Orderings of coherent systems with randomized dependent components

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Abstract

Consider a general coherent system with independent or dependent components, and assume that the components are randomly chosen from two different stocks, with the components of the first stock having better reliability than the others. Then here we provide sufficient conditions on the component's lifetimes and on the random numbers of components chosen from the two stocks in order to improve the reliability of the whole system according to different stochastic orders. We also discuss several examples in which such conditions are satisfied and an application to the study the optimal random allocation of components in series and parallel systems. As a novelty, our study includes the case of coherent systems with dependent components by using basic mathematical tools (and copula theory).

Keywords: Reliability, coherent systems, stochastic orders, domination function, copula, Parrondo's paradox.

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1   Introduction

Coherent systems are basic concepts in reliability theory (see, e.g., Barlow and Proschan [2] and Kuo and Zhu [13] for a detailed introduction to coherent systems, related properties and applications). Series systems, parallel systems and k-out-of-n systems are particular cases of coherent systems. In the theory of coherent systems, it is important to study the performance of a system composed by different kinds of units, and to define the optimal allocation of these units in the system. Some results on this topic are given in da Costa Bueno [5], da Costa Bueno and do Carmo [6], Li and Ding [15], Brito et al. [4], Misra et al. [18], Navarro and Rychlik [23], Zhang [27], Eryilmaz [10], Kuo and Zhu [13], Zhao et al. [28, 29], Belzunce et al. [3], Levitin et al. [14] and Hazra and Nanda [11]. In this context, a translation of the Parrondo's paradox was proposed in Di Crescenzo [7]. Parrondo's paradox shows that, in game theory, sometimes a random strategy might be a better option than any deterministic strategy. Analogously, Di Crescenzo [7] proves that in a series system with independent non-identically distributed components, the random choice of these components is a better option than to use components with different distributions in the system. Indeed, Di Crescenzo [7] proved that, in some situations, a random choice is the best option when we have to choose between two kinds of units having different behaviors. This result was extended to other system structures and to the case of dependent components in Navarro and Spizzichino [24]. More recently, the analysis of series and parallel systems formed by components having independent lifetimes and randomly chosen from two different stocks has been performed in Di Crescenzo and Pellerey [8]. In that paper, they assume that the components can be chosen from two stocks, where the items of a batch are better than those of the other one in the usual stochastic order. Under these assumptions, they obtain conditions on the random numbers of components chosen from the two stocks such that the reliability of system's lifetime improves. Similar results were obtained recently in Hazra and Nanda [11] for other stochastic orders and for series and parallel systems with independent components.

In this paper we extend the results given in [8, 11] for series and parallel systems with independent components to general coherent systems with arbitrary structures and with possibly dependent components. We also show that these results can be used to study the optimal random allocation in a coherent system of components chosen from two different stocks.

The rest of the paper is organized as follows. In Section 2 we recall useful notions and definitions, such as the definitions of the stochastic orders used to compare the performances of systems, and the notion of copula, which is used to formally describe the dependence between the component lifetimes. We also describe the model of coherent systems with a random number of components chosen from different stocks. Section 3 contains the main results, which are centered on conditions to improve these systems in the stochastic, hazard rate, reversed hazard rate and likelihood ratio orders. In Section 4 we study ordering properties for systems in which the units are chosen randomly by means of Bernoulli trials. Several examples are provided in Section 5 to illustrate our theoretical results. In particular, certain general results are obtained for series systems when the dependence between the components is modeled by an Archimedean copula. An application to the study of the optimal random allocation of components in series and parallel systems is given in Section 6. Finally, some concluding remarks are presented in Section 7.

Hereafter some conventions and notations used throughout the paper are given. The notation $=_{st}$ stands for equality in distribution. We write "increasing" instead of "non-decreasing" and "decreasing" instead of "non-increasing". Also, we will denote by $D_i g(x_1, \ldots, x_m)$ the partial derivative

with respect to x_i of the function g, i.e.,

$$D_i g(x_1, \ldots, x_m) = \frac{\partial}{\partial x_i} g(x_1, \ldots, x_m)$$

whenever it exists. Moreover, we shall use the usual notation (a, b) for the open interval of real numbers $(a, b) = \{x \in \mathbb{R} : a < x < b\}$. Analogously, $[a, b]$ represents the closed interval.

2 Preliminaries

Firstly, we briefly recall the definitions of the stochastic orders that will be used throughout the paper to compare random lifetimes or the random numbers of components chosen from the two stocks. For further details, basic properties and applications of these orders, we refer the reader to Shaked and Shanthikumar [26], or to Barlow and Proschan [2] where, in particular, a list of applications in reliability theory is described.

Let X and Y be two absolutely continuous random variables having common support $[0, \infty)$, distribution functions F and G, and reliability (or survival) functions $\overline{F} = 1 - F$ and $\overline{G} = 1 - G$, respectively. Let f and g be their probability density functions, $\lambda_X = f/\overline{F}$ and $\lambda_Y = g/\overline{G}$ be their hazard functions and $\tilde{\lambda}_X = f/F$ and $\tilde{\lambda}_Y = g/G$ be their reversed hazard functions, respectively. We say that X is smaller than Y

- in the *usual stochastic order* (denoted by $X \leq_{st} Y$) if $\overline{F}(t) \leq \overline{G}(t)$ for all $t \in [0, \infty)$,
- in the *hazard rate order* (denoted by $X \leq_{hr} Y$) if $\lambda_X(t) \geq \lambda_Y(t)$ for all $t \in [0, \infty)$, i.e., if the ratio $\overline{F}(t)/\overline{G}(t)$ is decreasing in $[0, \infty)$,
- in the *reversed hazard rate order* (denoted by $X \leq_{rhr} Y$) if $\tilde{\lambda}_X(t) \leq \tilde{\lambda}_Y(t)$ for all $t \in [0, \infty)$, i.e., if the ratio $F(t)/G(t)$ is decreasing in $[0, \infty)$,
- in the *likelihood ratio order* (denoted by $X \leq_{lr} Y$) if the ratio $f(t)/g(t)$ is decreasing in $[0, \infty)$,
- in the *convex order* (denoted by $X \leq_{cx} Y$) if $E(\psi(X)) \leq E(\psi(Y))$) for all convex functions ψ for which both expectations exist,
- in the *increasing convex order* (denoted by $X \leq_{icx} Y$) if $E(\psi(X)) \leq E(\psi(Y))$) for all increasing convex functions ψ for which both expectations exist,
- in the *increasing concave order* (denoted by $X \leq_{icv} Y$) if $E(\psi(X)) \leq E(\psi(Y))$) for all increasing concave functions ψ for which both expectations exist.

In particular here we just point out that

- $X \leq_{hr} Y$ if, and only if, $[X - t \mid X > t] \leq_{st} [Y - t \mid Y > t]$ for all $t \in [0, \infty)$,
- $X \leq_{rhr} Y$ if, and only if, $[t - X \mid X \leq t] \geq_{st} [t - Y \mid Y \leq t]$ for all $t \in [0, \infty)$,
- $X \leq_{lr} Y$ if, and only if, $[X \mid a \leq X \leq b] \leq_{st} [Y \mid a \leq X \leq b]$ for all $a \leq b$.

Hence, the hazard rate order and the reversed hazard rate order are often employed to compare residual lifetimes and inactivity times of systems, respectively, and the likelihood ratio order can be used to compare both residual lifetimes and inactivity times, while this is not the case for the weaker usual stochastic order. Moreover, the following relationships are well known:

$$
\begin{array}{ccc}
X \leq_{lr} Y & \Rightarrow & X \leq_{hr} Y \\
\Downarrow & & \Downarrow \\
X \leq_{rhr} Y & \Rightarrow & X \leq_{st} Y \quad \Rightarrow \quad X \leq_{icx, icv} Y.
\end{array}
$$

Dealing with vectors of possibly dependent lifetimes, a common tool to describe the dependence is by means of its copula. Given a vector $\boldsymbol{X} = (X_1, \ldots, X_m)$ of lifetimes, having joint distribution F and marginal distributions $F_1, \ldots, F_m$, the function $C : [0,1]^m \to \mathbb{R}^+$ is said to be the *copula* of $\boldsymbol{X}$ if

$$F(x_1, \ldots, x_m) = C(F_1(x_1), \ldots, F_m(x_m)), \quad \text{for all } (x_1, \ldots, x_m) \in \mathbb{R}^m.$$

If the marginal distributions F_i, for $i = 1, \ldots, m$, are continuous, then the copula C is unique and it is defined as

$$C(u_1, \ldots, u_m) = F(F_1^{-1}(u_1), \ldots, F_m^{-1}(u_m)) = P(F_1(X_1) \le u_1, \ldots, F_m(X_m) \le u_m)$$

for $u_1, \ldots, u_m \in (0,1)$. Copulas entirely describe the dependence between the components of a random vector; for example, concordance indexes like the Spearman's ρ or Kendall's τ of a vector $\boldsymbol{X}$ can be defined by means of its copula (see Nelsen [25] for a monograph on copulas and their properties).

In reliability, often the survival copula S instead of the copula C is considered. Let $\boldsymbol{X}$ have joint reliability function $\overline{F}$ and marginal reliability functions $\overline{F}_1, \ldots, \overline{F}_m$; then the function $S : [0,1]^m \to \mathbb{R}^+$ is said to be the *survival copula* of $\boldsymbol{X}$ if

$$\overline{F}(x_1, \ldots, x_m) = S(\overline{F}_1(x_1), \ldots, \overline{F}_m(x_m)), \quad \text{for all } (x_1, \ldots, x_m) \in \mathbb{R}^m.$$

Among copulas (or survival copulas), particularly interesting is the class of Archimedean copulas. A copula is said to be *Archimedean* if it can be written as

$$C(u_1, \ldots, u_m) = \overline{W}\left(\sum_{i=1}^m \overline{W}^{-1}(u_i)\right) \quad \text{for all } u_i \in [0,1] \tag{2.1}$$

for a suitable decreasing and m-monotone function $\overline{W} : [0, \infty) \to [0, 1]$ such that $\overline{W}(0) = 1$ and with inverse function $\overline{W}^{-1}$. The function $\overline{W}$ is usually called the *generator* of the Archimedean copula C. As pointed out in Nelsen [25], many standard distributions (such as the ones in Gumbel, Frank, Clayton and Ali-Mikhail-Haq families) are special cases of this class. We also refer the reader to Müller and Scarsini [19] or McNeil and Nešlehová [17], and references therein, for details, properties and recent applications of Archimedean copulas. All the Archimedean copulas are *exchangeable*, that is,

$$C(u_1, \ldots, u_m) = C(u_{\sigma(1)}, \ldots, u_{\sigma(m)})$$

for any permutation σ.

The model considered in this paper is described here. We consider a coherent system formed by m components, whose (possibly dependent) random lifetimes are denoted by $X_1, \ldots, X_k, Y_{k+1}, \ldots, Y_m$. These lifetimes come from two distinct classes $\mathcal{C}_X = \{X_1, \ldots, X_k\}$ and $\mathcal{C}_Y = \{Y_{k+1}, \ldots, Y_m\}$, having sizes k and $m - k$, respectively. Given two preassigned random lifetimes X and Y, we assume that the lifetimes belonging to $\mathcal{C}_X$ are identically distributed to X and those in $\mathcal{C}_Y$ are identically distributed to Y. Denote by $\overline{F}_X(t) = P(X > t)$ and $\overline{F}_Y(t) = P(Y > t)$ their respective reliability functions.

From Barlow and Proschan [2], we know that the system lifetime can be written as a function $\phi : \mathbb{R}^m \to \mathbb{R}$ (which only depends on the structure of the system) of the component lifetimes. Thus we denote by $T_0 = \phi(Y_1, \ldots, Y_m)$ and $T_m = \phi(X_1, \ldots, X_m)$ the system lifetimes obtained from

identically distributed (i.d.) components from classes $\mathcal{C}_Y$ and $\mathcal{C}_X$, respectively. In a similar way, we denote by $T_k = \phi(X_1, \ldots, X_k, Y_{k+1}, \ldots, Y_m)$ the system lifetime with k i.d. components from $\mathcal{C}_X$ and $m - k$ i.d. components from $\mathcal{C}_Y$, for $k = 1, 2, \ldots, m - 1$. Note that, without loss of generality, we are assuming that the first k components are in $\mathcal{C}_X$ and the last ones are in $\mathcal{C}_Y$ (in other cases, we just need to change the function ϕ).

