbers are combined in a desirable way to produce a single fuzzy number (Klir and Yuan 1995). The literature reflects numerous ways and operators to aggregate fuzzy sets, e.g., *intersection*, *minimum*, *product* (also known as fuzzy *t-norms*) and *union*, *maximum*, *summation* (also known as *s-norms*). Other common operators for aggregation are *arithmetic*, *geometric* and *harmonic means*. In addition, there is a class of *generalized mean* operator developed by Yager (1988) called *ordered weighted averaging* (OWA) operators. The OWA provides flexible aggregation operation ranging between the *minimum* and the *maximum* operators. Detailed discussions on the selection of appropriate aggregation operators can be found in Klir and Yuan (1995), and Smolikova and Wachowiak (2001). There is an increasing reported applications of OWA operators in the disciplines of civil and environmental engineering (Makropoulos and Butler 2006, 2005, 2004; Smith 2006, 2002; Makropoulos *et al.* 2003).

This paper explores the potential use of OWA operators to develop WQI. The motive behind selecting the OWA for aggregation of sub-indices is its capability to encompass a range of operators from *minimum* to *maximum* including various averaging aggregation operations like arithmetic mean (see Figure 1). The OWA weight generation provides a flexibility to incorporate decision maker's attitude or tolerance, which can be related to an intended use of water. The OWA operation procedure involves three steps – (1) reordering of the input arguments (sub-indices), (2) determining the weights associated with the OWA operators, and (3) aggregation process. The concept of OWA is explored in various disciplines of engineering and artificial intelligence and many nuances and extensions have been proposed. In this paper, the basic concept of Xu (2005) to generate OWA weights using the normal probability density function (PDF) is extended to more probability distributions. To explain our approach we used a data set of raw water quality (Table 1) provided by Swamee and Tyagi (2000).

Swamee and Tyagi (2000) data set comprises nine water quality indicators (sub-indices) including BOD_5, fecal coliform, dissolved oxygen (DO) proportion with respect to saturation, nitrates, pH, phosphates, temperature, total solids, and turbidity. Table 1 summarizes the results of this study. As the units of various water quality indicators (sub-indices) are non-commensurate, transformation functions are used to translate the actual values into an interval of [0, 1], where "0" corresponds to worst value and "1" corresponds to the best value. Transformation functions including uniform deceasing sub-indices (UDS), unimodal sub-indices (US), and non-uniformly decreasing sub-indices (NDS) are proposed by Swamee and Tyagi (2000) for various water quality indicators. For example for DO, 'higher' proportion means a 'higher' value of sub-index (i.e. a *benefit criterion*). But contrarily for fecal coliform, 'higher' concentration refers to a 'lower' value of sub-index and vice versa (i.e. a *cost criterion*). Therefore, an appropriate transformation function is required for each water quality indicator to translate or map actual values over an interval [0, 1] before the implementation of aggregation process.

Water quality indices were determined for a data set comprised of nine water quality indicators using eight aggregation methods. The formulations of these methods are also provided in Table 1. The values of WQI ranged from 0.12 to 0.866 using different methods. Interestingly, the index proposed by Swamee and Tyagi (2000) is very sensitive, and may predict value lower than the *minimum* value of sub-index (0.125). Interested readers can refer to Swamee and Tyagi (2000) for detailed discussion on their approach.

This paper is organized in five sections. Section 2 explains the basic concept and its nuances of OWA operators proposed by Yager (1988). Section 3 discusses the formulation of probabilistic OWA and regular increasing monotone (RIM) functions. Section 4 provides a discussion on the use of probability density functions for generating OWA weights and results of sensitivity analyses. The final section provides a summary and concludes the paper.

2. ORDERED WEIGHTED AVERAGE (OWA) OPERATOR

2.1 Basic concepts

Most multi-criteria decision analysis problems require neither strict "*anding*" (minimum) nor strict "*oring*" of the *s*-norm (maximum). For mutually exclusive and independent probabilities in the fault tree analysis, these two extremes correspond to multiplication (*and*-gate) and summation (*or*-gate). To generalize this idea, Yager (1988) introduced a new family of aggregation techniques called the ordered weighted average (OWA) operators, which form general mean type aggregators. The OWA operator provides flexibility to utilize the range of "*anding*" or "*oring*" to include the attitude of a decision maker in the aggregation process.

An OWA operator of dimension n is a mapping of $R^n \rightarrow R$ (where R = [0, 1]), which has an associated n number of criteria $\mathrm{w} = (w_1, w_2, \cdots, w_n)^T$, where $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$. Hence, for a given n criteria (sub-indices) vector ($a_1, a_2, ..., a_n$), the OWA aggregation is performed as follows

$$OWA(a_1, a_2, ..., a_n) = \sum_{j=1}^{n} w_j b_j \qquad (1)$$

where b_j is the j^{th} largest element in the vector ($a_1, a_2, ..., a_n$), and $b_1 \geq b_2 \geq ... \geq b_n$. Therefore, the weights w_j of OWA are not associated with any particular value a_j, rather they are associated with the ordinal position of b_j. The linear form of OWA equation aggregates multiple criteria vector ($a_1, a_2, ..., a_n$) and provides a nonlinear solution (Yager and Filev 1999; Filev and Yager 1998).

The OWA operator is bounded between *minimum* (a conjunctive operator or a *t*-norm) and *maximum* (a disjunctive operator or an *s*-norm) operators (Figure 1). The range between these two extremes can be expressed through the concept of *orness* (α) function, which is defined by Yager (1988) as follows

$$\alpha = \frac{1}{n-1} \sum_{i=1}^{n} w_i (n - i), \qquad \text{and} \qquad \alpha \in [0, 1] \qquad (2)$$

The *orness* function characterizes the degree to which the aggregation is like an *or* operation. An α = 0, corresponds to a scenario where OWA vector w becomes (0, 0,…, 1), i.e., the element with the *minimum* value in the multiple criteria vector ($a_1, a_2, ..., a_n$) gets the complete weight, which implies that the OWA becomes a *minimum* operator. Similarly, α = 1, corresponds to a scenario where OWA vector w becomes (1, 0,…, 0), i.e., an element with a *maximum* value in the multiple criteria vector ($a_1, a_2, ..., a_n$) is assigned complete weight, which implies that the OWA becomes *maximum* operator. Further, if all elements in the multiple criteria vector ($a_1, a_2, ..., a_n$) are assigned equal weights (arithmetic average), i.e., w = (1/*n*, 1/*n*,…, 1/*n*), the *orness* becomes α = 0.5. With the exception of the two extreme cases of α = 0 or 1, an infinite number OWA weight distributions are possible for any α value. For example, α = 0.5 does not guarantee that weights are uniformly distributed (i.e., $w_i = 1/n$), rather it hints that weights are distributed symmetrically on both sides of the median ordinal position. Therefore, for any symmetric probability density function (PDF) either uniform or normal, α will be 0.5.

To differentiate weight distribution at a given α, a 'measure' called dispersion $Disp(w)$ was introduced by Yager (1988). The concept of dispersion is similar to Shanon's entropy, and can be computed by:

$$Disp(w) = -\sum_{i=1}^{n} w_i \ln(w_i) \tag{3}$$

where $0 \le Disp(w) \le \ln(n)$. The measure $Disp(w)$ provides a degree to which the information in the arguments is used. Therefore, when α = 0 or 1 (i.e., $w_i = 1$), for *minimum* or *maximum* operators respectively, the dispersion is "zero" and when $w_i = 1/n$ (a uniform distribution), the dispersion is maximum, i.e., "ln(*n*)".

2.2 Determination of OWA weights

One of the major challenges in OWA method is to generate weights. Since the introduction of OWA operators by Yager (1988), different methods of OWA weight generation and extension of OWA operators have been proposed in the literature. A comprehensive, though not exhaustive list of OWA extensions has been provided in Table 2. Xu (2005) and Filev and Yager (1998) have carried a treatise on the generation for OWA weights. A few of these methods are discussed in the following paragraphs. Further, motivated by the work of Xu (2005), a new method of generating OWA weights using different, well know probability distributions is also discussed.

