

# Subexponential Asymptotics of the Stationary Distributions of GI/G/1-Type Markov Chains\*

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## Abstract

This paper considers the subexponential asymptotics of the stationary distributions of GI/G/1-type Markov chains in two cases: (i) the phase transition matrix in non-boundary levels is stochastic; and (ii) it is strictly substochastic. For the case (i), we present a weaker sufficient condition for the subexponential asymptotics than those given in the literature. As for the case (ii), the subexponential asymptotics has not been studied, as far as we know. We show that the subexponential asymptotics in the case (ii) is different from that in the case (i). We also study the locally subexponential asymptotics of the stationary distributions in both cases (i) and (ii).

**Keywords:** GI/G/1-type Markov chain; (Locally) subexponential; Markov additive process (MAdP); Stationary distribution.

**Mathematics Subject Classification:** Primary 60K25; Secondary 60J10.

## 1 Introduction

This paper studies the subexponential asymptotics of the stationary distribution of an irreducible and positive recurrent Markov chain of GI/G/1 type [10]. The GI/G/1-type Markov chain includes M/G/1- and GI/M/1-type ones as special cases and plays an important role in studying the stationary queue-length and/or waiting-time distributions in various Markovian queues such as continuous-time BMAP/GI/1, BMAP/D/c, SMAP/MSP/c queues, and discrete-time SMAP/GI/1 queues, where BMAP, SMAP and MSP represent batch Markovian arrival process, semi-Markovian arrival process and Markovian service process, respectively.

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Let  $\{(X_n, S_n); n = 0, 1, \dots\}$  denote a GI/G/1-type Markov chain such that  $X_n \in \mathbb{Z}_+ := \{0, 1, 2, \dots\}$  and

$$\begin{aligned} S_n &\in \mathbb{M}_0 := \{1, 2, \dots, M_0\}, \quad \text{if } X_n = 0, \\ S_n &\in \mathbb{M} := \{1, 2, \dots, M\}, \quad \text{otherwise,} \end{aligned}$$

where  $M_0$  and  $M$  are positive integers. The state space of  $\{(X_n, S_n)\}$  is given by  $\mathbb{S} = (\{0\} \times \mathbb{M}_0) \cup (\mathbb{N} \times \mathbb{M})$ , where  $\mathbb{N} = \{1, 2, 3, \dots\}$ . Further, the sub-state spaces  $\{(0, j); j \in \mathbb{M}_0\}$  and  $\{(k, j); j \in \mathbb{M}\}$  ( $k \in \mathbb{N}$ ) are called level 0 and level  $k$ , respectively.

Let  $\mathbf{T}$  denote the transition probability matrix of the GI/G/1-type Markov chain  $\{(X_n, S_n)\}$ , which can be partitioned as follows [10]:

$$\mathbf{T} = \begin{array}{c} \text{lev. } 0 \quad 1 \quad 2 \quad 3 \quad \dots \\ \left( \begin{array}{ccccc} \mathbf{B}(0) & \mathbf{B}(1) & \mathbf{B}(2) & \mathbf{B}(3) & \dots \\ \mathbf{B}(-1) & \mathbf{A}(0) & \mathbf{A}(1) & \mathbf{A}(2) & \dots \\ \mathbf{B}(-2) & \mathbf{A}(-1) & \mathbf{A}(0) & \mathbf{A}(1) & \dots \\ \mathbf{B}(-3) & \mathbf{A}(-2) & \mathbf{A}(-1) & \mathbf{A}(0) & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array} \right), \end{array}$$

where  $\mathbf{A}(k)$  ( $k \in \mathbb{Z} := \{0, \pm 1, \pm 2, \dots\}$ ) is an  $M \times M$  matrix,  $\mathbf{B}(0)$  is an  $M_0 \times M_0$  matrix,  $\mathbf{B}(k)$  ( $k \in \mathbb{N}$ ) is an  $M_0 \times M$  matrix, and  $\mathbf{B}(k)$  ( $k \in \mathbb{Z} \setminus \mathbb{Z}_+$ ) is an  $M \times M_0$  matrix. Throughout the paper, we assume the following, unless otherwise stated.

**Assumption 1** (a)  $\mathbf{T}$  is an irreducible and positive-recurrent stochastic matrix; (b)  $\mathbf{A} := \sum_{k=-\infty}^{\infty} \mathbf{A}(k)$  is irreducible.

Under Assumption 1,  $\mathbf{T}$  has a unique and positive stationary distribution (see, e.g., [6, Chapter 3, Theorem 3.1]), which is denoted by  $\mathbf{x} = (x_j(k))_{(k,j) \in \mathbb{S}}$ . For later use, we define  $\mathbf{x}(0) = (x_j(0))_{j \in \mathbb{M}_0}$  and  $\mathbf{x}(k) = (x_j(k))_{j \in \mathbb{M}}$  for  $k \in \mathbb{N}$ . Further, let  $\bar{\mathbf{x}}(k) = \sum_{l=k+1}^{\infty} \mathbf{x}(l)$  for  $k \in \mathbb{Z}_+$ .

Some researchers have studied the subexponential asymptotics of the stationary distribution  $\mathbf{x} = (\mathbf{x}(0), \mathbf{x}(1), \mathbf{x}(2), \dots)$  of the GI/G/1-type Markov chain (including the M/G/1-type one). The previous studies assume that  $\mathbf{A}$  is stochastic, though  $\mathbf{A}$  is not stochastic in general. In fact,

$$\lim_{k \rightarrow \infty} \mathbf{B}(-k) \neq \mathbf{O} \text{ if and only if } \mathbf{A}\mathbf{e} \neq \mathbf{e},$$

where  $\mathbf{e}$  denotes a column vector of ones with an appropriate dimension according to the context.

We briefly review the literature related to this paper. For this purpose, let  $Y$  denote a random variable in  $\mathbb{Z}_+$ , and for a while, assume that

$$\lim_{k \rightarrow \infty} \frac{\sum_{l=k+1}^{\infty} \mathbf{A}(l)}{\mathbf{P}(Y > k)} = \mathbf{C}_1 \geq \mathbf{O}, \quad \lim_{k \rightarrow \infty} \frac{\sum_{l=k+1}^{\infty} \mathbf{B}(l)}{\mathbf{P}(Y > k)} = \mathbf{C}_2 \geq \mathbf{O},$$

with  $C_1 \neq \mathbf{O}$  or  $C_2 \neq \mathbf{O}$ . Asmussen and Møller [2] consider two cases: (a)  $Y$  is regularly varying; and (b)  $Y$  belongs to both the subexponential class  $\mathcal{S}$  (see Definition A.1.2) and the maximum domain of attraction of the Gumbel distribution (see, e.g., [9, Section 3.3]). For the two cases, they show that under some additional conditions,

$$\lim_{k \rightarrow \infty} \frac{\bar{x}(k)}{P(Y_e > k)} = c_1 > 0, \quad Y_e \in \mathcal{S}, \quad (1.1)$$

where  $Y_e$  denotes the discrete equilibrium random variable of  $Y$ , distributed with  $P(Y_e = k) = P(Y > k)/E[Y]$  ( $k \in \mathbb{Z}_+$ ). Note here that  $Y \in \mathcal{S}$  does not necessarily imply  $Y_e \in \mathcal{S}$  and vice versa (see [21, Remark 3.5]).

Li and Zhao [17] show the subexponential tail asymptotics (1.1) under the condition that  $C_2 = \mathbf{O}$  and  $Y$  belongs to a subclass  $\mathcal{S}^*$  of  $\mathcal{S}$  (see Definition A.1.3). Note here that  $Y \in \mathcal{S}^*$  implies  $Y \in \mathcal{S}$  and  $Y_e \in \mathcal{S}$  (see Proposition A.2 in [19]). Although Li and Zhao [17] derive some other asymptotic formulae for  $\{\bar{x}(k)\}$ , those formulae are incorrect due to “the inverse of a singular matrix” (for details, see [19]).

Takine [22] proves that the subexponential tail asymptotics (1.1) holds for an M/G/1-type Markov chain, assuming that  $Y_e \in \mathcal{S}$  but not necessarily  $Y \in \mathcal{S}$ . Thus Takine’s result shows that  $Y \in \mathcal{S}$  is not a necessary condition for the subexponential decay of  $\{\bar{x}(k)\}$ . However, Masuyama [19] points out that Takine’s proof needs an additional condition that the  $G$ -matrix is aperiodic. Further, Masuyama [19] presents a weaker sufficient condition for (1.1) than those presented in the literature [2, 17, 22], though his result is limited to the M/G/1-type Markov chain. Recently, Kim and Kim [13] improve Masuyama [19]’s sufficient condition in the case where the  $G$ -matrix is periodic.

In this paper, we study the subexponential decay of the tail probabilities  $\{\bar{x}(k)\}$  in two cases: (i)  $\mathbf{A}$  is stochastic (i.e.,  $\mathbf{A}\mathbf{e} = \mathbf{e}$ ); and (ii)  $\mathbf{A}$  is strictly substochastic (i.e.,  $\mathbf{A}\mathbf{e} \leq \mathbf{e}$ ,  $\neq \mathbf{e}$ ). For the case (i), we generalize Masuyama [19]’s and Kim and Kim [13]’s results to the GI/G/1-type Markov chain. The obtained sufficient condition for the subexponential tail asymptotics (1.1) is weaker than those presented in Asmussen and Møller [2] and Li and Zhao [17]. As for the case (ii), we present a subexponential asymptotic formula such that

$$\lim_{k \rightarrow \infty} \frac{\bar{x}(k)}{P(Y > k)} = c_2 > 0, \quad Y \in \mathcal{S}.$$

It should be noted that the embedded queue length process of a BMAP/GI/1 queue with disasters falls into the case (ii) (see, e.g., [24]). As far as we know, the subexponential asymptotics in the case (ii) has not been studied in the literature. Therefore, this paper is the first report on the subexponential asymptotics in the case (ii).

We also study the locally subexponential asymptotics of the stationary probabilities  $\{x(k)\}$ . In the case (i) (i.e.,  $\mathbf{A}$  is stochastic), we prove the following formula under some technical

conditions:

$$\lim_{k \rightarrow \infty} \frac{\mathbf{x}(k)}{\mathbb{P}(Y_e = k)} = \mathbf{c}_3 > \mathbf{0}, \quad Y \in \mathcal{S}^*.$$

Further, in the case (ii) (i.e.,  $\mathbf{A}$  is strictly substochastic), we assume that  $Y$  is *locally subexponential with span one* (i.e.,  $Y \in \mathcal{S}_{\text{loc}}(1)$ ; see Definition A.2.2). We then show that

$$\lim_{k \rightarrow \infty} \frac{\mathbf{x}(k)}{\mathbb{P}(Y = k)} = \mathbf{c}_4 > \mathbf{0}, \quad Y \in \mathcal{S}_{\text{loc}}(1),$$

with some technical conditions. For the reader's convenience, Appendix C presents simple examples of the case where the stationary distribution is locally subexponential.

The rest of this paper is divided into three sections. Section 2 describes some basic results on the GI/G/1-type Markov chain and its related Markov additive process (MAdP). In Sections 3 and 4, we studied the subexponential tail asymptotics and locally subexponential asymptotics, respectively, of the stationary distribution.

## 2 The GI/G/1-Type Markov Chain and Its Related Markov Additive Process

Throughout this paper, we use the following conventions. Let  $\mathbf{I}$  denote the identity matrix with an appropriate dimension. For any matrix  $\mathbf{M}$ ,  $[\mathbf{M}]_{i,j}$  represents the  $(i, j)$ th element of  $\mathbf{M}$ . For any matrix sequence  $\{\mathbf{M}(k); k \in \mathbb{Z}_+\}$ , let  $\overline{\mathbf{M}}(k) = \sum_{l=k+1}^{\infty} \mathbf{M}(l)$  ( $k \in \mathbb{Z}_+$ ). For any two matrix sequences  $\{\mathbf{M}(k); k \in \mathbb{Z}_+\}$  and  $\{\mathbf{N}(k); k \in \mathbb{Z}_+\}$  such that their products are well-defined, let  $\mathbf{M} * \mathbf{N}(k) = \sum_{l=0}^k \mathbf{M}(k-l)\mathbf{N}(l)$  for  $k \in \mathbb{Z}_+$ . Further, for any square matrix sequence  $\{\mathbf{M}(k); k \in \mathbb{Z}_+\}$ , let  $\{\mathbf{M}^{*n}(k); k \in \mathbb{Z}_+\}$  denote the  $n$ -fold convolution of  $\{\mathbf{M}(k)\}$  with itself, i.e.,  $\mathbf{M}^{*n}(k) = \sum_{l=0}^k \mathbf{M}^{*(n-1)}(k-l)\mathbf{M}(l)$ , where  $\mathbf{M}^{*0}(0) = \mathbf{I}$  and  $\mathbf{M}^{*0}(k) = \mathbf{O}$  for  $k \in \mathbb{N}$ . The conventions for matrices are also used for vectors and scalars in an appropriate manner. Finally, the superscript “t” represents the transpose operator for vectors and matrices.

### 2.1 $R$ - and $G$ -matrices

In this subsection, we assume that  $\mathbf{T}$  is irreducible and stochastic, but do not necessarily assume the recurrence of  $\mathbf{T}$ .

We consider a censored Markov chain obtained by observing  $\{(X_n, S_n)\}$  only when it is in levels 0 through  $k$  ( $k \in \mathbb{Z}_+$ ). Let  $\mathbf{T}^{[k]}$  ( $k \in \mathbb{Z}_+$ ) denote the transition probability matrix of the censored Markov chain, which is irreducible due to the irreducibility of the original chain. Let  $\mathbf{T}_{\nu,\eta}^{[k]}$  ( $\nu, \eta \in \{0, 1, \dots, k\}$ ) denote a submatrix of  $\mathbf{T}^{[k]}$  such that  $[\mathbf{T}_{\nu,\eta}^{[k]}]_{i,j}$  represents the probability that the censored Markov chain moves from state  $(\nu, i) \in \mathbb{S}$  to  $(\eta, j) \in \mathbb{S}$  in one step.

From the block Toeplitz-like structure of  $\mathbf{T}$ , we see that  $\mathbf{T}_{k-l,k}^{[k]}$  and  $\mathbf{T}_{k,k-l}^{[k]}$  are independent of  $k$  if  $l \in \{0, 1, \dots, k-1\}$  and  $k \in \mathbb{N}$ . We thus define  $\Phi(l)$  ( $l \in \mathbb{Z}$ ) as

$$\begin{aligned}\Phi(l) &= \mathbf{T}_{k-l,k}^{[k]}, & l \in \{0, 1, \dots, k-1\}, k \in \mathbb{N}, \\ \Phi(-l) &= \mathbf{T}_{k,k-l}^{[k]}, & l \in \{0, 1, \dots, k-1\}, k \in \mathbb{N}.\end{aligned}\quad (2.1)$$

Note here that for any fixed  $\nu \in \mathbb{N}$ ,  $[\Phi(0)]_{i,j}$  represents the probability of hitting state  $(\nu, j)$  for the first time before entering the levels  $0, 1, \dots, \nu-1$ , given that it starts with state  $(\nu, i)$ , i.e.,

$$[\Phi(0)]_{i,j} = \mathbb{P}(S_{T_{\downarrow\nu}} = j \mid X_0 = \nu, S_0 = i),$$

where  $T_{\downarrow l} = \inf\{n \in \mathbb{N}; X_n = l < X_m \ (m = 1, 2, \dots, n-1)\}$ . Thus  $\sum_{n=0}^{\infty} (\Phi(0))^n = (\mathbf{I} - \Phi(0))^{-1}$  exists because  $\mathbf{T}^{[k]}$  is irreducible.

**Proposition 2.1.1 (Theorem 1 in [10])**  $\{\Phi(k); k \in \mathbb{Z}\}$  is the minimal nonnegative solution of the following equations.

$$\begin{aligned}\Phi(k) &= \mathbf{A}(k) + \sum_{m=1}^{\infty} \Phi(k+m)(\mathbf{I} - \Phi(0))^{-1} \Phi(-m), & k \in \mathbb{Z}_+, \\ \Phi(-k) &= \mathbf{A}(-k) + \sum_{m=1}^{\infty} \Phi(m)(\mathbf{I} - \Phi(0))^{-1} \Phi(-k-m), & k \in \mathbb{Z}_+.\end{aligned}$$

**Remark 2.1.1** The proof of Theorem 1 in [10] is based on induction and probabilistic interpretation, which are valid without the recurrence of  $\mathbf{T}$ .

Let  $\mathbf{G}$  and  $\mathbf{G}(k)$  ( $k \in \mathbb{N}$ ) denote

$$\mathbf{G} = \sum_{k=1}^{\infty} \mathbf{G}(k), \quad \mathbf{G}(k) = (\mathbf{I} - \Phi(0))^{-1} \Phi(-k), \quad k \in \mathbb{N}, \quad (2.2)$$

respectively. Note that for any fixed  $\nu \in \mathbb{N}$ ,  $[\mathbf{G}(k)]_{i,j}$  represents the probability of hitting state  $(\nu, j)$  when the Markov chain  $\{(X_n, S_n)\}$  enters the levels  $0, 1, \dots, \nu+k-1$  for the first time, given that it starts with state  $(\nu+k, i)$ , i.e.,

$$[\mathbf{G}(k)]_{i,j} = \mathbb{P}(X_{T_{<k+\nu}} = \nu, S_{T_{<k+\nu}} = j \mid X_0 = k+\nu, S_0 = i), \quad k \in \mathbb{N},$$

where  $T_{<l} = \inf\{n \in \mathbb{N}; X_n < l \leq X_m \ (m = 1, 2, \dots, n-1)\}$ .

Let  $\mathbf{L}(k)$  ( $k \in \mathbb{N}$ ) denote

$$\mathbf{L}(k) = \sum_{i=1}^k \sum_{\substack{(n_1, n_2, \dots, n_i) \in \mathbb{N}^i \\ n_1 + n_2 + \dots + n_i = k}} \mathbf{G}(n_1) \mathbf{G}(n_2) \cdots \mathbf{G}(n_i), \quad k \in \mathbb{N}. \quad (2.3)$$

For any fixed  $\nu \in \mathbb{N}$ ,  $[\mathbf{L}(k)]_{i,j}$  represents the probability of hitting state  $(\nu, j)$  when the Markov chain  $\{(X_n, S_n)\}$  enters the levels  $0, 1, \dots, \nu$  for the first time, given that it starts with state  $(\nu + k, i)$ , i.e.,

$$[\mathbf{L}(k)]_{i,j} = \mathbb{P}(S_{T_{\downarrow\nu}} = j \mid X_0 = k + \nu, S_0 = i).$$

It follows from (2.3) that

$$\widehat{\mathbf{L}}(z) := \sum_{k=1}^{\infty} z^{-k} \mathbf{L}(k) = (\mathbf{I} - \widehat{\mathbf{G}}(z))^{-1} \widehat{\mathbf{G}}(z), \quad (2.4)$$

where  $\widehat{\mathbf{G}}(z) = \sum_{k=1}^{\infty} z^{-k} \mathbf{G}(k)$ .

Let  $\mathbf{R}_0(k)$  and  $\mathbf{R}(k)$  ( $k \in \mathbb{Z}_+$ ) denote  $M_0 \times M$  and  $M \times M$  matrices, respectively, such that

$$\begin{aligned} \mathbf{R}_0(0) &= \mathbf{O}, & \mathbf{R}(0) &= \mathbf{O}, \\ \mathbf{R}_0(k) &= \mathbf{T}_{0,k}^{[k]} (\mathbf{I} - \Phi(0))^{-1}, & \mathbf{R}(k) &= \Phi(k) (\mathbf{I} - \Phi(0))^{-1}, \quad k \in \mathbb{N}. \end{aligned} \quad (2.5)$$

For any fixed  $\nu \in \mathbb{N}$ ,  $[\mathbf{R}(k)]_{i,j}$  ( $k \in \mathbb{N}$ ) represents the expected number of visits to state  $(\nu + k, j)$  before entering the levels  $0, 1, \dots, \nu + k - 1$ , given that the Markov chain  $\{(X_n, S_n)\}$  starts with state  $(\nu, i)$ . Further,  $\mathbf{R}_0(k)$  ( $k \in \mathbb{N}$ ) can be interpreted in the same way though  $\nu \in \mathbb{N}$  is replaced by zero. Formally, for  $k \in \mathbb{N}$ ,

$$\begin{aligned} [\mathbf{R}_0(k)]_{i,j} &= \mathbb{E} \left[ \sum_{n=1}^{T_{<k}} \mathbb{1}(X_n = k, S_n = j) \mid X_0 = 0, S_0 = i \right], \\ [\mathbf{R}(k)]_{i,j} &= \mathbb{E} \left[ \sum_{n=1}^{T_{<k+\nu}} \mathbb{1}(X_n = k + \nu, S_n = j) \mid X_0 = \nu \in \mathbb{N}, S_0 = i \right], \end{aligned}$$

where  $\mathbb{1}(\chi)$  denotes the indicator function of an event  $\chi$ . It follows from the definitions of  $\mathbf{R}_0(k)$ ,  $\mathbf{R}(k)$ ,  $\mathbf{L}(k)$  and  $\Phi(0)$  that

$$\mathbf{R}_0(k) = \left[ \mathbf{B}(k) + \sum_{m=1}^{\infty} \mathbf{B}(k+m) \mathbf{L}(m) \right] (\mathbf{I} - \Phi(0))^{-1}, \quad k \in \mathbb{N}, \quad (2.6)$$

$$\mathbf{R}(k) = \left[ \mathbf{A}(k) + \sum_{m=1}^{\infty} \mathbf{A}(k+m) \mathbf{L}(m) \right] (\mathbf{I} - \Phi(0))^{-1}, \quad k \in \mathbb{N}, \quad (2.7)$$

which hold without the recurrence of  $\mathbf{T}$ .