We assume that the joint reliability function of the component lifetimes can be written as

$$P(X_1 > t_1, \ldots, X_k > t_k, Y_{k+1} > t_{k+1}, \ldots, Y_m > t_m) = S\left(\overline{F}_X(t_1), \ldots, \overline{F}_X(t_k), \overline{F}_Y(t_{k+1}), \ldots, \overline{F}_Y(t_m)\right)$$
(2.2)

for $t_1, \ldots, t_m \geq 0$ and for $k = 0, 1, \ldots, m$, where S is a fixed survival copula of dimension m, that is, we assume that the copula does not depend on k. Hence, due to Eq. (2.2), the possible dependence among the components is expressed by the survival copula solely. In other words, the dependence is due to the system (by effect of the common environment, of the position of the components, etc.), and not by the number k of components which came from $\mathcal{C}_X$. The independence case can be included in this general model by choosing the product copula $S(u_1, \ldots, u_m) = u_1 \ldots u_m$.

The reliability function of the system can then be written as

$$\overline{F}_{T_k}(t) = H\big(\underbrace{\overline{F}_X(t), \ldots, \overline{F}_X(t)}_{k-\text{times}}, \underbrace{\overline{F}_Y(t), \ldots, \overline{F}_Y(t)}_{(m-k)-\text{times}}\big), \qquad t > 0, \tag{2.3}$$

where $H : [0,1]^m \to [0,1]$ is called *domination function* (see, e.g. Navarro and Spizzichino [24] or Navarro et al. [20, 21]). The domination function H is increasing in $[0,1]^m$ and satisfies $H(0, \ldots, 0) = 0$ and $H(1, \ldots, 1) = 1$. However, in general, H is not necessarily a copula. Moreover, it only depends on the function ϕ (the structure of the system) and on the survival copula S (the dependence model). As example,

(i) for the series systems we have $H(\mathbf{u}) = S(\mathbf{u})$ for any $\mathbf{u} = (u_1, \ldots, u_m) \in [0,1]^m$;

(ii) for a 2-out-of-3 system (i.e., a system with three components which works if at least two components work), the domination function is

$$H(u_1, u_2, u_3) = S(u_1, u_2, 1) + S(u_1, 1, u_3) + S(1, u_2, u_3) - 2S(u_1, u_2, u_3),$$

so that the system reliability function is

$$\overline{F}_T(t) = S(\overline{F}_1(t), \overline{F}_2(t), 1) + S(\overline{F}_1(t), 1, \overline{F}_3(t)) + S(1, \overline{F}_2(t), \overline{F}_3(t)) - 2S(\overline{F}_1(t), \overline{F}_2(t), \overline{F}_3(t)),$$

where $\overline{F}_i$, $i = 1, 2, 3$, are the reliability functions of the component lifetimes.

If the system components are independent, then H only depends on the system structure. For example, in a 2-out-of-3 system with independent components we have

$$H(u_1, u_2, u_3) = u_1 u_2 + u_1 u_3 + u_2 u_3 - 2u_1 u_2 u_3.$$

We note that if $X \geq_{st} Y$, as H is increasing, then

$$T_k \leq_{st} T_{k+1} \qquad \text{for } k = 0, 1, \ldots, m - 1. \tag{2.4}$$

This expresses the intuitive result by which the system with more 'good' components is always more reliable. Hereafter we aim to generalize this result to the case when the size of the two classes $\mathcal{C}_X$

and $\mathcal{C}_Y$ is random. To this purpose we remark that if k is chosen randomly according to a discrete random variable K with support included in $\{0, 1, \ldots, m\}$, then the system reliability function is

$$\overline{F}_{T_K}(t) = \sum_{k=0}^{m} H\big(\underbrace{\overline{F}_X(t), \ldots, \overline{F}_X(t)}_{k-\text{times}}, \underbrace{\overline{F}_Y(t), \ldots, \overline{F}_Y(t)}_{(m-k)-\text{times}}\big) P(K = k). \tag{2.5}$$

Similarly, one can consider the cumulative distribution functions instead of the reliability (survival) functions, i.e., expressing the possible dependence in the component's vector by their joint distribution function which can be written as

$$P(X_1 \le t_1, \ldots, X_k \le t_k, Y_{k+1} \le t_{k+1}, \ldots, Y_m \le t_m) = C(F_X(t_1), \ldots, F_X(t_k), F_Y(t_{k+1}), \ldots, F_Y(t_m))$$

for $k = 0, 1, \ldots, m$, where C is a fixed copula of dimension m. We recall again that the copula is always the same, independently on k. Thus, the cumulative distribution function of the system with lifetime T_k can be written as

$$F_{T_k}(t) = \widetilde{H}\big(\underbrace{F_X(t), \ldots, F_X(t)}_{k-times}, \underbrace{F_Y(t), \ldots, F_Y(t)}_{(m-k)-times}\big),$$

where $\widetilde{H}$ only depends on ϕ and on C. For instance, for a parallel system, we have $\widetilde{H} = C$. We remark that the function $\widetilde{H}$ is increasing in $[0, 1]^m$ and satisfies $\widetilde{H}(0, \ldots, 0) = 0$ and $\widetilde{H}(1, \ldots, 1) = 1$. However, $\widetilde{H}$ is not necessarily a copula.

If k is chosen randomly according to a discrete random variable K with support included in $\{0, 1, \ldots, m\}$, then, similarly to (2.5), we have

$$F_{T_K}(t) = \sum_{k=0}^{m} \widetilde{H}\big(\underbrace{F_X(t), \ldots, F_X(t)}_{k-times}, \underbrace{F_Y(t), \ldots, F_Y(t)}_{(m-k)-times}\big) P(K = k). \tag{2.6}$$

3 Comparisons of coherent systems with two classes of components

For what it concerns conditions on the system and on the random numbers of components chosen from the two stocks such that stochastic comparisons between the lifetimes of the corresponding systems are satisfied, a first result immediately follows from Theorem 1.A.6 of Shaked and Shanthikumar [26]. Hereafter we denote by T_{K_i} the lifetime of the system obtained when the size k of the class $\mathcal{C}_X$ is chosen randomly according to the random variable K_i with support included in $\{0, 1, \ldots, m\}$, for $i = 1, 2$.

Proposition 3.1 *Let $X \ge_{st} Y$; if k is chosen randomly according to two different random variables K_1 and K_2 with supports included in $\{0, 1, \ldots, m\}$, and if $K_1 \le_{st} K_2$, then*

$$T_{K_1} \le_{st} T_{K_2}.$$

Theorem 1 in [8] proves that condition $K_1 \le_{st} K_2$ in the preceding proposition can be replaced by the weaker condition $K_1 \le_{icx} K_2$ when the system is a series system with independent components. Analogously, Theorem 2 in [8] proves that condition $K_1 \le_{st} K_2$ in the preceding proposition can be replaced by the weaker condition $K_1 \le_{icv} K_2$ when the system is a parallel system with independent

components. In order to extend these properties to other coherent systems and other dependence models, we need the following lemma. First we introduce the following notation

$$\mathbf{z}_j = \mathbf{z}_j(u, v) = (\underbrace{u, \ldots, u,}_{j-times} \underbrace{v, \ldots, v}_{(m-j)-times}) \tag{3.1}$$

for $j = 0, \ldots, m$.

Lemma 3.1 *If $\varphi(k) = H(\mathbf{z}_k)$ is convex (concave) in $\{0, \ldots, m\}$ for all $u, v \in (0, 1)$ and $\psi : \mathbb{R} \to \mathbb{R}$ is an increasing function, then $\eta(k) = E(\psi(T_k))$ is convex (concave) in $\{0, \ldots, m\}$.*

Proof. To prove that $\eta(k)$ is convex (concave), let us verify that

$$\eta(k+1) - \eta(k) \geq \eta(k) - \eta(k-1) \quad (\leq)$$

for $k = 1, \ldots, m - 1$. This is equivalent to

$$\eta(k+1) + \eta(k-1) \geq 2\eta(k) \quad (\leq)$$

for $k = 1, \ldots, m - 1$. From (2.3), the reliability function of T_k is given by $\overline{F}_{T_k}(t) = H(\mathbf{z}_k)$ when $u = \overline{F}_X(t)$ and $v = \overline{F}_Y(t)$. Let us define the random variable Z as T_{k+1} with probability $1/2$ and as T_{k-1} with probability $1/2$. Then the reliability function of Z is

$$\overline{F}_Z(t) = \frac{1}{2}\overline{F}_{T_{k-1}}(t) + \frac{1}{2}\overline{F}_{T_{k+1}}(t) = \frac{1}{2}H(\mathbf{z}_{k-1}) + \frac{1}{2}H(\mathbf{z}_{k+1}).$$

Hence, as $\varphi(k) = H(\mathbf{z}_k)$ is convex (concave), we have

$$H(\mathbf{z}_{k+1}) - H(\mathbf{z}_k) \geq H(\mathbf{z}_k) - H(\mathbf{z}_{k-1}) \quad (\leq).$$

Therefore, $\overline{F}_Z(t) \geq \overline{F}_{T_k}(t)$ $(\leq)$ for all t, i.e. $Z \geq_{st} T_k$ $(\leq_{st})$. Being ψ increasing, it follows

$$\frac{1}{2}E(\psi(T_{k-1})) + \frac{1}{2}E(\psi(T_{k+1})) = E(\psi(Z)) \geq E(\psi(T_k)) \quad (\leq).$$

Then $\eta(k)$ is convex (concave). ∎

Now we can obtain the following result.

Proposition 3.2 *If k is chosen randomly according to two different random variables K_1 and K_2 with supports included in $\{0, 1, \ldots, m\}$ and $\varphi(k) = H(\mathbf{z}_k)$ is convex (concave) in $\{0, 1, \ldots, m\}$ for all $u, v \in (0, 1)$, then:*

(i) $K_1 \leq_{cx} K_2$ implies $T_{K_1} \leq_{st} T_{K_2}$ $(\geq_{st})$.

(ii) $X \geq_{st} Y$ and $K_1 \leq_{icx} K_2$ $(\leq_{icv})$ imply $T_{K_1} \leq_{st} T_{K_2}$.

Proof. (*i*) If $\varphi(k) = H(\mathbf{z}_k)$ is convex, then, from Lemma 3.1, $\eta(k) = E(\psi(T_k))$ is convex for any increasing function ψ. Therefore, from the definition of the convex order, $K_1 \leq_{cx} K_2$ implies

$$E(\psi(T_{K_1})) = E(\eta(K_1)) \leq E(\eta(K_2)) = E(\psi(T_{K_1})) \tag{3.2}$$

for any increasing function ψ, that is, $T_{K_1} \leq_{st} T_{K_2}$. If $\varphi(k)$ is concave, then $-\varphi(k)$ is convex and with a similar reasoning we obtain $T_{K_1} \geq_{st} T_{K_2}$.