Yager (1988) proposed generation of OWA weight using a *linguistic quantifier* as a fuzzy subset $Q(r)$ in the unit interval [0, 1]. The classic binary logic allows the representation of two linguistic quantifiers "there exists" $Q^{*}(r)$ (or), and "for all" $Q_{*}(r)$ (and) (Figure 1). But in natural language many additional quantifiers such as "many", most", "few", "almost all", "very few" are possible. Zadeh (1983) suggested two types of quantifiers – the first related to number of elements and the second related to the proportion of the elements. Yager (1996) further distinguished the relative quantifiers into three classes; regular increasing monotone (RIM), regular decreasing monotone (RDM) and regular unimodal (RUM) quantifier. A discussion on RUM and RIM functions is provided in Section 3.

O'Hagan (1988) developed a procedure to generate the OWA weights for given degree of *orness* α, *maximizing* the entropy (ME-OWA) (Equation 3). Yager (1993) expressed the measure of entropy as $1-\max(w_i)$ and generated weights by *minimizing* $\max_i(w_i)$ for predefined level of *orness* α. Yager and Filev (1994) further introduced different OWA weight generation methods based on given levels of *orness* α. Filev and Yager (1998) developed procedures based on the "exponential smoothing" to generate the OWA weights. Yager and Filev (1999) suggested an algorithm to obtain the OWA weights from a collection of samples with the relevant aggregated data. Fullér and Majlender (2001) used the method of Lagrange multipliers to solve O'Hagan's procedure analytically. Xu and Da (2002) established a linear objective-programming model to obtain the weights of the OWA operator under partial weight information. Recently, Xu (2005) proposed OWA weight generation method by equating weight distribution to normal probability density function, which is similar to RUM function.

2.3 Nuances and extensions of OWA operators

The OWA operators are weighted sums of the ordered elements, bounded between *minimum* (the biggest *t*-norm) and *maximum* (the smallest *s*-norm) operators. Yager and Filev (1994) extended the concept of OWA weights generation and aggregation technique (S-OWA-AND, S-OWA-OR, NOWA). Herrera *et al.* (1996) and Bordogna *et al.* (1997) introduced the concept of aggregation of linguistic values in OWA, called LOWA operator. Herrera *et al.* (1996) used the convex method to aggregate the linguistic values, whereas, Bordogna *et al.* (1997) proposed the use of MAX-MIN composition operator in OWA aggregation.

Traditionally in OWA, input criteria (sub-indices) are assumed to be equally important and OWA weights are assigned based on the ordinal position. To deal with the criteria of varying importance in the aggregation process, Tora (1997) introduced the concept of weighted OWA (WOWA) operators, which initially assigns significance weights to the input values and then OWA aggregation is performed in a regular way. Yager and Filev (1999) introduced induced OWA (IOWA) operator, which unlike the OWA operator, allows ordering by an inducing parameter that is associated with the input values. The utility of the inducing parameter is only for ordering, and not in the aggregation process. Xu and Da (2002) later proposed a generalized IOWA operator. Schaefer and Mitchell (1999) proposed a generalized OWA operator. Xu and Da (2002) introduced uncertain OWA operator (UOWA) for interval-valued numbers. Chen and Chen (2003, 2005) introduced the concept of fuzzy input values, called fuzzy number IOWA (FN-IOWA). Table 2 provides a summary of some of these important studies in chronological order.

3. PROPOSED APPROACH FOR GENERATING OWA WEIGHTS

The weights distribution of OWA operator can synonymously be viewed as probability density function (PDF), because OWA weights satisfy the basic axioms of probability, i.e., $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$. Due to this resemblance, probability distributions can be used to generate OWA weights. The PDFs are continuous functions, in which the area under the curve is “1”. The OWA weights are discrete and similar to probability mass function, in which the sum of the weights (probabilities) is equal to “1”. Therefore, PDFs can be discretized to accommodate OWA weights. The advantages of using PDFs for generating OWA weights include: (1) simplicity, (2) familiarity with the properties of statistical distributions, and (3) availability of diverse shapes of PDFs to obtain desired degree of *orness* and *dispersion.*

3.1 OWA weight generation using Normal distribution

Xu (2005) proposed the Normal probability density function to generate OWA weights. The normal distribution is one of the best known and most widely used two-parameter distributions. It is also called Gaussian or Laplacian distribution. According to the Central Limit Theorem, a sum of independent random variables always converges towards normality, regardless of the distribution of the individual random variables. The Normal distribution is a unimodal symmetric function about its mean μ with a standard deviation σ. The PDF of normal distribution for a continuous random variable x is defined as:

$$f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-[(x-\mu)^2/2\sigma^2]}, \qquad -\infty < x < +\infty \tag{4}$$

For n number of sub-indices (water quality indicators or criteria) to be aggregated, the OWA weights can be computed as:

$$w_i'^N=\frac{1}{\sigma_n\sqrt{2\pi}}e^{-[(i-\mu_n)^2/2\sigma_n^2]}, \qquad i=1,2,...,n \tag{5}$$

where μ_n and σ_n (> 0) are the mean and standard deviation of the collection $1,2,...,n$, respectively. The mean μ_n and standard deviation σ_n can be computed by:

$$\mu_n=\frac{1}{n}\frac{n(1+n)}{2}=\lambda\,(1+n) \tag{6}$$

$$\sigma_n=\sqrt{\frac{1}{n}\sum_{i=1}^{n}(i-\mu_n)^2} \tag{7}$$

where parameter λ can be referred to as fractile or quantile representing the location of the maximum weight, which is assigned to the median ordinal position (50th percentile) for a symmetric unimodal function (a traditional normal PDF). For example, if there are nine sub-indices to be aggregated (i.e., n = 9) using OWA, the modal location will be at the 5th criterion (in descending order value) according to Equation 6. This concept can be generalized for other fractiles or quantiles (λ). Therefore, if λ < 0.5, a positively skewed distribution (leaning towards left) can be generated using Equations (6) and (7). Similarly, if λ > 0.5, a negatively skewed distribution (leaning towards right) can be generated using Equations (6) and (7). Parameter λ therefore defines the ordinal position to which a decision-maker intends to assign the maximum value of OWA weight.

As mentioned earlier, PDFs need to be discretized to generate OWA weight distribution. Therefore, normalization of Equation (5) is required to obtain OWA weight vector $w=(w_1,w_2,...,w_n)^T$:

$$w_i^N=\frac{w_i'^N}{\sum_{j=1}^{n}w_i'^N}, \qquad i=1,2,...,n \tag{8}$$

The OWA weight distribution w_i^N can be generated using Equations 6-8 and are plotted in Figure 2 for parameter λ = {0.1, 0.3, 0.5, 0.7, 0.9}. As mentioned earlier, λ = 0.5 is a special case of Xu (2005) where OWA weights are symmetrical about the median ordinal position. These weight distributions represent OWA weights generated for n = 9. The area under the curve has no specific meaning; rather the heights corresponding to ordinal positions (i) represent OWA weights (w_i). Therefore, the sum of these heights is equal to 1 i.e., $\sum_{i=1}^{n} w_i = 1$, not the area under the curve.

The Normal PDF with λ = 0.5 provides compromising OWA weight distribution, i.e., sub-indices close to minimum and maximum ordinal positions get the lower values of OWA weights. In case of positively skewed distribution (i.e., $\lambda \in [0, 0.5]$), OWA weight distribution approaches towards RDM function, whereas in case of negatively skewed distribution (i.e., $\lambda \in [0.5, 1]$), OWA weight distribution becomes similar to RIM function. Table 3 provides a summary of *orness* and *dispersion* values for five fractiles λ = {0.1, 0.3, 0.5, 0.7, 0.9} using Normal distribution function. The corresponding *orness* and *dispersion* are {0.627, 0.618, 0.5, 0.382, 0.373}, {2.112, 2.099, 2.119, 2.099, 2.112}, respectively. The degree of *orness,* which refers to *optimistic* to *pessimistic* attitude of a decision-maker, decreases with the increase in fractile values. The values of *dispersion* measures are consistently close to maximum value of ln(9) ≈ 2.2 for all cases. These high *dispersion* values hint that OWA weights generated through Normal PDF use information from "most" of the sub-indices.