We now define  $\widehat{\mathbf{R}}_0(z)$ ,  $\widehat{\mathbf{R}}(z)$  and  $\widehat{\mathbf{B}}(z)$  as

$$\widehat{\mathbf{R}}_0(z) = \sum_{k=1}^{\infty} z^k \mathbf{R}_0(k), \quad \widehat{\mathbf{R}}(z) = \sum_{k=1}^{\infty} z^k \mathbf{R}(k), \quad \widehat{\mathbf{B}}(z) = \sum_{k=1}^{\infty} z^k \mathbf{B}(k),$$

respectively. We then have the following result.

**Proposition 2.1.2 (Theorem 1 and Lemma 3 in [18])** *Let  $r_{R_0}$ ,  $r_R$ ,  $r_G$ ,  $r_{A_+}$ ,  $r_{A_-}$  and  $r_B$  denote the convergence radii of  $\widehat{R}_0(z)$ ,  $\widehat{R}(z)$ ,  $\widehat{G}(1/z) = \sum_{k=1}^{\infty} z^k \mathbf{G}(k)$ ,  $\sum_{k=1}^{\infty} z^k \mathbf{A}(k)$ ,  $\sum_{k=1}^{\infty} z^k \mathbf{A}(-k)$  and  $\widehat{B}(z)$ , respectively. Then  $r_{R_0} = r_B \geq 1$ ,  $r_R = r_{A_+} \geq 1$  and  $r_G = r_{A_-} \geq 1$ .*

**Proposition 2.1.3 (Theorem 14 in [27])** *Let  $\widehat{A}(z) = \sum_{k \in \mathbb{Z}} z^k \mathbf{A}(k)$ . We then have*

$$\mathbf{I} - \widehat{A}(z) = (\mathbf{I} - \widehat{R}(z))(\mathbf{I} - \Phi(0))(\mathbf{I} - \widehat{G}(z)), \quad |z| \in I_A, \quad (2.8)$$

where  $I_A = (1/r_{A_-}, r_{A_+}) \cup \{1\}$ .

**Remark 2.1.2** Although Theorem 14 in [27] assume that  $\mathbf{A}$  is irreducible and stochastic, these conditions are not necessarily required by the algebraic proof of the theorem.

**Proposition 2.1.4** *Let  $\mathbf{R} = \sum_{k=1}^{\infty} \mathbf{R}(k)$ . If  $\mathbf{A}$  is irreducible and strictly substochastic, then (i)  $\text{sp}(\mathbf{G}) < 1$ ; (ii)  $\text{sp}(\mathbf{R}) < 1$ ; and (iii)  $\text{sp}(\sum_{l=0}^{\infty} \Phi(-l)) < 1$ , where  $\text{sp}(\cdot)$  denotes the spectral radius of a matrix in parentheses.*

*Proof.* See Appendix B.1. □

## 2.2 Sufficient conditions for positive recurrence

In this subsection, we provide two sets of sufficient conditions for Assumption 1. For later use, let  $\boldsymbol{\pi} > \mathbf{0}$  denote a left eigenvector of  $\mathbf{A}$  such that  $\boldsymbol{\pi} \mathbf{A} = \text{sp}(\mathbf{A}) \boldsymbol{\pi}$  and  $\boldsymbol{\pi} \mathbf{e} = 1$  (see Theorem 8.4.4 in [11]). Let  $\sigma$  denote

$$\sigma = \boldsymbol{\pi} \sum_{k \in \mathbb{Z}} k \mathbf{A}(k) \mathbf{e}. \quad (2.9)$$

If  $\mathbf{A}$  is stochastic, then  $\boldsymbol{\pi}$  is the unique invariant probability vector of  $\mathbf{A}$  and  $\sigma$  is the conditional mean drift of the level process  $\{X_n; n \in \mathbb{Z}_+\}$  with  $X_n \geq 1$ .

**Proposition 2.2.1 (Proposition 3.1 in Chapter XI of [3])** *Suppose  $\mathbf{T}$  and  $\mathbf{A}$  are irreducible and stochastic. Then  $\mathbf{T}$  is positive recurrent if and only if  $\sigma < 0$  and  $\sum_{k=1}^{\infty} k \mathbf{B}(k) \mathbf{e} < \infty$ .*

**Proposition 2.2.2** *Suppose  $\mathbf{T}$  is irreducible and stochastic. Then if  $\mathbf{A}$  is irreducible and strictly substochastic,  $\mathbf{T}$  is positive recurrent.*

*Proof.* Proposition 2.1.4 implies that  $\lim_{k \rightarrow \infty} \mathbf{R}^k = \mathbf{O}$  and  $(\mathbf{I} - \mathbf{G})^{-1}$  exists. Further from (2.6), we have

$$\begin{aligned}
 \mathbf{R}_0 &:= \sum_{k=1}^{\infty} \mathbf{R}_0(k) \\
 &= \left[ \sum_{k=1}^{\infty} \mathbf{B}(k) + \sum_{m=1}^{\infty} \left( \sum_{k=1}^{\infty} \mathbf{B}(k+m) \right) \mathbf{L}(m) \right] (\mathbf{I} - \Phi(0))^{-1} \\
 &\leq \sum_{k=1}^{\infty} \mathbf{B}(k) \left[ \mathbf{I} + \sum_{m=1}^{\infty} \mathbf{L}(m) \right] (\mathbf{I} - \Phi(0))^{-1} \\
 &= \sum_{k=1}^{\infty} \mathbf{B}(k) (\mathbf{I} - \mathbf{G})^{-1} (\mathbf{I} - \Phi(0))^{-1} < \infty,
 \end{aligned}$$

where the last equality follows from (2.4). As a result, it follows from Theorem 3.4 in [25] that  $\mathbf{T}$  is positive recurrent.  $\square$

### 2.3 Matrix-product form of the stationary distribution

This subsection discusses the stationary distribution  $\{\mathbf{x}(k)\}$  under Assumption 1. It is easy to see that  $(\mathbf{x}(0), \mathbf{x}(1), \dots, \mathbf{x}(k))$  is an invariant measure vector of the censored transition probability matrix  $\mathbf{T}^{[k]}$ , i.e.,

$$(\mathbf{x}(0), \mathbf{x}(1), \dots, \mathbf{x}(k)) \mathbf{T}^{[k]} = (\mathbf{x}(0), \mathbf{x}(1), \dots, \mathbf{x}(k)),$$

which leads to

$$\mathbf{x}(k) = \left[ \mathbf{x}(0) \mathbf{T}_{0,k}^{[k]} + \sum_{l=1}^{k-1} \mathbf{x}(l) \mathbf{T}_{l,k}^{[k]} \right] (\mathbf{I} - \mathbf{T}_{k,k}^{[k]})^{-1}, \quad k \in \mathbb{N}. \quad (2.10)$$

In terms of  $\mathbf{R}(k)$  and  $\mathbf{R}_0(k)$ , we can rewrite (2.10) as

$$\mathbf{x}(k) = \mathbf{x}(0) \mathbf{R}_0(k) + \sum_{l=1}^k \mathbf{x}(l) \mathbf{R}(k-l), \quad k \in \mathbb{N}, \quad (2.11)$$

where we use  $\mathbf{R}(0) = \mathbf{O}$ . It then follows from (2.11) that

$$\mathbf{x}(k) = \mathbf{x}(0) \mathbf{R}_0 * \mathbf{F}(k), \quad k \in \mathbb{N}, \quad (2.12)$$

where  $\mathbf{F}(k)$  ( $k \in \mathbb{Z}_+$ ) is given by

$$\mathbf{F}(k) = \sum_{n=0}^{\infty} \mathbf{R}^{*n}(k). \quad (2.13)$$

Thus we have

$$\bar{\mathbf{x}}(k) = \mathbf{x}(0) \overline{\mathbf{R}_0 * \mathbf{F}}(k), \quad k \in \mathbb{Z}_+. \quad (2.14)$$

Further let  $\hat{\mathbf{x}}(z) = \sum_{k=1}^{\infty} z^k \mathbf{x}(k)$ . We then have

$$\hat{\mathbf{x}}(z) = \mathbf{x}(0) \hat{\mathbf{R}}_0(z) (\mathbf{I} - \hat{\mathbf{R}}(z))^{-1}. \quad (2.15)$$

Letting  $z = 1$  in (2.15) yields

$$\bar{\mathbf{x}}(0) = \mathbf{x}(0) \mathbf{R}_0 (\mathbf{I} - \mathbf{R})^{-1}, \quad (2.16)$$

where  $\mathbf{R}_0 = \sum_{k=1}^{\infty} \mathbf{R}_0(k)$ .

## 2.4 Period of the related Markov additive process

We consider a MAdP  $\{(\check{X}_n, \check{S}_n); n \in \mathbb{Z}_+\}$  with state space  $\mathbb{Z} \times \mathbb{M}$  and kernel  $\{\mathbf{A}(k); k \in \mathbb{Z}\}$ . The stochastic behavior of the MAdP  $\{(\check{X}_n, \check{S}_n)\}$  is equivalent to that of the GI/GI/1-type Markov chain  $\{(X_n, S_n)\}$  while the latter is being in non-boundary levels, i.e., for any  $i, j \in \mathbb{M}$ ,

$$\begin{aligned} & \mathbb{P}(\check{X}_{n+1} = k, \check{S}_{n+1} = j \mid \check{X}_n = l, \check{S}_n = i) \\ &= \mathbb{P}(X_{n+1} = k, S_{n+1} = j \mid X_n = l, S_n = i), \quad k, l \in \mathbb{N}. \end{aligned} \quad (2.17)$$

The period of the MAdP  $\{(\check{X}_n, \check{S}_n)\}$ , denoted by  $\tau$ , is the largest positive integer such that

$$[\mathbf{A}(k)]_{i,j} > 0 \text{ only if } k \equiv p(j) - p(i) \pmod{\tau}, \quad (2.18)$$

where  $p$  is some function  $p$  from  $\mathbb{M}$  to  $\{0, 1, \dots, \tau - 1\}$  (see Appendix B in [14] and its revised version [15]).

**Remark 2.4.1** Lemma B.2 in [14] states that function  $p$  satisfying (2.18) is *injective*, which is not true in general. This error is corrected in the revised version [15].

**Remark 2.4.2** If the Markov chain  $\{(X_n, S_n)\}$  is of M/G/1 type, the period  $\tau$  is less than or equal to  $M$  (see, e.g., Proposition 2.9 in [14]).

**Remark 2.4.3** We suppose

$$\begin{aligned} \mathbf{A}(0) &= \mathbf{O}, \quad \mathbf{A}(1) = \begin{pmatrix} 0 & 0 & \frac{1}{6} \\ 0 & 0 & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & 0 \end{pmatrix}, \\ \mathbf{A}(-2) &= \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}, \quad \mathbf{A}(-1) = \begin{pmatrix} 0 & 0 & \frac{1}{6} \\ 0 & 0 & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & 0 \end{pmatrix}. \end{aligned}$$

Let  $p(0) = p(1) = 1$  and  $p(2) = 0$ . It then follows that

$$[\mathbf{A}(k)]_{i,j} > 0 \text{ only if } k \equiv p(j) - p(i) \pmod{2},$$

and thus the period of MAdP with kernel  $\{\mathbf{A}(k)\}$  is equal to two.

We now introduce the following notation.

**Definition 2.4.1** For any finite square matrix  $\mathbf{X}$  with possibly complex elements, let  $\delta(\mathbf{X})$  denote an eigenvalue of  $\mathbf{X}$ , which satisfies  $\delta(\mathbf{X}) = \text{sp}(\mathbf{X})e^{\iota\xi}$  and

$$\xi = \inf\{0 \leq x < 2\pi; \det(\text{sp}(\mathbf{X})e^{\iota x}\mathbf{I} - \mathbf{X}) = 0\},$$

where  $\iota$  denotes the imaginary unit, i.e.,  $\iota = \sqrt{-1}$ .

**Remark 2.4.4** Suppose  $\mathbf{X}$  is nonnegative. We then have  $\delta(\mathbf{X}) = \text{sp}(\mathbf{X})$  (see Theorem 8.3.1 in [11]). Further, if  $\mathbf{X}$  is irreducible,  $\delta(\mathbf{X})$  is the Perron-Frobenius eigenvalue of  $\mathbf{X}$  (see Theorem 8.4.4 in [11]).

Let  $\boldsymbol{\mu}(z)$  and  $\mathbf{v}(z)$  denote the left- and right-eigenvectors of  $\widehat{\mathbf{A}}(z)$  corresponding to the eigenvalue  $\delta(\widehat{\mathbf{A}}(z))$ , normalized such that

$$\boldsymbol{\mu}(z)\boldsymbol{\Delta}_M(z/|z|)\mathbf{e} = 1, \quad \boldsymbol{\mu}(z)\mathbf{v}(z) = 1,$$

where  $\boldsymbol{\Delta}_M(z)$  denotes an  $M \times M$  diagonal matrix as follows:

$$\boldsymbol{\Delta}_M(z) = \begin{pmatrix} z^{-p(1)} & & & \\ & z^{-p(2)} & & \\ & & \ddots & \\ & & & z^{-p(M)} \end{pmatrix}.$$

Note that  $\boldsymbol{\mu}(1) = \boldsymbol{\pi}$  and  $\mathbf{v}(1) = \mathbf{e}$ . Further, let  $\omega$  denote an arbitrary complex number such that  $|\omega| = 1$ . We then have the following results.

**Proposition 2.4.1 (Lemma B.3 in [14])** Suppose Assumption 1 holds and let  $\omega_x = \exp(2\pi\iota/x)$  for  $x \geq 1$ . Then the following are true for all  $y \in I_A$  and  $\nu = 0, 1, \dots, \tau - 1$ .

- (i)  $\delta(\widehat{\mathbf{A}}(y\omega_\tau^\nu)) = \delta(\widehat{\mathbf{A}}(y))$ , both of which are simple eigenvalues; and
- (ii)  $\boldsymbol{\mu}(y\omega_\tau^\nu) = \boldsymbol{\mu}(y)\boldsymbol{\Delta}_M(\omega_\tau^\nu)^{-1}$  and  $\mathbf{v}(y\omega_\tau^\nu) = \boldsymbol{\Delta}_M(\omega_\tau^\nu)\mathbf{v}(y)$ .

**Proposition 2.4.2 (Theorem B.1 in [14])** Suppose Assumption 1 holds and  $\delta(\widehat{\mathbf{A}}(y)) = 1$  for some  $y \in I_A$ . Then  $\delta(\widehat{\mathbf{A}}(y\omega)) = 1$  if and only if  $\omega^\tau = 1$ . Therefore

$$\tau = \max\{n \in \mathbb{N}; \delta(\widehat{\mathbf{A}}(y\omega_n)) = 1\}.$$

Further if  $\delta(\widehat{\mathbf{A}}(y\omega)) = 1$ , the eigenvalue is simple.

## 2.5 Spectral analysis of $G$ -matrices from stochastic $A$

In this subsection, we assume that Assumption 1 holds and  $A$  is stochastic. Under the assumptions,  $G$  is stochastic, i.e.,  $\delta(G) = 1$  (see Theorem 3.4 in [25]).

We first provide a basic result on the structure of  $G$ .

**Proposition 2.5.1** *Suppose Assumption 1 holds and  $A$  is stochastic. Then  $G$  has an exactly one irreducible class, denoted by  $\mathbb{M}_\bullet \subseteq \mathbb{M}$ . Thus,  $G$  is irreducible, or after some permutations it takes a form such that*

$$\begin{array}{cc} & \mathbb{M}_\bullet & \mathbb{M}_T \\ \begin{array}{c} \mathbb{M}_\bullet \\ \mathbb{M}_T \end{array} & \begin{pmatrix} G_\bullet & O \\ G_\circ & G_T \end{pmatrix}, & \mathbb{M}_T := \mathbb{M} \setminus \mathbb{M}_\bullet, \end{array}$$

where  $G_\bullet$  is irreducible,  $G_T$  is strictly lower triangular and  $G_\circ$  does not have, in general, a special structure.

*Proof.* See Appendix B.2. □

**Remark 2.5.1**<sup>‡1</sup> The proof of Proposition 2.5.1 relies only on the facts that (i)  $A$  is irreducible; and (ii)  $G$  is not a nilpotent matrix. Thus, the irreducibility of  $A$  and  $\text{sp}(G) > 0$  imply that Proposition 2.5.1 holds. On the other hand, Proposition 2.1.3 yields

$$\det(I - A) = \det(I - R) \det(I - \Phi(0)) \det(I - G).$$

Note here that if  $T$  is positive recurrent then  $\det(I - R) \neq 0$  (see the proof of Theorem 3.4 of [25]). Therefore, if the conditions of Proposition 2.5.1 are satisfied, then  $\det(I - A) = 0$ ,  $\det(I - \Phi(0)) \neq 0$  and  $\det(I - R) \neq 0$  and thus  $\det(I - G) = 0$ , which implies that  $\text{sp}(G) = 1$ .

Let  $G_\bullet(k)$  ( $k \in \mathbb{N}$ ) denote the square submatrix of  $G(k)$  ( $k \in \mathbb{N}$ ) corresponding to the irreducible class  $\mathbb{M}_\bullet \subseteq \mathbb{M}$ , i.e.,  $G_\bullet = \sum_{k=1}^{\infty} G_\bullet(k)$ . Further let  $\hat{G}_\bullet(z) = \sum_{k=1}^{\infty} z^{-k} G_\bullet(k)$ . It then follows from Proposition 2.5.1 that

$$\delta(\hat{G}(z)) = \delta(\hat{G}_\bullet(z)), \tag{2.19}$$

because  $G_T$  (if any) is a nilpotent matrix.

We now consider a MAdP  $\{(\check{X}_n^{(G)}, \check{S}_n^{(G)}); n \in \mathbb{Z}_+\}$  with state space  $\mathbb{Z} \times \mathbb{M}_\bullet$  and kernel  $\{I^{(G)}(k); k \in \mathbb{Z}\}$ , where

$$I^{(G)}(k) = \begin{cases} O, & k \in \mathbb{Z}_+, \\ G_\bullet(k), & k \in \mathbb{Z} \setminus \mathbb{Z}_+. \end{cases} \tag{2.20}$$

<sup>‡1</sup>This remark is added in the revised version.

Equation (2.20) and the irreducibility of  $\sum_{k \in \mathbb{Z}} \mathbf{I}^{(G)}(k) = \mathbf{G}_\bullet$  imply that the period of the MAdP  $\{\check{X}_n^{(G)}, \check{S}_n^{(G)}\}$  is well-defined (see Definition B.1 in [14]) and denoted by  $\tau_G$ . Combining (2.19), (2.20) and Theorem B.1 in [14], we obtain

$$\tau_G = \max\{n \in \mathbb{N}; \delta(\widehat{\mathbf{G}}(\omega_n)) = 1\}. \quad (2.21)$$

**Proposition 2.5.2** *Suppose Assumption 1 holds and  $\mathbf{A}$  is stochastic. Then the following are true.*

- (i)  $\tau_G = \tau$ ;
- (ii)  $\delta(\widehat{\mathbf{G}}(\omega)) = 1$  if and only if  $\omega^\tau = 1$ ;
- (iii) if  $\delta(\widehat{\mathbf{G}}(\omega)) = 1$ , the eigenvalue is simple; and
- (iv) for  $y > 1/r_{A_-}$ ,

$$\delta(\widehat{\mathbf{G}}(y\omega_\tau^\nu)) = \delta(\widehat{\mathbf{G}}(y)), \quad \nu = 0, 1, \dots, \tau - 1,$$

which are simple eigenvalues of  $\widehat{\mathbf{G}}(y\omega_\tau^\nu)$  and  $\widehat{\mathbf{G}}(y)$ , respectively.