(*ii*) If $\varphi(k) = H(\mathbf{z}_k)$ is convex (concave), then, from Lemma 3.1, $\eta(k) = E(\psi(T_k))$ is convex (concave) for any increasing function ψ. Moreover, as $X \geq_{st} Y$, from (2.4), we have that $\eta(k)$ is increasing. Therefore, from the definition of the increasing convex (concave) order, $K_1 \leq_{icx} K_2$ $(\leq_{icv})$ implies relation (3.2) for any increasing function ψ, that is, $T_{K_1} \leq_{st} T_{K_2}$. ∎

Remark 3.1 *Note that in (i) we do not need the condition $X \geq_{st} Y$. So, the property holds even if $X \leq_{st} Y$ or if X and Y are not ordered. If $X \geq_{st} Y$, then in (ii), we need the convexity (concavity) of $\varphi(k) = H(\mathbf{z}_k)$ just for $u \geq v$. In the case of a series system with independent components, we have*

$$\varphi(k) = H(\mathbf{z}_k) = u^k v^{m-k},$$

which is a convex function for all $u, v \in (0,1)$. Thus, from (ii), we obtain Theorem 1 in [8]. Analogously, from (i), we obtain that $K_1 \leq_{cx} K_2$ implies $T_{K_1} \leq_{st} T_{K_2}$ even without the condition $X \geq_{st} Y$. In the case of an arbitrary series system, the function H is equal to the survival copula S of the joint distribution of the component lifetimes. Hence $\varphi(k)$ is convex if and only if

$$\varphi(k+1) - \varphi(k) \leq \varphi(k) - \varphi(k-1)$$

holds, that is, if and only if

$$S(\mathbf{z}_{k+1}) - 2S(\mathbf{z}_k) + S(\mathbf{z}_{k-1}) \geq 0$$

holds. If S is exchangeable and $u \leq v$, then the left-hand-side of the preceding expression is equal to

$$P(U_1 \leq u, \ldots, U_{k-1} \leq u, u < U_k \leq v, u < U_{k+1} \leq v, U_{k+2} \leq v, \ldots, U_m \leq v),$$

where $(U_1, \ldots, U_m)$ is a random vector with distribution function equal to the survival copula S. Analogously, if S is exchangeable and $u \geq v$, then it is equal to

$$P(U_1 \leq u, \ldots, U_{k-1} \leq u, v < U_k \leq u, v < U_{k+1} \leq u, U_{k+2} \leq v, \ldots, U_m \leq v).$$

Hence, in any case, if S is exchangeable, then $\varphi(k)$ is convex for any series system (with this dependence model). Thus, from (ii) in the preceding proposition, we obtain that Theorem 1 in [8] can be applied to series systems with dependent components having an arbitrary exchangeable survival copula S. Analogously, from (i), we obtain that $K_1 \leq_{cx} K_2$ implies $T_{K_1} \leq_{st} T_{K_2}$ even without the condition $X \geq_{st} Y$. In the case of a parallel system with independent components, we have

$$\varphi(k) = H(\mathbf{z}_k) = 1 - (1-u)^k (1-v)^{m-k}$$

which is a concave function for all $u, v \in (0,1)$. Thus, from (ii) we obtain Theorem 2 in [8]. Analogously, from (i), we obtain that $K_1 \leq_{cx} K_2$ implies $T_{K_1} \geq_{st} T_{K_2}$ even without the condition $X \geq_{st} Y$. The same happens for parallel systems with dependent components having an exchangeable survival (or distribution) copula.

Next our aim is to obtain similar results for the stronger hazard rate, reverse hazard rate and likelihood ratio orders. The next one deals with the hazard rate order.

Proposition 3.3 *Let $X \geq_{hr} Y$, with X and Y absolutely continuous. If for a fixed $k \in \{0, 1, \ldots, m-1\}$ it holds that*

$$\frac{u_i D_i H(u_1, \ldots, u_m)}{H(u_1, \ldots, u_m)} \quad \text{is decreasing in } u_{k+1}, \tag{3.3}$$

for all $i = 1, 2, \ldots, m$ and for all $u_1, \ldots, u_m \in (0,1)$, then

$$T_k \leq_{hr} T_{k+1}, \tag{3.4}$$

where $T_i = \phi(X_1, \ldots, X_i, Y_{i+1}, \ldots, Y_m)$, $i = k, k+1$.

Proof. From (2.3) we have that the reliability function of the system with lifetime T_k can be expressed as

$$\overline{F}_{T_k}(t) = H(\mathbf{z}_k(t)), \qquad t > 0, \tag{3.5}$$

where

$$\mathbf{z}_k(t) = \big(\underbrace{\overline{F}_X(t), \ldots, \overline{F}_X(t)}_{k-\text{times}}, \underbrace{\overline{F}_Y(t), \ldots, \overline{F}_Y(t)}_{(m-k)-\text{times}} \big). \tag{3.6}$$

Then its density function is

$$f_{T_k}(t) = f_X(t) \sum_{i=1}^{k} D_i H(\mathbf{z}_k(t)) + f_Y(t) \sum_{i=k+1}^{m} D_i H(\mathbf{z}_k(t)),$$

where $f_X(t) = -\overline{F}'_X(t)$ and $f_Y(t) = -\overline{F}'_Y(t)$. Hence, its hazard rate can be written as

$$\lambda_{T_k}(t) = \lambda_X(t) \sum_{i=1}^{k} \frac{\overline{F}_X(t) D_i H(\mathbf{z}_k(t))}{H(\mathbf{z}_k(t))} + \lambda_Y(t) \sum_{i=k+1}^{m} \frac{\overline{F}_Y(t) D_i H(\mathbf{z}_k(t))}{H(\mathbf{z}_k(t))},$$

where $\lambda_X(t) = f_X(t)/\overline{F}_X(t)$ and $\lambda_Y(t) = f_Y(t)/\overline{F}_Y(t)$ are the component hazard rate functions. The hazard rate $\lambda_{T_{k+1}}(t)$ of T_{k+1} has a similar representation. Therefore,

$$
\begin{aligned}
\lambda_{T_k}(t) - \lambda_{T_{k+1}}(t) = {} & \lambda_X(t) \sum_{i=1}^{k} \overline{F}_X(t) \left[\frac{D_i H(\mathbf{z}_k(t))}{H(\mathbf{z}_k(t))} - \frac{D_i H(\mathbf{z}_{k+1}(t))}{H(\mathbf{z}_{k+1}(t))} \right] \\
& + \lambda_Y(t) \frac{\overline{F}_Y(t) D_{k+1} H(\mathbf{z}_k(t))}{H(\mathbf{z}_k(t))} - \lambda_X(t) \frac{\overline{F}_X(t) D_{k+1} H(\mathbf{z}_{k+1}(t))}{H(\mathbf{z}_{k+1}(t))} \\
& + \lambda_Y(t) \sum_{i=k+2}^{m} \overline{F}_Y(t) \left[\frac{D_i H(\mathbf{z}_k(t))}{H(\mathbf{z}_k(t))} - \frac{D_i H(\mathbf{z}_{k+1}(t))}{H(\mathbf{z}_{k+1}(t))} \right].
\end{aligned}
$$

Now noting that $X \geq_{hr} Y$ implies $\overline{F}_X(t) \geq \overline{F}_Y(t)$ for all t, and recalling (3.3) and (3.6), we have

$$\frac{\overline{F}_Y(t) D_{k+1} H(\mathbf{z}_k(t))}{H(\mathbf{z}_k(t))} \geq \frac{\overline{F}_X(t) D_{k+1} H(\mathbf{z}_{k+1}(t))}{H(\mathbf{z}_{k+1}(t))}$$

and

$$\frac{D_i H(\mathbf{z}_k(t))}{H(\mathbf{z}_k(t))} \geq \frac{D_i H(\mathbf{z}_{k+1}(t))}{H(\mathbf{z}_{k+1}(t))}$$

for all $i = 1, 2, \ldots, m$, $i \neq k+1$. Then, since $\lambda_Y(t) \geq \lambda_X(t)$ by assumption, and H is increasing, we have

$$\lambda_Y(t) \frac{\overline{F}_Y(t) D_{k+1} H(\mathbf{z}_k(t))}{H(\mathbf{z}_k(t))} \geq \lambda_X(t) \frac{\overline{F}_X(t) D_{k+1} H(\mathbf{z}_{k+1}(t))}{H(\mathbf{z}_{k+1}(t))}.$$

Hence $\lambda_{T_k}(t) - \lambda_{T_{k+1}}(t) \geq 0$ for all $t \geq 0$, that is, (3.4) holds. ∎

In the following proposition we obtain a result similar to that given in Proposition 3.1 for the hazard rate order.

Proposition 3.4 *Let $X \geq_{hr} Y$, with X and Y absolutely continuous. If k is chosen randomly according to two different random variables K_1 and K_2 with supports included in $\{0, 1, \ldots, m\}$, $K_1 \leq_{hr} K_2$ and the condition*

$$\frac{u_i D_i H(u_1, \ldots, u_m)}{H(u_1, \ldots, u_m)} \quad \text{is decreasing in } u_j, \tag{3.7}$$

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holds for all $i, j = 1, 2, \ldots, m$ and for all $u_1, \ldots, u_m \in (0,1)$, then

$$T_{K_1} \leq_{hr} T_{K_2}.$$

The proof is obtained from (2.5), Proposition 3.3 and Theorem 1.B.14 in Shaked and Shanthikumar [26]. In Example 5.1 we provide a simple sufficient condition for (3.7) in the special case of series systems and Archimedean copulas. In particular, for series systems with independent components we have $H(u_1, \ldots, u_m) = u_1 \ldots u_m$ and

$$\frac{u_i D_i H(u_1, \ldots, u_m)}{H(u_1, \ldots, u_m)} = 1.$$

Hence (3.7) holds for all $i, j = 1, 2, \ldots, m$ and for all $u_1, \ldots, u_m \in (0,1)$ and thus we obtain Theorem 2.2 in [11].

Let us now perform suitable comparisons involving the reversed hazard rate order. We proceed similarly as above, but working with the cumulative distribution functions instead of working with the reliability functions. The proof of the following statement is similar to that of Proposition 3.3 and thus it is omitted.

Proposition 3.5 *Let $X \geq_{rhr} Y$, with X and Y absolutely continuous. If for a fixed $k \in \{0, 1, \ldots, m-1\}$ it holds that*

$$\frac{u_i D_i \widetilde{H}(u_1, \ldots, u_m)}{\widetilde{H}(u_1, \ldots, u_m)} \text{ is decreasing in } u_{k+1} \tag{3.8}$$

for all $i = 1, 2, \ldots, m$ and for all $u_1, \ldots, u_m \in (0,1)$, then

$$T_k \leq_{rhr} T_{k+1}, \tag{3.9}$$

where $T_i = \phi(X_1, \ldots, X_i, Y_{i+1}, \ldots, Y_m)$, $i = k, k+1$.

The analogous result when k is random can be stated as follows.

Proposition 3.6 *Let $X \geq_{rhr} Y$, with X and Y absolutely continuous. Assume that k is chosen randomly according to two different random variables K_1 and K_2 with supports included in $\{0, 1, \ldots, m\}$. If $K_1 \leq_{rhr} K_2$ and the condition*

$$\frac{u_i D_i \widetilde{H}(u_1, \ldots, u_m)}{\widetilde{H}(u_1, \ldots, u_m)} \text{ is decreasing in } u_j, \tag{3.10}$$

holds for all $i, j = 1, 2, \ldots, m$ and for all $u_1, \ldots, u_m \in (0,1)$, then

$$T_{K_1} \leq_{rhr} T_{K_2}.$$

The proof is obtained from Eq. (2.6), Proposition 3.5, assumption $K_1 \leq_{rhr} K_2$ and Theorem 1.B.52 in Shaked and Shanthikumar [26].

In particular, for parallel systems with independent components, we have $\widetilde{H}(u_1, \ldots, u_m) = u_1 \ldots u_m$ and

$$\frac{u_i D_i \widetilde{H}(u_1, \ldots, u_m)}{\widetilde{H}(u_1, \ldots, u_m)} = 1.$$

Hence (3.10) holds for all $i, j = 1, 2, \ldots, m$ and for all $u_1, \ldots, u_m \in (0,1)$ and thus we obtain Theorem 3.2 in [11].

We pinpoint that the system comparisons given in Propositions 3.3 and 3.4 can be obtained also under different assumptions. This is shown below, where hazard rate orderings are established under new conditions which do not involve partial derivatives.

Proposition 3.7 *For a fixed $k \in \{0, 1, \ldots, m-1\}$, let us consider the function*

$$g_k(u, v) := \frac{H(\mathbf{z}_k)}{H(\mathbf{z}_{k+1})}, \tag{3.11}$$

where $\mathbf{z}_j$ is defined by (3.1) for $j = k, k+1$ and for $u, v \in (0, 1)$. Let us assume that g_k can be expressed as

$$g_k(u, v) = \psi_k(u, v/u) \qquad \text{for all } u, v \in (0, 1), \tag{3.12}$$

where $\psi_k : (0, 1)^2 \to (0, \infty)$ is increasing in each of its arguments. Then, if $X \geq_{hr} Y$, we have

$$T_k \leq_{hr} T_{k+1},$$

where $T_j = \phi(X_1, \ldots, X_j, Y_{j+1}, \ldots, Y_m)$, $j = k, k+1$.