After generating OWA weights, the water quality index can be obtained using Equation (1). The sub-indices values used in the analysis are the same as provided in Column 4 of Table 1. For example, for λ = 0.1, the following analysis is performed to obtain water quality index I:

Step 1: Reordering of the input arguments (sub-indices, s_i):

a = (0.125, 0.423, 0.622, 0.233, 0.866, 0.251, 0.396, 0.168, 0.269)

After reordering (in descending order)

b = (0.866, 0.622, 0.423, 0.396, 0.269, 0.251, 0.233, 0. 168, 0.125)

Step 2: Determining the weights associated with the OWA operators:

Using normal distribution for λ = 0.1 and n = 9

w = (0.166, 0.162, 0.152, 0.136, 0.117, 0.096, 0.075, 0.056, 0.04)

Step 3: Aggregating using Equation (1):

$$I^{N}_{0.1} = 0.166 \times 0.866 + 0.423 \times 0.622 + \ldots + 0.04 \times 0.125 = 0.45$$

where "N" refers to OWA weight distribution using normal PDF and "0.1" refers to fractiles λ. Similarly, water quality indices for other fractiles can be obtained and summary of results are provided in Table 3.

3.2 OWA weight generation using inverse form of normal distribution

The RUM functions (e.g., normal distribution) are convex, which assign maximum OWA weight to a predefined ordinal position as discussed above. Contrarily, decision-maker may want to assign minimum value to predefined ordinal position. This can be achieved, using the inverse form of normal distribution function. For example, the normal distribution at $\lambda = 0.5$ assigns maximum weight to the median position, while the inverse form of normal distribution assigns minimum value to the median ordinal position. This reflects a situation, where the decision maker ignores the values at predefined ordinal position and gives more weight to sub-indices at the extremes. The OWA weights for the inverse form of normal distribution function can be generated from Equation (8) as follows:

$$w'^{IN}_{i} = \mathrm{MAX} - \left(w_i^N - \mathrm{MIN}\right) \qquad i = 1,2,\ldots,n \tag{9a}$$

and

$$w^{IN}_{i} = \frac{w'^{IN}_{i}}{\sum_{j=1}^{n}\left(w'^{IN}_{j}\right)} \tag{9b}$$

where MAX and MIN are the *maximum* and *minimum* values of OWA weights w_i^N generated using Equation (8), i.e., MAX($w_1, w_2 \ldots, w_n$), MIN($w_1, w_2 \ldots, w_n$), respectively. For example, the following steps are used to generate OWA weights using the inverse form of normal PDF for $\lambda = 0.1$

Step 1: Generate OWA using normal PDF (w_i^N) for given n (= 9) and predefined λ (= 0.1):

w = (0.166, 0.162, 0.152, 0.136, 0.117, 0.096, 0.075, 0.056, 0.04)

Step 2: Determine MAX and MIN values:

MAX = MAX(0.166, 0.162, 0.152, 0.136, 0.117, 0.096, 0.075, 0.056, 0.04) = 0.166

MIN = MIN(0.166, 0.162, 0.152, 0.136, 0.117, 0.096, 0.075, 0.056, 0.04) = 0.04

Step 3) Generate OWA weights for inverse form of normal PDF using Equation (9):

w_1^{IN} = [0.166 – (0.166 – 0.04)]/0.854 = 0.047

w_2^{IN} = [0.166 – (0.162 – 0.04)]/0.854 = 0.051

$\vdots$

w_9^{IN} = [0.166 – (0.04 – 0.04)]/0.854 = 0.194

where denominator 0.854 is the factor $\sum_{j=1}^{n}\left(w_j'^{IN}\right)$ in Equation (9).

The OWA weight distribution can be generated using Equations 6-9. These OWA weight distributions are plotted in Figure 3 for fractiles λ = {0.1, 0.3, 0.5, 0.7, 0.9}. Table 3 provides a summary of *orness* and *dispersion* for these five fractiles using Inverse form of normal PDF. The corresponding *orness* and *dispersion* are {0.352, 0.336, 0.5, 0.664, 0.648}, {2.088, 2.039, 2.117, 2.039, 2.088}, respectively. The degree of *orness* increases with the increase in fractile values, i.e., moving from *pessimistic* to *optimistic* attitude of a decision-maker. There is no noticeable variation in the dispersion measures. Water quality indices are calculated using the same procedure as explained earlier for normal PDF generated OWA weights. The variation in WQI ranges from 0.282 to 0.471 for pessimistic to optimistic attitude, respectively. It is worth noting that neither normal nor Inverse normal PDF captures the extreme cases of "OR" and "And" scenarios, i.e. for degree of *orness* α approaches to either 1 or 0, respectively. The dispersion *Disp* approximately equals to the maximum possible value of ln(9) = 2.2 in all cases. These high values refer to a case of maximum use of available information of the sub-indices.

3.3 OWA weight generation using exponential distribution

The exponential distribution is a memory-less continuous distribution. The exponential distribution is often used to model the time between random arrivals of events that occur at a constant average rate. The Exponential distribution is defined by a simple parameter μ, which is the mean time between failures. The PDF of exponential distribution for a variable x is defined as:

$$f(x) = \frac{1}{\mu} e^{-x/\mu}, \qquad (x > 0) \tag{10}$$

To generate OWA weights $w_i'^E$ using the PDF of exponential distribution, Equation (10) can be re-written as following:

$$w_i'^E = \frac{1}{\mu_n} e^{-i/\mu_n}, \qquad i = 1,2,...,n \tag{11}$$

As mentioned earlier, PDFs need to be discretized to generate OWA weight distribution. Therefore, normalization of Equation (11) is required to obtain OWA weight vector $w = (w_1, w_2, ..., w_n)^T$:

$$w_i^E = \frac{w_i'^E}{\sum_{j=1}^{n} w_j'^E}, \qquad i = 1,2,...,n \tag{12}$$

The exponential PDFs are extreme positive skewed distribution, i.e., similar to RDM function. Figure 4 shows the OWA weights distribution for fractiles λ = {0.1, 0.3, 0.5, 0.7, 0.9}. The increase in the value of a fractile reduces the steepness of weight distribution and a flat curve is achieved. This phenomenon can also be observed by comparing the degree of *orness* α at varying λ. For λ = 0.1 the α = 0.927 (a strictly Or-type operator), whereas for λ = 0.9, the α equals to 0.591 (a compromising operator). This phenomenon can be further elaborated through *Disp* measure. For λ = 0.1, the *Disp* = 1.04 (a little information is used), whereas for λ = 0.9, the α value equals to 2.16 (high entropy entails maximum information). The degree of *orness* α for exponential PDF for OWA weight generation is bounded by an interval [0.5, 1]. It implies that OWA weights generated through an exponential distribution provide Or-type solutions only. After generating OWA weights using exponential distribution PDF, the water quality index can be obtained using Equation (1). The results are also summarized in Table 3.

3.4 OWA weight generation using inverse type of exponential distribution

To generate OWA weights using inverse type of exponential distribution, the following equation is used:

$$w_i'^{IE} = \frac{1}{\mu_n} e^{i/\mu_n}, \qquad i = 1,2,...,n \tag{13}$$

Therefore, to obtain OWA weight vector $w = (w_1, w_2, ..., w_n)^T$, normalization of Equation (13) can be performed as follows

$$w_i^{IE} = \frac{w_i'^{IE}}{\sum_{j=1}^{n} w_j'^{IE}}, \qquad i = 1,2,\ldots,n \tag{14}$$

The inverse type of exponential distribution is highly negatively skewed distribution, i.e., similar to RIM function. Figure 5 shows the OWA weights distribution for fractiles λ = {0.1, 0.3, 0.5, 0.7, 0.9}. Increase in the value of a fractile reduces the steepness of weight distribution and a flat curve is achieved for λ = 1. The degree of *orness* α for inverse type of exponential PDF for OWA weight distribution is bounded by an interval [0, 0.5]. It implies that OWA weights generated through inverse exponential PDF provide And-type solutions. After generating OWA weights using inverse exponential distribution PDF, the water quality index can be obtained using Equation (1). The results are also summarized in Table 3.