*Proof.* See Appendix B.3. □

We define  $\lambda_i^{(G)}(z)$ 's ( $i = 2, 3, \dots, M$ ) as the eigenvalues of  $\widehat{\mathbf{G}}(z)$  such that  $\delta(\widehat{\mathbf{G}}(z)) \geq |\lambda_i^{(G)}(z)|$  (see Definition 2.4.1). We then have

$$\det(\mathbf{I} - \widehat{\mathbf{G}}(z)) = (1 - \delta(\widehat{\mathbf{G}}(z))) \prod_{i=2}^M (1 - \lambda_i^{(G)}(z)). \quad (2.22)$$

**Proposition 2.5.3** *Suppose Assumption 1 holds and  $\mathbf{A}$  is stochastic. Let*

$$\boldsymbol{\psi}(\omega_\tau^\nu) = \frac{\pi(\mathbf{I} - \mathbf{R})(\mathbf{I} - \Phi(0))}{\pi(\mathbf{I} - \mathbf{R})(\mathbf{I} - \Phi(0))\mathbf{e}} \Delta_M(\omega_\tau^\nu)^{-1}, \quad \nu = 0, 1, \dots, \tau - 1, \quad (2.23)$$

$$\mathbf{y}(\omega_\tau^\nu) = \Delta_M(\omega_\tau^\nu)\mathbf{e}, \quad \nu = 0, 1, \dots, \tau - 1. \quad (2.24)$$

Then the following hold for  $\nu = 0, 1, \dots, \tau - 1$ : (i)  $\boldsymbol{\psi}(\omega_\tau^\nu)$  and  $\mathbf{y}(\omega_\tau^\nu)$  are the left- and right-eigenvectors of  $\widehat{\mathbf{G}}(\omega_\tau^\nu)$  corresponding to the eigenvalue  $\delta(\widehat{\mathbf{G}}(\omega_\tau^\nu)) = 1$ ; and (ii)

$$\text{adj}(\mathbf{I} - \widehat{\mathbf{G}}(\omega_\tau^\nu)) = \prod_{i=2}^M (1 - \lambda_i^{(G)}(\omega_\tau^\nu)) \mathbf{y}(\omega_\tau^\nu) \boldsymbol{\psi}(\omega_\tau^\nu),$$

where  $\text{adj}(\mathbf{Y})$  denotes the adjugate matrix of a square matrix  $\mathbf{Y}$ .

*Proof.* See Appendix B.4. □

### 3 Subexponential Tail Asymptotics

This section studies the subexponential decay of the tail probabilities  $\{\bar{x}(k)\}$ , under the following assumption.

**Assumption 2** Either of (I) and (II) is satisfied:

(I) Assumption 1 holds,  $\mathbf{A}$  is stochastic, and  $\sum_{k \in \mathbb{Z}} |k| \mathbf{A}(k) < \infty$ ; or

(II) Assumption 1 holds and  $\mathbf{A}$  is strictly substochastic.

Assumption 2 (I) and (II) are considered in subsections 3.1 and 3.2, respectively.

#### 3.1 Case of stochastic $\mathbf{A}$

**Lemma 3.1.1** Under Assumption 2 (I),

$$\sigma = -\pi(\mathbf{I} - \mathbf{R})(\mathbf{I} - \Phi(0)) \sum_{k=1}^{\infty} k \mathbf{G}(k) \mathbf{e} \in (-\infty, 0), \quad (3.1)$$

where  $\sigma$  is defined in (2.9).

*Proof.* We have  $-\infty < \sigma < 0$  due to (2.9), Proposition 2.2.1 and the third condition of Assumption 2 (I). Further since  $\sigma = \pi(d/dz)\hat{\mathbf{A}}(z)|_{z=1}\mathbf{e}$  and  $(d/dz)\hat{\mathbf{G}}(z)|_{z=1} = -\sum_{k=1}^{\infty} k \mathbf{G}(k)$ , we obtain (3.1) by differentiating (2.8) with respect to  $z$ , pre-multiplying by  $\pi$ , post-multiplying by  $\mathbf{e}$  and letting  $z = 1$ .  $\square$

Using Lemma 3.1.1, Propositions 2.5.2 and 2.5.3, we obtain the following result.

**Lemma 3.1.2** If Assumption 2 (I) holds, then for  $l = 0, 1, \dots, \tau - 1$ ,

$$\lim_{n \rightarrow \infty} \mathbf{L}(n\tau + l) = \sum_{\nu=0}^{\tau-1} \frac{1}{(\omega_{\tau}^{\nu})^l} \Delta_M(\omega_{\tau}^{-\nu}) \mathbf{e} \psi \Delta_M(\omega_{\tau}^{-\nu})^{-1}, \quad (3.2)$$

where

$$\psi = \pi(\mathbf{I} - \mathbf{R})(\mathbf{I} - \Phi(0))/(-\sigma). \quad (3.3)$$

*Proof.* See Appendix B.5.  $\square$

For  $l = 0, 1, \dots, \tau - 1$ , let  $\mathbb{M}^{(l)} = \{j \in \mathbb{M}; p(j) = l\}$  and  $|\mathbb{M}^{(l)}|$  denote the cardinality of  $\mathbb{M}^{(l)}$ . Further, let  $\psi^{(l)}$  denote a subvector of  $\psi$  corresponding to  $\mathbb{M}^{(l)}$ , and  $\mathbf{e}^{(l)}$  denote an  $|\mathbb{M}^{(l)}| \times 1$  vector of ones. Note here that  $\sum_{\nu=0}^{\tau-1} (\omega_{\tau}^m)^{\nu} = 0$  for all  $m = 1, 2, \dots, \tau - 1$  because  $\omega_{\tau}, \omega_{\tau}^2, \dots, \omega_{\tau}^{\tau-1}$  are the solutions of the equation  $\sum_{\nu=0}^{\tau-1} z^{\nu} = 0$ . It then follows from

Lemma 3.1.2 that

$$\begin{aligned} \lim_{n \rightarrow \infty} [\mathbf{L}(n\tau + l)]_{i,j} &= [\boldsymbol{\psi}]_j \sum_{\nu=0}^{\tau-1} (\omega_{\tau}^{-\nu})^{l-p(i)+p(j)} \\ &= \begin{cases} \tau[\boldsymbol{\psi}]_j, & \text{if } p(i) \equiv p(j) + l \pmod{\tau}, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

This equation can be rewritten as

$$\lim_{n \rightarrow \infty} \mathbf{L}(n\tau + l) = \tau \mathbf{E} \mathbf{H}_l, \quad (3.4)$$

where

$$\mathbf{E} = \begin{matrix} \mathbb{M}^{(0)} \\ \mathbb{M}^{(1)} \\ \vdots \\ \mathbb{M}^{(\tau-2)} \\ \mathbb{M}^{(\tau-1)} \end{matrix} \begin{pmatrix} \mathbf{e}^{(0)} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{e}^{(1)} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{e}^{(\tau-2)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{e}^{(\tau-1)} \end{pmatrix}, \quad (3.5)$$

and

$$\mathbf{H}_l = \begin{matrix} \mathbb{M}^{(0)} & \mathbb{M}^{(1)} & \cdots & \mathbb{M}^{(\tau-l-1)} & \mathbb{M}^{(\tau-l)} & \mathbb{M}^{(\tau-l+1)} & \cdots & \mathbb{M}^{(\tau-1)} \\ \left( \begin{array}{cccccccc} \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \boldsymbol{\psi}^{(\tau-l)} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \boldsymbol{\psi}^{(\tau-l+1)} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \boldsymbol{\psi}^{(\tau-1)} \\ \boldsymbol{\psi}^{(0)} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\psi}^{(1)} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \boldsymbol{\psi}^{(\tau-l-1)} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \end{array} \right) \end{matrix}.$$

**Remark 3.1.1** Suppose the Markov chain  $\{(X_n, S_n)\}$  is of M/G/1-type. It then follows that  $\mathbf{L}(n) = \mathbf{G}^n$  for  $n = 1, 2, \dots$ . Further it is easy to see that  $\boldsymbol{\psi}$  is a stationary probability vector of  $\mathbf{G}$  and therefore  $[\boldsymbol{\psi}]_j = 0$  for all  $j \in \mathbb{M}_T$  (see Proposition 2.5.1). We now define  $\boldsymbol{\psi}_{\bullet}^{(l)}$  ( $l = 0, 1, \dots, \tau - 1$ ) as a subvector of  $\boldsymbol{\psi}$  corresponding to  $\mathbb{M}_{\bullet}^{(l)} := \{j \in \mathbb{M}_{\bullet} \cap \mathbb{M}^{(l)}\}$ . As a

result, (3.4) yields

$$\lim_{n \rightarrow \infty} \frac{1}{\tau} \mathbf{G}^{n\tau} = \begin{matrix} & \mathbb{M}_{\bullet}^{(0)} & \mathbb{M}_{\bullet}^{(1)} & \cdots & \mathbb{M}_{\bullet}^{(\tau-1)} & \mathbb{M}_{\mathbf{T}} \\ \begin{matrix} \mathbb{M}_{\bullet}^{(0)} \\ \mathbb{M}_{\bullet}^{(1)} \\ \vdots \\ \mathbb{M}_{\bullet}^{(\tau-1)} \\ \mathbb{M}_{\mathbf{T}}^{(0)} \\ \mathbb{M}_{\mathbf{T}}^{(1)} \\ \vdots \\ \mathbb{M}_{\mathbf{T}}^{(\tau-1)} \end{matrix} & \begin{pmatrix} e\psi_{\bullet}^{(0)} & \mathbf{O} & \cdots & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & e\psi_{\bullet}^{(1)} & \cdots & \mathbf{O} & \mathbf{O} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{O} & \mathbf{O} & \cdots & e\psi_{\bullet}^{(\tau-1)} & \mathbf{O} \\ e\psi_{\bullet}^{(0)} & \mathbf{O} & \cdots & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & e\psi_{\bullet}^{(1)} & \cdots & \mathbf{O} & \mathbf{O} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{O} & \mathbf{O} & \cdots & e\psi_{\bullet}^{(\tau-1)} & \mathbf{O} \end{pmatrix} \end{matrix}, \quad (3.6)$$

where  $\mathbb{M}_{\mathbf{T}}^{(l)} = \mathbb{M}^{(l)} \setminus \mathbb{M}_{\bullet}^{(l)}$  ( $l = 0, 1, \dots, \tau - 1$ ). Note here that  $\psi_{\bullet}^{(l)}e = 1/\tau$  for all  $l = 0, 1, \dots, \tau - 1$  because  $(1/\tau)\mathbf{G}^{n\tau}e = e/\tau$  for all  $n = 1, 2, \dots$ . As a result, the limit (3.6) is consistent with the equation (14) in [19], where  $\sum_{\nu=1}^{\tau} \mathbf{f}_{\nu} = e$  and each element of  $\mathbf{f}_{\nu}$  ( $\nu = 1, 2, \dots, \tau$ ) is equal to one or zero.

**Lemma 3.1.3** *If Assumption 2 (I) holds, then*

$$\lim_{n \rightarrow \infty} \sum_{l=0}^{\tau-1} \mathbf{L}(n\tau + l) = \tau e\psi. \quad (3.7)$$

*Proof.* We obtain (3.7) by combining (3.4) and

$$\sum_{l=0}^{\tau-1} \mathbf{H}_l = e\psi. \quad (3.8)$$

□

We now make the following assumption.

**Assumption 3** There exists some random variable  $Y$  in  $\mathbb{Z}_+$  with positive finite mean such that

$$\lim_{k \rightarrow \infty} \frac{\overline{\mathbf{A}}(k)e}{\mathbb{P}(Y > k)} = \frac{\mathbf{c}_A}{\mathbb{E}[Y]}, \quad \lim_{k \rightarrow \infty} \frac{\overline{\mathbf{B}}(k)e}{\mathbb{P}(Y > k)} = \frac{\mathbf{c}_B}{\mathbb{E}[Y]}, \quad (3.9)$$

where  $\mathbf{c}_A$  and  $\mathbf{c}_B$  are nonnegative  $M \times 1$  and  $M_0 \times 1$  vectors, respectively, satisfying  $\mathbf{c}_A \neq \mathbf{0}$  or  $\mathbf{c}_B \neq \mathbf{0}$ .

**Lemma 3.1.4** *Suppose Assumptions 2 (I) and 3 hold. If  $Y_e$  is long-tailed (i.e.,  $Y_e \in \mathcal{L}$ ; see Definition A.1.1). We then have*

$$\lim_{k \rightarrow \infty} \sum_{m=1}^{\infty} \frac{\overline{\mathbf{A}}(k+m)\mathbf{L}(m)}{\mathbb{P}(Y_e > k)} = \frac{\mathbf{c}_A \pi(\mathbf{I} - \mathbf{R})(\mathbf{I} - \Phi(0))}{-\sigma}, \quad (3.10)$$

$$\lim_{k \rightarrow \infty} \sum_{m=1}^{\infty} \frac{\overline{\mathbf{B}}(k+m)\mathbf{L}(m)}{\mathbb{P}(Y_e > k)} = \frac{\mathbf{c}_B \pi(\mathbf{I} - \mathbf{R})(\mathbf{I} - \Phi(0))}{-\sigma}. \quad (3.11)$$

*Proof.* See Appendix B.6. □

**Lemma 3.1.5** *Suppose Assumptions 2 (I) and 3 hold. If  $Y_e \in \mathcal{L}$ , then*

$$\lim_{k \rightarrow \infty} \frac{\overline{\mathbf{R}}(k)}{\mathbb{P}(Y_e > k)} = \frac{\mathbf{c}_A \boldsymbol{\pi} (\mathbf{I} - \mathbf{R})}{-\sigma}, \quad (3.12)$$

$$\lim_{k \rightarrow \infty} \frac{\overline{\mathbf{R}}_0(k)}{\mathbb{P}(Y_e > k)} = \frac{\mathbf{c}_B \boldsymbol{\pi} (\mathbf{I} - \mathbf{R})}{-\sigma}. \quad (3.13)$$

*Proof.* It follows from (2.7) that

$$\overline{\mathbf{R}}(k) = \left[ \overline{\mathbf{A}}(k) + \sum_{m=1}^{\infty} \overline{\mathbf{A}}(k+m) \mathbf{L}(m) \right] (\mathbf{I} - \boldsymbol{\Phi}(0))^{-1}. \quad (3.14)$$

Note that Corollary 3.3 in [21] and (3.9) yield

$$\limsup_{k \rightarrow \infty} \frac{\overline{\mathbf{A}}(k)}{\mathbb{P}(Y_e > k)} \leq \limsup_{k \rightarrow \infty} \frac{\overline{\mathbf{A}}(k) \mathbf{e} \mathbf{e}^t}{\mathbb{P}(Y > k)} \limsup_{k \rightarrow \infty} \frac{\mathbb{P}(Y > k)}{\mathbb{P}(Y_e > k)} = \mathbf{O}.$$

Thus from (3.14), we have

$$\lim_{k \rightarrow \infty} \frac{\overline{\mathbf{R}}(k)}{\mathbb{P}(Y_e > k)} = \lim_{k \rightarrow \infty} \sum_{m=1}^{\infty} \frac{\overline{\mathbf{A}}(k+m) \mathbf{L}(m)}{\mathbb{P}(Y_e > k)} (\mathbf{I} - \boldsymbol{\Phi}(0))^{-1}. \quad (3.15)$$

Substituting (3.10) into (3.15), we obtain (3.12). Similarly, we can prove (3.13). □

The following theorem presents a subexponential asymptotic formula for  $\{\overline{\mathbf{x}}(k)\}$ .

**Theorem 3.1.1** *Suppose Assumptions 2 (I) and 3 hold. If  $Y_e \in \mathcal{S}$ , then*

$$\lim_{k \rightarrow \infty} \frac{\overline{\mathbf{x}}(k)}{\mathbb{P}(Y_e > k)} = \frac{\mathbf{x}(0) \mathbf{c}_B + \overline{\mathbf{x}}(0) \mathbf{c}_A}{-\sigma} \cdot \boldsymbol{\pi} \quad (3.16)$$

*Proof.* It follows from (2.13) that

$$\sum_{k=0}^{\infty} \mathbf{F}(k) = (\mathbf{I} - \mathbf{R})^{-1}. \quad (3.17)$$

Thus using (3.17) and Lemma 6 in [12], we have

$$\begin{aligned} \lim_{k \rightarrow \infty} \frac{\overline{\mathbf{F}}(k)}{\mathbb{P}(Y_e > k)} &= \lim_{k \rightarrow \infty} \sum_{n=0}^{\infty} \frac{\overline{\mathbf{R}}^{*n}(k)}{\mathbb{P}(Y_e > k)} \\ &= (\mathbf{I} - \mathbf{R})^{-1} \lim_{k \rightarrow \infty} \frac{\overline{\mathbf{R}}(k)}{\mathbb{P}(Y_e > k)} (\mathbf{I} - \mathbf{R})^{-1}. \end{aligned}$$

Substituting (3.12) into the above equation yields

$$\lim_{k \rightarrow \infty} \frac{\overline{\mathbf{F}}(k)}{\mathbb{P}(Y_e > k)} = \frac{(\mathbf{I} - \mathbf{R})^{-1} \mathbf{c}_A \boldsymbol{\pi}}{-\sigma}. \quad (3.18)$$

Finally, applying Proposition A.3 in [19] to (2.14) and using (3.13) and (3.18) lead to

$$\lim_{k \rightarrow \infty} \frac{\bar{\mathbf{x}}(k)}{\mathbb{P}(Y_e > k)} = \frac{\mathbf{x}(0)}{-\sigma} [\mathbf{c}_B \boldsymbol{\pi} + \mathbf{R}_0 (\mathbf{I} - \mathbf{R})^{-1} \mathbf{c}_A \boldsymbol{\pi}],$$

from which and (2.16) we have (3.16).  $\square$

**Remark 3.1.2** Theorem 3.1.1 is a generalization of Theorem 1 in [13] to the GI/G/1-type Markov chain. In fact, the latter extends the corollary of Theorem 3.1 in [19] (Corollary 3.1 therein) to the case where the  $G$ -matrix is periodic.

### 3.2 Case of strictly substochastic $A$

In this subsection, we make the following assumption in addition to Assumption 2 (II):

**Assumption 4** There exists some random variable  $Y$  in  $\mathbb{Z}_+$  such that

$$\lim_{k \rightarrow \infty} \frac{\bar{\mathbf{A}}(k)}{\mathbb{P}(Y > k)} = \mathbf{C}_A, \quad \lim_{k \rightarrow \infty} \frac{\bar{\mathbf{B}}(k)}{\mathbb{P}(Y > k)} = \mathbf{C}_B, \quad (3.19)$$

where  $\mathbf{C}_A$  and  $\mathbf{C}_B$  are nonnegative  $M \times M$  and  $M_0 \times M$  matrices, respectively, satisfying  $\mathbf{C}_A \neq \mathbf{O}$  or  $\mathbf{C}_B \neq \mathbf{O}$ .

**Lemma 3.2.1** Suppose Assumptions 2 (II) and 4 hold. If  $Y \in \mathcal{L}$ , then

$$\lim_{k \rightarrow \infty} \frac{\bar{\mathbf{R}}(k)}{\mathbb{P}(Y > k)} = \mathbf{C}_A \left( \mathbf{I} - \sum_{l=0}^{\infty} \boldsymbol{\Phi}(-l) \right)^{-1}, \quad (3.20)$$

$$\lim_{k \rightarrow \infty} \frac{\bar{\mathbf{R}}_0(k)}{\mathbb{P}(Y > k)} = \mathbf{C}_B \left( \mathbf{I} - \sum_{l=0}^{\infty} \boldsymbol{\Phi}(-l) \right)^{-1}. \quad (3.21)$$

*Proof.* From (2.7) and (3.19), we have

$$\lim_{k \rightarrow \infty} \frac{\bar{\mathbf{R}}(k)}{\mathbb{P}(Y > k)} = \left[ \mathbf{C}_A + \lim_{k \rightarrow \infty} \sum_{m=1}^{\infty} \frac{\bar{\mathbf{A}}(k+m)}{\mathbb{P}(Y > k)} \mathbf{L}(m) \right] (\mathbf{I} - \boldsymbol{\Phi}(0))^{-1}. \quad (3.22)$$

Note here that under Assumption 2 (II),  $\text{sp}(\mathbf{G}) < 1$  (see Proposition 2.1.4) and thus (2.4) yields

$$\sum_{m=1}^{\infty} \mathbf{L}(m) = (\mathbf{I} - \mathbf{G})^{-1} \mathbf{G} < \infty, \quad (3.23)$$

from which and (3.19) it follows that for  $k = 0, 1, \dots$ ,

$$\sum_{m=1}^{\infty} \frac{\bar{\mathbf{A}}(k+m)}{\mathbb{P}(Y > k)} \mathbf{L}(m) \leq \sup_{k \in \mathbb{Z}_+} \frac{\bar{\mathbf{A}}(k)}{\mathbb{P}(Y > k)} \sum_{m=1}^{\infty} \mathbf{L}(m) < \infty.$$

Therefore applying the dominated convergence theorem to (3.22) and using (3.19) and  $Y \in \mathcal{L}$ , we obtain

$$\begin{aligned}
& \lim_{k \rightarrow \infty} \frac{\overline{\mathbf{R}}(k)}{\mathbf{P}(Y > k)} \\
&= \left[ \mathbf{C}_A + \sum_{m=1}^{\infty} \lim_{k \rightarrow \infty} \frac{\overline{\mathbf{A}}(k+m)}{\mathbf{P}(Y > k+m)} \frac{\mathbf{P}(Y > k+m)}{\mathbf{P}(Y > k)} \mathbf{L}(m) \right] (\mathbf{I} - \Phi(0))^{-1} \\
&= \mathbf{C}_A [\mathbf{I} + (\mathbf{I} - \mathbf{G})^{-1} \mathbf{G}] (\mathbf{I} - \Phi(0))^{-1} \\
&= \mathbf{C}_A (\mathbf{I} - \mathbf{G})^{-1} (\mathbf{I} - \Phi(0))^{-1}.
\end{aligned} \tag{3.24}$$

From (2.2), we have

$$\begin{aligned}
(\mathbf{I} - \mathbf{G})^{-1} &= \left[ \mathbf{I} - (\mathbf{I} - \Phi(0))^{-1} \sum_{l=1}^{\infty} \Phi(-l) \right]^{-1} \\
&= \left( \mathbf{I} - \sum_{l=0}^{\infty} \Phi(-l) \right)^{-1} (\mathbf{I} - \Phi(0)).
\end{aligned} \tag{3.25}$$

Finally, substituting (3.25) into (3.24) yields (3.20). Equation (3.21) can be proved in the same way.  $\square$

**Theorem 3.2.2** *Suppose Assumptions 2 (II) and 4 hold. If  $Y \in \mathcal{S}$ , then*

$$\lim_{k \rightarrow \infty} \frac{\overline{\mathbf{x}}(k)}{\mathbf{P}(Y > k)} = [\mathbf{x}(0)\mathbf{C}_B + \overline{\mathbf{x}}(0)\mathbf{C}_A](\mathbf{I} - \mathbf{A})^{-1} > \mathbf{0}. \tag{3.26}$$

*Proof.* Applying Proposition A.3 in [19] to (2.14) and using (3.17) and (3.21), we have

$$\begin{aligned}
\lim_{k \rightarrow \infty} \frac{\overline{\mathbf{x}}(k)}{\mathbf{P}(Y > k)} &= \mathbf{x}(0)\mathbf{C}_B \left( \mathbf{I} - \sum_{l=0}^{\infty} \Phi(-l) \right)^{-1} (\mathbf{I} - \mathbf{R})^{-1} \\
&\quad + \mathbf{x}(0)\mathbf{R}_0 \lim_{k \rightarrow \infty} \frac{\overline{\mathbf{F}}(k)}{\mathbf{P}(Y > k)},
\end{aligned} \tag{3.27}$$

where  $\mathbf{F}(k)$  is given in (2.13). Further it follows from Lemma 6 in [12] and (3.20) that

$$\lim_{k \rightarrow \infty} \frac{\overline{\mathbf{F}}(k)}{\mathbf{P}(Y > k)} = (\mathbf{I} - \mathbf{R})^{-1} \mathbf{C}_A \left( \mathbf{I} - \sum_{l=0}^{\infty} \Phi(-l) \right)^{-1} (\mathbf{I} - \mathbf{R})^{-1}.$$

Substituting the above equation into (3.27) and using (2.16), we have

$$\lim_{k \rightarrow \infty} \frac{\overline{\mathbf{x}}(k)}{\mathbf{P}(Y > k)} = [\mathbf{x}(0)\mathbf{C}_B + \overline{\mathbf{x}}(0)\mathbf{C}_A] \left( \mathbf{I} - \sum_{l=0}^{\infty} \Phi(-l) \right)^{-1} (\mathbf{I} - \mathbf{R})^{-1}. \tag{3.28}$$

Note here that (3.25) yields

$$\begin{aligned} & \left( \mathbf{I} - \sum_{l=0}^{\infty} \Phi(-l) \right)^{-1} (\mathbf{I} - \mathbf{R})^{-1} \\ &= (\mathbf{I} - \mathbf{G})^{-1} (\mathbf{I} - \Phi(0))^{-1} (\mathbf{I} - \mathbf{R})^{-1} = (\mathbf{I} - \mathbf{A})^{-1}, \end{aligned} \quad (3.29)$$

where the second equality follows from Proposition 2.1.3. As a result, we obtain (3.26) by combining (3.28) with (3.29).

