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Interval availability analysis of a two-echelon, multi-item system

Ahmad Al Hanbali *, Matthieu van der Heijden

University of Twente, School of Management and Governance, P.O. Box 217, 7500 AE Enschede, The Netherlands

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ABSTRACT

In this paper we analyze the interval availability of a two-echelon, multi-item spare part inventory system. We consider a scenario inspired by a situation that we encountered at Thales Netherlands, a manufacturer of naval sensors and naval command and control systems. Modeling the complete system as a Markov chain we analyze the interval availability and we compute in closed and exact form the expectation and the variance of the availability during a finite time interval $[0,T]$. We use these characteristics to approximate the survival function using a Beta distribution, together with the probability that the interval availability is equal to one. Comparison of our approximation with simulation shows excellent accuracy, especially for points of the distribution function below the mean value. The latter points are practically most relevant.

1. Introduction

Nowadays, the aftersales service business represents a considerable part of the economy and, moreover, is continuously growing (Aberdeen Group, 2005; Deloitte., 2006). Advanced capital goods such as MRI scanners, lithography systems, baggage handling systems, and radar systems, are highly downtime critical. The high criticality in these cases is due to lost production, missions that need to be aborted, patients that cannot be treated, and flights that are delayed or canceled. So the customers of these advanced goods are not just interested in acquiring these systems at an affordable price, but far more in a good balance between the resulting Total Cost of Ownership (TCO) and system productivity throughout the life cycle, including the limitation of downtime. It is often the case that the system upkeep costs during the life cycle of the system constitute a large part of the TCO. However, the core business of customers is the usage of the system and not its upkeep. Therefore, a major part of the system upkeep is preferably outsourced to the manufacturer or to an intermediate service provider that can offer a good balance between the downtime and costs. For that reason, service contracts are made between the service provider and customers. These contracts specify the services provided by the supplier with their corresponding Service Level Agreements (SLAs), such as the time between system failure and time of fault resolution, and the system availability.

The SLAs are measured over a predetermined time window, e.g., a quarter or a year. For the service providers, it is essential that the service levels are attained, because in some cases penalties apply if an SLA target is violated. In case of a large scale service contract (the average performance over many systems is measured), the average performance should meet the target. If the number of systems covered by a contract is relatively small, we have inherent statistical variability and we need an additional buffer in performance to assure that the probability of not meeting the SLAs over the time window is still acceptable. We encountered such a situation at Thales Netherlands, a manufacturer of naval sensors and naval command and control systems. There, a service contract typically covers a few systems only. In the literature, this issue is usually neglected. In this paper, we are mainly interested in the logistical delay due to the unavailability of spare parts, since this is the basis of current service contracts at Thales Netherlands. Moreover, the focus will be on SLAs that are based on the system availability during a predetermined period of time.

In service parts logistics there is usually a tradeoff between the cost involved in keeping the stocks very close to the customers sites or at a central depot, which can supports multiple customers at the same time. Due to the risk pooling effect, it is more desirable for a service provider to position the stocks of spare parts centrally. However, having a strict SLA, e.g., 99% availability in a quarter, forces the service provider to move some spare parts closer to the customer sites. In addition, in order to reduce the system downtime and its critical consequences, the repair of a failed system is usually done by replacing the failed part with a new part. The failed part is sent to the repair shop, i.e., the inventory is managed using one-for-one replenishment, so an $(s-1,s)$-policy. This policy is justified by the fact that most parts are slow movers for which a replenishment order of size one is usually (near) optimal.

* Corresponding author. Tel.: +31 534894033.
E-mail address: a.alhanbali@utwente.nl (A. Al Hanbali).

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Sherbrooke (1968) was among the first to tackle the spare part optimization problem. He proposed the METRIC model that is based on the maximization of system availability subject to a constraint on the invested budget in spare parts. The main decision in METRIC is how much to keep in stock at each of the locations in the supply network. The METRIC model provides good approximations for multi-echelon spare part networks, especially in case of a high availability. Graves (1985) and Slay (1984) extended the METRIC model and proposed an improved approach called the VARI-METRIC. We note that the VARI-METRIC model is the approach used in most commercial software tools on spare parts optimization.

It is worth to mention that both METRIC and VARI-METRIC and most spare parts management theory are based on finding an optimal balance between the initial spare part investment and the steady state system availability, i.e., the fraction of time the system is operational during a very long (infinite) period of time. However, in practice we often see that the agreed upon availability SLA is the average availability during a finite period, e.g., month, quarter, or year. Moreover, if the availability during a period of time is lower than a specific percentage, penalty rules apply. This motivates us to analyze the availability during a finite period of time, the so-called interval availability that is defined in reliability theory as follows, see, e.g., (Nakagawa & Goel, 1973):

Definition. The system interval availability is defined as the fraction of time a system is operational during a period of time $[0,T]$.

Note that in (Barlow, Proschan, & Hunter, 1965; Hosford, 1960) the interval availability is defined as the *expected* fraction of time a system is operational during $[0,T]$. To avoid confusion in this paper and according to the previous definition the interval availability is a random variable that has a distribution. In addition, this probability distribution has a finite support between zero and one with probability masses at the points zero and one: There are strictly positive probabilities that (i) an operational system will not face any lack of spare parts during $[0,T]$, and (ii) a failed system waiting for a specific spare part will not be repaired by replacement during $[0,T]$. In practical instances, the first probability will be significant, and the second probability will be close to zero.

Our main contribution in this paper consists of the following points:

- We propose a computational efficient and accurate approximation for the interval availability of a multi-item system supported by a two echelon supply network. More specifically, our approximation is accurate in the practical case of systems with high average availability.
- As part of this approximation, we derive in closed-form the variance and the third moment of the cumulative sojourn time in a subset of states of Markov chain in a finite interval. In principle, we can also derive all the higher moments using the same approach.
- Using simulation we show that the survival function of the interval availability is not very sensitive to the order-and-ship time distribution at the points of survival function that are below the expected availability. This justifies our Markovian approach, specifically, the assumption of exponential order-and-ship times.

The paper is organized as follows. In Section 2 we briefly review the related literature. Section 3 describes our model and the assumptions used to analyze the interval availability distribution of a two echelon, multi item supply network. In Section 4 we report our approximation where our key results are reported in a set of Theorems. In Section 5 we validate our approximation using simulation and evaluate the impact of the order-and-ship time on the interval availability. Finally, in Section 6, we conclude the paper and give some directions for further research.

2. Related literature

In this section we shall review the existing literature on interval availability. Takács (1957) is among the first to analyze the interval availability distribution function of an on–off stochastic process. Takács result is in the form of an infinite sum of terms, each consisting of multiple convolutions. This result is hard to compute numerically. van der Heijden (1988) approximates the interval availability distribution using two-moment approximations for the on and off periods, which yields accurate results within small computation times. Another approximation based on fitting the *approximated* first two moments, the hundred percent, and the nil probability of the interval availability in a Beta distribution is proposed in (Smith, 1997). For an on–off two states Markov chain the first two moments of the interval availability are derived exactly in (Kirmani & Hood, 2008). We note that in all these previously mentioned studies the underlying assumption is that the on periods are independent and the off periods are independent, moreover, all the on and off period are independent of each other, i.e., the on–off process can be represented by a renewal process.

De Souza e Silva and Gail (1986) derive in closed-form the cumulative sojourn time distribution in a subset of states of a Markov chain during a finite period of time. The subset of states can, for example, represent the operational states of a system. Therefore, the division of the cumulative sojourn time by the period length gives right away the system interval availability. We note that computing the full curve of the interval availability distribution using the method of De Souza e Silva and Gail (1986) or its improved version in (Rubino & Sericola, 1995) is time consuming. Carrasco (2004) proposes an efficient algorithm to compute the interval availability distribution for the special case of the systems which can be modeled by an *absorbing* Markov chain. In the latter three papers the renewal assumption of the on–off process is not necessary.