3.5 OWA weight generation using RIM functions

A class of function to generate OWA weights, called regularly increasing monotone (RIM) quantifier was first proposed by Yager (1988). The RIM functions are bounded by two linguistic quantifiers "there exists" $Q^*(r)$ (OR) and "for all", $Q_*(r)$ (AND) as described earlier. Thus, for any RIM quantifier $Q(r)$, the limit $Q_*(r) \le Q(r) \le Q^*(r)$ holds true (Yager and Filev 1994). The OWA weights can be generated for a given RIM quantifier $Q(r)$ as follows

$$w_i = Q\left(\frac{i}{n}\right) - Q\left(\frac{i-1}{n}\right) \qquad i = 1,2,\ldots,n \tag{15}$$

Yager (1988, 1996) defined a parameterized class of fuzzy subsets, which provide families of RIM quantifiers that change continuously between $Q_*(r)$ and $Q^*(r)$:

$$Q(r) = r^{\beta} \qquad r \ge 0 \tag{16}$$

(1) For β =1; $Q(r) = r$ (a linear function) called the *unitor* quantifier

(2) For $\beta \to \infty$; $Q_*(r)$, the universal quantifier

(3) For $\beta \to 0$; $Q^*(r)$, the existential quantifier

Therefore Equation (15) can be generalized as

$$w_i = \left(\frac{i}{n}\right)^{\beta} - \left(\frac{i-1}{n}\right)^{\beta}, \qquad i = 1,2,\ldots,n \tag{17}$$

where is β is a degree of a polynomial function. Figure 6 demonstrates the variation in the OWA weight distributions with respect to various degrees of polynomial (β). For $\beta = 1$, the RIM function becomes a uniform distribution, i.e., weight distribution becomes similar to an arithmetic mean, i.e., $w_i = 1/n$. For $\beta > 1$, the RIM function leans towards right, i.e., "And-type" operators manifesting negatively skewed OWA weight distributions. Similarly, for $\beta < 1$, the RIM function leans towards left (it becomes RDM), i.e., "Or-type" operators manifesting positively skewed OWA weight distributions.

4. DISCUSSION

The normalization process used for the inverse form of the normal distribution assures that the maximum value of OWA weights generated through the normal distribution will be assigned the minimum value not "zero". Due to this normalization process, these two weight distributions are not exactly the mirror image of each other with respect to a straight line drawn at minimum weight parallel to *x*-axis. The method proposed to derive OWA weight distribution using inverse form of exponential PDF is rather different. The OWA weight distribution using inverse form of Exponential PDF is completely independent from exponential OWA weight distribution. These two weight distributions are mirror image of each other if a vertical line is drawn at the median ordinal position. The authors acknowledge that similar methodology can be opted for inverse normal distribution OWA weights. The motive here is to show various alternatives to generate OWA weights through PDFs. The OWA weights generated through various methods discussed in this paper comply with basic properties of

1) *idempotency*, i.e., $w_i + w_i = w_i$,

2) *commutativity*, i.e., $w_i + w_j = w_j + w_i$ and

3) *associativity*, i.e., $w_i + (w_j + w_k) = (w_i + w_j) + w_k$.

The details of these properties can be found in Yager (1988).

Normal distribution and its inverse form provide OWA weight distributions with a limited variation in the degree of *orness*, which ranges in the interval [0.3, 0.7]. Exponential distribution and its inverse form provide OWA weight distributions in the intervals [0.5, 1] and [0, 0.5], respectively.

Figure 7(A) compares the degree of *orness* of these four PDFs. Figure 7(B) provides a comparison of *Dispersion* measures for these OWA weight distributions. A limited variation in *Dispersion* measures can be observed in the case of normal and its inverse form due to thick tails. This implies that more information is being used, i.e., all sub-indices contribute to the final index value. Figure 7(C) compares the WQI variations at various fractiles for four PDFs using raw water quality data provided by Swamee and Tyagi (2000). Very limited variations in WQI are observed for various fractiles in case of normal and its inverse form. On the other hand, in case of exponential distribution and its inverse form, both provide more variations in WQI values due to larger interval of degree of *orness*. The variations in the values of WQIs (Figure 7C) and corresponding *orness* (Figure 7A) follow a similar trend due to their linear relationship.

In case of RIM functions, increasing the power of polynomial (β) from 0.1 to 9, varies the WQI values from *maximum* (Or-type) to *minimum* (And-type) type operators. Figure 8 shows this variation, in which arithmetic mean value of WQI corresponds to $\beta = 1$. Figure 9 compares domain of degree of *orness* for 4 PDFs as discussed above and various RIM functions. Figure 9 can also be interpreted by comparing with Figure 1 as described earlier. The normal and its Inverse can be used for generating OWA weights for compromising decisions, in which the whole spectrum of information is important. Contrarily, exponential and its inverse can be used to generate OWA weight distribution representing risk-averse and pro-risk attitudes of the decision-maker, respectively. In the water quality context, normal and its inverse distributions can be used where decision-makers think that for overall water quality “some” of the attributes (sub-indices or indicators) have to be complied with. Similarly, exponential and its inverse distributions can be used where decision-makers think that to describe overall water quality either “few” or “most” of the attributes (sub-indices) have to be complied with, respectively.

Table 4 provides results of sensitivity analyses for three scenarios for WQI evaluation. In the first scenario, a sub-index (s_1) has the best possible value (i.e., 1) and all remaining sub-indices ($s_2, s_3, \ldots, s_9$) have the worst possible value (i.e., 0). To evaluate the change (δ) in WQI (I) for any given method the following relationship is used

$$\delta(\%) = \frac{\left(I - I^{AM}\right)}{I^{AM}} 100 \qquad (18)$$

where I^{AM} is a value of WQI using arithmetic mean operator. A positive value of δ suggests that the degree of *orness* is more than 0.5, while a negative value of δ suggests that the degree of *orness* is less than 0.5. Therefore higher positive δ value account for those cases when results are "exaggerated" and higher negative values of WQI results refer to "eclipsed" cases. Similar interpretation can be made for second scenario, when a sub-index (s_1) has the worst possible value (i.e., 0) and all remaining sub-indices (s_2, s_3, …, s_9) have the best possible value (i.e., 1). A third scenario highlights an interesting point in which a constant value of water quality index is achieved regardless of the change in the type of functions used at any given fractile levels (in case of PDFs) or power of polynomial (in case of RIM functions). This confirms that OWA weights generated through different methods are *idempotent* in nature.

5. SUMMARY AND CONCLUSION

In our daily life and in many engineering problems, we are often confronted with the issue of aggregating various non-commensurate attributes to make a certain decision. Multi criteria decision-making (MCDM) is not simply a problem of mathematics rather it is a problem of judgment (Halpern and Fagin 1992). Therefore, it deals not only with objective uncertainty (related to random phenomenon), but also with the subjective (related to decision-maker) uncertainty. Various mathematical and statistical techniques are available in the literature, which deals with complex problem of decision-making under various types of uncertainties.

Most MCDM problems neither require strict conjunctive or disjunctive logic for the aggregation of multiple criteria. Yager (1988) introduced a new family of aggregation techniques called the ordered weighted average (OWA) operators. The OWA operator provides a flexibility to utilize the degree of *orness* as a surrogate representing the attitude of a decision maker in the aggregation process. This paper proposes a new approach for generating OWA weight distributions using various types of commonly used probability density functions. An example of water quality index is used to explain the proposed approach. The specific conclusions of this paper are:

- Probability density functions (normal, exponential and their inverse forms) can be used to generate OWA weight distributions.
- Selection of OWA weight distributions using PDFs of normal and its inverse form represents normative or compromising attitude of a decision maker. The degree of *orness* α for both cases varies approximately in an interval [0.3, 0.7].

- Higher values of *dispersion* measure in case of normal and its inverse form imply that most of the information is being incorporated in the aggregation.
- Selection of OWA weight distributions using PDFs of exponential and its inverse form represents optimistic (Or-type) and pessimistic (And-type) attitudes of the decision maker, respectively. The degree of *orness* α for these cases varies in an intervals [0.5, 1] and [0, 0.5], respectively.
- Higher variations in *dispersion* measure for exponential and its inverse form imply that the level of information used depends on the fractiles level chosen by the decision-maker.
- For $\beta = 1$, the RIM function becomes a uniform distribution, i.e., an arithmetic average operator. For $\beta > 1$, the RIM function become "And-type" operators. Similarly, for $\beta < 1$, the RIM function becomes RDM, i.e., "Or-type" operators.
- The method of OWA weight generation can be expanded to other PDFs due to its simplicity and familiarity with the statistical properties.

In the water quality context, normal and its inverse forms can be used for OWA weight generation where the intended use of water requires that "some" of the attributes (sub-indices or water quality indicators) have to comply with regulatory limits. Similarly, exponential and its inverse distributions can be used where intended use of water requires that either "few" or "most" of the attributes (sub-indices or water quality indicators) have to comply with regulatory limits, respectively.