Proof. Let $\varphi_k : (0, \infty) \to (0, 1)^2$ be defined as $\varphi_k(t) = (\overline{F}_X(t), \overline{F}_Y(t)/\overline{F}_X(t))$. Recall that the reliability functions $\overline{F}_X(t)$ and $\overline{F}_Y(t)$ are decreasing. Moreover, the assumption $X \geq_{hr} Y$ implies that $\overline{F}_Y(t)/\overline{F}_X(t)$ is decreasing for all $t \geq 0$, and that $X \geq_{st} Y$, i.e. $\overline{F}_Y(t)/\overline{F}_X(t) \leq 1$ for all $t \geq 0$. Thus, the function $\varphi_k(t)$ is decreasing in t. Hence, since ψ_k is increasing in each of its arguments, the composition $\psi_k \circ \varphi_k$ is decreasing in t, i.e., the function in (3.11) is decreasing in $t \geq 0$. Due to (3.5) and (3.6) this monotonicity is equivalent to $T_k \leq_{hr} T_{k+1}$. ∎

Let us obtain now a similar comparison result when the size k of the first class $\mathcal{C}_X$ is random.

Proposition 3.8 *Let K_1 and K_2 be random variables with supports included in $\{0, 1, \ldots, m\}$ and such that $K_1 \leq_{hr} K_2$. If for all $k \in \{0, 1, \ldots, m-1\}$ the functions (3.11) can be expressed as in (3.12), where $\psi_k : (0, 1)^2 \to (0, \infty)$ is increasing in each of its arguments, and if $X \geq_{hr} Y$, then*

$$T_{K_1} \leq_{hr} T_{K_2}.$$

The proof of Proposition 3.8 is immediately obtained from Eq. (2.5), Proposition 3.7 and Theorem 1.B.14 in Shaked and Shanthikumar [26]. Hereafter we come to similar comparison results involving the reversed hazard rate order.

Proposition 3.9 *For a fixed $k \in \{0, 1, \ldots, m-1\}$, we consider the function*

$$h_k(u, v) := \frac{\widetilde{H}(\mathbf{z}_k)}{\widetilde{H}(\mathbf{z}_{k+1})}, \tag{3.13}$$

where $\mathbf{z}_j$ is defined by (3.1) for $j = k, k+1$ and for $u, v \in (0, 1)$. Let us assume that h_k can be expressed as

$$h_k(u, v) = \zeta_k(v, u/v) \qquad \text{for all } u, v \in (0, 1), \tag{3.14}$$

where $\zeta_k : (0, 1)^2 \to (0, \infty)$ is decreasing in each of its arguments. Then, if $X \geq_{rhr} Y$, we have

$$T_k \leq_{rhr} T_{k+1},$$

where $T_j = \phi(X_1, \ldots, X_j, Y_{j+1}, \ldots, Y_m)$, $j = k, k+1$.

Proposition 3.10 *Let K_1 and K_2 be random variables with supports included in $\{0, 1, \ldots, m\}$ and such that $K_1 \leq_{rhr} K_2$. If for all $k \in \{0, 1, \ldots, m-1\}$ the functions (3.13) can be expressed as in (3.14), where $\zeta_k : (0, 1)^2 \to (0, \infty)$ is increasing in each of its arguments, and if $X \geq_{rhr} Y$ then*

$$T_{K_1} \leq_{rhr} T_{K_2}.$$

The proofs of Propositions 3.9 and 3.10 are similar to those of Propositions 3.7 and 3.8, and thus they are omitted. Let us now consider the case of comparisons based on the likelihood ratio order.

Proposition 3.11 *For a fixed $k \in \{0, 1, \ldots, m-1\}$, we consider the function*

$$\rho_k(u, v, w) := \frac{\sum_{i=1}^{k} D_i H(\mathbf{z}_k) + w \sum_{i=k+1}^{m} D_i H(\mathbf{z}_k)}{\sum_{i=1}^{k+1} D_i H(\mathbf{z}_{k+1}) + w \sum_{i=k+2}^{m} D_i H(\mathbf{z}_{k+1})}, \tag{3.15}$$

where $\mathbf{z}_j$ is defined by (3.1) for $j = k, k+1$ and where $u, v \in (0, 1)$ and $w \in (0, \infty)$. Let us assume that ρ_k can be expressed as

$$\rho_k(u, v, w) = \xi_k(u, v/u, w) \qquad \text{for all } u, v \in (0, 1) \text{ and all } w \in (0, \infty), \tag{3.16}$$

where $\xi_k : (0, 1)^2 \times (0, \infty) \to (0, \infty)$ is increasing in each of its arguments. Then, if $X \geq_{lr} Y$, we have

$$T_k \leq_{lr} T_{k+1},$$

where $T_j = \phi(X_1, \ldots, X_j, Y_{j+1}, \ldots, Y_m)$, $j = k, k+1$.

Proof. As the reliability function of T_k is given in (2.3), then its probability density function is

$$f_{T_k}(t) = f_X(t) \sum_{i=1}^{k} D_i H(\mathbf{z}_k(t)) + f_Y(t) \sum_{i=k+1}^{m} D_i H(\mathbf{z}_k(t)), \qquad t \geq 0,$$

where $\mathbf{z}_k(t)$ is defined in (3.6). Hence, due to (3.15), the ordering $T_k \leq_{lr} T_{k+1}$ is equivalent to the condition that the function

$$\frac{f_{T_k}(t)}{f_{T_{k+1}k}(t)} = \frac{\sum_{i=1}^{k} D_i H(\mathbf{z}_k(t)) + [f_Y(t)/f_X(t)] \sum_{i=k+1}^{m} D_i H(\mathbf{z}_k(t))}{\sum_{i=1}^{k+1} D_i H(\mathbf{z}_{k+1}(t)) + [f_Y(t)/f_X(t)] \sum_{i=k+2}^{m} D_i H(\mathbf{z}_{k+1}(t))} \tag{3.17}$$

is decreasing in t. Consider now a function $\varphi_k : (0, \infty) \to (0, 1)^2 \times (0, \infty)$ defined as $\varphi_k(t) = (\overline{F}_X(t), \overline{F}_Y(t)/\overline{F}_X(t), f_Y(t)/f_X(t))$. Recall that the reliability functions $\overline{F}_X(t)$ and $\overline{F}_Y(t)$ are decreasing in t. Furthermore, $X \geq_{lr} Y$ means that $f_Y(t)/f_X(t)$ is decreasing in t. In addition, $X \geq_{lr} Y$ implies $X \geq_{hr} Y$ and hence $\overline{F}_Y(t)/\overline{F}_X(t)$ is decreasing in t. Moreover, $X \geq_{lr} Y$ implies $X \geq_{st} Y$ and hence $\overline{F}_Y(t)/\overline{F}_X(t) \leq 1$. Thus $\varphi_k(t)$ is decreasing in $t \geq 0$. Therefore, due to (3.16) and since ξ_k is increasing in each of its arguments by assumption, then the function in (3.17), which can be written as the composition $\xi_k \circ \varphi_k(t)$, is decreasing in t and thus $T_k \leq_{lr} T_{k+1}$ holds. ∎

The following result is analogous to Proposition 3.8.

Proposition 3.12 *Let K_1 and K_2 be random variables with supports included in $\{0, 1, \ldots, m\}$ and such that $K_1 \leq_{lr} K_2$. If for all $k \in \{0, 1, \ldots, m-1\}$ the functions (3.15) can be expressed as in (3.16), where $\xi_k : (0, 1)^2 \times (0, \infty) \to (0, \infty)$ is increasing in each of its arguments, and if $X \geq_{lr} Y$, with X and Y absolutely continuous, then*

$$T_{K_1} \leq_{lr} T_{K_2}.$$

The proof of Proposition 3.12 can be obtained from Proposition 3.11 and Theorem 1.C.17 in Shaked and Shanthikumar [26].

Various examples related to the above results are discussed in Section 5.

4 Choosing components via Bernoulli trials

For the coherent systems with two classes of components introduced in Section 2, hereafter we investigate the case when the m components are chosen at each position via independent Bernoulli trials with possibly varying probabilities. Thus we consider the two coherent systems with lifetimes

$$T_{\mathbf{p}} = \phi(Z_{p_1}, \dots, Z_{p_m}) \quad \text{and} \quad T_{\mathbf{q}} = \phi(Z_{q_1}, \dots, Z_{q_m}), \tag{4.1}$$

where $\mathbf{p} = (p_1, \dots, p_m)$ and $\mathbf{q} = (q_1, \dots, q_m)$ for $p_i, q_i \in [0, 1]$. Here we assume that $(Z_{p_1}, \dots, Z_{p_m})$ and $(Z_{q_1}, \dots, Z_{q_m})$ have the same copula. Given the reliability functions $\overline{F}_X(t)$ and $\overline{F}_Y(t)$, we also assume that the outcome of the i-th trial establishes whether the i-th system component follows the distribution of X or Y. Hence, the marginal reliability functions of Z_{p_i} and Z_{q_i} are given respectively by the following mixtures:

$$P(Z_{p_i} > t) = p_i \, \overline{F}_X(t) + (1 - p_i) \, \overline{F}_Y(t), \qquad P(Z_{q_i} > t) = q_i \, \overline{F}_X(t) + (1 - q_i) \, \overline{F}_Y(t), \tag{4.2}$$

for $t \in \mathbb{R}$ and for $i = 1, 2, \dots, m$. Let us assume $X \geq_{st} Y$. Of course, if $p_i \geq q_i$ for $i = 1, 2, \dots, m$, then the systems with lifetimes (4.1) satisfy $T_{\mathbf{p}} \geq_{st} T_{\mathbf{q}}$ (since H is increasing). It should be noted that we are dealing with two systems whose structures are identical whereas their components are possibly different, as their nature depends on the outcome of the Bernoulli trials. Hence, since $(Z_{p_1}, \dots, Z_{p_m})$ and $(Z_{q_1}, \dots, Z_{q_m})$ have the same copula, the two systems have the same domination function H. We also recall that the copula of the system does not depend on the inserted components, but it relies on the common environment.

In order to state a more general result for the usual stochastic order, we first need to recall the following two well-known majorization orders (see, e.g., Marshall et al. [16], p. 12). Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$ be real vectors, and let $x_{(1)} \leq \dots \leq x_{(m)}$ and $y_{(1)} \leq \dots \leq y_{(m)}$ be the ordered values obtained from $\mathbf{x}$ and $\mathbf{y}$, respectively; then

(i) $\mathbf{x}$ said to be *weakly submajorized* by $\mathbf{y}$, denoted as $\mathbf{x} \leq_w \mathbf{y}$, if

$$\sum_{i=j}^m x_{(i)} \leq \sum_{i=j}^m y_{(i)} \qquad \text{for } j = 1, 2, \dots, m;$$

(ii) $\mathbf{x}$ said to be *weakly supermajorized* by $\mathbf{y}$, denoted as $\mathbf{x} \leq^w \mathbf{y}$, if

$$\sum_{i=1}^j x_{(i)} \geq \sum_{i=1}^j y_{(i)} \qquad \text{for } j = 1, 2, \dots, m.$$

Note that these two orders are equivalent whenever

$$\sum_{i=1}^m x_i = \sum_{i=1}^m y_i. \tag{4.3}$$

So, $\mathbf{x}$ is said to be *majorized* by $\mathbf{y}$, denoted as $\mathbf{x} \preceq \mathbf{y}$, if conditions (4.3) and $\mathbf{x} \leq_w \mathbf{y}$ hold.

We will also need the concept of Schur-convexity (Schur-concavity) defined as follows. A function $H : \mathbb{R}^m \to \mathbb{R}$ is *Schur-convex* (*Schur-concave*) in $\mathcal{D}$ if

$$H(\mathbf{x}) \leq H(\mathbf{y}) \quad (\geq) \qquad \text{for all } \mathbf{x} \preceq \mathbf{y},$$

with $\mathbf{x}, \mathbf{y} \in \mathcal{D} \subset \mathbb{R}^m$.

Now we can state the following preliminary result.

Proposition 4.1 *Let us consider two coherent systems with the same structure and respective lifetimes* $\phi(X_1, \ldots, X_m)$ *and* $\phi(Y_1, \ldots, Y_m)$, *where* $(X_1, \ldots, X_m)$ *and* $(Y_1, \ldots, Y_m)$ *have the same copula and have marginal reliability functions* $\overline{F}_i(t)$ *and* $\overline{G}_i(t)$, *respectively, for* $i = 1, 2, \ldots, m$. *If the common domination function* H *is Schur-convex (Schur-concave) on* $(0, 1)^m$ *and*

$$\left(\overline{F}_1(t), \ldots, \overline{F}_m(t)\right) \leq_w \left(\overline{G}_1(t), \ldots, \overline{G}_m(t)\right) \quad (\leq^w) \tag{4.4}$$

for all t, *then*

$$T_X \leq_{st} T_Y \quad (\geq_{st}).$$

The proof can be obtained from Theorem A.8 in Marshall et al. [16, p. 87].