It is easy to show that the right hand side of (3.26) is positive. Indeed,  $(\mathbf{I} - \mathbf{A})^{-1} > \mathbf{O}$  due to the irreducibility of  $\mathbf{A}$ . In addition,  $\mathbf{x}(0)\mathbf{C}_B + \bar{\mathbf{x}}(0)\mathbf{C}_A \geq \mathbf{0}, \neq \mathbf{0}$  because  $\mathbf{x}(0) > \mathbf{0}$  and  $\bar{\mathbf{x}}(0) > \mathbf{0}$ ; and  $\mathbf{C}_A \neq \mathbf{O}$  or  $\mathbf{C}_B \neq \mathbf{O}$ . Therefore,  $(\mathbf{x}(0)\mathbf{C}_B + \bar{\mathbf{x}}(0)\mathbf{C}_A)(\mathbf{I} - \mathbf{A})^{-1} > \mathbf{0}$ .  $\square$

## 4 Locally Subexponential Asymptotics

This section considers the locally subexponential asymptotics of the stationary distribution.

### 4.1 Case of stochastic $\mathbf{A}$

In this subsection, we proceed under Assumption 2 (I) and the following assumption:

**Assumption 5** There exists some random variable  $Y$  in  $\mathbb{Z}_+$  with positive finite mean such that

$$\lim_{k \rightarrow \infty} \frac{\mathbf{A}(k)\mathbf{E}}{\mathbf{P}(Y = k)} = \frac{\mathbf{C}_A^{\mathbf{E}}}{\mathbf{E}[Y]}, \quad \lim_{k \rightarrow \infty} \frac{\mathbf{B}(k)\mathbf{E}}{\mathbf{P}(Y = k)} = \frac{\mathbf{C}_B^{\mathbf{E}}}{\mathbf{E}[Y]}, \quad (4.1)$$

where  $\mathbf{E}$  is given in (3.5), and where  $\mathbf{C}_A^{\mathbf{E}}$  and  $\mathbf{C}_B^{\mathbf{E}}$  are nonnegative  $M \times \tau$  and  $M_0 \times \tau$  matrices, respectively, satisfying  $\mathbf{C}_A^{\mathbf{E}} \neq \mathbf{O}$  or  $\mathbf{C}_B^{\mathbf{E}} \neq \mathbf{O}$ .

**Lemma 4.1.1** Suppose Assumptions 2 (I) and 5 hold. Further, suppose either of the following is satisfied:  $Y$  is locally long-tailed with span one (i.e.,  $Y \in \mathcal{L}_{\text{loc}}(1)$ ; see Definition A.2.1); or  $Y \in \mathcal{L}$  and  $\{\mathbf{P}(Y = k)\}$  is eventually nonincreasing. Then

$$\lim_{k \rightarrow \infty} \sum_{m=1}^{\infty} \frac{\mathbf{A}(k+m)\mathbf{L}(m)}{\mathbf{P}(Y_e = k)} = \mathbf{C}_A^{\mathbf{E}} e^{\frac{\pi(\mathbf{I} - \mathbf{R})(\mathbf{I} - \Phi(0))}{-\sigma}}, \quad (4.2)$$

$$\lim_{k \rightarrow \infty} \sum_{m=1}^{\infty} \frac{\mathbf{B}(k+m)\mathbf{L}(m)}{\mathbf{P}(Y_e = k)} = \mathbf{C}_B^{\mathbf{E}} e^{\frac{\pi(\mathbf{I} - \mathbf{R})(\mathbf{I} - \Phi(0))}{-\sigma}}. \quad (4.3)$$

*Proof.* See Appendix B.7.  $\square$

**Remark 4.1.1** Lemma 4.1.1 is proved by using Proposition A.2.1, which requires either that  $Y \in \mathcal{L}_{\text{loc}}(1)$  or that  $Y \in \mathcal{L}$  and  $\{\mathbf{P}(Y = k)\}$  is eventually nonincreasing.

**Lemma 4.1.2** *Under the same assumptions as in Lemma 4.1.1,*

$$\lim_{k \rightarrow \infty} \frac{\mathbf{R}(k)}{\mathbb{P}(Y_e = k)} = \mathbf{C}_A^E e \frac{\boldsymbol{\pi}(\mathbf{I} - \mathbf{R})}{-\sigma}, \quad (4.4)$$

$$\lim_{k \rightarrow \infty} \frac{\mathbf{R}_0(k)}{\mathbb{P}(Y_e = k)} = \mathbf{C}_B^E e \frac{\boldsymbol{\pi}(\mathbf{I} - \mathbf{R})}{-\sigma}. \quad (4.5)$$

*Proof.* It follows from  $\mathbf{E}e = e$ , (4.1) and  $Y \in \mathcal{L}$  that

$$\lim_{k \rightarrow \infty} \frac{\mathbf{A}(k)}{\mathbb{P}(Y_e = k)} \leq \mathbb{E}[Y] \lim_{k \rightarrow \infty} \frac{\mathbf{A}(k) \mathbf{E}e e^\top \mathbb{P}(Y = k)}{\mathbb{P}(Y > k) \mathbb{P}(Y > k)} = \mathbf{O}.$$

Thus from (2.7), we have

$$\lim_{k \rightarrow \infty} \frac{\mathbf{R}(k)}{\mathbb{P}(Y_e = k)} = \lim_{k \rightarrow \infty} \sum_{m=1}^{\infty} \frac{\mathbf{A}(k+m) \mathbf{L}(m)}{\mathbb{P}(Y_e = k)} (\mathbf{I} - \boldsymbol{\Phi}(0))^{-1}. \quad (4.6)$$

Substituting (4.2) into (4.6) yields (4.4). Similarly, we can readily show (4.5).  $\square$

We now obtain a locally subexponential asymptotic formula for  $\{\mathbf{x}(k)\}$ .

**Theorem 4.1.1** *Suppose Assumptions 2 (I) and 5 hold. Further, suppose (i)  $Y_e$  is locally subexponential with span one (i.e.,  $Y_e \in \mathcal{S}_{\text{loc}}(1)$ ; see Definition A.2.2); and (ii)  $Y \in \mathcal{L}_{\text{loc}}(1)$  or  $\{\mathbb{P}(Y = k)\}$  is eventually nonincreasing. Then*

$$\lim_{k \rightarrow \infty} \frac{\mathbf{x}(k)}{\mathbb{P}(Y_e = k)} = \frac{\mathbf{x}(0) \mathbf{C}_B^E e + \bar{\mathbf{x}}(0) \mathbf{C}_A^E e}{-\sigma} \cdot \boldsymbol{\pi}. \quad (4.7)$$

**Remark 4.1.2** According to Definition A.2.2 and Proposition A.2.2,  $Y_e \in \mathcal{S}_{\text{loc}}(1)$  is equivalent to  $Y \in \mathcal{S}^*$ . Thus since  $\mathcal{S}^* \subset \mathcal{S} \subset \mathcal{L}$ , the assumptions of Theorem 4.1.1 are sufficient for those of Lemma 4.1.1.

*Proof of Theorem 4.1.1.* Proposition A.2.6 yields

$$\begin{aligned} \lim_{k \rightarrow \infty} \frac{\mathbf{F}(k)}{\mathbb{P}(Y_e = k)} &= \lim_{k \rightarrow \infty} \sum_{n=0}^{\infty} \frac{\mathbf{R}^{*n}(k)}{\mathbb{P}(Y_e = k)} \\ &= (\mathbf{I} - \mathbf{R})^{-1} \lim_{k \rightarrow \infty} \frac{\mathbf{R}(k)}{\mathbb{P}(Y_e = k)} (\mathbf{I} - \mathbf{R})^{-1}, \end{aligned}$$

from which and (4.4) it follows that

$$\lim_{k \rightarrow \infty} \frac{\mathbf{F}(k)}{\mathbb{P}(Y_e = k)} = \frac{(\mathbf{I} - \mathbf{R})^{-1} \mathbf{C}_A^E e \boldsymbol{\pi}}{-\sigma}. \quad (4.8)$$

Further applying Proposition A.2.5 to (2.12) and using (4.5) and (4.8), we obtain

$$\lim_{k \rightarrow \infty} \frac{\mathbf{x}(k)}{\mathbb{P}(Y_e = k)} = \frac{\mathbf{x}(0)}{-\sigma} [\mathbf{C}_B^E e \boldsymbol{\pi} + \mathbf{R}_0 (\mathbf{I} - \mathbf{R})^{-1} \mathbf{C}_A^E e \boldsymbol{\pi}].$$

Substituting (2.16) into the above equation yields (4.7).  $\square$

We present another asymptotic formula.

**Assumption 6** There exists some random variable  $Y$  in  $\mathbb{Z}_+$  with positive finite mean such that

$$\lim_{k \rightarrow \infty} \frac{\mathbf{A}(k)\mathbf{e}}{\mathbf{P}(Y = k)} = \frac{\mathbf{c}_A}{\mathbf{E}[Y]}, \quad \lim_{k \rightarrow \infty} \frac{\mathbf{B}(k)\mathbf{e}}{\mathbf{P}(Y = k)} = \frac{\mathbf{c}_B}{\mathbf{E}[Y]},$$

where  $\mathbf{c}_A$  and  $\mathbf{c}_B$  are nonnegative  $M \times 1$  and  $M_0 \times 1$  vectors, respectively, satisfying  $\mathbf{c}_A \neq \mathbf{0}$  or  $\mathbf{c}_B \neq \mathbf{0}$ .

**Theorem 4.1.2** Suppose Assumptions 2 (I) and 6 hold. Further, suppose (i)  $Y_e \in \mathcal{S}_{\text{loc}}(1)$ ; (ii)  $Y \in \mathcal{L}_{\text{loc}}(1)$  or  $\{\mathbf{P}(Y = k)\}$  is eventually nonincreasing; and (iii)  $\{\mathbf{A}(k); k \in \mathbb{Z}_+\}$  and  $\{\mathbf{B}(k); k \in \mathbb{N}\}$  are eventually nonincreasing. Then

$$\lim_{k \rightarrow \infty} \frac{\mathbf{x}(k)}{\mathbf{P}(Y_e = k)} = \frac{\mathbf{x}(0)\mathbf{c}_B + \bar{\mathbf{x}}(0)\mathbf{c}_A}{-\sigma} \cdot \boldsymbol{\pi}.$$

*Proof.* This theorem can be proved in a very similar way to Theorem 3.1.1. For doing this, we require an additional condition that  $\{\mathbf{A}(k); k \in \mathbb{Z}_+\}$  and  $\{\mathbf{B}(k); k \in \mathbb{N}\}$  are eventually nonincreasing, i.e., there exists some  $k_* \in \mathbb{N}$  such that  $\mathbf{A}(k) \geq \mathbf{A}(k+1)$  and  $\mathbf{B}(k) \geq \mathbf{B}(k+1)$  for all  $k \geq k_*$ . The details are omitted.  $\square$

**Remark 4.1.3** Since  $\mathbf{E}\mathbf{e} = \mathbf{e}$ , Assumption 6 is sufficient for Assumption 5. Thus Theorem 4.1.2 is not a collorary of Theorem 4.1.1.

## 4.2 Case of strictly substochastic $\mathbf{A}$

In addition to Assumption 2 (II), we assume the following:

**Assumption 7** There exists some random variable  $Y$  in  $\mathbb{Z}_+$  such that

$$\lim_{k \rightarrow \infty} \frac{\mathbf{A}(k)}{\mathbf{P}(Y = k)} = \mathbf{C}_A, \quad \lim_{k \rightarrow \infty} \frac{\mathbf{B}(k)}{\mathbf{P}(Y = k)} = \mathbf{C}_B, \quad (4.9)$$

where  $\mathbf{C}_A$  and  $\mathbf{C}_B$  are nonnegative  $M \times M$  and  $M_0 \times M$  matrices, respectively, satisfying  $\mathbf{C}_A \neq \mathbf{O}$  or  $\mathbf{C}_B \neq \mathbf{O}$ .

**Lemma 4.2.1** Suppose Assumptions 2 (II) and 7 hold. If  $Y \in \mathcal{L}_{\text{loc}}(1)$ ; and  $r_{A_-} > 1$  or  $\{\mathbf{P}(Y = k)\}$  is eventually nonincreasing, then

$$\lim_{k \rightarrow \infty} \frac{\mathbf{R}(k)}{\mathbf{P}(Y = k)} = \mathbf{C}_A \left( \mathbf{I} - \sum_{l=0}^{\infty} \Phi(-l) \right)^{-1}, \quad (4.10)$$

$$\lim_{k \rightarrow \infty} \frac{\mathbf{R}_0(k)}{\mathbf{P}(Y = k)} = \mathbf{C}_B \left( \mathbf{I} - \sum_{l=0}^{\infty} \Phi(-l) \right)^{-1}. \quad (4.11)$$

*Proof.* From (2.7) and (4.9), we have

$$\lim_{k \rightarrow \infty} \frac{\mathbf{R}(k)}{\mathbf{P}(Y = k)} = \left[ \mathbf{C}_A + \lim_{k \rightarrow \infty} \sum_{m=1}^{\infty} \frac{\mathbf{A}(k+m)}{\mathbf{P}(Y = k)} \mathbf{L}(m) \right] (\mathbf{I} - \Phi(0))^{-1}. \quad (4.12)$$

To apply the dominated convergence theorem to (4.12), we show that for all sufficiently large  $k$ ,

$$\sum_{m=1}^{\infty} \frac{\mathbf{A}(k+m)}{\mathbf{P}(Y = k)} \mathbf{L}(m) < \infty.$$

Suppose  $\{\mathbf{P}(Y = k)\}$  is eventually nonincreasing. We then have for all sufficiently large  $k$ ,

$$\sum_{m=1}^{\infty} \frac{\mathbf{A}(k+m)}{\mathbf{P}(Y = k)} \mathbf{L}(m) \leq \sup_{m' \in \mathbb{N}} \frac{\mathbf{A}(k+m')}{\mathbf{P}(Y = k+m')} \sum_{m=1}^{\infty} \mathbf{L}(m) < \infty,$$

where the last inequality is due to (3.23) and (4.9). On the other hand, suppose  $r_{A-} > 1$ . It then follows from Proposition 2.1.2 that  $\{\mathbf{G}(k)\}$  is light-tailed, i.e.,

$$\sum_{k=1}^{\infty} r^k \mathbf{G}(k) < \infty \quad \text{for all } 1 < r < r_{A-}. \quad (4.13)$$

Note here that  $\widehat{\mathbf{G}}(1/z) = \sum_{k=1}^{\infty} z^k \mathbf{G}(k)$  and  $\text{sp}(\widehat{\mathbf{G}}(1)) < 1$  (see Proposition 2.1.4). Thus according to Theorem 8.1.18 in [11],

$$\text{sp}(\widehat{\mathbf{G}}(1/z)) = 1 \quad \text{only if } 1 < z \leq r_{A-}. \quad (4.14)$$

The equations (2.4), (4.13) and (4.14) imply that there exists some  $r > 1$  such that

$$\sum_{m=1}^{\infty} r^m \mathbf{L}(m) < \infty.$$

Further it follows from Assumption 7 and  $Y \in \mathcal{L}_{\text{loc}}(1)$  that for any  $\varepsilon > 0$  there exists some  $k_0 \in \mathbb{Z}_+$  such that for all  $k \geq k_0$ ,

$$\frac{\mathbf{A}(k+m)}{\mathbf{P}(Y = k)} \leq (\mathbf{C}_A + \varepsilon \mathbf{e} \mathbf{e}^t) \frac{\mathbf{P}(Y = k+m)}{\mathbf{P}(Y = k)} \leq (1 + \varepsilon)^m (\mathbf{C}_A + \varepsilon \mathbf{e} \mathbf{e}^t), \quad m \in \mathbb{Z}_+.$$

Therefore, for  $0 < \varepsilon \leq r - 1$  and  $k \geq k_0$ ,

$$\sum_{m=1}^{\infty} \frac{\mathbf{A}(k+m)}{\mathbf{P}(Y = k)} \mathbf{L}(m) \leq (\mathbf{C}_A + \varepsilon \mathbf{e} \mathbf{e}^t) \sum_{m=1}^{\infty} (1 + \varepsilon)^m \mathbf{L}(m) < \infty.$$

As a result, applying the dominated convergence theorem to (4.12) and following the proof of Lemma 3.2.1, we can prove (4.10). Equation (4.11) can be proved in the same way.  $\square$

Using Lemma 4.2.1, we can readily prove the following theorem. The proof is very similar to that of Theorem 3.2.2 and thus is omitted.

**Theorem 4.2.1** *Suppose Assumptions 2 (II) and 7 hold. If  $Y \in \mathcal{S}_{\text{loc}}(1)$ ; and  $r_{A_-} > 1$  or  $\{P(Y = k)\}$  is eventually nonincreasing, then*

$$\lim_{k \rightarrow \infty} \frac{x(k)}{P(Y = k)} = [x(0)C_B + \bar{x}(0)C_A](I - A)^{-1} > 0.$$

## 5 Discussion on Assumptions

This section discusses the assumptions of the theorems presented in Sections 3 and 4.

We first consider the case of stochastic  $A$ , for which Theorems 3.1.1, 4.1.1 and 4.1.2 are shown. The assumptions of these theorems are summarized in Table 1, where “*eventually non-increasing*” is abbreviated as “ENI”. Note here that Assumption 5 implies Assumption 3 due

Table 1: The assumptions of the theorems in case of stochastic  $A$

Theorem 3.1.1	Theorem 4.1.1	Theorem 4.1.2
Assumption 2 (I)	Assumption 2 (I)	Assumption 2 (I)
Assumption 3	Assumption 5	Assumption 6
$Y_e \in \mathcal{S}$	$Y_e \in \mathcal{S}_{\text{loc}}(1)$	$Y_e \in \mathcal{S}_{\text{loc}}(1)$
	$Y \in \mathcal{L}_{\text{loc}}(1)$ or $\{P(Y = k)\}$ is ENI	$Y \in \mathcal{L}_{\text{loc}}(1)$ or $\{P(Y = k)\}$ is ENI
		$\{A(k)\}$ and $\{B(k)\}$ are ENI

to  $Ee = e$ . Recall also that if  $Y_e \in \mathcal{S}_{\text{loc}}(1)$ , then  $Y_e \in \mathcal{S}$  (see Remark A.2.2). Thus the assumptions of Theorem 4.1.1 are more restrictive than those of Theorem 3.1.1. Similarly, we can readily confirm that the assumptions of Theorem 4.1.2 imply those of Theorem 3.1.1. It should be noted that Theorem 4.1.2 is not a corollary of Theorem 4.1.1 because Assumption 6 is weaker than Assumption 5.