In this paper, we propose a numerically efficient approach to compute the distribution function of the interval availability. Our approach builds on the result of De Souza e Silva and Gail (1986) extensively in order to compute in closed-form the first two moments of the interval availability. These two moments have not been derived previously in the literature for a Markov chain with more than two states. Moreover, we follow a similar approach to (Smith, 1997) to approximate the interval availability by a Beta distribution using the first two moments in addition to the hundred percent probability of the interval availability.

Finally, we note that the analysis of a service level over a finite period of time is not only of interest in reliability theory, but also in inventory management of fast moving products where demand is typically modeled by a Normal distribution. See, e.g., (Banerjee & Paul, 2005; Chen, Lin, & Thomas, 2003) in which the interest is on the expected fill rate over a finite period of time T for a single site, single item system. In these papers, it is proven that the expected fill rate over a finite period is larger than over the infinite period case. Thomas (2005) evaluate the impact of T and the demand distribution on the fill rate distribution over T. In the latter paper, simulation is used due to the difficulty in explicitly computing the fill rate distribution during T. Tactical decisions on stock level to meet the time-based SLA in the case of multi-echelon, single item scenario are considered in (Cohen, Kleindorfer, & Lee, 1986) and for the multi-item scenario in (Ettl, Feigin, Lin, & Yao, 2000). The restriction in the analysis is that the time period should be equal to the supply lead time of the part. More recently, the model in the latter two papers is extended and a scalability analysis is added in (Caggiano, Jackson, Muckstadt, & Rappold, 2007).

3. Model

We consider a two-echelon, multi-item supply network. There is a single depot that supports multiple identical systems which are located at different bases. There is a single system per base. A system consists of multiple items that are subject to breakdown. These items are repairable and belong to the class of expensive slow-movers, i.e., they have low failure rates. The depot and the bases hold stocks of spares for each item. Upon an item failure, the item is immediately sent to the depot for repair and at the same time a replenishment order is issued according to the $(s-1, s)$ policy, where s denotes the order-up-to level. The unsatisfied demand of parts is backordered. When the replenishment order arrives at the base it is used to fill backorders, if any. Otherwise, it is added to stock. The time needed to transfer a spare from the depot to the base is assumed to be exponentially distributed. Although these transfer times tend to show little variation in practice, we need this assumption to facilitate Markov analysis. Moreover, it is known that the steady state availability tends to show little sensitivity to the distributional form of repair times and order-and-ship times only, see (Alfredsson & Verrijdt, 1999). In Section 5, we use simulation to explore whether the same holds for the interval availability distribution. It turns out that the survival function of the interval availability is not very sensitive to the order-and-ship time distribution at the points of survival function that are below the expected availability. We say that the system is operational if all the items are operational. Obviously, if an item fails and no spare is available at the base, the system will be malfunctioning and unavailable for use.

We consider a scenario inspired by a situation that we encountered at Thales Netherlands. There is one naval radar system at each of the N frigates (bases in our model). A system consists of M items. We assume that the jth item fails according to a Poisson process with rate λ_j, $j = 1, \ldots, M$. Moreover, the failure of item j is independent of the other items. We assume that the replenishment lead time of the jth item at the depot is exponentially distributed with rate μ_j. The replenishment lead time includes the time to transport the failed item from the base to the depot and the time to repair the item at the depot. We model the depot repair shop as an ample server queueing system, i.e., it has infinite repair capacity. We also assume that the order-and-ship time of a spare part from the depot to base i is exponentially distributed with rate μ_{0i}. This means that all items at base i have the same transshipment time, however, the transshipment time may depend on the base where items are located. For sake of simplicity, we shall consider in the following the case where $\mu_{0i} = \mu_0$ for all i. Let s_{ij}, $i = 0, \ldots, N, j = 1, \ldots, M$, denote the base stock level of item j at base i, where $i = 0$ represents the depot and $i = 1, \ldots, N$ represents the ith base. Under the above assumption it is easily seen that the behavior of the system over time can be modeled as a continuous-time Markov chain. More precisely, since there is a finite number of spare parts in the network the continuous-time Markov chain is of finite size. Compared to (VARI-)METRIC, we use different assumptions for the order-and-ship times and the replenishment lead times. In our model both are random variables having an exponential distribution, whereas they may have a general probability distribution in VARI-METRIC. We need the exponential assumption in order to facilitate the Markov chain analysis. In addition (VARI-)METRIC assumes an infinite population, e.g., the number of jobs in a repair shop or the number of backorders can grow infinitely large. This of course occurs with a very small probability for scenarios of high expected availability. In the contrary with (VARI-)METRIC, we explicitly model the size of the installed base and the stock in the supply network, so that the demand for a spare part stops if the number of items in repair exceeds the total number of that spare parts in the network. This is more realistic and also facilitates the model numerical analysis for the interval availability distribution, since we limit the size of our continuous-time Markov chain.

Let $A_i(T)$, $i = 1, \ldots, N$, denote the interval availability of the system at base i during $[0, T]$. Our objective is to find the survival function of $A_i(T)$, i.e., the complementary cumulative distribution function of $A_i(T)$. For this reason, we first compute the mean and the second moment of the interval availability as well as the probability that the interval availability equals 1, i.e., $P(A_i(T) = 1)$. Although we may also compute the probability mass in the point zero, $P(A_i(T) = 0)$, this is not really useful since for practical relevant problem instances it will be very close to zero. Next, using the three performance metrics as mentioned above, we approximate the survival function of $A_i(T)$ by a mixture of a probability mass at one and a Beta distribution (so assuming zero probability mass in the point 0). Throughout this paper, we shall only focus on the interval availability of a tagged system, since we can analyze each system separately using the same method. For this reason, we shall drop the index i in $A_i(T)$ and refer to it as $A(T)$: the interval availability of a tagged system at one of the bases. In addition, we shall refer to the stock level of item j in the tagged system as s_j.

Since the failure processes of the different items are not coupled, i.e., the failure of an item does not cause the failure of other items, and the repair capacity is infinite, the different items on the tagged system can be assumed to behave mutually independent over time. Let $X_j(t)$ denote the state of item j in the tagged system at time t, i.e., $X_j(t) = 1$ if the item is operational at time t and zero otherwise. Note that $X_j(t) = 0$ if item j fails and there is no spare part available at the base to replace the malfunctioning item. Let $BO_j(t)$ denote the number of item j backorders of the tagged system at the depot. Let $TR_j(t)$ denote the number of items of type j in transport from the depot to the tagged system. Therefore, the pipeline of item j in the tagged system, denoted by $PL_j(t)$, is equal to $BO_j(t) + TR_j(t)$. Note that $PL_j(t)$ depends on the stock on-hand at the depot. Furthermore, the depot stock depends on the failure processes of item j in *all* the systems in the installed base including the tagged system. Let us denote $R_j(t)$ as the total number of failed items of type j in the depot repair shop. Note that backorders at the depot are served according to a FIFO discipline. Therefore, if $R_j(t) \geqslant s_{0j}$, i.e., the on-hand stock in the depot is equal to zero, it is also necessary to keep track of the position of the tagged system backorders in the depot backorders list. Moreover, it is also necessary to know how many items of type j are in transport from the depot to the tagged system. This is a complication that arises when computing the interval availability distribution which is not encountered in the (VARI-)METRIC model for the steady state average availability. The previous complication makes a detailed Markov chain analysis difficult. For this reason, we shall propose an approximate three-dimensional finite-size Markov chain to represent the state evolution of item j over time in the next section.