Durante and Sempi [9] (see also Nelsen [25], p. 134) proved that every Archimedean copula is Schur-concave. So the preceding proposition can be applied to series systems with dependent components having Archimedean survival copulas since, in this case, $H = S$.

As a consequence of the preceding proposition, we have the following result for coherent systems with components chosen randomly via Bernoulli trials. Again we remark that the systems have the same structure and the components have the same dependence structure (copula), independently on the outcomes of the Bernoulli trials.

Proposition 4.2 *Let* $T_{\mathbf{p}}$ *and* $T_{\mathbf{q}}$ *be defined as in (4.1). Assume that* $(Z_{p_1}, \ldots, Z_{p_m})$ *and* $(Z_{q_1}, \ldots, Z_{q_m})$ *have the same copula, and that* Z_{p_i} *and* Z_{q_i} *have the reliability functions given in (4.2). Let the common domination function* H *be Schur-convex (Schur-concave) on* $(0, 1)^m$. *If* $X \geq_{st} Y$ *and*

$$\mathbf{p} \leq_w \mathbf{q} \quad (\leq^w), \tag{4.5}$$

then

$$T_{\mathbf{p}} \leq_{st} T_{\mathbf{q}} \quad (\geq_{st}).$$

Proof. The reliability functions of Z_{p_i} and Z_{q_i} are given in (4.2), for $i = 1, 2, \ldots, m$. As assumption $X \geq_{st} Y$ implies $\overline{F}_X(t) \geq \overline{F}_Y(t)$ for all t, then we have

$$\overline{F}_{(i)}(t) = p_{(i)}\overline{F}_X(t) + (1 - p_{(i)})\overline{F}_Y(t), \qquad \overline{G}_{(i)}(t) = q_{(i)}\overline{F}_X(t) + (1 - q_{(i)})\overline{F}_Y(t)$$

for all t, where $\overline{F}_{(1)}(t) \leq \cdots \leq \overline{F}_{(m)}(t)$ and $\overline{G}_{(1)}(t) \leq \cdots \leq \overline{G}_{(m)}(t)$ are the ordered reliability functions obtained from the reliability functions of $(Z_{p_1}, \ldots, Z_{p_m})$ and $(Z_{q_1}, \ldots, Z_{q_m})$, respectively, and where $p_{(1)} \leq \cdots \leq p_{(m)}$ and $q_{(1)} \leq \cdots \leq q_{(m)}$ are the ordered values of $(p_1, \ldots, p_m)$ and $(q_1, \ldots, q_m)$, respectively. Hence

$$\sum_{i=j}^{m} \overline{F}_{(i)}(t) = \left(\sum_{i=j}^{m} p_{(i)}\right)\overline{F}_X(t) + \left(\sum_{i=j}^{m}(1 - p_{(i)})\right)\overline{F}_Y(t)$$

and

$$\sum_{i=j}^{m} \overline{G}_{(i)}(t) = \left(\sum_{i=j}^{m} q_{(i)}\right)\overline{F}_X(t) + \left(\sum_{i=j}^{m}(1 - q_{(i)})\right)\overline{F}_Y(t)$$

for all t and for $j = 1, 2, \ldots, m$. Therefore, from assumption (4.5), we have

$$\sum_{i=j}^{m} \overline{F}_{(i)}(t) \leq \sum_{i=j}^{m} \overline{G}_{(i)}(t)$$

for $j = 1, 2, \ldots, m$, that is, (4.4) holds. Then, if H is Schur-convex, the stated result is obtained from Proposition 4.1. The proof is similar when H is Schur-concave and $\mathbf{p} \leq^w \mathbf{q}$ holds. ∎

Let us now provide a similar result for the hazard rate order. We first need the following lemma (see, e.g., Proposition 2.3 in Navarro and Rubio [22]).

Lemma 4.1 *Let X and Y be two random variables with respective reliability functions $\overline{F}_X$ and $\overline{F}_Y$ and such that $X \geq_{hr} Y$. If Z_p has reliability function $p\overline{F}_X(t) + (1-p)\overline{F}_Y(t)$, with $0 \leq p \leq 1$, then*

$$Z_{p_1} \geq_{hr} Z_{p_2}$$

for all $1 \geq p_1 \geq p_2 \geq 0$.

Now we can prove the following result.

Proposition 4.3 *Let $T_{\mathbf{p}}$ and $T_{\mathbf{q}}$ be defined as in (4.1), where $(Z_{p_1}, \ldots, Z_{p_m})$ and $(Z_{q_1}, \ldots, Z_{q_m})$ have the same copula, and where Z_{p_i} and Z_{q_i} have reliability functions given in (4.2). Let $X \geq_{hr} Y$, with X and Y absolutely continuous. If*

$$\frac{u_i D_i H(u_1, \ldots, u_m)}{H(u_1, \ldots, u_m)} \quad \text{is decreasing in } u_1, \ldots, u_m \tag{4.6}$$

for all $(u_1, \ldots, u_m) \in (0, 1)^m$ and for all $i = 1, 2, \ldots, m$ and

$$p_i \leq q_i \tag{4.7}$$

for all $i = 1, 2, \ldots, m$, then

$$T_{\mathbf{p}} \leq_{hr} T_{\mathbf{q}}.$$

Proof. Let us consider the reliability function of the system with lifetime $T_{\mathbf{p}} = \phi(Z_{p_1}, \ldots, Z_{p_m})$, given by

$$\overline{F}_{\mathbf{p}}(t) = H(\overline{F}_1(t), \ldots, \overline{F}_m(t)),$$

where

$$\overline{F}_i(t) = p_i \overline{F}_X(t) + (1 - p_i)\overline{F}_Y(t),$$

with $0 \leq p_i \leq 1$, $i = 1, 2, \ldots, m$. Its density function is

$$f_{\mathbf{p}}(t) = \sum_{i=1}^m f_i(t) D_i H(\overline{F}_1(t), \ldots, \overline{F}_m(t)),$$

where $f_i(t) = -\overline{F}_i'(t)$. Hence its hazard rate can be written as

$$\lambda_{\mathbf{p}}(t) = \sum_{i=1}^m \lambda_i(t) \frac{\overline{F}_i(t) D_i H(\overline{F}_1(t), \ldots, \overline{F}_m(t))}{H(\overline{F}_1(t), \ldots, \overline{F}_m(t))},$$

where $\lambda_i(t) = f_i(t)/\overline{F}_i(t)$ is the hazard rate function of $\overline{F}_i(t)$. Analogously, the hazard rate of $T_{\mathbf{q}} = \phi(Z_{q_1}, \ldots, Z_{q_m})$ can be written as

$$\lambda_{\mathbf{q}}(t) = \sum_{i=1}^m \eta_i(t) \frac{\overline{G}_i(t) D_i H(\overline{G}_1(t), \ldots, \overline{G}_m(t))}{H(\overline{G}_1(t), \ldots, \overline{G}_m(t))},$$

where

$$\overline{G}_i(t) = q_i \overline{F}_X(t) + (1 - q_i)\overline{F}_Y(t),$$

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$\eta_i(t) = -\overline{G}'_i(t)/\overline{G}_i(t)$ and $0 \le q_i \le 1$, $i = 1, 2, \ldots, m$. Now, by using Lemma 4.1 and $p_i \le q_i$, we have

$$\lambda_i(t) \ge \eta_i(t)$$

and

$$\overline{F}_i(t) \le \overline{G}_i(t)$$

for all $i = 1, 2, \ldots, m$. Therefore, using that H is increasing and (4.6) is decreasing, we have

$$
\begin{aligned}
\lambda_{\mathbf{p}}(t) &= \sum_{i=1}^{m} \lambda_i(t) \frac{\overline{F}_i(t) D_i H(\overline{F}_1(t), \ldots, \overline{F}_m(t))}{H(\overline{F}_1(t), \ldots, \overline{F}_m(t))} \\
&\ge \sum_{i=1}^{m} \eta_i(t) \frac{\overline{F}_i(t) D_i H(\overline{F}_1(t), \ldots, \overline{F}_m(t))}{H(\overline{F}_1(t), \ldots, \overline{F}_m(t))} \\
&\ge \sum_{i=1}^{m} \eta_i(t) \frac{\overline{G}_i(t) D_i H(\overline{G}_1(t), \ldots, \overline{G}_m(t))}{H(\overline{G}_1(t), \ldots, \overline{G}_m(t))} \\
&= \lambda_{\mathbf{q}}(t)
\end{aligned}
$$

for all $t \ge 0$, this giving the proof. ∎

We pinpoint that even though the condition (4.6) used above is quite strong, some cases in which it is satisfied will be shown in Example 5.1.

Remark 4.1 If H is exchangeable (i.e. $H(u_1, \ldots, u_m) = H(u_{\sigma(1)}, \ldots, u_{\sigma(m)})$ for every permutation σ), then a simpler proof of Proposition 4.3 can be given. Indeed, let $N_{\mathbf{p}}$ and $N_{\mathbf{q}}$ be the number of components of kind X chosen according to $\mathbf{p} = (p_1, \ldots, p_m)$ and $\mathbf{q} = (q_1, \ldots, q_m)$, respectively. Hence, such variables can be expressed as sums of independent random variables, i.e.

$$N_{\mathbf{p}} = \sum_{i=1}^{m} U_i, \qquad N_{\mathbf{q}} = \sum_{i=1}^{m} V_i,$$

where U_i and V_i are Bernoulli random variables with parameters p_i and q_i, $i = 1, 2, \ldots, m$. Assuming that $p_i \le q_i$, then we have $U_i \le_{hr} V_i$, for $i = 1, 2, \ldots, m$. Thus Theorem 1.B.4 of Shaked and Shantikumar [26] yields $N_{\mathbf{p}} \ge_{hr} N_{\mathbf{q}}$, since the Bernoulli variables are IFR. By the symmetry of H, the distribution of the system only depends on the number of components from $\mathcal{C}_X$ and not on where are located in the system. Therefore we can put all the components of class $\mathcal{C}_X$ at the beginning of the system, and consider $T_i = \phi(X_1, \ldots, X_i, Y_{i+1}, \ldots, Y_m)$, so that $T_{\mathbf{p}} =_{st} T_{N_{\mathbf{p}}}$ and $T_{\mathbf{q}} =_{st} T_{N_{\mathbf{q}}}$. If $X \ge_{hr} Y$ and if (4.6) holds, then Proposition 3.4 and condition $N_{\mathbf{p}} \le_{hr} N_{\mathbf{q}}$ give the assertion $T_{\mathbf{p}} \le_{hr} T_{\mathbf{q}}$. Actually, in this case, the condition (4.7), can be replaced by the weaker condition $p_{(i)} \le q_{(i)}$ for $i = 1, 2, \ldots, m$.

In the following proposition the systems are compared in the reversed hazard rate order. The proof can be given by using arguments similar to that used in the above results.

Proposition 4.4 Let $T_{\mathbf{p}}$ and $T_{\mathbf{q}}$ be defined as in (4.1), where $(Z_{p_1}, \ldots, Z_{p_m})$ and $(Z_{q_1}, \ldots, Z_{q_m})$ have the same copula, and where Z_{p_i} and Z_{q_i} have reliability functions given in (4.2). Let $X \ge_{rhr} Y$, with X and Y absolutely continuous. If

$$\frac{u_i D_i \widetilde{H}(u_1, \ldots, u_m)}{\widetilde{H}(u_1, \ldots, u_m)}, \quad \text{is decreasing in } u_1, \ldots, u_m, \tag{4.8}$$

for all $(u_1, \ldots, u_m) \in (0, 1)^m$ and for all $i = 1, 2, \ldots, m$ and

$$p_i \leq q_i$$

for all $i = 1, 2, \ldots, m$, then

$$T_{\mathbf{p}} \leq_{rhr} T_{\mathbf{q}}.$$

5   Theoretical examples

In this section some theoretical examples are included to illustrate the application of the results given in the foregoing. These examples show that the different conditions stated in the preceding sections might hold, or not, depending on the system structure and on the dependence between the components. Therefore, in some cases, to increase the number of good components does not necessarily lead to an improvement of the system performance (under different comparison criteria).