Next we consider the case of substochastic  $A$ , for which Theorems 3.2.2 and 4.2.1 are shown. It is easy to see that Assumption 7 implies Assumption 4. Further if  $Y \in \mathcal{S}_{\text{loc}}(1)$ , then  $Y \in \mathcal{S}$  (see Remark A.2.2). Therefore the assumptions of Theorem 4.2.1 are more restrictive than those of Theorem 3.2.2 (see Table 2).

## A Subexponential Distributions

This section provides a brief overview of two classes of subexponential distributions on  $\mathbb{Z}_+$ . One is the class of “ordinal” subexponential distributions introduced by Chistyakov [7], and the other one is the class of “locally” subexponential distributions introduced by Chover et al. [8] and generalized by Asmussen et al. [4].

Table 2: The assumptions of the theorems in case of strictly substochastic  $A$ 

Theorem 3.2.2	Theorem 4.2.1
Assumption 2 (II)	Assumption 2 (II)
Assumption 4	Assumption 7
$Y \in \mathcal{S}$	$Y \in \mathcal{S}_{\text{loc}}(1)$
	$r_{A_-} > 1$ or $\{\mathbf{P}(Y = k)\}$ is ENI

In what follows, let  $U$  denote a random variable in  $\mathbb{Z}_+$  and  $U_j$  ( $j \in \mathbb{Z}_+$ ) denote independent copies of  $U$ . Let  $U_e$  denote the discrete equilibrium random variable of  $U$ , distributed with  $\mathbf{P}(U_e = k) = \mathbf{P}(U > k)/\mathbf{E}[U]$  ( $k \in \mathbb{Z}_+$ ). Further, for any  $h \in \mathbb{N} \cup \{\infty\}$ , let  $\Delta_h = (0, h]$  and  $k + \Delta_h = \{x \geq 0; k < x \leq k + h\}$  for  $k \in \mathbb{Z}_+$ .

## A.1 Ordinal subexponential class

We begin with the definition of the long-tailed class, which covers the subexponential class.

**Definition A.1.1 ([3, 9, 21])** A random variable  $U$  in  $\mathbb{Z}_+$  and its distribution are said to be long-tailed if  $\mathbf{P}(U > k) > 0$  for all  $k \in \mathbb{Z}_+$  and

$$\lim_{k \rightarrow \infty} \frac{\mathbf{P}(U > k + 1)}{\mathbf{P}(U > k)} = 1.$$

The class of long-tailed distributions is denoted by  $\mathcal{L}$ .

The following result is used to derive some of the asymptotic results presented in this paper.

**Proposition A.1 (Proposition A.1 in [19])** If  $U_e \in \mathcal{L}$ , then for any  $h \in \mathbb{N}$ ,  $l_0 \in \mathbb{Z}_+$  and  $\nu = 0, 1, \dots, h - 1$ ,

$$\frac{1}{\mathbf{E}[U]} \lim_{k \rightarrow \infty} \frac{\sum_{l=l_0}^{\infty} \mathbf{P}(U > k + lh + \nu)}{\mathbf{P}(U_e > k)} = \frac{1}{h}.$$

We now introduce the definition of the subexponential class.

**Definition A.1.2 ([7, 9, 21])** A random variable  $U$  and its distribution are said to be subexponential if  $\mathbf{P}(U > k) > 0$  for all  $k \in \mathbb{Z}_+$  and

$$\lim_{k \rightarrow \infty} \frac{\mathbf{P}(U_1 + U_2 > k)}{\mathbf{P}(U > k)} = 2.$$

The class of subexponential distributions is denoted by  $\mathcal{S}$ .

**Remark A.1.1**  $\mathcal{S} \subset \mathcal{L}$  (see, e.g., [21]), and there exists an example of not subexponential but long-tailed distributions (see [20]).

The following is a discrete analog of class  $\mathcal{S}^*$  introduced by Klüppelberg [16].

**Definition A.1.3** A random variable  $U$  and its distribution belong to class  $\mathcal{S}^*$  if  $P(U > k) > 0$  for all  $k \in \mathbb{Z}_+$  and

$$\lim_{k \rightarrow \infty} \sum_{l=0}^k \frac{P(U > k-l)P(U > l)}{P(U > k)} = 2E[U] < \infty. \quad (\text{A.1})$$

**Remark A.1.2** If  $U \in \mathcal{S}^*$ , then  $U \in \mathcal{S}$  and  $U_e \in \mathcal{S}$  (see Proposition A.2 in [19]).

## A.2 Locally subexponential class

We first introduce the locally long-tailed class, which is required by the definition of the locally subexponential class.

**Definition A.2.1 (Definition 1 in [4])** A random variable  $U$  and its distribution  $F$  are called *locally long-tailed* with span  $h \in \mathbb{N} \cup \{\infty\}$  if  $P(U \in k + \Delta_h) > 0$  for all sufficiently large  $k$  and

$$\lim_{k \rightarrow \infty} \frac{P(U \in k + 1 + \Delta_h)}{P(U \in k + \Delta_h)} = 1.$$

We denote by  $\mathcal{L}_{\text{loc}}(h)$  the class of locally long-tailed distributions with span  $h$  hereafter.

**Remark A.2.1** By definition,  $\mathcal{L}_{\text{loc}}(\infty) = \mathcal{L}$ . Further, if  $U \in \mathcal{L}_{\text{loc}}(1)$ , then  $U \in \mathcal{L}_{\text{loc}}(n)$  for all  $n = 2, 3, \dots$  and  $U \in \mathcal{L}$ .

The following proposition is a locally asymptotic version of Proposition A.1.

**Proposition A.2.1** Suppose (i)  $U \in \mathcal{L}_{\text{loc}}(1)$ ; or (ii)  $U \in \mathcal{L}$  and  $\{P(U = k)\}$  is eventually nonincreasing. Then for any  $h \in \mathbb{N}$ ,  $l_0 \in \mathbb{Z}_+$  and  $\nu = 0, 1, \dots, h-1$ ,

$$\lim_{k \rightarrow \infty} \frac{\sum_{l=l_0}^{\infty} P(U = k + lh + \nu)}{P(U > k)} = \frac{1}{h}. \quad (\text{A.2})$$

*Proof.* See Appendix B.8. □

**Definition A.2.2 (Definition 2 in [4])** A random variable  $U$  and its distribution  $F$  are called *locally subexponential* with span  $h \in \mathbb{N} \cup \{\infty\}$  if  $U \in \mathcal{L}_{\text{loc}}(h)$  and

$$\lim_{k \rightarrow \infty} \frac{P(U_1 + U_2 \in k + \Delta_h)}{P(U \in k + \Delta_h)} = 2.$$

We denote by  $\mathcal{S}_{\text{loc}}(h)$  the class of locally subexponential distributions with span  $h$ . Obviously,  $\mathcal{S}_{\text{loc}}(\infty)$  is equivalent to (ordinal) subexponential class  $\mathcal{S}$  (see Definition A.1.2). Further, Definition A.2.2 shows that  $\mathcal{S}_{\text{loc}}(h) \subset \mathcal{L}_{\text{loc}}(h)$ .

**Remark A.2.2** If  $U \in \mathcal{S}_{\text{loc}}(h)$  for some  $h \in \mathbb{N}$ , then  $U \in \mathcal{S}_{\text{loc}}(nh)$  for all  $n \in \mathbb{N}$  and  $U \in \mathcal{S}$  (see Remark 2 in [4]).

**Proposition A.2.2**  $U \in \mathcal{S}^*$  if and only if  $U_e \in \mathcal{S}_{\text{loc}}(1)$ .

*Proof.* The if-part is obvious. Indeed, since  $P(U_e = k) = P(U > k)/E[U]$  for  $k \in \mathbb{Z}_+$ , it follows that if  $U_e \in \mathcal{S}_{\text{loc}}(1)$ , then (A.1) holds, i.e.,  $U \in \mathcal{S}^*$ .

On the other hand, suppose (A.1) holds for  $h = 1$ . We then have

$$\lim_{k \rightarrow \infty} \sum_{l=0}^k \frac{P(U_e = k-l)P(U_e = l)}{P(U_e = k)} = 2.$$

Further  $U \in \mathcal{S} \subset \mathcal{L}$  (see Proposition A.2 in [19]) and thus

$$\lim_{k \rightarrow \infty} \frac{P(U > k+1)}{P(U > k)} = \lim_{k \rightarrow \infty} \frac{P(U_e = k+1)}{P(U_e = k)} = 1.$$

As a result,  $U_e \in \mathcal{S}_{\text{loc}}(1)$ . □

**Proposition A.2.3 (Proposition 3 in [4])** Suppose  $U \in \mathcal{S}_{\text{loc}}(h)$  for some  $h \in \mathbb{N} \cup \{\infty\}$  and let  $U^{(j)}$  ( $j \in \mathbb{N}$ ) denote independent random variables in  $\mathbb{Z}_+$  such that

$$\lim_{k \rightarrow \infty} \frac{P(U^{(j)} \in k + \Delta_h)}{P(U \in k + \Delta_h)} = c_j \in \mathbb{R}_+.$$

Then for  $n \in \mathbb{N}$ ,

$$\lim_{k \rightarrow \infty} \frac{P(U^{(1)} + U^{(2)} + \dots + U^{(n)} \in k + \Delta_h)}{P(U \in k + \Delta_h)} = \sum_{j=1}^n c_j.$$

Further, if  $\sum_{j=1}^n c_j > 0$ , then  $U^{(1)} + U^{(2)} + \dots + U^{(n)} \in \mathcal{S}_{\text{loc}}(h)$ .

**Proposition A.2.4** Let  $\{F(k); k \in \mathbb{Z}_+\}$  and  $\{F_j(k); k \in \mathbb{Z}_+\}$  ( $j = 1, 2, \dots, m$ ) denote probability mass functions. Suppose (i)  $F \in \mathcal{S}_{\text{loc}}(1)$ ; and (ii) for  $j = 1, 2, \dots, m$ ,

$$\lim_{k \rightarrow \infty} \frac{F_j(k)}{F(k)} = c_j \in \mathbb{R}_+. \quad (\text{A.3})$$

Then for any  $\varepsilon > 0$  there exists some  $C_\varepsilon \in (0, \infty)$  such that

$$F_1^{*n_1} * F_2^{*n_2} * \dots * F_m^{*n_m}(k) \leq C_\varepsilon (1 + \varepsilon)^{n_1 + n_2 + \dots + n_m} F(k), \quad (\text{A.4})$$

for all  $k > \sup\{k \in \mathbb{Z}_+; F(k) = 0\}$  and  $n_1, n_2, \dots, n_m \in \mathbb{N}$ .

*Proof.* See Appendix B.9.  $\square$

**Proposition A.2.5** For  $d_i \in \mathbb{N}$  ( $i = 0, 1, 2$ ), let  $\{\mathbf{P}(k); k \in \mathbb{Z}_+\}$  and  $\{\mathbf{Q}(k); k \in \mathbb{Z}_+\}$  denote nonnegative  $d_0 \times d_1$  and  $d_1 \times d_2$  matrix sequences, respectively, such that  $\mathbf{P} := \sum_{k=0}^{\infty} \mathbf{P}(k)$  and  $\mathbf{Q} := \sum_{k=0}^{\infty} \mathbf{Q}(k)$  are finite. Suppose that for some  $U \in \mathcal{S}_{\text{loc}}(1)$ ,

$$\lim_{k \rightarrow \infty} \frac{\mathbf{P}(k)}{\mathbf{P}(U = k)} = \tilde{\mathbf{P}} \geq \mathbf{O}, \quad \lim_{k \rightarrow \infty} \frac{\mathbf{Q}(k)}{\mathbf{P}(U = k)} = \tilde{\mathbf{Q}} \geq \mathbf{O}.$$

We then have

$$\lim_{k \rightarrow \infty} \frac{\mathbf{P} * \mathbf{Q}(k)}{\mathbf{P}(U = k)} = \tilde{\mathbf{P}}\mathbf{Q} + \mathbf{P}\tilde{\mathbf{Q}}.$$

*Proof.* This proposition can be proved in the same way as Proposition A.3 in [19], and thus the proof is omitted.  $\square$

**Proposition A.2.6** Let  $\{\mathbf{W}(k); k \in \mathbb{Z}_+\}$  denote a sequence of (finite dimensional) nonnegative square matrices such that  $\sum_{n=0}^{\infty} \mathbf{W}^n = (\mathbf{I} - \mathbf{W})^{-1} < \infty$ , where  $\mathbf{W} = \sum_{k=0}^{\infty} \mathbf{W}(k)$ . If there exists some  $U \in \mathcal{S}_{\text{loc}}(1)$  such that

$$\lim_{k \rightarrow \infty} \frac{\mathbf{W}(k)}{\mathbf{P}(U = k)} = \tilde{\mathbf{W}} \geq \mathbf{O},$$

then

$$\lim_{k \rightarrow \infty} \frac{\sum_{n=0}^{\infty} \mathbf{W}^{*n}(k)}{\mathbf{P}(U = k)} = (\mathbf{I} - \mathbf{W})^{-1} \tilde{\mathbf{W}} (\mathbf{I} - \mathbf{W})^{-1}.$$

*Proof.* Using Proposition A.2.5, we can readily prove, by induction, that

$$\lim_{k \rightarrow \infty} \frac{\mathbf{W}^{*n}(k)}{\mathbf{P}(U = k)} = \sum_{l=0}^{n-1} \mathbf{W}^l \tilde{\mathbf{W}} \mathbf{W}^{n-l-1}. \quad (\text{A.5})$$

Further it follows from Proposition A.2.4 that for any  $\varepsilon > 0$  there exist some  $k_0 \in \mathbb{Z}_+$  and some  $C_\varepsilon \in (0, \infty)$  such that for all  $k \geq k_0$  and  $n \in \mathbb{N}$ ,

$$\frac{[\mathbf{W}^{*n}(k)]_{i,j}}{\mathbf{P}(U = k)} \leq C_\varepsilon (1 + \varepsilon)^n [\mathbf{W}^n]_{i,j}.$$

Note here that  $\text{sp}(\mathbf{W}) < 1$  and thus  $\sum_{n=1}^{\infty} (1 + \varepsilon)^n \mathbf{W}^n < \infty$  for any sufficiently small  $\varepsilon > 0$ .

As a result, using the dominated convergence theorem and (A.5), we obtain

$$\begin{aligned} \lim_{k \rightarrow \infty} \frac{\sum_{n=0}^{\infty} \mathbf{W}^{*n}(k)}{\mathbf{P}(U = k)} &= \lim_{k \rightarrow \infty} \frac{\mathbf{W}^{*0}(k)}{\mathbf{P}(U = k)} + \sum_{n=1}^{\infty} \lim_{k \rightarrow \infty} \frac{\mathbf{W}^{*n}(k)}{\mathbf{P}(U = k)} \\ &= \sum_{n=1}^{\infty} \sum_{l=0}^{n-1} \mathbf{W}^l \tilde{\mathbf{W}} \mathbf{W}^{n-l-1} \\ &= (\mathbf{I} - \mathbf{W})^{-1} \tilde{\mathbf{W}} (\mathbf{I} - \mathbf{W})^{-1}. \end{aligned}$$

$\square$

## B Proofs

### B.1 Proof of Proposition 2.1.4

Equation (2.8) yields

$$\det(\mathbf{I} - \mathbf{A}) = \det(\mathbf{I} - \mathbf{R}) \det(\mathbf{I} - \Phi(0)) \det(\mathbf{I} - \mathbf{G}).$$

It thus follows from  $\text{sp}(\mathbf{A}) < 1$  that

$$\det(\mathbf{I} - \mathbf{G}) \neq 0, \quad \det(\mathbf{I} - \mathbf{R}) \neq 0. \quad (\text{B.1})$$

Note here that by definition,

$$\sum_{k=1}^N \sum_{j \in \mathbb{M}} [\mathbf{G}(k)]_{i,j} = \mathbb{P}(T_{<N} < \infty \mid X_0 = N, S_0 = i), \quad \text{for all } N \in \mathbb{N},$$

which shows that  $\mathbf{G}\mathbf{e} \leq \mathbf{e}$  and thus  $\text{sp}(\mathbf{G}) \leq 1$  (see Theorem 8.1.22 in [11]). Further,  $\text{sp}(\mathbf{R}) \leq 1$  due to the duality of the  $R$ - and  $G$ -matrices (see [26]). Therefore, it follows from Theorem 8.3.1 in [11] and (B.1) that (i)  $\text{sp}(\mathbf{G}) < 1$  and (ii)  $\text{sp}(\mathbf{R}) < 1$ .

Finally, we prove (iii). From (2.1), we have

$$\Phi(-k) \geq \mathbf{O}, \quad \mathbf{0} \leq \sum_{l=0}^{k-1} \Phi(-l)\mathbf{e} \leq \mathbf{e}, \quad \text{for all } k \in \mathbb{N},$$

which implies that  $\text{sp}(\sum_{l=0}^{\infty} \Phi(-l)) \leq 1$  (see Theorem 8.1.22 in [11]). Thus it suffices to prove that  $\sum_{l=0}^{\infty} \Phi(-l)$  does not have the eigenvalue *one*. Indeed, (2.2) yields

$$(\mathbf{I} - \Phi(0))(\mathbf{I} - \mathbf{G}) = \mathbf{I} - \sum_{l=0}^{\infty} \Phi(-l).$$

Therefore we have  $\det(\mathbf{I} - \sum_{l=0}^{\infty} \Phi(-l)) \neq 0$  because  $\mathbf{I} - \Phi(0)$  is nonsingular and  $\text{sp}(\mathbf{G}) < 1$ .

### B.2 Proof of Proposition 2.5.1

We prove this proposition by reduction to absurdity. To do so, we suppose either (i)  $\mathbf{G}$  is strictly lower triangular, or (ii)  $\mathbf{G}$  takes a form such that

$$\mathbf{G} = \begin{pmatrix} \mathbf{G}_1 & \mathbf{O} & \mathbf{O} \\ \mathbf{G}_{2,1} & \mathbf{G}_2 & \mathbf{O} \\ \mathbf{G}_{3,1} & \mathbf{G}_{3,2} & \mathbf{G}_3 \end{pmatrix}, \quad (\text{B.2})$$

where  $\mathbf{G}_i$  ( $i = 1, 2$ ) is irreducible and  $\mathbf{G}_2$  can be equal to  $\mathbf{G}_T$  (in that case, the last block row and column vanish). If (i) is true, then  $\mathbf{G}$  is a nilpotent matrix, which is inconsistent with  $\delta(\mathbf{G}) = 1$ .

In what follows, we consider case (ii). For simplicity, we partition the phase set  $\mathbb{M}$  into subsets  $\mathbb{M}_1$ ,  $\mathbb{M}_2$  and  $\mathbb{M}_3$  corresponding to  $\mathbf{G}_1$ ,  $\mathbf{G}_2$  and  $\mathbf{G}_3$ , respectively. Further we write  $(k, i) \xrightarrow{\neq(0,*)} (l, j)$  ( $k, l \in \mathbb{N}; i, j \in \mathbb{M}$ ) when state  $(l, j)$  can be reached from state  $(k, i)$  avoiding level zero.

Let  $\mathbf{G}_2(k)$  denote a submatrix of  $\mathbf{G}(k)$  such that  $\sum_{k=1}^{\infty} \mathbf{G}_2(k) = \mathbf{G}_2$ . The irreducibility of  $\mathbf{G}_2$  shows that  $\sum_{k=1}^{K_G} \mathbf{G}_2(k)$  is irreducible for some  $K_G \in \mathbb{N}$ . Thus for any  $i_2 \in \mathbb{M}_2$ , there exists some  $(k'_2, i'_2) \in \mathbb{N} \times \mathbb{M}_2$  such that

$$(k'_2, i'_2) \xrightarrow{\neq(0,*)} (1, i_2).$$

Similarly,  $\sum_{k=-K_A}^{\infty} \mathbf{A}(k)$  is irreducible for some  $K_A \in \mathbb{N}$  due to the irreducibility of  $\mathbf{A}$ , and thus there exists some  $(k_1, i_1) \in \mathbb{N} \times \mathbb{M}_1$  such that

$$(k_1, i_1) \xrightarrow{\neq(0,*)} (k'_2, i'_2).$$

As a result,

$$(k_1, i_1) \xrightarrow{\neq(0,*)} (k'_2, i'_2) \xrightarrow{\neq(0,*)} (1, i_2), \quad i_1 \in \mathbb{M}_1, \quad i_2, i'_2 \in \mathbb{M}_2,$$

which contradicts to the structure of  $\mathbf{G}$  shown in (B.2).