The tagged system is operational at time t if $X_j(t) = 1$, for all $j = 1, \ldots, M$. Let $O(T)$ denote the total sojourn time of the joint process $(X_1(t), X_2(t), \ldots, X_M(t))$ in state $(1, \ldots, 1)$ during $[0, T]$. The interval availability of the tagged system can be seen as the fraction of time that the tagged system is operational, i.e., $A(T) = O(T)/T$. Note that the processes $X_j(t)$, for $j = 1, \ldots, M$, are mutually independent by approximation and can be modeled as a Markov chain. Therefore, the joint process $(X_1(t), X_2(t), \ldots, X_M(t))$ is also a Markov chain.

A word on notation: Given that A is a matrix, $A(i, j)$ denotes the (i, j)-entry of A. We use I to denote the identity matrix of an appropriate size, and use $\otimes$ as the Kronecker product operator defined as follows. Let A and B be two matrices then $A \otimes B$ is a block matrix where the (i, j)-block is equal to $A(i, j)B$. If A is a square matrix, we denote its number of rows by $||A||$. We use e to denote a column vector with an appropriate size and with all entries equal to one.

4. Approximation

In this section, we first approximate $X_j(t)$ with a three-dimensional continuous-time finite-state Markov chain. The main advantages of this approximation are that it gives accurate results and that it can be solved efficiently, see Section 5. Second, we represent the transition generator of the joint process $(X_1(t), X_2(t), \ldots, X_M(t))$ as function of the generators of $X_j(t)$, $j = 1, \ldots, M$. The main approximations are as follows:

a. All the systems in the installed base, excluding the tagged system, have a constant annual demand for spare parts. This means, regardless of the state of these systems, each item failure rate is constant over time. We note that the latter failure rate can be adjusted by the availability of item j, but numerical experiments show that this yields a minor improvement of the results.
b. A depot repair completion at time t of an item of type j that is used to replenish a backorder of the tagged system occurs with a rate equal to $R_j(t)\mu_j * BO_j(t)/(R_j(t) - s_{0j})^+$, where $(\cdot)^+ = \max(\cdot, 0)$. Note that $R_j(t)\mu_j$ is the depot repair completion rate at time t and $(R_j(t) - s_{0j})^+$ is the total number of backorders of item j at the depot at time t.

Let us consider the finite-state three-dimensional approximate Markov chain $\{(BO_j(t), TR_j(t), R_j(t)) : t \geqslant 0\}$, referred to as AMC_j. We note that the chain has a finite state space because of the finite number of stocks in the network. Recall that the pipeline of item j in the tagged system equals $PL_j(t) = BO_j(t) + TR_j(t)$, and that $R_j(t)$ is the total number of type j items in the depot repair shop. Note that $PL_j(t) \in \{0, \ldots, s_j + 1\}$, $R_j(t) \in \{0, \ldots, s_{0j} + s_{1j} + \cdots + s_{Mj} + M\}$, and $BO_j(t) \leqslant (R_j(t) - s_{0j})^+$, the total number of depot backorders of item j. The process AMC_j has the following transitions:

1. A failure of item j in the tagged system if the on-hand stock at the depot is positive, i.e., $R_j(t) < s_{0j}$. It represents the transition from $(0, TR_j(t), R_j(t))$ to $(0, TR_j(t) + 1, R_j(t) + 1)$ with rate λ_j.
2. A failure of item j in the tagged system if no on-hand stock is available at the depot, i.e., $R_j(t) \geqslant s_{0j}$. It represents the transition from $(BO_j(t), TR_j(t), R_j(t))$ to $(BO_j(t) + 1, TR_j(t), R_j(t) + 1)$ with rate λ_j.
3. A failure of item j in one of the systems in the installed based excluding the tagged system. It represents the transition from $(BO_j(t), TR_j(t), R_j(t))$to $(BO_j(t), TR_j(t), R_j(t) + 1)$, which occurs, by approximation a, with rate $(N - 1)\lambda_j$.
4. A depot repair completion of item j if the on-hand stock at the depot is non-negative, i.e., $R_j(t) \leqslant s_{0j}$. It represents the transition from $(0, TR_j(t), R_j(t))$ to $(0, TR_j(t), R_j(t) - 1)$ with rate $R_j(t)\mu_j$.
5. A depot repair completion of item j that is used to replenish a backorder of the tagged system, i.e., $R_j(t) > s_{0j}$. It represents the transition from $(BO_j(t), TR_j(t), R_j(t))$ to $(BO_j(t) - 1, TR_j(t) + 1, R_j(t) - 1)$, which occurs, by approximation b, with rate $R_j(t)\mu_j * BO_j(t)/(R_j(t) - s_{0j})^+$.
6. A depot repair completion of item j that is used to fill a backorder of the systems in the installed based excluding the tagged system. It represents the transition from $(BO_j(t), TR_j(t), R_j(t))$ to $(BO_j(t), TR_j(t), R_j(t) - 1)$, which due to approximation b occurs with rate $R_j(t)\mu_j * \left(1 - \frac{BO_j(t)}{(R_j(t) - s_{0j})^+}\right)$.
7. An arrival of an item j from the depot to the base of the tagged system. It represents the transition from $(BO_j(t), TR_j(t), R_j(t))$ to $(BO_j(t), TR_j(t) - 1, R_j(t))$ with rate $TR_j(t)\mu_O$.

We emphasize that some transition rates are approximations. We evaluate the accuracy of these approximations numerically in Section 5 using simulation.

Let G_j denote the transition generator of AMC_j, which represents the evolution over time of item j that is used in the tagged system. Note that AMC_j is a continuous-time finite-state Markov chain that is *irreducible*, because all states are connected directly or indirectly via other states. Moreover, it is positive recurrent in the sense that, starting in any state, the mean time to return to that state is finite. Therefore, we deduce that AMC_j has a steady state probability distribution. Let $\pi_{m,n,l}(j)$ denote the steady state probability that AMC_j is in state (m, n, l). Let $\pi(j)$ denote the steady state probability distribution vector of AMC_j. The tagged system is operational if $m + n \leqslant s_j$ for all items $j = 1, \ldots, M$, since there is no backorder of any item at the base then. On the other hand, when $m + n = s_j + 1$, there is one item j backorder at the base, and so item j is not available in the tagged system. Let Ω_j denote the state space of AMC_j. We split Ω_j into to two disjoint subsets: Ω_j^o is the subset of operational states with $(m + n \leqslant s_j)$, and Ω_j^m is the subset of malfunctioning states $(m + n = s_j + 1)$. Note that $\Omega_j = \Omega_j^o \cup \Omega_j^m$.

The steady state probability that item j is operational in the tagged system equals

$$P(X_j = 1) = \sum_{n=0}^{s_j} \sum_{l=0}^{M + s_{0j} + \sum_{l=1}^{M} s_{0j}} \sum_{m=0}^{\min(s_j - n, (l - s_{0j})^+)} \pi_{m,n,l}(j),$$

where X_j is the steady state of the process $X_j(t)$, i.e., $X_j = \lim_{t\to\infty} X_j(t)$. Note that the upper bound of m in the previous equation is due to the fact that $m + n$ should be smaller than s_j and the number of item j backorders at the depot destined for the tagged system cannot exceed the total number of backorders at the depot $(l - s_{0j})^+$. Throughout this paper, we shall assume that the AMC_j starts in steady state at time 0. Therefore, for all $t \in [0, T]$ the chain AMC_j, $j = 1, \ldots, M$, will remain in steady state, i.e., $P(X_j = x) = P(X_j(t) = x)$, $\forall t \in [0, T]$.

In the following, we shall use the uniformization method, which is extremely useful for computational purposes. The uniformization method transforms a continuous-time Markov chain with non-identical state leaving rates to an equivalent process in which the transition epochs are generated by a Poisson process at a uniform rate over all states. This is done by introducing additional virtual transitions from a certain state to the same state with the required rate. For more details see (Tijms, 2003) and the references therein. Let P_j denote the transition probability matrix of the uniformized process of $X_j(t)$, $t \geqslant 0$. The matrix P_j reads

$$P_j = I + \frac{1}{v} G_j,$$

where I is the identity matrix of size equal to the size of G_j, and v is given by:

$$v > \max(|G_j(l, l)|, l = 1, \ldots, ||G_j||),$$

where $||G_j||$ is the number of rows in square matrix G_j.