The first example shows that in the case that the dependence among components is described by Archimedean survival copulas, then the condition (3.7), which can be restated in terms of the generator of the copula, is not always satisfied even considering simple series systems. Moreover this example also illustrated the fact that whenever the components have a negative dependence based on an Archimedean structure, then such dependence plays in favor of condition (3.7). The example also shows that the conditions in Propositions 3.4 and 3.8 might hold for a series system with dependent components.

Example 5.1 Let us consider a series system with two dependent components having an Archimedean survival copula (2.1) given by

$$S(u, v) = \overline{W}\big(\overline{W}^{-1}(u) + \overline{W}^{-1}(v)\big), \qquad (5.1)$$

where $\overline{W} : [0, \infty) \to [0, 1]$ is a strictly convex reliability function, and where $\overline{W}^{-1}$ is the inverse of $\overline{W}$. Observe that, since $H = S$, then

$$\frac{u D_1 H(u, v)}{H(u, v)} = \frac{u}{w\big(\overline{W}^{-1}(u)\big)} \cdot \frac{w\big(\overline{W}^{-1}(u) + \overline{W}^{-1}(v)\big)}{\overline{W}\big(\overline{W}^{-1}(u) + \overline{W}^{-1}(v)\big)}, \qquad (5.2)$$

where w is the probability density function corresponding to $\overline{W}$. Let $b = \overline{W}^{-1}(v)$ and $a = \overline{W}^{-1}(u)$, i.e. $u = \overline{W}(a)$. Then we can rewrite the ratio above as

$$\frac{\overline{W}(a)}{w(a)} \cdot \frac{w(a + b)}{\overline{W}(a + b)} \equiv \frac{\lambda_W(a + b)}{\lambda_W(a)},$$

where $\lambda_W = w/\overline{W}$ is the hazard rate function corresponding to reliability function $\overline{W}$. Now, observe that the function (5.2) is decreasing both in u and v if and only if $\lambda_W(a + b)/\lambda_W(a)$ is increasing both in a and b. This last condition is satisfied if λ_W is increasing and logconvex. Thus, condition (3.7) holds if the hazard rate function λ_W corresponding to the generator of the copula is increasing and logconvex. It is interesting to observe that the increasing hazard rate property of the generator corresponds to a copula with negative dependence (see, for instance, Averous and Dortet-Bernadet [1]). So that negative dependence plays in favor of this ordering result. For related results, see also Chapter 11 in Jaworski et al. [12].

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For instance, consider the series system with two dependent components having the Gumbel-Barnett Archimedean survival copula given by

$$S(u, v) = H(u, v) = uv \exp(-\theta \ln u \ln v), \qquad u, v \in (0, 1), \quad \theta \in (0, 1], \tag{5.3}$$

which is obtained from the generator $\overline{W}(u) = \exp\{(1 - e^u)/\theta\}$ (see copula n. 9 of Table 4.1 in Nelsen [25], p. 116). Therefore, in this case, we have

$$\frac{uD_1H(u, v)}{H(u, v)} = 1 - \theta \ln v, \qquad \frac{vD_2H(u, v)}{H(u, v)} = 1 - \theta \ln u. \tag{5.4}$$

Such functions are decreasing both in u and v in $(0, 1)^2$ and so condition (3.7) holds for $i, j = 1, 2$. Notice that $\lambda_W(u) = (1/\theta)e^u$ is increasing and logconvex and so (3.7) holds. Hence, from Proposition 3.3, we have

$$T_0 \leq_{hr} T_1 \leq_{hr} T_2$$

and thus Proposition 3.4 can be applied to this system. Moreover, from (5.3), we have

$$\frac{H(v, v)}{H(u, v)} = \frac{v}{u} \exp\left\{-\theta \ln\left(\frac{v}{u}\right) \ln\left(u\frac{v}{u}\right)\right\}, \qquad \frac{H(u, v)}{H(u, u)} = \frac{v}{u} \exp\left\{-\theta \ln(u) \ln\left(\frac{v}{u}\right)\right\}.$$

So the functions in (3.11) can be expressed as in (3.12) when $k = 0, 1$, with

$$\psi_0(x, z) = z \exp\{-\theta \ln(z) \ln(xz)\}, \qquad \psi_1(x, z) = z \exp\{-\theta \ln(x) \ln(z)\}.$$

Both such functions are increasing in each of their arguments on $(0, 1)^2$, for all $\theta \in (0, 1]$, and thus the condition in Proposition 3.8 holds. ◇

The second example proves that the sufficient condition (3.7) in Proposition 3.4 is not a necessary condition.

Example 5.2 Consider the series system with component lifetimes having the Clayton Archimedean copula

$$S(u, v) = \left(u^{-\theta} + v^{-\theta} - 1\right)^{-1/\theta} \qquad (\theta > 0),$$

which is obtained from (5.1) by taking

$$\overline{W}(u) = (1 + \theta u)^{-1/\theta}$$

(see copula n. 1 of Table 4.1 in Nelsen [25], p. 116). The corresponding hazard rate is $\lambda_W(u) = (1 + \theta u)^{-1}$, which is decreasing for $u > 0$. In this case, the condition (3.7) holds if and only if

$$\frac{u^{-\theta}}{u^{-\theta} + v^{-\theta} - 1} \quad \text{is decreasing in } u, v$$

for all $u, v \in (0, 1)$. However, this function is strictly decreasing in $u \in (0, 1)$ and is strictly increasing in $v \in (0, 1)$ for all $\theta > 0$. Hence the condition (3.7) does not hold. Then, the series system with lifetimes

$$T_0 = \min(Y_1, Y_2), \qquad T_1 = \min(X_1, Y_2), \qquad T_2 = \min(X_1, X_2), \tag{5.5}$$

having dependent components with a Clayton survival copula and lifetimes $X_1 =_{st} X_2$ and $Y_1 =_{st} Y_2$ satisfying $X_1 \geq_{hr} Y_1$ are not necessarily hr-ordered. However, when the component lifetimes have exponential distributions with means 2 (in class $\mathcal{C}_X$) and 1 (in class $\mathcal{C}_Y$), respectively, it holds $T_0 \leq_{hr} T_1 \leq_{hr} T_2$, as shown in Figure 1, where the hazard rate functions of the system lifetimes in (5.5) are plotted for $t \in (0, 10)$. ◇
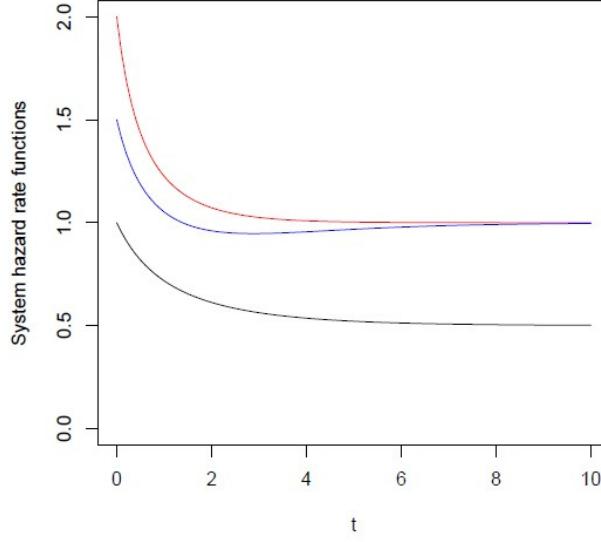
Figure 1: Hazard rate functions for the system lifetimes T_0, T_1 and T_2 (from the top to the bottom) considered in Eq. (5.5).

In the next example we consider a parallel system with two independent components in which the condition in Proposition 3.4 does not hold, whereas that in Proposition 3.8 is fulfilled.

Example 5.3 Let us consider a parallel system with two independent components. Since the domination function is

$$H(u,v) = u + v - uv,$$

we have

$$\frac{uD_1H(u,v)}{H(u,v)} = 1 - \frac{v}{H(u,v)} = 1 - \frac{1}{1+(-1+1/v)u},$$

which is increasing on u and decreasing on v in $(0,1)^2$. Then, condition (3.7) does not hold for this system. However, from (5.3), we have

$$\frac{H(v,v)}{H(u,v)} = \frac{2-u(v/u)}{1+(1-u)(v/u)}\frac{v}{u}, \qquad \frac{H(u,v)}{H(u,u)} = \frac{1+(1-u)(v/u)}{2-u},$$

and thus, the functions given in (3.11) can be expressed as in (3.12) when $k = 0, 1$, with

$$\psi_0(x,z) = \frac{2-xz}{1-x+1/z}, \qquad \psi_1(x,z) = \frac{1+(1-x)z}{2-x}.$$

Both such functions are increasing in each of their arguments on $(0,1)^2$, and thus the condition in Proposition 3.8 is fulfilled. The hazard rate functions of lifetimes

$$T_0 = \max(Y_1, Y_2), \qquad T_1 = \max(X_1, Y_2), \qquad T_2 = \max(X_1, X_2) \qquad (5.6)$$

can be seen in Figure 2 in the case of components with exponential distribution having means 2 and 1 (in class $\mathcal{C}_X$ and $\mathcal{C}_Y$, respectively). Analogously, it can be seen that condition (3.8) holds and
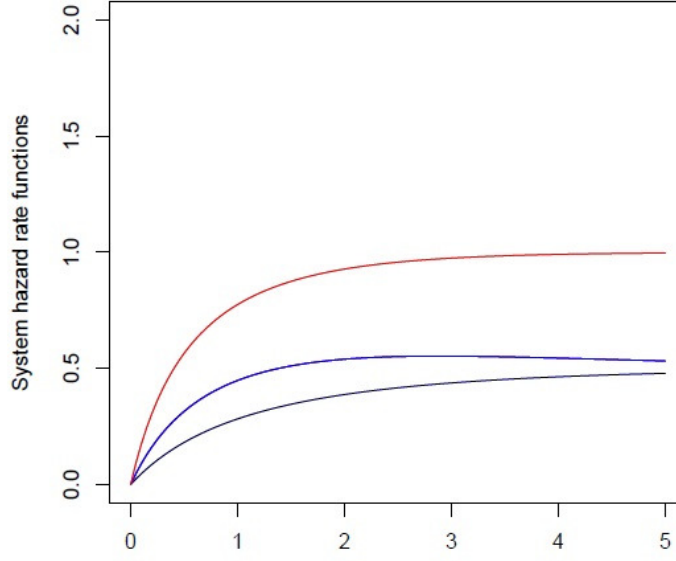
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Figure 2: Hazard rate functions of lifetimes T_0, T_1 and T_2 (from the top to the bottom), given in Eq. (5.6) for the parallel systems considered in Example 5.3.

hence Proposition 3.6 can be applied to this system, this leading to an analogous result involving the reversed hazard rate order. ◇

Next we give a paradoxical example. It confirms that the system lifetimes T_k, $k = 0, 1, \ldots, m$, are not necessarily hr-ordered (even in the case of independent components). Hence, if a component is improved in the hr sense, then the system performance is not necessarily improved (in the hr-order). This means that a used system with age $t > 0$ having k good components might be better (more reliable) than a used system with age t having $k + 1$ good components.

Example 5.4 Let us consider the system lifetime $T = \max(Z_1, \min(Z_2, Z_3))$ with three independent components having lifetimes Z_1, Z_2, Z_3 (see Figure 3), and domination function

$$H(u_1, u_2, u_3) = u_1 + u_2 u_3 - u_1 u_2 u_3.$$

For $k = 0, 1, 2, 3$, we assume that the lifetimes of the first k components belong to class $\mathcal{C}_X$, whereas those of the last $3 - k$ components belong to class $\mathcal{C}_Y$, with exponential reliability functions $\overline{F}_X(t) = \exp(-t)$, $t \geq 0$, and $\overline{F}_Y(t) = \exp(-2t)$, $t \geq 0$, respectively. We recall that T_k denotes the system lifetime when the size of $\mathcal{C}_X$ is k, for $k = 0, 1, 2, 3$. The hazard rate functions of T_k, for all k, are plotted in Figure 4. We have that $T_0 \leq_{hr} T_k$ for $k = 1, 2, 3$, but T_1, T_2, T_3 are not hr-ordered. Hence, improving the components does not improve such systems in the hr order. For instance, the hazard rate of the system with the best components (T_3) is better than the system with one of the worse components (T_2) for $t \in (0, 1.1676)$. However, this ordering is reversed for $t \in (1.1676, \infty)$. As a consequence, we have

$$[T_3 - t | T_3 > t] \leq_{st} [T_2 - t | T_2 > t] \qquad \text{for } t \in (1.1676, \infty).$$
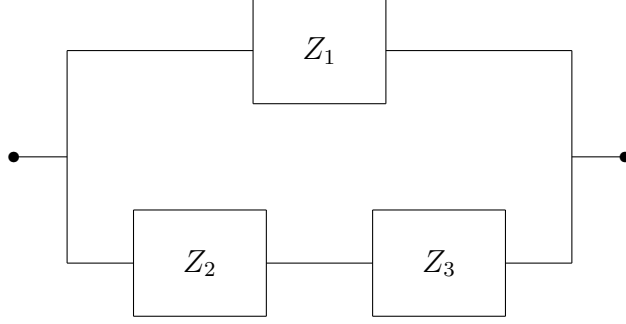
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Figure 3: Schematic representation of the system of Example 5.4.