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Table 1 Comparison of various water quality indices to evaluate raw water quality (reproduced from Swamee and Tyagi 2000)

Water Quality indicators (1)	Observed values (q) (2)	*Transformation function (3)	Sub-index (s_i) (4)	§Weight (w_i) (5)	Aggregation methods for developing WQI (6)	WQI (I) (7)
BOD_5 (mg/L)	20	UDS (3, 20)	0.125	0.10	1. Swamee and Tyagi (2000)	0.120
Fecal coliforms (MPN/100 mL)	66	UDS (0.3, 4)	0.423	0.15	2. Weighted arithmetic mean	0.409
Dissolved oxygen (proportion)	0.6	US (1, 3, 1, 0)	0.622	0.17	3. Arithmetic mean	0.373
Nitrates (mg/L)	25	UDS (3, 40)	0.233	0.10	4. Weighted product	0.427
pH	7.8	US (7, 4, 6, 0)	0.866	0.12	5. Geometric mean	0.396
Phosphates (mg/L)	2	UDS (1, 0.67)	0.251	0.10	6. Minimum	0.125
Temperature (^{o}C)	32	US (20, 0.5, 7, 0)	0.396	0.10	7. Maximum	0.866
Total solids (mg/L)	1,000	US (75, 1, 1, 0.8)	0.168	0.08	8. Squared root of harmonic mean	0.235
Turbidity (JTU)	70	UDS (1.5, 50)	0.269	0.08		

* Sub-indices (s_i) in column (4) are obtained by corresponding transformation function. Two types of transformation functions are used

$$UDS(m, q_c) = \left(1 + \frac{q}{q_c}\right)^{-m}$$; UDS: uniform decreasing sub-indices

$$US(q^*, n, p, r) = \frac{pr + (n+p)(1-r)\left(\frac{q}{q^*}\right)}{p + n(1-r)\left(\frac{q}{q^*}\right)}$$; US: unimodal sub-indices

§ these are significance (importance) weights, not the OWA weights

1. $I^{ST} = \left(1 - N + \sum_{i=1}^{N} s_i^{-1/k}\right)^{-k}$ where $k = 0.4$

2. $I^{WAM} = \sum_{i=1}^{N} w_i s_i$

3. $I^{AM} = \frac{1}{N}\sum_{i=1}^{N} s_i$

4. $I^{WP} = \prod_{i=1}^{N} s_i^{w_i}$

5. $I^{GM} = \left(\prod_{i=1}^{N} s_i\right)^{1/N}$

6. $I^{Min} = \min(s_1, s_2, \ldots, s_N)$

7. $I^{Max} = \max(s_1, s_2, \ldots, s_N)$

8. $I^{Sq-HM} = \left(\frac{1}{N}\sum_{i=1}^{N} s_i^{-2}\right)^{-0.5}$

Table 2 A summary of some important studies in the development of OWA operators

Reference	Aggregation algorithm	Type of OWA	Weight generation algorithm
Yager (1988)	$OWA(a_1, a_2,..., a_n) = \sum_{j=1}^{n} w_j b_j$	OWA operator	$w_i = Q\left(\frac{i}{n}\right) - Q\left(\frac{i-1}{n}\right),\ i = 1,2,\dots,n$
Yager and Filev (1994)	$SOWA(a_1, a_2,..., a_n) = \alpha \underset{j}{Max}(a_j) + \frac{1}{n}(1-\alpha)\sum_{j=1}^{n} x_j$	"orlike" S-OWA-OR operator	$w_i = \begin{cases} \frac{1}{n}(1-\alpha)+\alpha, & i=1; \\ \frac{1}{n}(1-\alpha), & i=2,\dots,n; \end{cases} ,\alpha \in [0,1]$
	$SOWA(a_1, a_2,..., a_n) = \beta \underset{j}{Min}(a_j) + \frac{1}{n}(1-\beta)\sum_{j=1}^{n} x_j$	"andlike" S-OWA-AND operator	$w_i = \begin{cases} \frac{1}{n}(1-\beta)+\beta, & i \neq 1; \\ \frac{1}{n}(1-\beta), & i=n; \end{cases} ,\beta \in [0,1]$
Herrera *et al.* (1996)	$LOWA(a_1, a_2,..., a_n) = w \cdot B^T = \phi^n\{w_k, b_k, k = 1,2...,n\}$	Linguistic OWA operator	The weights w_j are generated using linguistic quantifiers.
Torra (1997)	$WOWA(a_1, a_2,..., a_n) = \sum_{j=1}^{n} w_j a_{\sigma(j)}$	Weighted OWA operator	$w_i = w^*\left(\sum_{j \le i} p_{\sigma(j)}\right) - w^*\left(\sum_{j<i} p_{\sigma(j)}\right)$
Bordogna *et al.* (1997)	$LOWA(a_1, a_2,..., a_n) = \underset{j}{\max} - \min\{w_j, b_j\}$	Linguistic OWA operator	The weights w_j are generated using linguistic quantifiers.
Yager and Filev (1999)	$IOWA(\langle u_1, a_1\rangle,...,\langle u_n, a_n\rangle) = \sum_{j=1}^{n} w_j b_j$	Induced OWA operator	The weights w_j can be generated by any of the methods discussed in section 2.2. u_1 is an order inducing parameter determined by the user.
Schaefer and Mitchell (1999)	$GOWA(a_1, a_2,..., a_n) = W^T PA$	Generalized OWA operator	The weights w_j can be generated by any of the methods discussed in section 2.2. P is a permutation matrix and $A = (a_1, a_2,..., a_n)$.
Xu and Da (2002)	$UOWA(\tilde{a}_1, \tilde{a}_2,..., \tilde{a}_n) = \sum_{j=1}^{n} w_j \tilde{b}_j$	Uncertain OWA operator	The weights w_j are generated by linear objective-programming model.
Xu and Da (2003)	$GIOWA(\langle v_1, u_1, a_1\rangle,...,\langle v_n, u_n, a_n\rangle) = \sum_{j=1}^{n} w_j b_j$	Generalized IOWA operator	The weights w_j can be generated by any of the methods discussed in section 2.2.
Chen and Chen (2003, 2005)	$FN\text{-}IOWA(\langle u_1, \tilde{a}_1\rangle,...,\langle u_n, \tilde{a}_n\rangle) = \sum_{j=1}^{n} \tilde{w}_j \tilde{b}_j$	Fuzzy number Induced OWA operator	Fuzzy weights are associated with linguistic levels, which vary from absolutely unimportant (AU) to absolutely important (AI). The weight $\tilde{w}_j$ and sub index $\tilde{b}_j$ are fuzzy numbers.

Table 3 Comparison of OWA weight distributions using PDFs to evaluate raw water quality

PDF	Fractile λ	OWA weights w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8	w_9	*Orness* α	*Dispersion Disp*	WQI *I*
Normal	0.1	0.166	0.162	0.152	0.136	0.117	0.096	0.075	0.056	0.040	0.627	2.112	0.450
	0.3	0.134	0.154	0.162	0.154	0.134	0.106	0.076	0.050	0.030	0.618	2.099	0.434
	0.5	0.051	0.085	0.124	0.156	0.168	0.156	0.124	0.085	0.051	0.500	2.119	0.345
	0.7	0.030	0.050	0.076	0.106	0.134	0.154	0.162	0.154	0.134	0.382	2.099	0.286
	0.9	0.040	0.056	0.075	0.096	0.117	0.136	0.152	0.162	0.166	0.373	2.112	0.288
Inverse Normal	0.1	0.047	0.051	0.064	0.082	0.105	0.129	0.153	0.175	0.194	0.352	2.088	0.282
	0.3	0.080	0.052	0.041	0.052	0.080	0.118	0.159	0.196	0.223	0.336	2.039	0.288
	0.5	0.174	0.138	0.097	0.065	0.052	0.065	0.097	0.138	0.174	0.500	2.117	0.401
	0.7	0.223	0.196	0.159	0.118	0.080	0.052	0.041	0.052	0.080	0.664	2.039	0.492
	0.9	0.194	0.175	0.153	0.129	0.105	0.082	0.064	0.051	0.047	0.648	2.088	0.471
Exponential	0.1	0.632	0.233	0.086	0.031	0.012	0.004	0.002	0.001	0.000	0.927	1.039	0.745
	0.3	0.298	0.214	0.153	0.110	0.079	0.056	0.040	0.029	0.021	0.743	1.895	0.552
	0.5	0.217	0.178	0.146	0.119	0.098	0.080	0.065	0.054	0.044	0.658	2.074	0.483
	0.7	0.184	0.159	0.138	0.120	0.104	0.090	0.078	0.068	0.059	0.616	2.132	0.452
	0.9	0.166	0.149	0.133	0.119	0.107	0.095	0.085	0.076	0.068	0.591	2.157	0.434
Inverse Exponential	0.1	0.000	0.001	0.002	0.004	0.012	0.031	0.086	0.233	0.632	0.073	1.039	0.152
	0.3	0.021	0.029	0.040	0.056	0.079	0.110	0.153	0.214	0.298	0.257	1.895	0.233
	0.5	0.044	0.054	0.065	0.080	0.098	0.119	0.146	0.178	0.217	0.342	2.074	0.278
	0.7	0.059	0.068	0.078	0.090	0.104	0.120	0.138	0.159	0.184	0.384	2.132	0.302
	0.9	0.068	0.076	0.085	0.095	0.107	0.119	0.133	0.149	0.166	0.409	2.157	0.316
Minimum		0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	1.000	0.000	0.000	0.125
Maximum		1.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	1.000	0.000	0.866
Arithmetic mean		0.111	0.111	0.111	0.111	0.111	0.111	0.111	0.111	0.111	0.500	2.197	0.357