Finally, let P_S denote the transition probability matrix of the joint uniformized process $((BO_1(t), TR_1(t), R_1(t)), \ldots, (BO_M(t), TR_M(t), R_M(t)))$. As an approximation, we consider that P_S is equal to $P_1 \otimes \cdots \otimes P_M$, see, e.g., (Rausand & Høyland, 2004). The latter approximation is accurate and very attractive from a computation point, see Section 5.

4.1. Performance metrics

In this section, we first derive closed form expressions for $E[A(T)]$, $Var[A(T)]$, and $P(A(T) = 1)$. Next, we fit a probability distribution to these three performance metrics.

Theorem 1. *The expected interval availability in [0,T] is equal to the steady state availability of the system and is given by:*

$$E[A(T)] = \prod_{j=1}^{M} \sum_{n=0}^{s_j} \sum_{l=0}^{M+s_{0j}+\sum_{l=1}^{M} s_{lj}} \sum_{m=0}^{\min(s_j-n,(l-s_{0j})^+)} \pi_{m,n,l}(j),$$

where s_j is the stock level of item j in the tagged system.

Proof. The system is operational at time t if $X_j(t) = 1$, $j = 1, \ldots, M$. Let $O(T)$ denote the total sojourn time of the joint process $(X_1(t), \ldots, X_M(t))$ in the state $(1, \ldots, 1)$ during $[0,T]$. Note that $A(T) = O(T)/T$. The expectation of $O(T)$ then reads

$$E[O(t)] = \int_0^T E[1_{\{(X_1(u),\ldots,X_M(u))=(1,\ldots,1)\}}]du = \int_0^T P[(X_1(u), \ldots, X_M(u)) = (1, \ldots, 1)]du = \int_0^T \prod_{j=1}^{M} P(X_j(u) = 1)du = \int_0^T \prod_{j=1}^{M} P(X_j = 1)du = T.\prod_{j=1}^{M} P(X_j = 1),$$

where the second equality in the first line is due to the independence of $X_1(t), \ldots, X_M(t)$, and the second equality in the second line follows from the fact the system starts in steady state at time zero. Therefore, the system will remain in steady state for all $t > 0$. □

Note that the result in Theorem 1 seems to be straightforward. However, this is not the case, since we know from the literature that the expected value of the fill rate in an inventory system over a finite period is larger than the steady state fill rate, see, e.g., (Thomas, 2005).

Before reporting our result on the variance of $A(T)$, let us introduce some notation. Let γ_j denote a row vector of size equal to the cardinality of the state space Ω_j. The vector γ_j is obtained from the steady state probability vector $\pi(j)$ of AMC_j by replacing the equilibrium probability of the malfunctioning states with zero. Let f_j denote a column vector of size equal to the cardinality of the state space Ω_j. The non-zero entries of f_j are equal to one and they represent the operational states.

Theorem 2. *The variance of the system interval availability in [0,T] is given by:*

$$Var[A(T)] = 2\sum_{n=1}^{\infty} e^{-\nu T} \frac{(\nu T)^n}{(n+2)!} \sum_{i=1}^{n} (n-i+1) \prod_{l=1}^{M} \gamma_l(P_l)^i f_l + 2E[A(T)] \times \frac{e^{-\nu t} + \nu T - 1}{(\nu T)^2} - E[A(T)]^2.$$

Proof. Recall that $O(T)$ denotes the total sojourn time of the joint process $(X_1(t), \ldots, X_M(t))$ in the state $(1, \ldots, 1)$ during $[0,T]$. Recall that P_S, the transition probability matrix of the joint uniformized process $((BO_1(t), TR_1(t), R_1(t)), \ldots, (BO_M(t), TR_M(t), R_M(t)))$, is approximately equal to $P_S = P_1 \otimes P_2 \otimes \cdots \otimes P_M$. Similarly, the probability vector that the joint process starts in an operational state at time zero reads

$$\gamma_s = \gamma_1 \otimes \gamma_2 \otimes \cdots \otimes \gamma_M.$$

Finally, let f_s denote the column vector defined as follows:

$$f_s = f_1 \otimes f_2 \otimes \cdots \otimes f_M.$$

It is well known that, see, e.g., (Horn & Johnson, 1985)

$$(A_1 \otimes A_2 \otimes \cdots \otimes A_k) \cdot (B_1 \otimes B_2 \otimes \cdots \otimes B_k) = A_1B_1 \otimes A_2B_2 \otimes \cdots \otimes A_kB_k.$$

Therefore, it is readily seen that

$$\gamma_s(P_s)^i f_s = \gamma_1(P_1)^i f_1 \otimes \cdots \otimes \gamma_M(P_M)^i f_M = \prod_{l=1}^{M} \gamma_l(P_l)^i f_l,$$

where the last equality follows from the fact that $\gamma_l(P_l)^i f_l$, for all l, are positive real numbers.

Let Ω_{so} denote the set of *system* operational states. According to Proposition 2 in Appendix I, we have that the variance of the interval availability of the joint process in Ω_{so} during $[0,T]$ is given by:

$$Var[A(T)] = 2\sum_{n=1}^{\infty} e^{-\nu T} \frac{(\nu T)^n}{(n+2)!} \sum_{i=1}^{n} (n-i+1)\gamma_s(P_s)^i f_s + 2E[A(T)] \times \frac{e^{-\nu T} + \nu T - 1}{(\nu T)^2} - E[A(T)]^2.$$

Inserting $\gamma_s(P_s)^i f_s$ by its value in the previous equation yields the desired result. □

Similarly we can derive the third moment of $A(T)$, see Proposition 3 in Appendix I. However, we do not include the third moment in our approximation for the purpose of keeping the analysis simple and, moreover, we have a satisfying result with the consideration of the first two moments only, see Section 5.

The interval availability is equal to one if for all items j the time until absorption of AMC_j into the subset Ω_j^m (malfunctioning states set) is larger than T, given that AMC_j starts at time 0 in Ω_j^o (operational states set). Let θ_j denote the row vector that only consists of the steady state probabilities of the operational states of AMC_j. We rearrange the generator G_j of AMC_j such that the operational states of Ω_j^o come first and then the malfunctioning states Ω_j^o. We assume that the states of Ω_j^m are absorbing. This newly constructed absorbing Markov chain is denoted by $AAMC_j$. Let O_j denote the transient generator of rearranged G_j. Let T_j^a denote the time until absorption into a state of Ω_j^m. It is then well known that, see, e.g., (Neuts, 1981)

$$P(T_j^a \geqslant T) = \theta_j \exp(TO_j)e.$$

Theorem 3. *The probability that $A(T) = 1$ is given by:*

$$P(A(T) = 1) = e^{-T\sum_{i=1}^{M} \nu_i} \prod_{j=1}^{M} \theta_j \sum_{n=0}^{\infty} \frac{(\nu_i T)^n}{n!} (P_j^a)^n e,$$

where $P_j^a = I + O_j/\nu_j$, e is a column vector of appropriate size with all elements equal to one, and $\nu_j > \max(|O_j(l,l)|, l = 1, \ldots, \|O_j\|)$.

Proof. The proof follows right away by noting that:

$$P(A(T) = 1) = \prod_{j=1}^{M} P(T_j^a \geqslant T) = \prod_{j=1}^{M} \theta_j \exp(TO_j)e,$$

and,

$$\exp(TO_j) = e^{-\nu_j T} \sum_{n=0}^{\infty} \frac{(\nu_j T)^n}{n!} (P_j^a)^n. \quad \square$$

Note that the infinite sum in Theorems 2 and 3 can be truncated with a predetermined truncation error bounds, see (De Souza e Silva & Gail, 1986; Tijms, 2003).