Hence, when $t \in (0, 1.1676)$ the residual lifetime of the system at age t for $k = 3$ is stochastically larger than the same for $k = 2$, whereas when $t \in (1.1676, \infty)$ such relation is reversed. This is intuitively in contrast with the ordering $T_3 \geq_{st} T_2$, which holds due to Proposition 3.1. Finally, note that T_1, T_2 and T_3 are asymptotically equivalent, in the sense that all their hazard rates tend to 1 when $t \to \infty$, whereas the hazard rate of T_0 tends to 2 when $t \to \infty$. $\diamond$

Next we show an example involving a system different from a series or a parallel system, and in which the conditions stated in (3.3) hold only for some choices of k.

Example 5.5 Let us consider the system lifetime $T = \min(Z_1, \max(Z_2, Z_3))$ with three independent components having lifetimes Z_1, Z_2, Z_3 (see Figure 5). Then, the domination function is

$$H(u_1, u_2, u_3) = u_1 u_2 + u_1 u_3 - u_1 u_2 u_3.$$

A straightforward calculation gives

$$\frac{u_i D_i H(u_1, u_2, u_3)}{H(u_1, u_2, u_3)} = \begin{cases} 1, & i = 1 \\ \dfrac{u_i - u_2 u_3}{u_2 + u_3 - u_2 u_3}, & i = 2, 3, \end{cases}$$

which are decreasing functions in u_1. Hence, condition (3.3) holds for $k = 0$. So, from Proposition 3.3, we have $T_0 \leq_{hr} T_1$, where $T_0 = \phi(Y_1, Y_2, Y_3)$ and $T_1 = \phi(X_1, Y_2, Y_3)$, under the assumptions that variables X_j and Y_j have reliability functions $\overline{F}_X(t)$ and $\overline{F}_Y(t)$, respectively, and satisfy $X \geq_{hr} Y$. However, it is easy to see that condition (3.3) does not hold for $k = 1, 2$. $\diamond$

In the next example we show that the conditions stated in Propositions 3.11 and 3.12 for the preservation of the lr order hold for series systems with independent components. It could be stated as a corollary of that propositions.

Example 5.6 Let us consider a series system with m independent components, so that the domination function is

$$H(u_1, \ldots, u_m) = u_1 \ldots u_m.$$

Hence, from (3.15), we have

$$\rho_k(u, v, w) = \frac{k u^{k-1} v^{m-k} + (m-k) w u^k v^{m-k-1}}{(k+1) u^k v^{m-k-1} + (m-k-1) w u^{k+1} v^{m-k-2}}$$
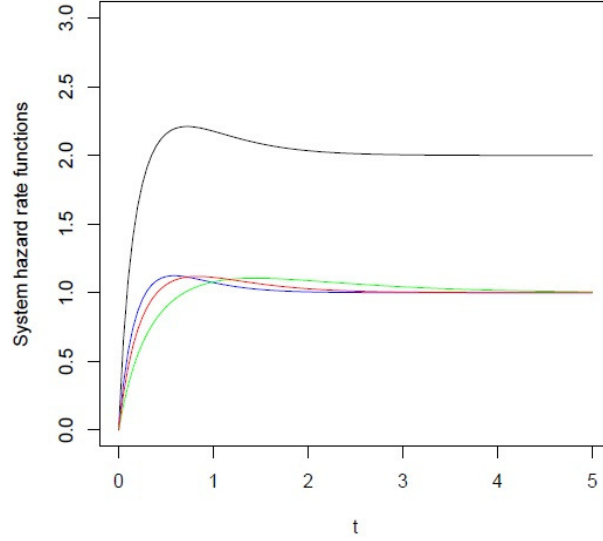$$= \frac{k(v/u)^2 + (m-k) w(v/u)}{(k+1)(v/u) + (m-k-1) w}.$$

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Figure 4: Hazard rate functions of the lifetimes T_k for the system of Example 5.4, when $k = 0, 1, 2, 3$ (from the top to the bottom at $t = 0.5$).

This function can be expressed as in (3.16), with

$$\xi_k(x, z, w) = \frac{kz^2 + (m-k)wz}{(k+1)z + (m-k-1)w}.$$

Let $a = kz^2$, $b = (m-k)w$, $c = (k+1)z$ and $d = (m-k-1)$. A straightforward calculation shows that the function

$$\alpha_1(w) = \frac{a+bw}{c+dw}$$

is increasing in w if $bc - ad \geq 0$. Therefore, ξ_k is increasing in w on $(0, \infty)$ for any fixed $x, z \in (0, 1)$, being

$$bc - ad = [(m-k)(k+1) - k(m-k-1)]z^2 = mz^2 \geq 0.$$

Analogously, it is easy to see that the function

$$\alpha_2(z) = \frac{az^2 + bz}{cz + d}$$

is increasing for $z \geq 0$ if $a, b, c, d \geq 0$, thus ξ_k is increasing in $z \in (0, 1)$ for any $x \in (0, 1)$ and $w \in (0, \infty)$. Moreover, ξ_k is clearly increasing in x. Therefore, Proposition 3.11 yields the following ordering for the system lifetimes: $T_k \leq_{lr} T_{k+1}$, for $k = 0, ..., m - 1$. Moreover, Proposition 3.12 can be applied to this system. ◇

In the next example we show that the conditions for the preservation of the lr order might also hold for series systems with two dependent components.

Example 5.7 Let us consider a series system with 2 dependent components having the Gumbel-Barnett Archimedean survival copula, already studied in Example 5.1. Making use of Eqs. (5.3) and
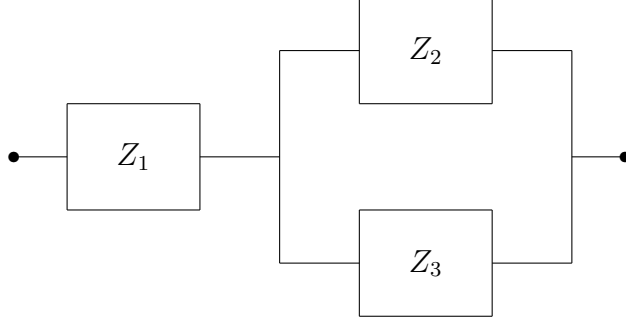
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Figure 5: Schematic representation of the system of Example 5.5.

(5.4), from (3.15) when $k = 0$, we have that $\rho_0(t)$ can be expressed as in (3.16), with

$$\xi_0(x, z, w) = \frac{2wz - 2wz \dfrac{\theta \ln z}{1 - \theta \ln x}}{w + z - z \dfrac{\theta \ln z}{1 - \theta \ln x}} \exp[-\theta \ln z \ln(xz)].$$

A direct calculation immediately shows that $\xi_0(x, z, w)$ is increasing in w on $(0, 1)$ for all $(x, z) \in (0, 1)^2$. Moreover, since $-(\theta \ln z)/(1 - \theta \ln x)$ is increasing in x and the function $\alpha_1(s) = (a + bs)/(c + ds)$ is increasing in s if $bc - ad \geq 0$, then, letting $a = b = 1$, $c = w + z$, $d = z$, $bc - ad = w \geq 0$ and $s = -(\theta \ln z)/(1 - \theta \ln x)$, we have that

$$\xi_0(x, z, w) = 2wz\alpha_1(s) \exp[-\theta \ln x \ln(xz)]$$

is increasing in x on $(0, 1)$ for all $z \in (0, 1)$ and all $w \in (0, \infty)$.

Analogously, we have

$$\xi_0(x, z, w) = 2w\alpha_3(z),$$

where

$$\alpha_3(z) = \frac{z - az \ln z}{w + z - az \ln z} \exp[-\theta \ln(xz) \ln z],$$

with $a = \theta/(1 - \theta \ln x) \geq 0$. Using arguments as above, it can shown that ξ_0 is increasing in z on $(0, 1)$ for any fixed $x \in (0, 1)$ and $w \in (0, \infty)$.

Then, from Proposition 3.11, we get

$$T_0 = \min(Y_1, Y_2) \leq_{lr} T_1 = \min(X_1, Y_2)$$

whenever $X \geq_{lr} Y$ and the components have a Gumbel-Barnett survival copula.

Similarly, when $k = 1$, (3.16) holds for

$$\xi_1(x, z, w) = \frac{xz(1 - \theta \ln(xz)) \exp[-\theta \ln x \ln(xz)] + wx(1 - \theta \ln x) \exp[-\theta \ln x \ln(xz)]}{2x(1 - \theta \ln x) \exp[-\theta \ln x \ln(xz)]},$$

that is,

$$\xi_1(x, z, w) = \left(w + z - z\frac{\theta \ln z}{1 - \theta \ln x}\right) \exp[-\theta \ln x \ln z].$$

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Therefore, ξ_1 is increasing in x and w on $(0,1) \times (0, \infty)$ for any $z \in (0,1)$. Moreover,

$$\xi_1(x, z, w) = \frac{1}{2}\left(w + z - az \ln z\right) \exp(b \ln z) = wz^b + z^{b+1} - az^{b+1} \ln z,$$

with $a = \frac{\theta}{1 - \theta \ln x} \geq 0$, and $b = -\theta \ln x \geq 0$. Then

$$\frac{\partial}{\partial z}\xi_1(x, z, w) = bwz^{b-1} + (b+1)z^b - az^b - a(b+1)z^b \ln z \geq 0$$

for $z \in (0,1)$ since

$$(b+1) - a = (1 - \theta \ln x) - \frac{\theta}{1 - \theta \ln x} = \frac{(1 - \theta \ln x)^2 - \theta}{1 - \theta \ln x} \geq 0$$

for all $\theta \in (0,1]$. Hence $\xi_1(x, z, w)$ is increasing in z on $(0,1)$ for any x and w in $(0,1) \times (0, \infty)$.

Then, from Proposition 3.11, we obtain that

$$T_1 = \min(X_1, Y_2) \leq_{lr} T_2 = \min(X_1, X_2)$$

whenever $X \geq_{lr} Y$ and the components have a Gumbel-Barnett survival copula. Hence, also Proposition 3.12 can be applied, i.e., given K_1 and K_2 with support $\{0, 1, 2\}$ and such that $K_1 \leq_{lr} K_2$, if $X \geq_{lr} Y$ then $T_{K_1} \leq_{lr} T_{K_2}$. $\diamond$

The last example describes an application of Proposition 4.1. Other applications are given in the following section.

Example 5.8 Let us consider two series systems, each one formed by 3 dependent components having lifetimes whose dependence is described by a Clayton Archimedean survival copula (see, e.g., Nelsen [25], p. 152). Thus the domination function is

$$H(u, v, w) = \left(u^{-\theta} + v^{-\theta} + w^{-\theta} - 2\right)^{-1/\theta},$$

with $\theta > 0$. Let $\overline{F}_i$ and $\overline{G}_i$, $i = 1, 2, 3$, denote the marginal reliability functions of the lifetimes of the components of the first and the second system, respectively. As stated in Section 4, H is Schur-concave since it is Archimedean, and Proposition 4.1 can be applied, proving that if

$$(\overline{F}_1(t), \overline{F}_2(t), \overline{F}_3(t)) \geq^w (\overline{G}_1(t), \overline{G}_2(t), \overline{G}_3(t)) \tag{5.7}$$

then $T_1 \leq_{st} T_2$.