Table 4 Sensitivity analyses of OWA operators (n = 9)

Scenario 1		$s_1 = 1$			$s_2, s_3,\ldots,s_9 = 0$			$I^{AM} = 0.111$; $I^{min} = 0$; $I^{max} = 1$			
	Normal		Inverse normal		Exponential		Inverse exponential			RIM	
λ	I^N	δ (%)	I^{IN}	δ (%)	I^E	δ (%)	I^{IE}	δ (%)	β	I^R	δ (%)
0.1	0.166	49	0.047	-58	0.632	469	0.000	-100	0.1	0.783	605
0.3	0.134	21	0.080	-28	0.298	168	0.021	-81	0.3	0.481	333
0.5	0.051	-55	0.174	56	0.217	95	0.044	-61	1	0.111	0
0.7	0.030	-73	0.223	101	0.184	66	0.059	-47	3	0.001	-99
0.9	0.040	-64	0.194	74	0.166	50	0.068	-38	9	0.000	-100

Scenario 2		$s_1 = 0$			$s_2, s_3,\ldots,s_9 = 1$			$I^{AM} = 0.889$; $I^{min} = 0$; $I^{max} = 1$			
	Normal		Inverse normal		Exponential		Inverse exponential			RIM	
λ	I^N	δ (%)	I^{IN}	δ (%)	I^E	δ (%)	I^{IE}	δ (%)	β	I^R	δ (%)
0.1	0.960	64	0.806	-74	1.000	100	0.368	-469	0.1	0.987	88
0.3	0.970	73	0.777	-101	0.979	81	0.702	-168	0.3	0.961	65
0.5	0.949	55	0.826	-56	0.956	61	0.783	-95	1	0.889	0
0.7	0.866	-21	0.920	28	0.941	47	0.816	-66	3	0.702	-168
0.9	0.834	-49	0.953	58	0.932	38	0.834	-50	9	0.346	-488

Scenario 3		$s_1, s_2,\ldots,s_9 = 0.5$			$I^{AM} = I^{min} = I^{max} = 0.5$						
	Normal		Inverse normal		Exponential		Inverse exponential			RIM	
λ	I^N	δ (%)	I^{IN}	δ (%)	I^E	δ (%)	I^{IE}	δ (%)	β	I^R	δ (%)
For all	0.500	0	0.500	0	0.500	0	0.500	0	For all	0.500	0

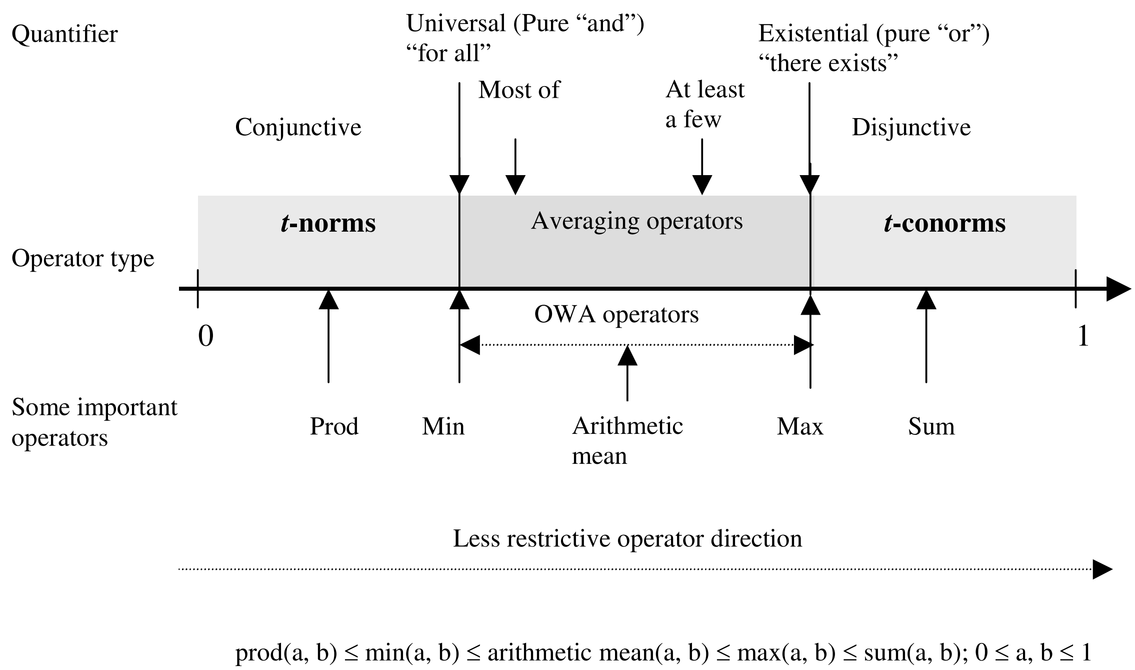


Fig. 1. Common aggregation operators (after Larsen, 2002)

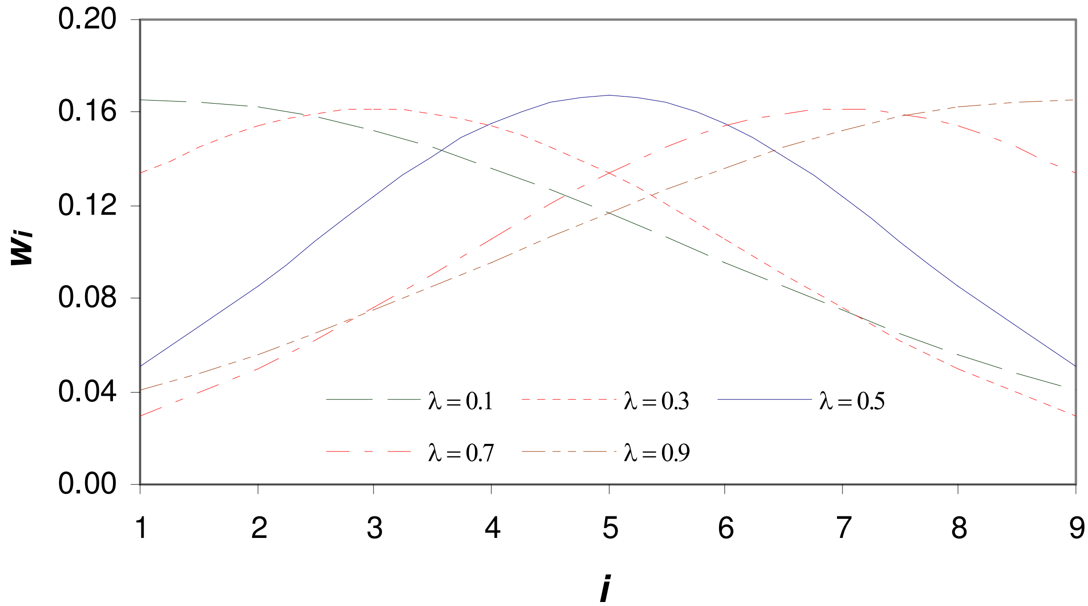


Fig. 2. OWA weights generated using normal distribution (n = 9) for fractiles λ = {0.1, 0.3, 0.5, 0.7, 0.9}