Remark 1. In the special case where there is no any stock on-hand of items at the bases, the event $A(T) = 1$ is only possible when there is no item failure during $[0,T]$. Therefore, $P(A(T) = 1)$ can be easily found as follows:

$$P(A(T) = 1) = E[A(T)] \prod_{j=1}^{M} e^{-\lambda_j T}.$$

Remark 2. It is hard to compute $P(A(T) = 0)$. This is because a system is on failure if at least one item is on failure for every $t \in [0, T]$. Moreover, $P(A(T) = 0)$ is negligible in practical problem instances of high expected availability. Therefore, we shall neglect this probability in the remainder of this paper.

4.2. Approximation of the survival probability function of A(T)

Until now we have computed the expectation and the variance of the interval availability $A(T)$ as well as the probability that the interval availability equals 1. We shall report now how to fit these metrics in a probability distribution that is a mixture of a probability mass at one and a Beta distribution. The choice for the Beta distribution is made for the main reason that the interval availability and a Beta distributed random variable both have finite support.

The interval availability has probability mass at zero and one. According to Remark 2, we neglect the probability mass at zero. Therefore, we approximate the interval availability as follows:

$$A(T) = (1 - P(A(T) = 1) * B + P(A(T) = 1),$$

where B is a Beta distributed random variable bounded between zero and one. From the latter equation, it is readily seen that

$$E[B] = \frac{E[A(T)] - P(A(T) = 1)}{1 - P(A(T) = 1)}, \quad E[B^2] = \frac{E[A(T)^2] - P(A(T) = 1)}{1 - P(A(T) = 1)}.$$

The probability density function of a Beta random variable reads

$$f(x; \alpha, \beta) = \frac{1}{B(\alpha, \beta)} x^{\alpha-1}(1 - x)^{\beta-1}, \tag{1}$$

where $B(\alpha, \beta)$ is the beta function. Given that the expectation and the variance of B are known, simple calculus gives that

$$\alpha = \frac{(1 - E[B]) * E[B]^2}{Var[B]} - E[B], \tag{2}$$

$$\beta = \alpha\left(\frac{1}{E[B]} - 1\right). \tag{3}$$

Theorem 4. *The survival function of the interval availability is given by:*

$$P(A(T) \geqslant y) = (1 - P(A(T) = 1) \int_y^1 f(x; \alpha, \beta) dx + P(A(T) = 1),$$

where P(A(T)=1) is given in Theorem 3 *and f(x; α, β), α, and β are given in* (1)–(3).

5. Numerical validation

To validate our approximations, we constructed a discrete event simulation model of the actual process. In this section, we compare the results of our model with the simulation. We consider different cases with different values for the average system availability and for the number of items per system (M). In the main scenario, one depot serves six bases. This scenario and its input parameters value are inspired by a case study done at Thales Netherlands. At each base, a single system is installed that is composed of $M = 10$, 30, 50 items. The repairs are done at the depot and there is no repair possible at the bases. The repair time of item j is exponentially distributed with rate $\mu_j = 1/MTTR_j$, where $MTTR_j$ is the mean time to repair item j. The order of magnitude of the $MTTR_j$ is between a few months to more than 1 year. In our model the order-and-ship time is exponentially distributed with mean 120 hour. Item j fails according to a Poisson process with mean time between failures ($MTBF_j$) equal to $1/\lambda_j, j = 1, \ldots, M$. The order of magnitude of λ_j is between a few times per year to a few times per 100 years. Each system is used for 3000 hour per year for missions. We are interested in the interval availability of a system during 1 year, i.e., $T = 8760$ hour. The implementation of the simulation is done in Plant Simulation v8.2. We used Matlab v7.8 to implement our model. We run the simulation and our model on a machine with dual core processor of 2.80 gigahertz with 4 gigabyte RAM. For details on the different stock allocation, $MTBF_j$, and $MTTR_j$ in the nine cases considered, we refer to Appendix II.

In the following, we shall first validate our model and then analyze in Section 5.2 the impact of non-exponential order-and-ship times on the interval availability distribution. Finally, in Section 5.3 we shall analyze the impact of the interval availability distribution on the stock allocation.

5.1. Model validation

In Figs. 1–3, we show the survival function of the interval availability using our model and the simulation with $M = 10$, 30, 50, respectively. Note that in both the simulation and the approximate model, the order-and-ship times are exponentially distributed. Observe that our model has the highest accuracy for the cases where $M = 10$ and 30 and where $E[A(T)]$ is larger than 80%. Our model predicts very well $E[A(T)]$ and $Var[A(T)]$ for all the cases, see the second and third row in Table 1 for details. Note that for all the different cases considered the difference of $P(A(T) \geqslant x)$, with $x \geqslant E[A(T)]$, obtained using the simulation and our model is larger than

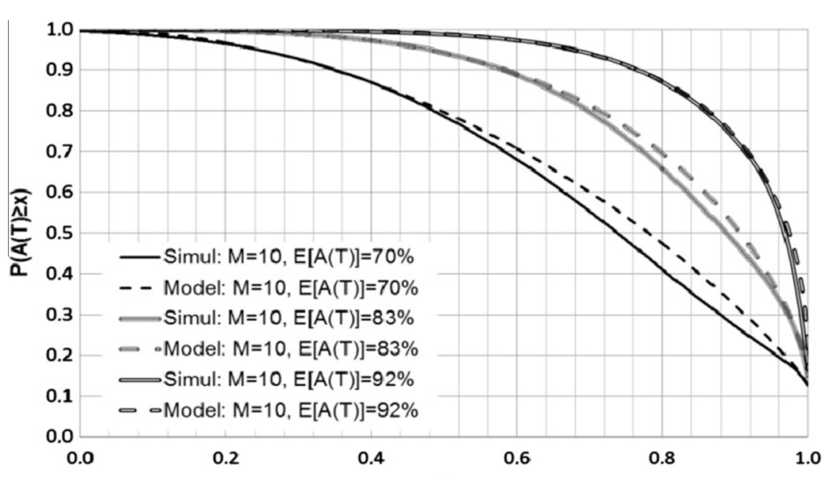

Fig. 1. Interval availability survival function with $M = 10$ in case: $E[A(T)] = 70\%$, $E[A(T)] = 83\%$, and $E[A(T)] = 92\%$.

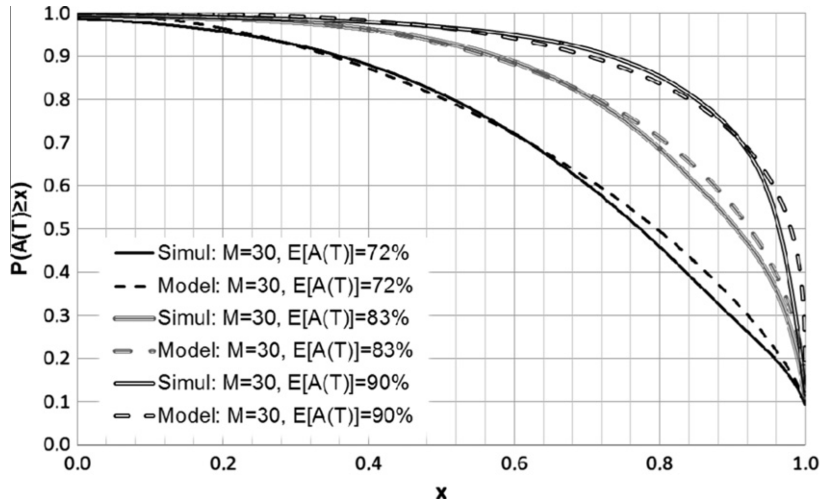

Fig. 2. Interval availability survival function with $M = 30$ in case: $E[A(T)] = 72\%$, $E[A(T)] = 83\%$, and $E[A(T)] = 90\%$.