### B.3 Proof of Proposition 2.5.2

From Theorem 8.1.18 in [11], we have

$$\text{sp}(\widehat{\mathbf{R}}(\omega)) \leq \delta(\widehat{\mathbf{R}}(1)) < 1, \tag{B.3}$$

where the second inequality is due to the positive-recurrence of  $\mathbf{T}$  (see Theorem 3.4 in [25]). It follows from (2.8), (B.3) and  $\text{sp}(\widehat{\Phi}(0)) < 1$  that

$$\det(\mathbf{I} - \widehat{\mathbf{A}}(\omega)) = 0 \iff \det(\mathbf{I} - \widehat{\mathbf{G}}(\omega)) = 0.$$

Note here that  $\text{sp}(\widehat{\mathbf{A}}(\omega)) \leq \delta(\widehat{\mathbf{A}}(1)) = 1$  and  $\text{sp}(\widehat{\mathbf{G}}(\omega)) \leq \delta(\widehat{\mathbf{G}}(1)) = 1$  (see Theorem 8.1.18 in [11]). Thus

$$\begin{aligned} \det(\mathbf{I} - \widehat{\mathbf{A}}(\omega)) = 0 &\iff \delta(\widehat{\mathbf{A}}(\omega)) = 1, \\ \det(\mathbf{I} - \widehat{\mathbf{G}}(\omega)) = 0 &\iff \delta(\widehat{\mathbf{G}}(\omega)) = 1. \end{aligned}$$

As a result,  $\delta(\widehat{\mathbf{G}}(\omega)) = 1$  if and only if  $\delta(\widehat{\mathbf{A}}(\omega)) = 1$ . Finally, the statement (i) follows from (2.21) and Proposition 2.4.2.

Since the statement (i) is proved, we readily obtain the statements (ii) and (iii) by applying Theorem B.1 in [14] to the MAdP  $\{(\check{X}_n^{(G)}, \check{S}_n^{(G)})\}$  and using (2.19). Further, the statement (iv) is an immediate consequence of (2.19) and Lemma B.3 in [14].

### B.4 Proof of Proposition 2.5.3

Since  $\mathbf{A}$  is stochastic, it follows from Propositions 2.4.1 and 2.5.2 that for  $\nu = 0, 1, \dots, \tau - 1$ ,

$$\delta(\widehat{\mathbf{A}}(\omega_\tau^\nu)) = \delta(\widehat{\mathbf{A}}(1)) = 1, \quad \delta(\widehat{\mathbf{G}}(\omega_\tau^\nu)) = \delta(\widehat{\mathbf{G}}(1)) = 1, \quad (\text{B.4})$$

$$\boldsymbol{\mu}(\omega_\tau^\nu) = \boldsymbol{\pi} \boldsymbol{\Delta}_M(\omega_\tau^\nu)^{-1}, \quad \mathbf{v}(\omega_\tau^\nu) = \boldsymbol{\Delta}_M(\omega_\tau^\nu) \mathbf{e}, \quad (\text{B.5})$$

where we use  $\boldsymbol{\mu}(1) = \boldsymbol{\pi}$  and  $\mathbf{v}(1) = \mathbf{e}$ . Therefore, (2.24) and the second equation in (B.5) yield  $\mathbf{v}(\omega_\tau^\nu) = \boldsymbol{\Delta}_M(\omega_\tau^\nu) \mathbf{e} = \mathbf{y}(\omega_\tau^\nu)$ .

We now define  $\tilde{\boldsymbol{\psi}}(\omega_\tau^\nu)$  as

$$\tilde{\boldsymbol{\psi}}(\omega_\tau^\nu) = \frac{\boldsymbol{\mu}(\omega_\tau^\nu)(\mathbf{I} - \widehat{\mathbf{R}}(\omega_\tau^\nu))(\mathbf{I} - \boldsymbol{\Phi}(0))}{\boldsymbol{\mu}(\omega_\tau^\nu)(\mathbf{I} - \widehat{\mathbf{R}}(\omega_\tau^\nu))(\mathbf{I} - \boldsymbol{\Phi}(0))\mathbf{v}(\omega_\tau^\nu)}, \quad \nu = 0, 1, \dots, \tau - 1. \quad (\text{B.6})$$

It can be shown that  $\tilde{\boldsymbol{\psi}}(\omega_\tau^\nu) = \boldsymbol{\psi}(\omega_\tau^\nu)$ , whose proof is given later. From (2.8), we have

$$\mathbf{I} - \mathbf{G}(\omega_\tau^\nu) = (\mathbf{I} - \boldsymbol{\Phi}(0))^{-1}(\mathbf{I} - \widehat{\mathbf{R}}(\omega_\tau^\nu))^{-1}(\mathbf{I} - \widehat{\mathbf{A}}(\omega_\tau^\nu)).$$

Pre-multiplying (resp. post-multiplying) the above equation by  $\tilde{\boldsymbol{\psi}}(\omega_\tau^\nu)$  (resp.  $\mathbf{v}(\omega_\tau^\nu)$ ) and using (B.4), we can readily verify that  $\tilde{\boldsymbol{\psi}}(\omega_\tau^\nu) (= \boldsymbol{\psi}(\omega_\tau^\nu))$  and  $\mathbf{v}(\omega_\tau^\nu) = \mathbf{y}(\omega_\tau^\nu)$  are the left- and right-eigenvectors of  $\widehat{\mathbf{G}}(\omega_\tau^\nu)$  corresponding to the eigenvalue  $\delta(\widehat{\mathbf{G}}(\omega_\tau^\nu)) = 1$ . As a result, the statement (i) holds.

As for the statement (ii), it follows from the second equation in (B.5) and (B.6) that

$$\tilde{\boldsymbol{\psi}}(\omega_\tau^\nu) \boldsymbol{\Delta}_M(\omega_\tau^\nu) \mathbf{e} = \tilde{\boldsymbol{\psi}}(\omega_\tau^\nu) \mathbf{v}(\omega_\tau^\nu) = 1.$$

Therefore the statement (ii) can be proved in the same way as the proof of Lemma 3.2 in [14].

In what follows, we prove  $\tilde{\boldsymbol{\psi}}(\omega_\tau^\nu) = \boldsymbol{\psi}(\omega_\tau^\nu)$ . For this purpose, we first show that

$$\sum_{l=0}^{\infty} (\omega_\tau^\nu)^l \boldsymbol{\Phi}(l) = \boldsymbol{\Delta}_M(\omega_\tau^\nu) \sum_{l=0}^{\infty} \boldsymbol{\Phi}(l) \boldsymbol{\Delta}_M(\omega_\tau^\nu)^{-1}. \quad (\text{B.7})$$

The definition of  $\boldsymbol{\Phi}(l)$  ( $l \in \mathbb{Z}_+$ ) implies

$$[\boldsymbol{\Phi}(l)]_{i,j} = \mathbf{P}(X_{T_{\downarrow l+1}} = l+1, S_{T_{\downarrow l+1}} = j \mid X_0 = 1, S_0 = i),$$

where  $T_{\downarrow l+1} = \inf\{n \in \mathbb{N}; X_n = l+1 < X_m \ (m = 1, 2, \dots, n-1)\}$ . Further (2.17) and (2.18) imply that for all  $n \in \mathbb{N}$ , the following probability is positive only if  $l \equiv p(j) - p(i) \pmod{\tau}$ :

$$\mathbf{P}(X_n = l+1, S_n = j, X_m \geq 1 \ (m = 1, 2, \dots, n-1) \mid X_0 = 1, S_0 = i).$$

Thus  $[\boldsymbol{\Phi}(l)]_{i,j} > 0$  only if  $l \equiv p(j) - p(i) \pmod{\tau}$ , which leads to

$$\sum_{l=0}^{\infty} z^l \boldsymbol{\Phi}(l) = \boldsymbol{\Delta}_M(z) \boldsymbol{\Lambda}_\Phi(z^\tau) \boldsymbol{\Delta}_M(z)^{-1}, \quad (\text{B.8})$$

where  $\Lambda_\Phi(z)$  denotes an  $M \times M$  matrix whose  $(i, j)$ th element is given by

$$[\Lambda_\Phi(z)]_{i,j} = \sum_{\substack{n \in \mathbb{Z}_+ \\ n\tau + p(j) - p(i) \geq 0}} z^n [\Phi(n\tau + p(j) - p(i))]_{i,j}.$$

As a result, (B.8) yields (B.7) because  $\Lambda_\Phi(1) = \sum_{l=0}^{\infty} \Phi(l)$ .

We now return to the proof of  $\tilde{\psi}(\omega_\tau^\nu) = \psi(\omega_\tau^\nu)$ . From (2.5) and (B.7), we have for  $\nu = 0, 1, \dots, \tau - 1$ ,

$$\begin{aligned} & (\mathbf{I} - \widehat{\mathbf{R}}(\omega_\tau^\nu))(\mathbf{I} - \Phi(0)) \\ &= \mathbf{I} - \sum_{l=0}^{\infty} (\omega_\tau^\nu)^l \Phi(l) \\ &= \Delta_M(\omega_\tau^\nu) \left( \mathbf{I} - \sum_{l=0}^{\infty} \Phi(l) \right) \Delta_M(\omega_\tau^\nu)^{-1} \\ &= \Delta_M(\omega_\tau^\nu) (\mathbf{I} - \mathbf{R})(\mathbf{I} - \Phi(0)) \Delta_M(\omega_\tau^\nu)^{-1}, \end{aligned} \tag{B.9}$$

where the last equality follows from the first equality with  $\nu = 0$ . Substituting (B.5) and (B.9) into (B.6) yields

$$\tilde{\psi}(\omega_\tau^\nu) = \frac{\pi(\mathbf{I} - \mathbf{R})(\mathbf{I} - \Phi(0))}{\pi(\mathbf{I} - \mathbf{R})(\mathbf{I} - \Phi(0))\mathbf{e}} \Delta_M(\omega_\tau^\nu)^{-1} = \psi(\omega_\tau^\nu).$$

## B.5 Proof of Lemma 3.1.2

From (2.4), we have

$$\widehat{\mathbf{L}}(1/z) = \frac{\text{adj}(\mathbf{I} - \widehat{\mathbf{G}}(1/z))}{\det(\mathbf{I} - \widehat{\mathbf{G}}(1/z))} - \mathbf{I}. \tag{B.10}$$

Note here that

$$\begin{aligned} \left| [\widehat{\mathbf{G}}(1/z)]_{i,j} \right| &= \left| \sum_{k=1}^{\infty} z^k [\mathbf{G}(k)]_{i,j} \right| \leq [\mathbf{G}]_{i,j}, & i, j \in \mathbb{M}, |z| \leq 1, \\ \text{sp}(\widehat{\mathbf{G}}(1/z)) &< \text{sp}(\mathbf{G}) = 1, & |z| < 1. \end{aligned}$$

It then follows from Proposition 2.5.2 that  $\{\omega_\tau^\nu; \nu = 0, 1, \dots, \tau - 1\}$  are the simple minimum-modulus poles of  $\widehat{\mathbf{L}}(1/z)$ . Therefore applying Theorem A.1 in [14] to (B.10), we obtain

$$\mathbf{L}(k) = \sum_{\nu=0}^{\tau-1} \frac{1}{(\omega_\tau^\nu)^k} \lim_{z \rightarrow \omega_\tau^\nu} \left( 1 - \frac{z}{\omega_\tau^\nu} \right) \frac{\text{adj}(\mathbf{I} - \widehat{\mathbf{G}}(1/z))}{\det(\mathbf{I} - \widehat{\mathbf{G}}(1/z))} + O((1 + \varepsilon_0)^{-k}) \mathbf{e}^\dagger \mathbf{e}, \tag{B.11}$$

for some  $\varepsilon_0 > 0$ , where  $f(x) = O(g(x))$  represents  $\limsup_{x \rightarrow \infty} |f(x)/g(x)| < \infty$ . Further it follows from l'Hôpital's rule and Proposition 2.5.3 that for  $\nu = 0, 1, \dots, \tau - 1$ ,

$$\begin{aligned}
& \lim_{z \rightarrow \omega_\tau^\nu} \left(1 - \frac{z}{\omega_\tau^\nu}\right) \frac{\text{adj}(\mathbf{I} - \widehat{\mathbf{G}}(1/z))}{\det(\mathbf{I} - \widehat{\mathbf{G}}(1/z))} \\
&= \lim_{z \rightarrow \omega_\tau^\nu} \frac{1 - \frac{z}{\omega_\tau^\nu}}{1 - \delta(\widehat{\mathbf{G}}(1/z))} \frac{\text{adj}(\mathbf{I} - \widehat{\mathbf{G}}(\omega_\tau^{-\nu}))}{\prod_{i=2}^M (1 - \lambda_i^{(G)}(\omega_\tau^{-\nu}))} \\
&= \frac{1}{\omega_\tau^\nu \cdot (d/dz)\delta(\widehat{\mathbf{G}}(1/z))|_{z=\omega_\tau^\nu}} \cdot \mathbf{y}(\omega_\tau^{-\nu}) \boldsymbol{\psi}(\omega_\tau^{-\nu}) \\
&= \frac{1}{\omega_\tau^\nu \cdot (d/dz)\delta(\widehat{\mathbf{G}}(1/z))|_{z=\omega_\tau^\nu}} \cdot \boldsymbol{\Delta}_M(\omega_\tau^{-\nu}) \frac{\mathbf{e}\boldsymbol{\psi}}{\boldsymbol{\psi}\mathbf{e}} \boldsymbol{\Delta}_M(\omega_\tau^{-\nu})^{-1}, \tag{B.12}
\end{aligned}$$

where the last equality is due to (2.23), (2.24) and (3.3). Letting  $y = 1/z$ , we have

$$\omega_\tau^\nu \frac{d}{dz} \delta(\widehat{\mathbf{G}}(1/z)) \Big|_{z=\omega_\tau^\nu} = - \frac{1}{\omega_\tau^\nu} \frac{d}{dy} \delta(\widehat{\mathbf{G}}(y)) \Big|_{y=1/\omega_\tau^\nu} = - \frac{d}{dy} \delta(\widehat{\mathbf{G}}(y)) \Big|_{y=1}, \tag{B.13}$$

where the second equality is due to Proposition 2.5.2 (iv). Applying (B.13) to (B.12) yields

$$\begin{aligned}
& \lim_{z \rightarrow \omega_\tau^\nu} \left(1 - \frac{z}{\omega_\tau^\nu}\right) \frac{\text{adj}(\mathbf{I} - \widehat{\mathbf{G}}(1/z))}{\det(\mathbf{I} - \widehat{\mathbf{G}}(1/z))} \\
&= \frac{-1}{(d/dy)\delta(\widehat{\mathbf{G}}(y))|_{y=1}} \cdot \boldsymbol{\Delta}_M(\omega_\tau^{-\nu}) \frac{\mathbf{e}\boldsymbol{\psi}}{\boldsymbol{\psi}\mathbf{e}} \boldsymbol{\Delta}_M(\omega_\tau^{-\nu})^{-1}. \tag{B.14}
\end{aligned}$$

In what follows, we calculate  $(d/dy)\delta(\widehat{\mathbf{G}}(y))|_{y=1}$ . Taking the derivative of both sides of (2.22) with  $z = y$ , letting  $y = 1$  and using  $\delta(\widehat{\mathbf{G}}(1)) = 1$ , we have

$$\frac{d}{dy} \delta(\widehat{\mathbf{G}}(y)) \Big|_{y=1} = - \frac{1}{\prod_{i=2}^M (1 - \lambda_i^{(G)}(1))} \cdot \frac{d}{dy} \det(\mathbf{I} - \widehat{\mathbf{G}}(y)) \Big|_{y=1}. \tag{B.15}$$

Similarly, from  $\det(\mathbf{I} - \widehat{\mathbf{G}}(y)) = \boldsymbol{\pi} \cdot \text{adj}(\mathbf{I} - \widehat{\mathbf{G}}(y))(\mathbf{I} - \widehat{\mathbf{G}}(y)) \cdot \mathbf{e}$ , we obtain

$$\frac{d}{dy} \det(\mathbf{I} - \widehat{\mathbf{G}}(y)) \Big|_{y=1} = \boldsymbol{\pi} \cdot \text{adj}(\mathbf{I} - \mathbf{G}) \sum_{k=1}^{\infty} k \mathbf{G}(k) \mathbf{e}, \tag{B.16}$$

where we use  $\mathbf{G}\mathbf{e} = \mathbf{e}$ . Note here that Proposition 2.5.3 and (3.3) imply

$$\text{adj}(\mathbf{I} - \mathbf{G}) = \frac{\mathbf{e}\boldsymbol{\psi}}{\boldsymbol{\psi}\mathbf{e}} \cdot \prod_{i=2}^M (1 - \lambda_i^{(G)}(1)).$$

It thus follows from (B.16) and Lemma 3.1.1 that

$$\begin{aligned} \frac{d}{dy} \det(\mathbf{I} - \widehat{\mathbf{G}}(y)) \Big|_{y=1} &= \frac{\psi}{\psi \mathbf{e}} \sum_{k=1}^{\infty} k \mathbf{G}(k) \mathbf{e} \cdot \prod_{i=2}^M (1 - \lambda_i^{(G)}(1)) \\ &= \frac{1}{\psi \mathbf{e}} \cdot \prod_{i=2}^M (1 - \lambda_i^{(G)}(1)), \end{aligned} \quad (\text{B.17})$$

where the second equality is due to  $\psi \sum_{k=1}^{\infty} k \mathbf{G}(k) \mathbf{e} = 1$  (see (3.1) and (3.3)). Further substituting (B.17) into (B.15) yields

$$\frac{d}{dy} \delta(\widehat{\mathbf{G}}(y)) \Big|_{y=1} = -\frac{1}{\psi \mathbf{e}},$$

from which and (B.14), we have

$$\lim_{z \rightarrow \omega_{\tau}^{\nu}} \left( 1 - \frac{z}{\omega_{\tau}^{\nu}} \right) \frac{\text{adj}(\mathbf{I} - \widehat{\mathbf{G}}(1/z))}{\det(\mathbf{I} - \widehat{\mathbf{G}}(1/z))} = \Delta_M(\omega_{\tau}^{-\nu}) \mathbf{e} \psi \Delta_M(\omega_{\tau}^{-\nu})^{-1}. \quad (\text{B.18})$$

Finally, we have (3.2) by substituting (B.18) into (B.11) and letting  $k = n\tau + l$ .

## B.6 Proof of Lemma 3.1.4

Equations (3.7) and (3.9) show that for any  $\varepsilon > 0$  there exists some  $m_* := m_*(\varepsilon) \in \mathbb{N}$  such that for all  $m \geq m_*$  and  $l = 0, 1, \dots, \tau - 1$ ,

$$\mathbf{e}(\tau \psi - \varepsilon \mathbf{e}^t) \leq \sum_{l=0}^{\tau-1} \mathbf{L}(\lfloor m/\tau \rfloor \tau + l) \leq \mathbf{e}(\tau \psi + \varepsilon \mathbf{e}^t), \quad (\text{B.19})$$

$$\frac{1}{\mathbb{E}[Y]} (\mathbf{c}_A - \varepsilon \mathbf{e}) \leq \frac{\overline{\mathbf{A}}(\lfloor m/\tau \rfloor \tau + l) \mathbf{e}}{\mathbb{P}(Y > m)} \leq \frac{1}{\mathbb{E}[Y]} (\mathbf{c}_A + \varepsilon \mathbf{e}). \quad (\text{B.20})$$

Further since  $Y_e \in \mathcal{L}$  and  $\mathbf{L}(m) \leq \mathbf{e} \mathbf{e}^t$  for all  $m = 1, 2, \dots$ , we have

$$\begin{aligned} & \limsup_{k \rightarrow \infty} \sum_{m=1}^{m_*-1} \frac{\overline{\mathbf{A}}(k+m) \mathbf{L}(m)}{\mathbb{P}(Y_e > k)} \\ & \leq \sum_{m=1}^{m_*-1} \limsup_{k \rightarrow \infty} \frac{\overline{\mathbf{A}}(k+m) \mathbf{e} \mathbf{e}^t}{\mathbb{P}(Y > k+m)} \limsup_{k \rightarrow \infty} \frac{\mathbb{P}(Y > k+m)}{\mathbb{P}(Y_e > k+m)} \\ & \quad \times \limsup_{k \rightarrow \infty} \frac{\mathbb{P}(Y_e > k+m)}{\mathbb{P}(Y_e > k)} \\ & = \mathbf{O}, \end{aligned} \quad (\text{B.21})$$

where the last equality follows from (3.9) and the fact that  $Y_e \in \mathcal{L}$  has a heavier tail than  $Y$  (see Corollary 3.3 in [21]).