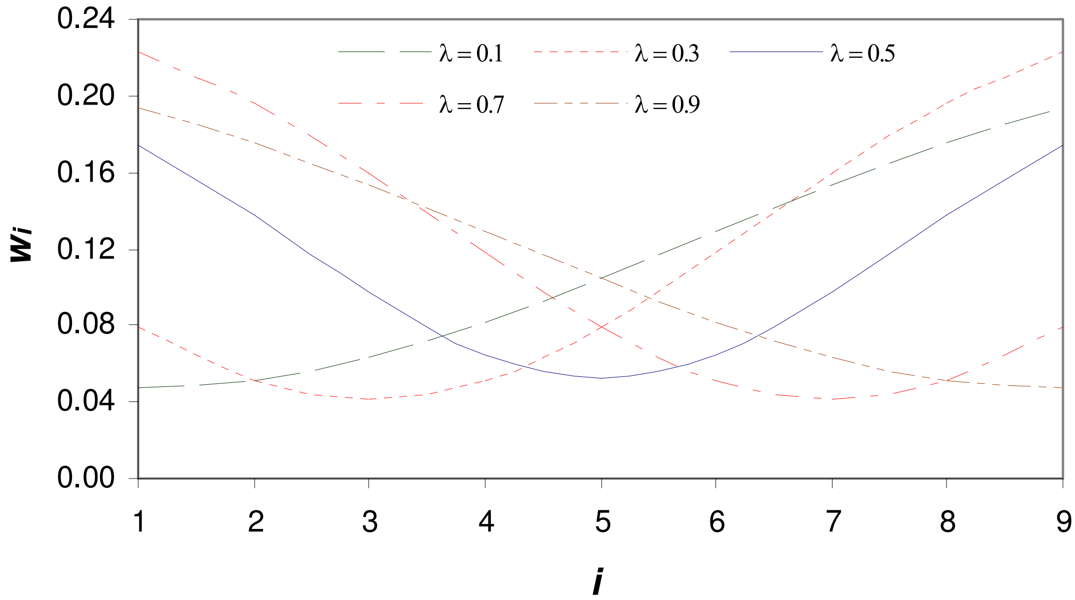


Fig. 3. OWA weights generated using inverse form of normal distribution (n = 9) for fractiles λ = {0.1, 0.3, 0.5, 0.7, 0.9}

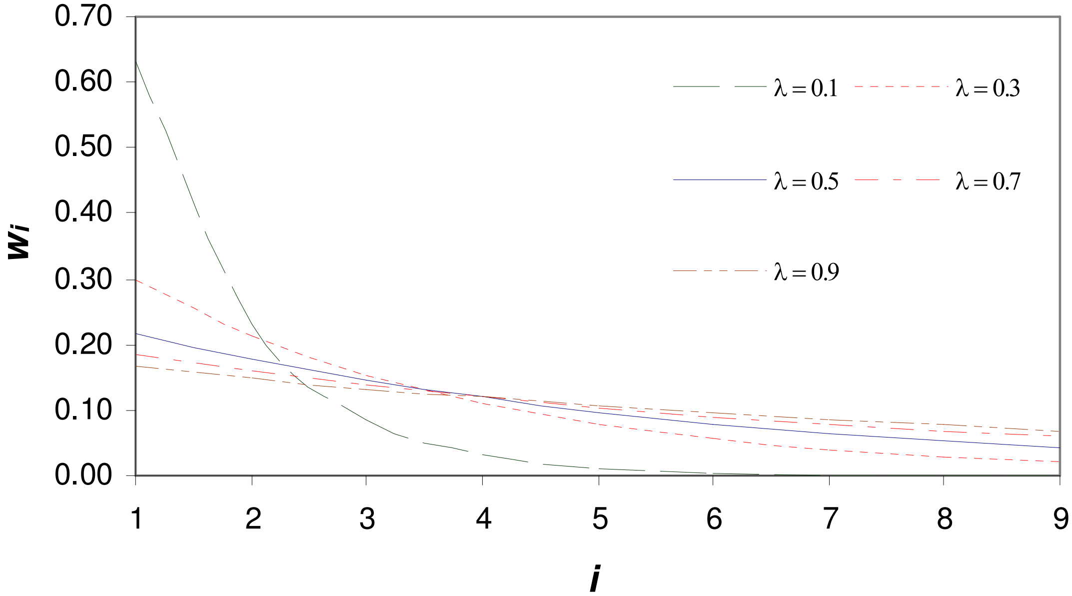


Fig. 4. OWA weights generated using exponential distribution (n = 9) for fractiles λ = {0.1, 0.3, 0.5, 0.7, 0.9}

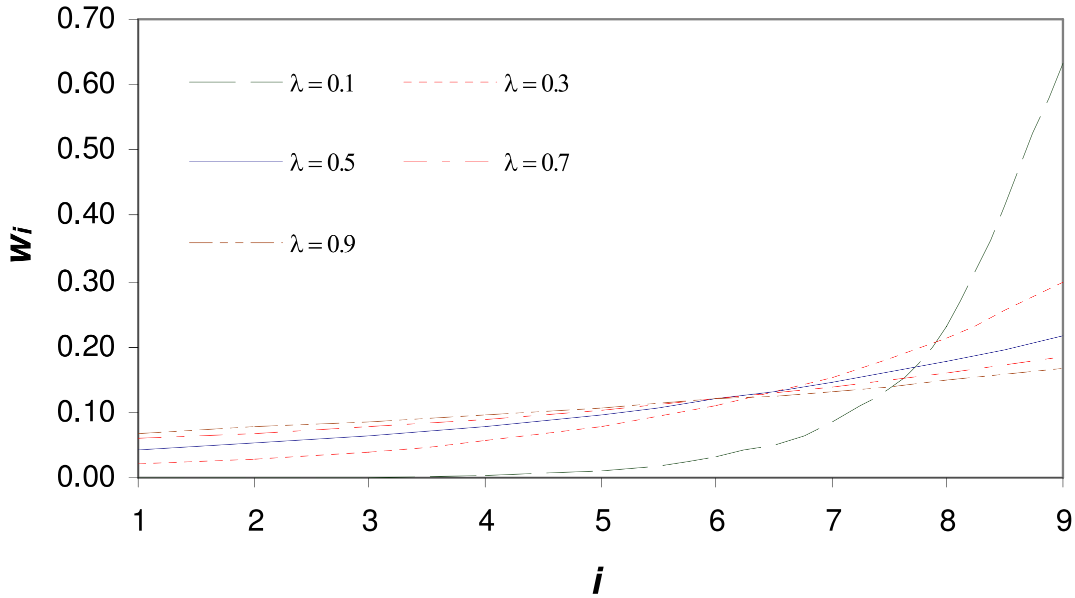


Fig. 5. OWA weights generated using inverse form of exponential distribution (n = 9) for fractiles λ = {0.1, 0.3, 0.5, 0.7, 0.9}