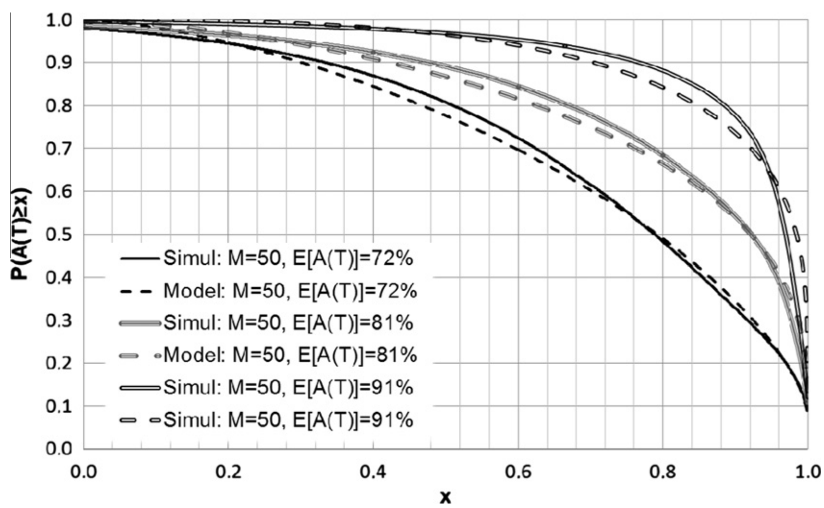

Fig. 3. Interval availability survival function with $M = 50$ in case: $E[A(T)] = 72\%$, $E[A(T)] = 81\%$, and $E[A(T)] = 91\%$.

Table 1
Relative absolute difference of $E[A(T)]$ (resp., $Var[A(T)]$) obtained using our model and simulation.

M	10	10	10	30	30	30	50	50	50
$E[A(T)](\%)$	70	83	92	72	83	90	72	81	91
Relative difference $E[A]$ (%)	3.03	1.23	0.42	1.48	1.22	0.48	0.81	0.92	0.44
Relative difference $Var[A(T)]$ (%)	0.98	2.72	4.47	0.03	3.92	4.63	2.65	3.70	2.76
Min difference of $P(A(T) > x)$, $x \leqslant E[A(T)]$	−0.05	−0.04	−0.01	−0.02	−0.03	−0.01	−0.02	−0.02	−0.01
Max difference $P(A(T) > x)$, $x \leqslant E[A(T)]$	0.00	0.00	0.00	0.01	0.01	0.02	0.03	0.03	0.04

−0.05 and smaller than 0.04, as indicated in Table 1. Finally, we should mention that the computation time of our model is less 500 ms for all the considered cases.

5.2. Exponential vs. deterministic and uniform order-and-ship time

In this section we show that the exponential assumption of the order-and-ship time considered in our model has almost no impact on the survival function of the interval availability, especially, on $P(A(T) \geqslant x)$ with $x \leqslant E[A(T)]$. Using simulations we compare the case with exponentially distributed order-and-ship times with deterministic and uniformly distributed order-and-ship times. All the three distributions have the same expectation equal to 120 hour. The uniform distribution has a finite support in the interval [108, 132]. Fig. 4 displays the survival function of the interval availability with the three distributions. Observe that the order-and-ship time distribution has only an impact on the tail of the survival function, i.e., on $P(A(T) \geqslant x)$ with $x \geqslant E[A(T)]$. In addition, the survival function in the deterministic and uniform case are almost the same and both have some discontinuity points in the tail. Furthermore, using simulation we also find that the repair time distribution has a minor impact on the survival function of the interval

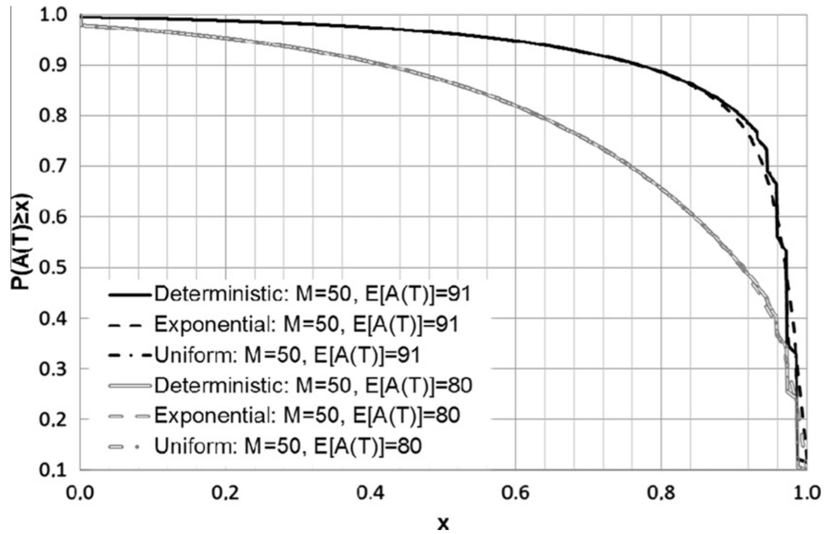

Fig. 4. Interval availability survival function for deterministic, exponential, and uniform order-and-ship time with $M = 50$ in case: $E[A(T)] = 80\%$, $E[A(T)] = 91\%$.

availability. This finding is in line with the literature (Alfredsson & Verrijdt, 1999).

5.3. Impact on stock allocation

In this section, we show that the use of the interval availability probabilities as goal instead of the expected availability may influence the stock allocation in the network. We shall consider two different stock allocations with the same expected availability and total number of items, however, with a different survival probability of the interval availability. We consider the following simple scenario: we have six systems, each of them consisting of two items with a mean time between failures equal to 3640 and 1905 hour. Both items have a mean time to repair equal to 2160. The mean order-and-ship time is equal to 120 hour. Both items prices are equal to one. For a given stock investment of 17 units, METRIC gives that the optimal stock allocation at depot is (6 10), at base one (0 1), and at the other bases (0 0). Using our model we find that for the previous stock allocation the expected system availability at base one is 94.9% and at the rest of bases is equal to 88.1%. Therefore, the expected system availability is equal to 89.2%. The survival probability that the interval availability of base one larger than 0.83, is equal to $P(A_1(T) \geqslant 0.83) = 95.55\%$ and for the rest of systems it is equal to 77.8%. Therefore, on average the survival probability at percentile 0.83 is equal to 80.8%. For the stock allocation at depot (6,11) and at bases (0,0), the expected system availability is equal to 88.9% and the survival availability of the systems at percentile 0.83 is equal to 81.9%. Observe that the two different stock allocations have almost the same expected availability and total number of items, however, the second allocation has a higher survival probability. Therefore, we conclude that the inclusion of the survival probability of the interval availability in the stock allocation optimization may lead to a different stock allocation compared to the METRIC approach.

6. Conclusion and directions for further research

In this paper we analyzed the interval availability of a two-echelon network that supports multi-item systems. We proposed an efficient analytical approximation that is based on a Markov chain analysis. From Markov chain analysis we computed in closed and exact form the expected, the variance, and the probability of hundred percent interval availability of the system. Using the previous three performance metrics we approximate the survival function of the interval availability with a mixture of a probability mass at one and a Beta distribution. The simulation result shows that our model has accurate results especially for high expected interval availability.

As a future research we plan to include our model in a optimization procedure of interval availability probability subject to a constraint on the total investment in the spare parts. Other constraints could also be added like the penalty costs of not meeting the SLA on interval availability. Besides, the extension of the model for multi-echelon supply network with more than two levels and possibly the multi-indenture case are also important. Since in some practical cases we encountered a case with multi-indenture, three-echelon network. Moreover, the restriction in our model for only repairing the failed items at the depot should be relaxed.