On the other hand,

$$\begin{aligned}
\sum_{m=m_*}^{\infty} \frac{\bar{\mathbf{A}}(k+m)\mathbf{L}(m)}{\mathbf{P}(Y_e > k)} &\leq \sum_{m'=\lfloor m_*/\tau \rfloor}^{\infty} \sum_{l=0}^{\tau-1} \frac{\bar{\mathbf{A}}(k+m'\tau+l)\mathbf{L}(m'\tau+l)}{\mathbf{P}(Y_e > k)} \\
&\leq \sum_{m'=\lfloor m_*/\tau \rfloor}^{\infty} \frac{\bar{\mathbf{A}}(k+m'\tau)}{\mathbf{P}(Y_e > k)} \sum_{l=0}^{\tau-1} \mathbf{L}(m'\tau+l) \\
&\leq \sum_{m'=\lfloor m_*/\tau \rfloor}^{\infty} \frac{\bar{\mathbf{A}}(k+m'\tau)e}{\mathbf{P}(Y_e > k)} (\tau\psi + \varepsilon e^t), \tag{B.22}
\end{aligned}$$

where the second inequality holds because  $\{\bar{\mathbf{A}}(k); k \in \mathbb{Z}_+\}$  is nonincreasing, and where the last inequality is due to (B.19). Note here that (3.9) implies for all sufficiently large  $k$ ,

$$\begin{aligned}
&\sum_{m'=\lfloor m_*/\tau \rfloor}^{\infty} \frac{\bar{\mathbf{A}}(k+m'\tau)e}{\mathbf{P}(Y_e > k)} \\
&\leq (c_A + \varepsilon e) \cdot \frac{1}{\mathbb{E}[Y]} \sum_{m'=\lfloor m_*/\tau \rfloor}^{\infty} \frac{\mathbf{P}(Y > k+m'\tau)}{\mathbf{P}(Y_e > k)},
\end{aligned}$$

from which and Proposition A.1 it follows that

$$\limsup_{k \rightarrow \infty} \sum_{m'=\lfloor m_*/\tau \rfloor}^{\infty} \frac{\bar{\mathbf{A}}(k+m'\tau)e}{\mathbf{P}(Y_e > k)} \leq \frac{c_A + \varepsilon e}{\tau}. \tag{B.23}$$

Combining (B.22) and (B.23) and letting  $\varepsilon \downarrow 0$  yield

$$\limsup_{k \rightarrow \infty} \sum_{m=m_*}^{\infty} \frac{\bar{\mathbf{A}}(k+m)\mathbf{L}(m)}{\mathbf{P}(Y_e > k)} \leq c_A \psi. \tag{B.24}$$

As a result, from (B.21) and (B.24), we have

$$\limsup_{k \rightarrow \infty} \sum_{m=1}^{\infty} \frac{\bar{\mathbf{A}}(k+m)\mathbf{L}(m)}{\mathbf{P}(Y_e > k)} \leq c_A \psi. \tag{B.25}$$

Next we consider the lower limit. It follows from (B.19) and (B.20) that

$$\begin{aligned}
\sum_{m=1}^{\infty} \frac{\bar{\mathbf{A}}(k+m)\mathbf{L}(m)}{\mathbf{P}(Y_e > k)} &\geq \sum_{m=m_*}^{\infty} \frac{\bar{\mathbf{A}}(k+m)\mathbf{L}(m)}{\mathbf{P}(Y_e > k)} \\
&\geq \sum_{m'=\lfloor m_*/\tau \rfloor+1}^{\infty} \sum_{l=0}^{\tau-1} \frac{\bar{\mathbf{A}}(k+m'\tau+l)\mathbf{L}(m'\tau+l)}{\mathbf{P}(Y_e > k)} \\
&\geq \sum_{m'=\lfloor m_*/\tau \rfloor+1}^{\infty} \frac{\bar{\mathbf{A}}(k+m'\tau+\tau)}{\mathbf{P}(Y_e > k)} \sum_{l=0}^{\tau-1} \mathbf{L}(m'\tau+l) \\
&\geq \sum_{m'=\lfloor m_*/\tau \rfloor+2}^{\infty} \frac{\bar{\mathbf{A}}(k+m'\tau)e}{\mathbf{P}(Y_e > k)} (\tau\psi - \varepsilon e^t), \tag{B.26}
\end{aligned}$$

where the third inequality requires the fact that  $\{\bar{\mathbf{A}}(k)\}$  is nonincreasing. Further the following can be shown in a very similar way to (B.23):

$$\liminf_{k \rightarrow \infty} \sum_{m'=\lfloor m_*/\tau \rfloor + 2}^{\infty} \frac{\bar{\mathbf{A}}(k + m'\tau)\mathbf{e}}{\mathbf{P}(Y_e > k)} \geq \frac{\mathbf{c}_A - \varepsilon \mathbf{e}}{\tau}.$$

Combining this with (B.26) and letting  $\varepsilon \downarrow 0$  yield

$$\liminf_{k \rightarrow \infty} \sum_{m=1}^{\infty} \frac{\bar{\mathbf{A}}(k + m)\mathbf{L}(m)}{\mathbf{P}(Y_e > k)} \geq \mathbf{c}_A \psi. \quad (\text{B.27})$$

Finally, (3.10) follows from (B.25), (B.27) and (3.3). Equation (3.11) can be proved in the same way, and thus the proof is omitted.

## B.7 Proof of Lemma 4.1.1

We give the proof of (4.2) only. Equation (4.3) can be proved in the same way. It follows from (3.4),  $\mathbf{E}\mathbf{e} = \mathbf{e}$  and (5) that for  $\varepsilon > 0$  there exists some  $m_* := m_*(\varepsilon) \in \mathbb{N}$  such that for all  $m \geq m_*$  and  $l = 0, 1, \dots, \tau - 1$ ,

$$\mathbf{E}(\tau \mathbf{H}_l - \varepsilon \mathbf{e} \mathbf{e}^t) \leq \mathbf{L}(m) \leq \mathbf{E}(\tau \mathbf{H}_l + \varepsilon \mathbf{e} \mathbf{e}^t), \quad m \equiv l \pmod{\tau}, \quad (\text{B.28})$$

$$\frac{1}{\mathbf{E}[Y]}(\mathbf{C}_A^{\mathbf{E}} - \varepsilon \mathbf{e} \mathbf{e}^t) \leq \frac{\mathbf{A}(m)\mathbf{E}}{\mathbf{P}(Y = m)} \leq \frac{1}{\mathbf{E}[Y]}(\mathbf{C}_A^{\mathbf{E}} + \varepsilon \mathbf{e} \mathbf{e}^t). \quad (\text{B.29})$$

Thus from (4.1),  $\mathbf{L}(m) \leq \mathbf{E} \mathbf{e} \mathbf{e}^t$  and  $Y \in \mathcal{L}$  (see Remark A.2.1), we have

$$\begin{aligned} & \lim_{k \rightarrow \infty} \sum_{m=1}^{m_*-1} \frac{\mathbf{A}(k+m)\mathbf{L}(m)}{\mathbf{P}(Y_e = k)} \\ & \leq \mathbf{E}[Y] \sum_{m=1}^{m_*-1} \lim_{k \rightarrow \infty} \frac{\mathbf{A}(k+m)\mathbf{E} \mathbf{e} \mathbf{e}^t}{\mathbf{P}(Y = k+m)} \frac{\mathbf{P}(Y = k+m)}{\mathbf{P}(Y > k)} = \mathbf{O}. \end{aligned}$$

Using this and (B.28), we obtain

$$\begin{aligned} & \limsup_{k \rightarrow \infty} \sum_{m=1}^{\infty} \frac{\mathbf{A}(k+m)\mathbf{L}(m)}{\mathbf{P}(Y_e = k)} \\ & = \limsup_{k \rightarrow \infty} \sum_{m=m_*}^{\infty} \frac{\mathbf{A}(k+m)\mathbf{L}(m)}{\mathbf{P}(Y_e = k)} \\ & = \limsup_{k \rightarrow \infty} \sum_{l=0}^{\tau-1} \sum_{\substack{m \geq m_* \\ m \equiv l \pmod{\tau}}} \frac{\mathbf{A}(k+m)\mathbf{L}(m)}{\mathbf{P}(Y_e = k)} \\ & \leq \sum_{l=0}^{\tau-1} \left[ \limsup_{k \rightarrow \infty} \sum_{\substack{m \geq m_* \\ m \equiv l \pmod{\tau}}} \frac{\mathbf{A}(k+m)\mathbf{E}}{\mathbf{P}(Y_e = k)} \right] (\tau \mathbf{H}_l + \varepsilon \mathbf{e} \mathbf{e}^t). \quad (\text{B.30}) \end{aligned}$$

Further it follows from (B.29) and Proposition A.2.1 that

$$\begin{aligned}
& \limsup_{k \rightarrow \infty} \sum_{\substack{m \geq m_* \\ m \equiv l \pmod{\tau}}} \frac{A(k+m)E}{P(Y_e = k)} \\
& \leq \frac{C_A^E + \varepsilon e e^t}{E[Y]} \limsup_{k \rightarrow \infty} \sum_{\substack{m \geq m_* \\ m \equiv l \pmod{\tau}}} \frac{P(Y = k+m)}{P(Y_e = k)} \\
& = \frac{C_A^E + \varepsilon e e^t}{\tau}.
\end{aligned} \tag{B.31}$$

Substituting (B.31) into (B.30) and letting  $\varepsilon \downarrow 0$ , we obtain

$$\limsup_{k \rightarrow \infty} \sum_{m=1}^{\infty} \frac{A(k+m)L(m)}{P(Y_e = k)} \leq C_A^E \sum_{l=0}^{\tau-1} H_l = C_A^E e \psi,$$

where we use (3.8) in the last equality. Similarly, we can show that

$$\liminf_{k \rightarrow \infty} \sum_{m=1}^{\infty} \frac{A(k+m)L(m)}{P(Y_e = k)} \geq C_A^E e \psi.$$

As a result,

$$\lim_{k \rightarrow \infty} \sum_{m=1}^{\infty} \frac{A(k+m)L(m)}{P(Y_e = k)} = C_A^E e \psi,$$

from which and (3.3) we have (4.2).

## B.8 Proof of Proposition A.2.1

We assume that condition (i) holds. It follows from  $U \in \mathcal{L}_{\text{loc}}(1)$  that for any  $\varepsilon > 0$  there exists  $k_0 \in \mathbb{N}$  such that for all  $k \geq k_0$  and  $l \in \mathbb{Z}_+$ ,

$$1 - \varepsilon \leq \frac{P(U = k + lh + \nu)}{P(U = k + lh)} \leq 1 + \varepsilon, \quad \nu = 0, 1, \dots, h-1.$$

Thus for all  $k \geq k_0$ , we have

$$1 - \varepsilon \leq \frac{\sum_{l=l_0}^{\infty} P(U = k + lh + \nu)}{\sum_{l=l_0}^{\infty} P(U = k + lh)} \leq 1 + \varepsilon, \quad \nu = 0, 1, \dots, h-1,$$

which leads to

$$\lim_{k \rightarrow \infty} \frac{\sum_{l=l_0}^{\infty} P(U = k + lh + \nu)}{\sum_{l=l_0}^{\infty} P(U = k + lh)} = 1, \quad \nu = 0, 1, \dots, h-1. \tag{B.32}$$

Therefore (B.32) yields for  $\nu = 0, 1, \dots, h-1$ ,

$$\begin{aligned}
& \lim_{k \rightarrow \infty} \frac{\sum_{l=l_0}^{\infty} \mathbb{P}(U = k + lh + \nu)}{\mathbb{P}(U > k + l_0 h - 1)} \\
&= \lim_{k \rightarrow \infty} \frac{\sum_{l=l_0}^{\infty} \mathbb{P}(U = k + lh + \nu)}{\sum_{m=l_0 h}^{\infty} \mathbb{P}(U = k + m)} \\
&= \lim_{k \rightarrow \infty} \frac{\sum_{l=l_0}^{\infty} \mathbb{P}(U = k + lh)}{\sum_{j=0}^{h-1} \sum_{l=l_0}^{\infty} \mathbb{P}(U = k + lh + j)} \\
&\quad \times \frac{\sum_{l=l_0}^{\infty} \mathbb{P}(U = k + lh + \nu)}{\sum_{l=l_0}^{\infty} \mathbb{P}(U = k + lh)} = \frac{1}{h}.
\end{aligned} \tag{B.33}$$

Note here that if  $U \in \mathcal{L}_{\text{loc}}(1)$ , then  $U \in \mathcal{L}$  and thus  $\lim_{k \rightarrow \infty} \mathbb{P}(U > k + l_0 h - 1) / \mathbb{P}(U > k) = 1$ . As a result, (B.33) implies (A.2).

Next we assume that condition (ii) holds. It then follows that for all sufficiently large  $k$ ,

$$\sum_{l=l_0}^{\infty} \mathbb{P}(U = k + lh) \geq \sum_{l=l_0}^{\infty} \mathbb{P}(U = k + lh + j), \quad j \in \mathbb{Z}_+. \tag{B.34}$$

Thus for any fixed (possibly negative) integer  $i$ ,

$$\begin{aligned}
& \lim_{k \rightarrow \infty} \frac{\mathbb{P}(U = k + l_0 h + i)}{h \sum_{l=l_0}^{\infty} \mathbb{P}(U = k + lh)} \\
&\leq \lim_{k \rightarrow \infty} \frac{\mathbb{P}(U = k + l_0 h + i)}{\sum_{j=0}^{h-1} \sum_{l=l_0}^{\infty} \mathbb{P}(U = k + lh + j)} \\
&= \lim_{k \rightarrow \infty} \frac{\mathbb{P}(U > k + l_0 h + i - 1) - \mathbb{P}(U > k + l_0 h + i)}{\mathbb{P}(U > k + l_0 h - 1)} = 0,
\end{aligned}$$

which implies that

$$\lim_{k \rightarrow \infty} \frac{\mathbb{P}(U = k + l_0 h + i)}{\sum_{l=l_0}^{\infty} \mathbb{P}(U = k + lh)} = 0. \tag{B.35}$$

Further (B.34) yields for all sufficiently large  $k$ ,

$$\begin{aligned}
1 &\geq \frac{\sum_{l=l_0}^{\infty} \mathbb{P}(U = k + lh + \nu)}{\sum_{l=l_0}^{\infty} \mathbb{P}(U = k + lh)} \\
&\geq 1 - \frac{\mathbb{P}(U = k + l_0 h)}{\sum_{l=l_0}^{\infty} \mathbb{P}(U = k + lh)}, \quad \nu = 0, 1, \dots, h-1,
\end{aligned}$$

from which and (B.35) it follows that (B.32) holds for  $\nu = 0, 1, \dots, h-1$ . Therefore we can prove (A.2) in the same way as the case of condition (i).

## B.9 Proof of Proposition A.2.4

The techniques for the proof are based on Lemma 4.2 in [1] and Lemma 10 in [12], though some modifications are required. For the reader's convenience, we provide a complete proof of this proposition.

We first prove the statement under an additional condition that  $c_j > 0$  for all  $j = 1, 2, \dots, m$ , and then remove the condition.

Let  $C = \max\{1, c_1, \dots, c_m\}$ ,  $d_0 = 1$  and  $d_j = c_j/C \leq 1$  for  $j = 1, 2, \dots, m$ . Let  $F_0(k)$  ( $k \in \mathbb{Z}_+$ ) denote a probability mass function such that  $F_0(k) = CF(k)$  for all sufficiently large  $k \geq k_0$ , where  $k_0$  is a positive integer such that  $F(k) > 0$  for all  $k \geq k_0$  (see Definitions A.2.1 and A.2.2).

From (A.3), we have

$$\lim_{k \rightarrow \infty} \frac{F_j(k)}{F_0(k)} = d_j \leq 1, \quad j = 0, 1, \dots, m. \quad (\text{B.36})$$

Further since  $F_j \in \mathcal{S}_{\text{loc}}(1) \subset \mathcal{L}_{\text{loc}}(1)$  (see Proposition A.2.3),

$$\lim_{n \rightarrow \infty} \lim_{k \rightarrow \infty} \frac{\sum_{l=0}^n F_i(l) F_j(k-l)}{F_j(k)} = \lim_{n \rightarrow \infty} \sum_{l=0}^n F_i(l) = 1, \quad (\text{B.37})$$

$$\lim_{k \rightarrow \infty} \frac{F_i * F_j(k)}{F_0(k)} = d_i + d_j, \quad (\text{B.38})$$

for all  $i, j = 0, 1, \dots, m$ . Thus any  $\varepsilon > 0$ , there exist some positive integers  $k'$  and  $k''$  such that  $k'' > 2k' \geq 2k_0$ ,  $F_0(k) = CF(k) \leq 1$  for all  $k \geq k'$  and for all  $i, j = 0, 1, \dots, m$ ,

$$\frac{F_0(k+1)}{F_0(k)} \geq 1 - \varepsilon, \quad \forall k \geq k', \quad (\text{B.39})$$

$$d_j - \frac{\varepsilon}{8} \leq \frac{F_j(k)}{F_0(k)} \leq 1 + \frac{\varepsilon}{2}, \quad \forall k \geq k', \quad (\text{B.40})$$

$$\frac{\sum_{l=0}^{k'-1} F_i(l) F_j(k-l)}{F_j(k)} \geq 1 - \frac{\varepsilon}{8d_j}, \quad \forall k \geq k'', \quad (\text{B.41})$$

$$F_i * F_j(k) \leq (d_i + d_j + \varepsilon/4) F_0(k), \quad \forall k \geq k''. \quad (\text{B.42})$$

Note here that (B.39), (B.40), (B.41) and (B.42) follow from  $F_0 \in \mathcal{L}_{\text{loc}}(1)$ , (B.36), (B.37) and (B.38), respectively.

We now show (A.4) for the convolution of two mass functions  $F_i$  and  $F_j$  ( $i, j = 0, 1, \dots, m$ ). Note that

$$F_i * F_j(k) = \sum_{l=0}^{k-k'} F_i(k-l) F_j(l) + \sum_{l=0}^{k'-1} F_i(l) F_j(k-l). \quad (\text{B.43})$$

It then follows from (B.40), (B.41) and (B.42) that for  $k \geq k'' > 2k'$ ,

$$\begin{aligned}
\sum_{l=0}^{k-k'} F_i(k-l) F_j(l) &= F_i * F_j(k) - \sum_{l=0}^{k'-1} F_i(l) F_j(k-l) \\
&\leq \left(d_i + d_j + \frac{\varepsilon}{4}\right) F_0(k) - \left(1 - \frac{\varepsilon}{8d_j}\right) F_j(k) \\
&\leq \left[\left(d_i + d_j + \frac{\varepsilon}{4}\right) - \left(1 - \frac{\varepsilon}{8d_j}\right) \left(d_j - \frac{\varepsilon}{8}\right)\right] F_0(k) \\
&\leq \left(d_i + \frac{\varepsilon}{2}\right) F_0(k) \leq \left(1 + \frac{\varepsilon}{2}\right) CF(k),
\end{aligned} \tag{B.44}$$

where the last inequality is due to  $d_j \leq 1$  and  $F_0(k) = CF(k)$  for all  $k \geq k'$ . Applying (B.44) to (B.43), we have for  $k \geq k'' > 2k'$ ,

$$\begin{aligned}
F_i * F_j(k) &\leq \left(1 + \frac{\varepsilon}{2}\right) CF(k) + \sum_{l=0}^{k'-1} F_i(l) F_j(k-l) \\
&\leq \left(1 + \frac{\varepsilon}{2}\right) CF(k) + \sup_{k-k'+1 \leq l \leq k} F_j(l).
\end{aligned} \tag{B.45}$$

Further for  $k \geq k'' > 2k'$ ,  $k - k' + 1 > k' + 1$  and thus (B.39) and (B.40) yield

$$\begin{aligned}
\sup_{k-k'+1 \leq l \leq k} F_j(l) &\leq \left(1 + \frac{\varepsilon}{2}\right) \sup_{k-k'+1 \leq l \leq k} F_0(l) \\
&= \left(1 + \frac{\varepsilon}{2}\right) \sup_{k-k'+1 \leq l \leq k} \frac{F_0(l)}{F_0(k)} \cdot CF(k) \\
&\leq \left(1 + \frac{\varepsilon}{2}\right) \frac{1}{(1-\varepsilon)^{k'-1}} \cdot CF(k) \\
&= \left(1 + \frac{\varepsilon}{2}\right) C'_\varepsilon \cdot CF(k), \quad k \geq k'' > 2k',
\end{aligned} \tag{B.46}$$

where  $C'_\varepsilon = 1/(1-\varepsilon)^{k'-1}$ . Substituting (B.46) into (B.45), we obtain

$$\begin{aligned}
F_i * F_j(k) &\leq \left(1 + \frac{\varepsilon}{2}\right) (1 + C'_\varepsilon) CF(k) \\
&\leq (1 + \varepsilon) \cdot 2C'_\varepsilon CF(k) \\
&\leq 2C'_\varepsilon \cdot (1 + \varepsilon)^2 CF(k), \quad k \geq k'',
\end{aligned} \tag{B.47}$$

where we use  $C'_\varepsilon \geq 1$ . Note here that  $F_i * F_j(k) \leq 1$  for all  $k \in \mathbb{Z}_+$  and

$$\sup_{k_0 \leq k \leq k''-1} F(k)/F(k'') \in (0, \infty).$$

Therefore there exists some  $C''_\varepsilon > 0$  such that

$$\begin{aligned}
F_i * F_j(k) &\leq C''_\varepsilon \frac{F(k)}{F(k'')} \\
&\leq \frac{C''_\varepsilon}{CF(k'')} \cdot (1 + \varepsilon)^2 CF(k), \quad k_0 \leq k \leq k'' - 1.
\end{aligned} \tag{B.48}$$

We now define  $K_\varepsilon$  as

$$K_\varepsilon = \max \left( 2C'_\varepsilon, \frac{C''_\varepsilon}{CF(k'')}, \frac{2+\varepsilon}{\varepsilon(1+\varepsilon)^2} C'_\varepsilon \right).$$

We then have the following inequality (which is used later).

$$\left(1 + \frac{\varepsilon}{2}\right) C'_\varepsilon \leq K_\varepsilon (1 + \varepsilon)^2 \frac{\varepsilon}{2}. \quad (\text{B.49})$$

Further combining (B.47) and (B.48) leads to

$$F_i * F_j(k) \leq K_\varepsilon (1 + \varepsilon)^2 CF(k), \quad k \geq k_0. \quad (\text{B.50})$$

Next we show (A.4) for the convolution of three mass functions  $F_i, F_j$  and  $F_\nu$  ( $i, j, \nu = 0, 1, \dots, m$ ). It follows from (B.50) and  $F_0(k) = CF(k)$  for all  $k \geq k'$  that

$$F_i * F_j(k) \leq K_\varepsilon (1 + \varepsilon)^2 F_0(k), \quad k \geq k'.$$

From this and (B.46), we have for  $k \geq k'' > 2k'$ ,

$$\begin{aligned} & F_i * F_j * F_\nu(k) \\ &= \sum_{l=0}^{k-k'} F_i * F_j(k-l) F_\nu(l) + \sum_{l=0}^{k'-1} F_i * F_j(l) F_\nu(k-l) \\ &\leq \sum_{l=0}^{k-k'} F_i * F_j(k-l) F_\nu(l) + \sup_{k-k'+1 \leq l \leq k} F_\nu(l) \\ &\leq K_\varepsilon (1 + \varepsilon)^2 \sum_{l=0}^{k-k'} F_0(k-l) F_\nu(l) + \left(1 + \frac{\varepsilon}{2}\right) C'_\varepsilon CF(k). \end{aligned} \quad (\text{B.51})$$

Applying (B.44) and (B.49) to (B.51) yields for  $k \geq k'' > 2k'$ ,

$$\begin{aligned} & F_i * F_j * F_\nu(k) \\ &\leq K_\varepsilon (1 + \varepsilon)^2 \left(1 + \frac{\varepsilon}{2}\right) CF(k) + K_\varepsilon (1 + \varepsilon)^2 \frac{\varepsilon}{2} CF(k) \\ &= K_\varepsilon (1 + \varepsilon)^2 \left(1 + \frac{\varepsilon}{2} + \frac{\varepsilon}{2}\right) CF(k) \\ &= K_\varepsilon (1 + \varepsilon)^3 CF(k). \end{aligned}$$

Further using  $C''_\varepsilon > 0$  such that (B.48) holds, we obtain

$$\begin{aligned} F_i * F_j * F_\nu(k) &\leq C''_\varepsilon \frac{F(k)}{F(k'')} \\ &\leq \frac{C''_\varepsilon}{CF(k'')} \cdot (1 + \varepsilon)^3 CF(k), \quad k_0 \leq k \leq k'' - 1. \end{aligned}$$

Therefore  $F_i * F_j * F_\nu(k) \leq K_\varepsilon(1 + \varepsilon)^3 CF(k)$  for  $k \geq k_0$ .