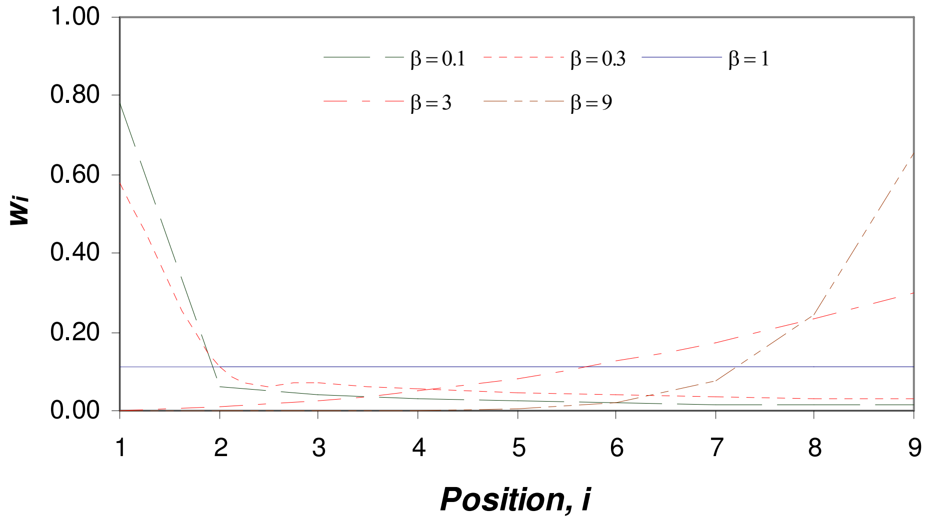
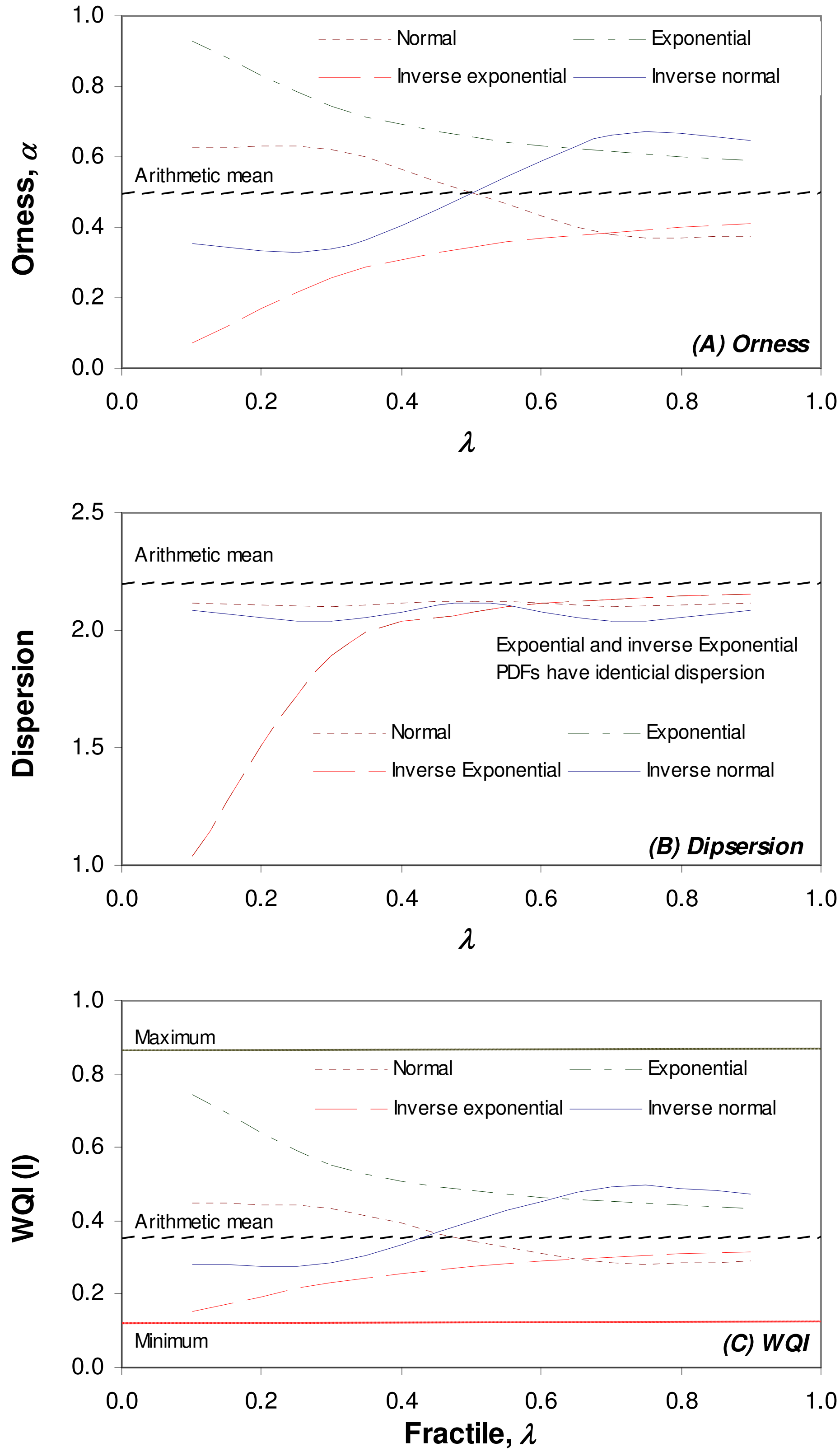


Fig. 6. OWA weights generated using the RIM function for power of polynomial $\beta = \{0.1, 0.3, 1, 3, 9\}$



Variation in *orness*, *dispersion* and *WQI* (for n = 9) with respect to fractiles of various OWA weight distributions

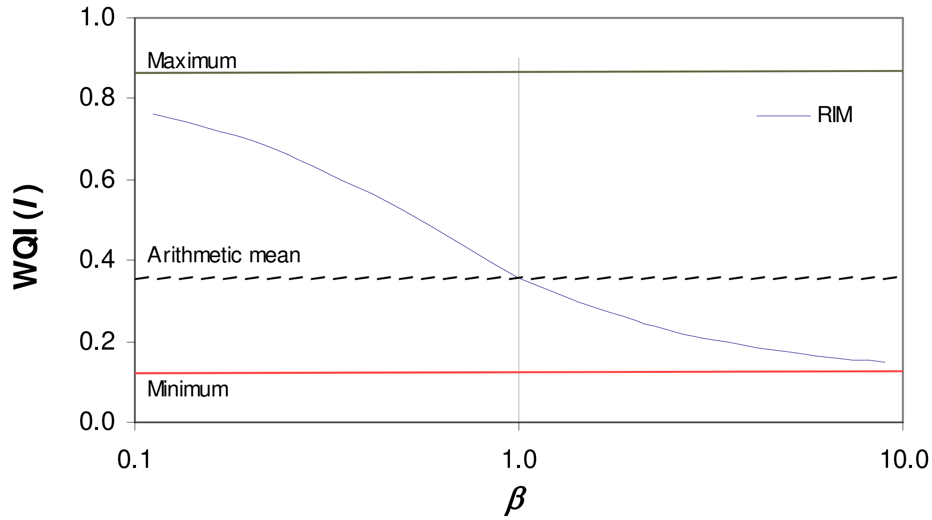


Fig. 7. Variation in WQI by varying degree of ploynomial (β) of RIM function

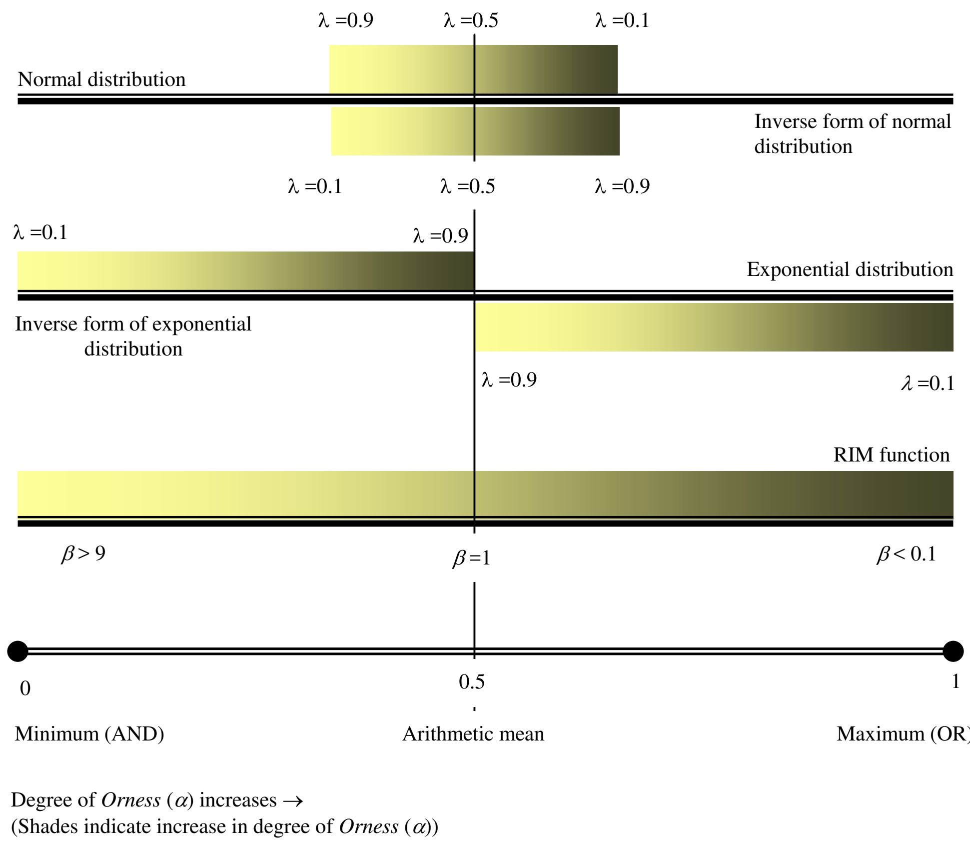


Fig. 8. Domains of degree of *orness* for various OWA weight distributions