Acknowledgements

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Appendix A. Technical proof

In this Appendix we derive the first three moment of the cumulative sojourn time in a subset of the states of a continuous time Markov chain during the interval $[0,T]$. Let G denote the transition rate generator of a continuous-time Markov chain $X(t)$ with finite state space Ω of size $|\Omega|$. We assume $X(t)$ has a steady state distribution and that $X(0)$ is distributed according to the steady state distribution. Let x_i denote the steady state probability $X(\infty) = i$. We denote by $G(l,m)$ the (l,m)-entry of G. Let us define the matrix P as follows:

$$P = I + \frac{1}{v}G,$$

where I is the identity matrix of size $|\Omega|^2$, and v is given by:

$$v > \max(|G_j(l,l)|, l = 1, \ldots, |\Omega|).$$

P can be interpreted as the transition probability matrix of the uniformized process of $X(t)$, $t \geqslant 0$, see, e.g., (Tijms, 2003).

Let $O(T)$ denote the total sojourn time during $[0,T]$ in a subset $\Omega_o \subset \Omega$. It is then well known that the cumulative distribution of the sojourn time in Ω_o reads, see, e.g., (De Souza e Silva & Gail, 1986; Tijms, 2003):

$$P(O(T) \leqslant x) = \sum_{n=0}^{\infty} e^{-\nu T}\frac{(\nu T)^n}{n!}\sum_{k=0}^{n}\alpha(n,k)\sum_{j=k}^{n}\binom{n}{j}\left(\frac{x}{T}\right)^j\left(1-\frac{x}{T}\right)^{n-j},$$
$$0 \leqslant x < T,$$

$$P(O(T) = T) = \sum_{n=0}^{\infty} e^{-\nu T}\frac{(\nu T)^n}{n!}\alpha(n,n+1),$$

where $\alpha(n,k)$ is the probability that the uniformized process visits k times the subset Ω_o during $[0,T]$ given that it makes n states transitions in $[0,T]$. Note that the interval availability equals $A(T)$=$O(T)/T$. Therefore, the derivation of the moment of $O(T)$ should immediately yield the moment of $A(T)$.

Let us denote P_o the probability matrix with jth column equal to the jth column of P if $j \in \Omega_o$ otherwise the jth column is equal to a vector of zeros. Let $P_f = P - P_o$. Conditioning on j, the state of the Markov chain at time T the probabilities $\alpha(n,k,j)$ can be computed recursively. Let $\Gamma(n,k)$ denote a row vector with jth entry equal to $\alpha(n,k,j)$. It is easy to see that $\Gamma(n,k)e = \alpha(n,k)$. The vector $\Gamma(n,k)$ then satisfies the following recursion, see (De Souza e Silva & Gail, 1986),

$$\Gamma(n,k) = \Gamma(n-1,k-1)P_o + \Gamma(n-1,k)P_f, n \geqslant 1, k \geqslant 1. \quad (4)$$

$$\Gamma(n,0) = \Gamma(n-1,0)P_f, n \geqslant 1, k = 0,$$

with the initial conditions:

$$\Gamma(0,0) = (x_1,\ldots,x_\Omega)P_f,$$

$$\Gamma(0,1) = (x_1,\ldots,x_\Omega)P_o,$$

Proposition 1. *The mth moment of A(T) is given by:*

$$E[A(T)^m] = \sum_{n=0}^{\infty} e^{-\nu T}\frac{(\nu T)^n}{(n+m)!}\sum_{k=1}^{n+1}\alpha(n,k)\prod_{l=k}^{k+m-1} l$$

Proof. The mth moment of $A(T)$ follows immediately by noting that the mth moment of $O(T)$ gives

$$E[O(T)^m] = m\int_0^T x^{m-1}P(O(T) > x)dx,$$

and that, $\int_0^T x^m\left(\frac{x}{T}\right)^j\left(1-\frac{x}{T}\right)^{n-j}dx = T^m\frac{(n-j)!(j+m-1)!}{(n+m)!}$, and $\sum_{j=k}^{n}\frac{(j+m-1)!}{j!} = \frac{1}{m}\left[\frac{(n+m)!}{n!} - \frac{k(m+k-1)!}{k!}\right]$. $\square$

We deduce from Proposition 1 that it remains to compute $\sum_{k=0}^{n+1}\alpha(n,k)\prod_{l=k}^{k+m-1} l$ in order to find $E[A(T)^m]$. In order to do so, we follow an approach based on generating functions. Let us multiply (2) by z^k and sum the result over all $k = 0, \ldots, n+1$. Then, we find that

$$\Delta_n(z) = \sum_{k=0}^{n+1}\Gamma(n,k)z^k = \Delta_0(z)(zP_o + P_f)^n$$
$$= (x_1,\ldots,x_{|\Omega|})(zP_o + P_f)^{n+1}, \quad n \geqslant 0. \quad (5)$$

In the following we restrict the derivation to the second and third moment of $A(T)$. Taking the first derivative of (3) according to z at point one and multiplying the result with the column vector e, we find that

$$\sum_{k=1}^{n+1}k\alpha(n,k) = (n+1)\sum_{i\in\Omega_o}x_i, \quad (6)$$

where we used that $P \cdot e = e$ and $(x_1,\ldots,x_{|\Omega|})P = (x_1,\ldots,x_{|\Omega|})$. Taking the second order derivative of (3) according to z at point one and multiplying the result with e, we find that

$$\sum_{k=1}^{n+1}k(k-1)\alpha(n,k) = 2p_0\sum_{i=1}^{n}(n-i+1)P^ie_0, \quad (7)$$

where $p_0 = (x_1,\ldots,x_{|\Omega|})P_o$, i.e., p_0 is the row vector with ith entry equal to x_i if $i \in \Omega_o$ and zero otherwise, and e_0 is the column vector with ith entry equal to 1 if $i \in \Omega_o$ and zero otherwise. Finally, the third derivative of (3) at point one gives that

$$\sum_{k=1}^{n+1}k(k-1)(k-2)\alpha(n,k) = 6p_0\sum_{i=1}^{n-1}\sum_{j=1}^{n-i}(n+1-i-j)P^{i-1}P_0P^je_0. \quad (8)$$

The sum of (5) with two times (4) gives $\sum_{k=0}^{n+1}k(k+1)\alpha(n,k)$ right away. Moreover, $\sum_{k=0}^{n+1}k(k+1)(k+2)\alpha(n,k) = (8) + 6*[(6)+(7)]$. We are now ready to show our main results.

Proposition 2. *The second moment of the fraction of time that the Markov chain X(t) sojourns in the subset Ω_0 during [0,T] is given by:*

$$E[A(T)^2] = 2\sum_{n=1}^{\infty}e^{-\nu T}\frac{(\nu T)^n}{(n+2)!}p_0\sum_{i=1}^{n}(n-i+1)P^ie_0$$
$$+ 2\sum_{l\in\Omega_0}x_l\frac{e^{-\nu T}+\nu T-1}{(\nu T)^2},$$

where x_i is the steady state probability of the Markov chain in state i, p_0 is the column vector with ith entry equal to x_i if $i \in \Omega_0$ and zero otherwise, and e_0 is the column vector with ith entry equal to 1 if $i \in \Omega_o$ and zero otherwise.