By repeating the above argument, we can prove that (A.4) holds under the additional condition that  $c_j > 0$  for all  $j = 1, 2, \dots, m$ . In what follows, we remove this condition.

Without loss of generality, we assume that  $c_j = 0$  for  $j = 1, 2, \dots, m'$  ( $1 \leq m' \leq m$ ) and  $c_j > 0$  for  $j = m' + 1, m' + 2, \dots, m$ . Then for any  $\delta > 0$ , there exists some positive integer  $k_* := k_*(\delta) \geq k_0$  such that for all  $k \geq k_*$ ,

$$F_j(k) \leq \delta F(k), \quad j = 1, 2, \dots, m'.$$

Let  $\{\tilde{F}_j(k); k \in \mathbb{Z}_+\}$  ( $j = 1, 2, \dots, m'$ ) denote a probability mass function such that

$$\tilde{F}_j(k) = \begin{cases} F_j(k)/\Theta_j, & k < k_*, \\ \delta F(k)/\Theta_j, & k \geq k_*, \end{cases}$$

where  $\Theta_j := \Theta_j(\delta) = \sum_{k=0}^{k_*-1} F_j(k) + \sum_{k=k_*}^{\infty} \delta F(k)$ . It then follows that  $F_j(k) \leq \Theta_j \tilde{F}_j(k)$  for all  $k \in \mathbb{Z}_+$  and  $j = 1, 2, \dots, m'$ . Thus we have

$$\begin{aligned} & F_1^{*n_1} * F_2^{*n_2} * \dots * F_m^{*n_m}(k) \\ & \leq \prod_{j=1}^{m'} \Theta_j^{n_j} \cdot \tilde{F}_1^{*n_1} * \dots * \tilde{F}_{m'}^{*n_{m'}} * F_{m'+1}^{*n_{m'+1}} * \dots * F_m^{*n_m}(k). \end{aligned} \quad (\text{B.52})$$

By definition,

$$\lim_{k \rightarrow \infty} \frac{\tilde{F}_j(k)}{F(k)} = \frac{\delta}{\Theta_j} > 0, \quad j = 1, 2, \dots, m'.$$

Therefore for any  $\varepsilon > 0$ , there exists some  $C_\varepsilon > 0$  such that

$$\begin{aligned} & \tilde{F}_1^{*n_1} * \dots * \tilde{F}_{m'}^{*n_{m'}} * F_{m'+1}^{*n_{m'+1}} * \dots * F_m^{*n_m}(k) \\ & \leq C_\varepsilon(1 + \varepsilon)^{n_1 + n_2 + \dots + n_m} F(k). \end{aligned} \quad (\text{B.53})$$

Note here that  $\lim_{\delta \downarrow 0} \Theta_j(\delta) = 1$  for all  $j = 1, 2, \dots, m'$ . Substituting (B.53) into (B.52) and letting  $\delta \downarrow 0$  yields (A.4).

## C Examples

### C.1 M/GI/1 queue with Pareto service-time distribution

We consider a stable M/GI/1 queue with a Pareto service-time distribution. Let  $\lambda$  denote the arrival rate of customers. Let  $H$  denote the service time distribution, which is given by

$$H(x) = 1 - (x + 1)^{-\gamma}, \quad x \geq 0,$$

with  $\gamma > 1$  and  $\gamma \notin \mathbb{N}$ . Note here that the mean service time is equal to  $1/(\gamma - 1)$  and thus the traffic intensity, denoted by  $\rho$ , is equal to  $\lambda/(\gamma - 1) < 1$ . Let  $\tilde{H}(s)$  denote the Laplace-Stieltjes transform (LST) of the service time distribution  $H$ . It then follows from Theorem 8.1.6 in [5] that

$$\tilde{H}(s) = \sum_{j=0}^{\lfloor \gamma \rfloor} h_j \frac{(-s)^j}{j!} - \Gamma(1 - \gamma) s^\gamma + o(s^\gamma), \quad (\text{C.1})$$

where  $h_j = \int_0^\infty x^j dH(x)$  ( $j = 1, 2, \dots$ ),  $f(x) = o(g(x))$  represents  $\lim_{x \rightarrow 0} f(x)/g(x) = 0$  and  $\Gamma$  denotes the Gamma function. Equation (C.1) yields

$$\tilde{H}(\lambda - \lambda z) = \sum_{j=0}^{\lfloor \gamma \rfloor} h_j \frac{(-\lambda)^j (1 - z)^j}{j!} - \Gamma(1 - \gamma) \lambda^\gamma (1 - z)^\gamma + o((1 - z)^\gamma). \quad (\text{C.2})$$

It is well-known that the stationary queue length distribution of the M/GI/1 queue, denoted by  $\{x(k); k \in \mathbb{Z}_+\}$ , is identical with the stationary distribution of the following stochastic matrix:

$$\begin{pmatrix} \alpha(0) & \alpha(1) & \alpha(2) & \alpha(3) & \cdots \\ \alpha(0) & \alpha(1) & \alpha(2) & \alpha(3) & \cdots \\ 0 & \alpha(0) & \alpha(1) & \alpha(2) & \cdots \\ 0 & 0 & \alpha(0) & \alpha(1) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where  $\{\alpha(k); k \in \mathbb{Z}_+\}$  satisfies  $\sum_{k=0}^\infty z^k \alpha(k) = \tilde{H}(\lambda - \lambda z)$  and thus  $\sum_{k=1}^\infty k \alpha(k) = \rho$ .

Let  $\bar{\alpha}(k) = \sum_{l=k+1}^\infty \alpha_l$  for  $k \in \mathbb{Z}_+$ . From (C.2), we then have

$$\begin{aligned} \sum_{k=0}^\infty z^k \bar{\alpha}(k) &= \frac{1 - \tilde{H}(\lambda - \lambda z)}{1 - z} \\ &= - \sum_{j=1}^{\lfloor \gamma \rfloor} h_j \frac{(-\lambda)^j (1 - z)^{j-1}}{j!} \\ &\quad + \Gamma(1 - \gamma) \lambda^\gamma (1 - z)^{\gamma-1} + o((1 - z)^{\gamma-1}). \end{aligned} \quad (\text{C.3})$$

Applying Lemma 5.3.2 in [23] to (C.2) and (C.3) yields

$$\alpha(k) \stackrel{k}{\sim} \gamma \lambda^\gamma k^{-\gamma-1}, \quad (\text{C.4})$$

$$\bar{\alpha}(k) \stackrel{k}{\sim} \lambda^\gamma k^{-\gamma}, \quad (\text{C.5})$$

where  $f(x) \stackrel{x}{\sim} g(x)$  represents  $\lim_{x \rightarrow \infty} f(x)/g(x) = 1$ . Note that (C.4) shows that the discrete distribution  $\{\alpha(k); k \in \mathbb{Z}_+\}$  is in the class  $\mathcal{L}_{\text{loc}}$ . In fact, as shown later,  $\{\alpha(k)\} \in \mathcal{S}^*$ , i.e.,  $\{\alpha_e(k)\} \in \mathcal{S}_{\text{loc}}(1)$ , where  $\alpha_e(k) = \bar{\alpha}(k)/\rho$  for  $k = 0, 1, \dots$ . Therefore it follows from Theorem 4.1.1 that

$$x(k) \stackrel{k}{\sim} \frac{\rho}{1 - \rho} \cdot \alpha_e(k) = \frac{\rho}{1 - \rho} \cdot \frac{\bar{\alpha}(k)}{\rho} \stackrel{k}{\sim} \frac{\lambda^\gamma}{1 - \rho} k^{-\gamma}.$$

In what follows, we prove that  $\{\alpha(k)\} \in \mathcal{S}^*$ , i.e.,

$$\sum_{l=0}^k \bar{\alpha}(l) \bar{\alpha}(k-l) \stackrel{k}{\sim} 2\rho \cdot \bar{\alpha}(k).$$

Let  $\nu := \nu(k)$  denote an integer such that  $k/3 \leq \nu(k) < k/2$ . For  $k \in \mathbb{Z}_+$ , we have

$$\sum_{l=0}^k \frac{\bar{\alpha}(l) \bar{\alpha}(k-l)}{\bar{\alpha}(k)} = 2 \sum_{l=0}^{\nu-1} \bar{\alpha}(l) \frac{\bar{\alpha}(k-l)}{\bar{\alpha}(k)} + \sum_{l=\nu}^{k-\nu} \bar{\alpha}(l) \frac{\bar{\alpha}(k-l)}{\bar{\alpha}(k)}. \quad (\text{C.6})$$

From (C.5), we obtain

$$\lim_{k \rightarrow \infty} \sum_{l=0}^{\nu-1} \bar{\alpha}(l) \frac{\bar{\alpha}(k-l)}{\bar{\alpha}(k)} = \sum_{l=0}^{\nu-1} \bar{\alpha}(l) \lim_{k \rightarrow \infty} \frac{\bar{\alpha}(k-l)}{\bar{\alpha}(k)} = \sum_{l=0}^{\nu-1} \bar{\alpha}(l). \quad (\text{C.7})$$

Further it follows from (C.5) that for any  $\varepsilon > 0$  there exists some  $k_* \in \mathbb{Z}_+$  such that for all  $k \geq k_*/3$ ,

$$1 - \varepsilon < \frac{\bar{\alpha}(k)}{\lambda^\gamma k^{-\gamma}} < 1 + \varepsilon,$$

which implies that for  $k \geq k_*$  and  $k/3 \leq \nu < k/2$ ,

$$\begin{aligned} \sum_{l=\nu}^{k-\nu} \bar{\alpha}(l) \frac{\bar{\alpha}(k-l)}{\bar{\alpha}(k)} &\leq \frac{(1+\varepsilon)^2}{1-\varepsilon} \sum_{l=\nu}^{k-\nu} \lambda^\gamma l^{-\gamma} \left( \frac{k-l}{k} \right)^{-\gamma} \\ &\leq \frac{(1+\varepsilon)^2}{1-\varepsilon} \lambda^\gamma (k-2\nu+1) \nu^{-\gamma} \left( \frac{\nu}{k} \right)^{-\gamma} \\ &\leq \frac{(1+\varepsilon)^2}{1-\varepsilon} \lambda^\gamma k \left( \frac{k}{3} \right)^{-\gamma} 3^\gamma \\ &\leq \frac{(1+\varepsilon)^2}{1-\varepsilon} (9\lambda)^\gamma k^{-\gamma+1} \rightarrow 0, \quad \text{as } k \rightarrow \infty. \end{aligned} \quad (\text{C.8})$$

Finally, applying (C.7) and (C.8) to (C.6) and letting  $\nu \rightarrow \infty$  yield

$$\lim_{k \rightarrow \infty} \sum_{l=0}^k \frac{\bar{\alpha}(l) \bar{\alpha}(k-l)}{\bar{\alpha}(k)} = 2 \sum_{l=0}^{\infty} \bar{\alpha}(l) = 2\rho.$$

## C.2 Discrete-time queue with disasters and Pareto-distributed batch arrivals

This subsection considers a discrete-time single-server queue with disasters and Pareto-distributed batch arrivals. The time interval  $[n, n+1)$  ( $n \in \mathbb{Z}_+$ ) is called slot  $n$ . Customers and disasters can arrive at the beginnings of respective slots, whereas departures of served customers can occur at the ends of respective slots.

We assume that the numbers of customer arrivals in respective slots are independent and identically distributed (i.i.d.) with a discrete Pareto distribution,  $\beta(k) = 1/(k+1)^\gamma - 1/(k+2)^\gamma$  ( $k \in \mathbb{Z}_+$ ), where  $\gamma > 1$ . Service times are i.i.d. with a geometric distribution with mean  $1/(1-q)$  ( $0 < q < 1$ ). We also assume that at most one disaster occurs at one slot with probability  $\phi$  ( $0 < \phi < 1$ ), which are independent of the arrival process of customers. If a disaster occurs in a slot, then both customers arriving in the slot and all the ones in the system are removed.

Let  $L_n$  ( $n \in \mathbb{Z}_+$ ) denote the number of customers at the middle of slot  $n$ . It then follows from Proposition 2.2.2 that  $\{L_n; n \in \mathbb{Z}_+\}$  is an ergodic Markov chain whose transition probability matrix is given by

$$\begin{pmatrix} b(0) & b(1) & b(2) & b(3) & b(4) & \cdots \\ \phi + a(0) & a(1) & a(2) & a(3) & a(4) & \cdots \\ \phi & a(0) & a(1) & a(2) & a(3) & \cdots \\ \phi & 0 & a(0) & a(1) & a(2) & \cdots \\ \phi & 0 & 0 & a(0) & a(1) & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix},$$

where

$$\begin{aligned} b(0) &= \phi + (1 - \phi)\beta(0), \\ b(k) &= (1 - \phi)\beta(k), & k = 1, 2, \dots, \\ a(0) &= (1 - \phi)\beta(0)(1 - q), \\ a(k) &= (1 - \phi)[\beta(k-1)q + \beta(k)(1 - q)], & k = 1, 2, \dots \end{aligned}$$

It is easy to see that  $\sum_{k=0}^{\infty} a(k) = 1 - \phi < 1$  and

$$\lim_{k \rightarrow \infty} \frac{a(k)}{\beta(k)} = 1 - \phi, \quad \lim_{k \rightarrow \infty} \frac{b(k)}{\beta(k)} = 1 - \phi.$$

Note here that  $\{\beta(k); k \in \mathbb{Z}_+\}$  is decreasing and

$$\beta(k) \stackrel{k}{\sim} \gamma k^{-\gamma-1}.$$

Thus as in subsection C.1, we can show that  $\{\beta(k); k \in \mathbb{Z}_+\} \in \mathcal{S}_{\text{loc}}(1)$ . As a result, Theorem 4.2.1 yields

$$\lim_{n \rightarrow \infty} \mathbb{P}(L_n = k) \stackrel{k}{\sim} \frac{1 - \phi}{\phi} \beta(k) \stackrel{k}{\sim} \frac{1 - \phi}{\phi} \gamma k^{-\gamma-1}.$$

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## References

- [1] Asmussen, S.; Henriksen, L.F.; Klüppelberg, C. Large claims approximations for risk processes in a Markovian environment. *Stochastic Processes and their Applications* **1994**, *54*, 29–43.
- [2] Asmussen, S.; Møller, J.R. Tail asymptotics for M/G/1 type queueing processes with subexponential increments. *Queueing Systems* **1999**, *33*, 153–176.
- [3] Asmussen, S. *Applied Probability and Queues*, 2nd Ed.; Springer: New York, 2003.
- [4] Asmussen, S.; Foss, S.; Korshunov, D. Asymptotics for sums of random variables with local subexponential behavior. *Journal of Theoretical Probability* **2003**, *16*, 489–518.
- [5] Bingham, N.H.; Goldie, C.M.; Teugels, J.L. *Regular Variation*, Cambridge University Press: New York, 1989.
- [6] Bremaud, P. *Markov Chains, Gibbs Fields, Monte Carlo Simulation and Queues*, Springer: New York, 1999.
- [7] Chistyakov, V.P. A theorem on sums of independent positive random variables and its applications to branching random processes. *Theory of Probability and Its Applications* **1964**, *9*, 640–648.
- [8] Chover, J.; Ney, P.; Wainger, S. Functions of probability measures. *Journal d'Analyse Mathématique* **1973**, *26*, 255–302.
- [9] Embrechts, P.; Klüppelberg, C.; Mikosch, T. *Modelling Extremal Events for Insurance and Finance*, Springer: Berlin, 1997.
- [10] Grassmann, W.K.; Heyman, D.P. Equilibrium distribution of block-structured Markov chains with repeating rows. *Journal of Applied Probability* **1990**, *27*, 557–576.
- [11] Horn, R.A.; Johnson, C.R. *Matrix Analysis*, 1st Paperback Ed.; Cambridge University Press: New York, 1990.
- [12] Jelenković, P.R.; Lazar, A.A. Subexponential asymptotics of a Markov-modulated random walk with queueing applications. *Journal of Applied Probability* **1998**, *35*, 325–347.
- [13] Kim, B.; Kim, J. A note on the subexponential asymptotics of the stationary distribution of M/G/1 type Markov chains. *European Journal of Operational Research* **2012**, *220*, 132–134.

- [14] Kimura, T.; Daikoku, K.; Masuyama, H.; Takahashi, Y. Light-tailed asymptotics of stationary tail probability vectors of Markov chains of M/G/1 type. *Stochastic Models* **2010**, *26*, 505–548.
- [15] Kimura, T.; Daikoku, K.; Masuyama, H.; Takahashi, Y. Light-tailed asymptotics of stationary tail probability vectors of Markov chains of M/G/1 type. A revised version of the published paper [14], arXiv:1110.4457.
- [16] Klüppelberg, C. Subexponential distributions and integrated tails. *Journal of Applied Probability* **1988**, *25*, 132–141.
- [17] Li, Q.-L.; Zhao, Y.Q. Heavy-tailed asymptotics of stationary probability vectors of Markov chains of GI/G/1 type. *Advances in Applied Probability* **2005**, *37*, 482–509.
- [18] Li, Q.-L.; Zhao, Y.Q. Light-tailed asymptotics of stationary probability vectors of Markov chains of GI/G/1 type. *Advances in Applied Probability* **2005**, *37*, 1075–1093.
- [19] Masuyama, H. Subexponential asymptotics of the stationary distributions of M/G/1-type Markov chains. *European Journal of Operational Research* **2011**, *213*, 509–516.
- [20] Pitman, E.J.G. Subexponential distribution functions. *Journal of the Australian Mathematical Society (Series A)* **1980**, *29*, 337–347.
- [21] Sigman, K. Appendix: A primer on heavy-tailed distributions. *Queueing Systems* **1999**, *33*, 261–275.
- [22] Takine, T. Geometric and subexponential asymptotics of Markov chains of M/G/1 type. *Mathematics of Operations Research* **2004**, *29*, 624–648.
- [23] Wilf, H.S. *Generatingfunctionology*, 2nd Ed.; Academic Press: Boston, 1994.
- [24] Shin, Y.W. BMAP/G/1 queue with correlated arrivals of customers and disasters. *Operations Research Letters* **2004**, *32*, 364–373.
- [25] Zhao, Y.Q.; Li, W.; Braun, W.J. Infinite block-structured transition matrices and their properties. *Advances in Applied Probability* **1998**, *30*, 365–384.
- [26] Zhao, Y.Q.; Li, W.; Alfa, A.S. Duality results for block-structured transition matrices. *Journal of Applied Probability* **1999**, *36*, 1045–1057.
- [27] Zhao, Y.Q.; Li, W.; Braun, W.J. Censoring, factorizations, and spectral analysis for transition matrices with block-repeating entries. *Methodology and Computing in Applied Probability* **2003**, *5*, 35–58.