Proposition 3. *The third moment of the fraction of time that the Markov chain X(t) sojourns in the subset Ω_0 during [0,T] is given by:*

$$E[A(T)^3] = 6\sum_{n=0}^{\infty}e^{-\nu T}\frac{(\nu T)^n}{(n+3)!}p_0\sum_{i=1}^{n-1}\sum_{j=1}^{n-i}(n-i-j+1)P^{i-1}P_0P^je_0$$
$$+ 12\sum_{n=0}^{\infty}e^{-\nu T}\frac{(\nu T)^n}{(n+3)!}p_0\sum_{i=1}^{n}(n-i+1)P^ie_0$$
$$+ 12\frac{\nu T-2+(\nu T+2)e^{-\nu T}}{(\nu T)^3}.$$

Proof. The results follow right away by replacing m by two and three in Proposition 1 and using Eqs. (5) and (6). $\square$

Appendix B. Simulation details

The mean time between failures, *MTBF*, of the items in the cases with $M = 50$ items are given as follows: $(MTBF_1 \cdots MTBF_{50})$ = (5280 3360000 38100 32400 3180 333000 185100 825000 339000 1095000 280200 726000 41400 223800 288300 195900 348000 56400 27780 265200 26520000 42900000 2652000 333000 13320000 13320000 26520000 3360000 6660000 666000 1095000 1110000 80100 300000000 300000000 300000000 300000000 150000 150000 150000 666000 666000 693000 693000 5280000 30000000 333000 600000 309000 1332000). Note that in the cases with $M = 10$ and 30 we respectively take the first 10 and 30 elements of the latter vector.

The mean time to repair, *MTTR*, of the items in the cases with $M = 50$ items are given as follows: $(MTTR_1 \cdots MTTR_{50})$ = (2160

5760 2160 2160 2160 2160 2160 4320 2160 2160 2160 2160 2160 2160 2160 2160 2160 4320 2160 2160 8640 7200 7920 8640 6480 6480 4320 4320 4320 5040 5760 4320 2160 3600 3600 5760 5040 4320 5040 5040 5760 5040 5040 5040 6480 6480 5040 6480 5760 2160). Note that in the cases with $M = 10$ and 30 we respectively take the first 10 and 30 elements of the latter vector.

The stock at the depot is a vector represented as $(s_{01}, \ldots, s_{0M})$.

Case 1: $M = 10$, $E[A(T)] = 70$, depot stock = (1 0 0 0 0 0 0 0 0 0), base stock = (0 0 0 0 0 0 0 0 0 1).
Case 2: $M = 10$, $E[A(T)] = 83$, depot stock = (1 1 1 1 1 1 0 0 0 0), base stock = (0 0 0 0 0 0 0 0 0 1).
Case 3: $M = 10$, $E[A(T)] = 92$, depot stock = (2 1 1 1 2 0 0 0 0 0), base stock = (0 0 0 0 0 0 0 0 0 1).
Case 4: $M = 30$, $E[A(T)] = 72$, depot stock = (1 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0), base stock = (0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 1 1).
Case 5: $M = 30$, $E[A(T)] = 83$, depot stock = (2 2 2 2 1 1 1 1 1 0 1 1 1 1 1 1 1 0 1 0 0 0 0 1 0 0 0 0 0 0), base stock = (0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 1 1).
Case 6: $M = 30$, $E[A(T)] = 90$, depot stock = (7 1 2 2 2 1 1 1 1 0 1 1 1 1 1 1 1 0 1 0 0 0 0 1 0 0 0 0 0 1), base stock = (0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 1 1).
Case 7: $M = 50$, $E[A(T)] = 72$, depot stock = (2 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0), base stock = (0 0 0 000 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 1 1 1 1 1 1 1).
Case 8: $M = 50$, $E[A(T)] = 81$, depot stock = (3 2 2 2 2 1 1 1 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0), base stock = (0 0 0 000 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 1 1 1 1 1 1 1).
Case 9: $M = 50$, $E[A(T)] = 91$, depot stock = (2 1 2 2 3 1 1 1 1 1 1 1 1 1 1 1 1 0 1 0 0 0 0 1 0 0 0 0 0 1 1 1 0 0 0 0 0 4 4 4 1 1 1 1 1 1 2 1 1 0), base stock = (0 0 0 000 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 1 1 1 1 1 1 1).

References

Aberdeen Group (Ed.). (2005). *The service parts management solution selection report* (Vol. Service Chain Management. Featured Research Series). Service Chain Management.

Alfredsson, P., Verrijdt, J., 1999. Modeling emergency supply flexibility in a two-echelon inventory system. Management Science 45 (10), 1416–1431.

Banerjee, A., Paul, A., 2005. Average fill rate and horizon length. Operations Research Letters 33 (5), 525–530.

Barlow, R.E., Proschan, F., Hunter, L.C., 1965. Mathematical theory of reliability. SIAM.

Caggiano, K.E., Jackson, P.L., Muckstadt, J.A., Rappold, J.A., 2007. Optimizing service parts inventory in a multiechelon, multi-item supply chain with time-based customer service-level agreements. Operations Research 55 (2), 303.

Carrasco, J.A., 2004. Solving large interval availability models using a model transformation approach. Computers & Operations Research 31 (6), 807–861.

Chen, J., Lin, D.K.J., Thomas, D.J., 2003. On the single item fill rate for a finite horizon. Operations Research Letters 31 (2), 119–123.

Cohen, M.A., Kleindorfer, P.R., Lee, H.L., 1986. Optimal stocking policies for low usage items in multi echelon inventory systems. Naval Research Logistics Quarterly 33 (1), 17–38.

De Souza e Silva, E., Gail, H.R., 1986. Calculating cumulative operational time distributions of repairable computer-systems. IEEE Transactions on Computers 35 (4), 322–332.

Deloitte. (2006). *The service revolution in global manufacturing industries*.

Ettl, M., Feigin, G.E., Lin, G.Y., Yao, D.D., 2000. A supply network model with base-stock control and service requirements. Operations Research, 216–232.

Graves, S.C., 1985. A multi-echelon inventory model for a repairable item with one-for-one replenishment. Management Science 31 (10), 1247–1256.

Horn, R.A., Johnson, C.R., 1985. Matrix analysis. Cambridge University Press, Cambridge Cambridgeshire; New York.

Hosford, J.E., 1960. Measures of dependability. Operations Research, 53–64.

Kirmani, E., & Hood, C. S., 2008. A new approach to analysis of interval availability. In *Paper presented at the IEEE international conference on availability, reliability and security* (ARES).

Nakagawa, T., Goel, A.L., 1973. A note on availability for a finite interval. IEEE Transactions on Reliability 22 (5), 271–272.

Neuts, M.F., 1981. Matrix-geometric solutions in stochastic models: An algorithmic approach. Dover Pubns..

Rausand, M., Høyland, A., 2004. System reliability theory: Models, statistical methods, and applications, 2nd ed. Wiley-Interscience, Hoboken, NJ.

Rubino, G., Sericola, B., 1995. Interval availability analysis using denumerable Markov processes: Application to multiprocessor subject to breakdowns and repair. IEEE Transactions on Computers 44 (2), 286–291.

Sherbrooke, C.C., 1968. METRIC: A multi-echelon technique for recoverable item control. Operations Research 16 (1), 122–141.

Slay, F. (1984). *VARI-METRIC: An approach to modelling multi-echelon resupply when the demand process is poisson with a gamma prior. report AF301-3*. Washington, DC: Logistics Management Institute.

Smith, M.A.J., 1997. An approximation of the interval availability distribution. Probability in the Engineering and Informational Sciences 11 (4), 451–467.

Takács, L., 1957. On certain sojourn time problems in the theory of stochastic processes. Acta Mathematica Hungarica 8 (1), 169–191.

Thomas, D.J., 2005. Measuring item fill-rate performance in a finite horizon. Manufacturing & Service Operations Management 7 (1), 74–80.

Tijms, H.C., 2003. A first course in stochastic models. Wiley, New York.

van der Heijden, A., 1988. Interval uneffectiveness distribution for a k-out-of-n multistate reliability system with repair. European Journal of Operational Research 36 (1), 66–77.