For example, if the components in the two series systems have exponential distributions with means $1, 2, 3$ and $4/3, 4/3, 3$, respectively. Then

$$(e^{-t}, e^{-t/2}, e^{-t/3}) \geq^w (e^{-3t/4}, e^{-3t/4}, e^{-t/3}) \qquad \text{for all } t \geq 0,$$

being

$$e^{-t/3} \geq e^{-t/3}, \quad e^{-t/2} + e^{-t/3} \geq e^{-3t/4} + e^{-t/3},$$

and

$$e^{-t} + e^{-t/2} + e^{-t/3} \geq e^{-3t/4} + e^{-3t/4} + e^{-t/3}$$

for all $t \geq 0$. Note that the last inequality is equivalent to $e^{-t} + e^{-t/2} - 2e^{-3t/4} \geq 0$, which holds because $e^{-t} + e^{-t/2} - 2e^{-3t/4} = (e^{-t/2} - e^{-t/4})^2 \geq 0$ for all t. Thus, from Proposition 4.1, we have

$$H(e^{-t}, e^{-t/2}, e^{-t/3}) \leq H(e^{-3t/4}, e^{-3t/4}, e^{-t/3})$$

for all $t \geq 0$, i.e., the lifetimes of the two systems are comparable in the usual stochastic order. It is interesting to observe that, in particular, letting $\overline{F}_1, \overline{F}_2, \overline{F}_3$ be any triplet of reliability functions, then

$$(\overline{F}_1(t), \overline{F}_2(t), \overline{F}_3(t)) \geq^w (\overline{G}(t), \overline{G}(t), \overline{G}(t)), \quad \text{for all } t \geq 0,$$

where $\overline{G}(t) = (\overline{F}_1(t) + \overline{F}_2(t) + \overline{F}_3(t))/3$, and the lifetime of the first system will be smaller, in the usual stochastic order, than the lifetime of the second one. Thus, in this series system and with this common dependence (copula), the homogeneity between components improves the reliability of the system. Hence it is better to choose them randomly at each position in the system. It should be pointed out that the same statement also follows from Theorem 5.4 in Navarro and Spizzichino [24]. $\diamond$

6 Optimal allocation of components in a system

In practice, there are several situations in which we have to use different kinds of units to build coherent systems. In such situations, we have to develop optimal procedures for the allocation of the units in the system to get the best coherent system. For example, if we have two kinds of components (in a similar number) to build series systems with two independent components then, surprisingly, a random choice of these components leads to a series system which is always better than the series system obtained by choosing one component from each kind. This is known as the Parrondo's paradox in the reliability theory, see [7]. Similar results were obtained in [8] for series and parallel systems with independent components and in [24] for arbitrary coherent systems with possibly dependent components.

The theoretical results obtained in this paper can be used to go further and analyze the optimal allocation of the components in a system when we only have two kinds of components and some are better than the other. Of course, the best systems are always those which include only the best components. However, we shall assume that this option is not possible, maybe simply because we do not know which are the best components, and that we want to use both kinds of components, with a fixed general proportion.

In these situations, as we have already mentioned before, random strategies may leed to the best systems. In this paper, two procedures for the selection of the components have been studied. In Section 3, we considered a random choice of the number k of good components in the systems through a discrete random variable K with support included in the set $\{0, 1, \ldots, m\}$. In section 4, we considered coherent systems obtained by choosing the component at position i in the system by using a Bernoulli trial with probability $p_i \in [0, 1]$, that is, by using a good component with probability p_i and a bad component with probability $1 - p_i$, for $i = 1, \ldots, m$. In both procedures, we considered both the cases of systems with independent or with dependent components (with a given copula for the dependency model).

In this general framework, which is the best strategy? Could the performance be improved by using random strategies? Will the best strategy depend on the system structure and/or on the dependence model between the components? To answer these questions, let us consider first the simple case of a series system with two independent components and let us assume that we have a 50% of good units and a 50% of bad units (the reasoning is similar with other proportions). Then, by using the notation introduced in Section 4, we can consider the series systems with lifetimes

$T_{(p_1,p_2)}$, where $p_i \in [0,1]$ is the probability of a good component at position i, for $i = 1, 2$. Here, to maintain the fifty-fifty proportion of good and bad units in the overall, we assume $p_1 + p_2 = 1$. If we do not include this assumption, the best option is obviously to take $p_1 = p_2 = 1$ (i.e. to use only good units). Note that $T_{(1,0)} =_{st} T_1$ represents the lifetime of the series system with a good component and a bad component. Analogously, $T_{(0.5,0.5)}$ represents the lifetime of the series system with components chosen at random in each position (i.e. the system considered in the Parrondo's paradox). For this system, we have $H(u, v) = uv$ which is a Schur-concave function in $(0, 1)^2$. Then, from Proposition 4.2, we have that if

$$(p_1, p_2) \leq^w (q_1, q_2), \tag{6.1}$$

then $T_{(p_1,p_2)} \geq_{st} T_{(q_1,q_2)}$. Hence, if we assume $p_1 + p_2 = q_1 + q_2 = 1$, then (6.1) holds if, and only if,

$$\min(p_1, p_2) \geq \min(q_1, q_2).$$

Therefore, the best option is $p_1 = p_2 = 1/2$ and the worst options are $q_1 = 1$ and $q_2 = 0$, or $q_1 = 0$ and $q_2 = 1$ (Parrondo's paradox).

If the components are dependent, then we have seen that H is equal to the survival copula S. From Proposition 4.2, $p_1 = p_2 = 1/2$ is also the best option for series systems of components with dependence determined by a Schur-concave survival copula S, as it is, for example, in the case of Archimedean copulas. However, this is not always the case (see [24], Example 6.2).

Analogously, if we consider a parallel system with two independent components, then the best options are $p_1 = 1$ and $p_2 = 0$, or $p_1 = 0$ and $p_2 = 1$, whereas the worst option is $q_1 = q_2 = 1/2$. From Proposition 4.2, the same happens if the components in the parallel system are dependent with a Schur-concave distribution copula C. Therefore, in general, the optimal choice will depend on both the system structure and the dependency model.

Now let us try to improve the system performance by using the random procedure studied in Section 3. We start again with the simple case of a series system with two independent components. In this procedure the number of good components in the system is determined by a random discrete variable K over the set $\{0, 1, 2\}$. By using the notation introduced in Section 3, the associated system lifetime is represented by T_K. Again we shall assume that there are 50% of good and bad units, that is, $E(K) = 1$ (the reasoning is similar with other values). Note that the system with a good component and a bad component can now be represented as $T_{K_1} =_{st} T_{(1,0)} =_{st} T_1$ with $P(K_1 = 1) = 1$. Analogously, the best option with the random procedure studied in Section 4, i.e. $T_{(0.5,0.5)}$, can now be represented as $T_{(0.5,0.5)} =_{st} T_{K_2}$ with $P(K_2 = 0) = P(K_2 = 2) = 1/4$ and $P(K_2 = 1) = 1/2$. Other reasonable options might be T_{K_3} with $P(K_3 = i) = 1/3$ for $i = 0, 1, 2$ or T_{K_4} with $P(K_4 = 0) = P(K_4 = 2) = 1/2$. Note that $E(K_i) = 1$ for $i = 1, 2, 3, 4$. Which one is the best option? Is T_{K_2} also the best option in this procedure? The answers to these questions can be obtained from Proposition 3.2, (i), and Remark 3.1. A straightforward calculation shows that

$$K_1 \leq_{cx} K_2 \leq_{cx} K_3 \leq_{cx} K_4.$$

Hence, as $\varphi(k)$ is convex (see Remark 3.1), we have

$$T_{K_1} \leq_{st} T_{K_2} \leq_{st} T_{K_3} \leq_{st} T_{K_4},$$

that is, K_4 is the best option. Actually, K_4 is the best option (the most convex) whenever $E(K) = 1$. Note again that here we do not need the condition $X \geq_{st} Y$, that is, it is the best option also if

$X \leq_{st} Y$ or if they are not ordered (for all reliability functions $\overline{F}_X$ and $\overline{F}_Y$). The same happens if the series system has two dependent components with an exchangeable survival copula (see Remark 3.1) but, in this case, $T_{(0.5,0.5)}$ and T_{K_2} are not necessarily equal in law. The plots of the respective reliability functions in the case of independent exponential components with means 5 or 1 can be seen in Figure 6. Of course, the best option (top) corresponds to T_{K_4}. The same will happen if the components are not ordered or if they are dependent with an exchangeable survival copula. Note that such conclusion is not surprising because this is what we usually do in the real life. For example, if a remote control has two batteries in series and we have two good batteries and two bad (or different) batteries, we usually put together the good batteries and the bad batteries (we do not mix them).

The conclusion is just the opposite when we consider a parallel system with two independent components (or with two dependent components having an exchangeable copula). In this case the best option is T_{K_1}, that is, to put a good component jointly with a bad component (put together different components). Thus, for example, if a plane has two engines and it can fly with just one of them, then, if we have two planes, two good engines and two engines not so good as the previous ones, the best option is, of course, put a good and a 'not so good' engine in each plane. The same conclusion holds if we just do not know which are the good engines or if the engines lifetimes are dependent with a symmetric dependence model (an exchangeable copula), which is a reasonable assumption in practice. The results included in this paper can be used to analyze other system structures and other dependency models. Moreover, there are also practical situations in which the random numbers K_i are comparable in the likelihood ratio order (thus also in the hazard and the reverse hazard orders), as those illustrated in Section 4 in [11]. In these cases all the statements in previous sections dealing with comparisons of systems in the lr, hr and rhr orders can be applied.

7   Conclusions

Surprisingly some coherent systems are not ordered (in different stochastic senses) when a unit is replaced by a better unit. Also, some systems might be improved if the components are chosen randomly. In this paper we have developed some procedures to compare coherent systems with two kinds of units having a fixed number or a random number of the best units. We also consider the case of systems in which the units are chosen randomly at each position by using independent Bernoulli trials. We consider various assumptions which include different system structures and different dependency between the components (including the independent case) but always assuming that the system structure and dependency model are the same (i.e. only the components are changed). Our results are distribution-free since they hold for any kind of components having stochastically ordered lifetimes (in different senses). The given examples show that the our findings can be applied to several situations. Specific results are obtained for series systems with dependent components having a common Archimedean copula. A practical application on the optimal allocation of components in a system structure is also given to illustrate the theoretical results.
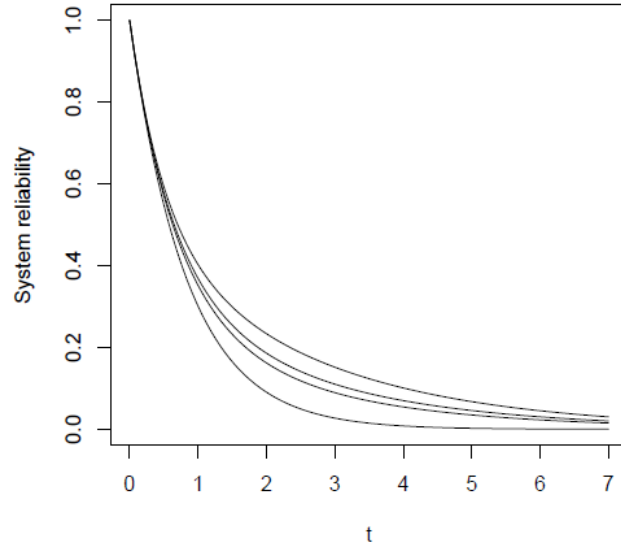
Figure 6: Reliability functions of the series systems with independent exponential components with means 5 or 1 when the number of good components is chosen randomly by using the random variables K_i, for $i = 1, 2, 3, 4$ (from below to the top) defined in Section 6.

References

[1] Averous, J.; Dortet-Bernadet, J.L. (2004) Dependence for Archimedean copulas and aging properties of their generating functions. Sankhyā 66, 607–620.

[2] Barlow, R.E.; Proschan, F. (1981) Statistical Theory of Reliability and Life Testing. Probability Models, To Begin With, Silver Spring, MD.

[3] Belzunce, F.; Martínez-Puertas, H.; Ruiz, J.M. (2013) On allocation of redundant components for systems with dependent components. European Journal of Operational Research 230, 573–580.

[4] Brito, G; Zequeira, R.I.; Valdés, J.E. (2011) On the hazard rate and reversed hazard rate orderings in two-component series systems with active redundancies. Statistics and Probability Letters 81, 201–206.

[5] da Costa Bueno, V. (2005) Minimal standby redundancy allocation in a k-out-of-n:F system of dependent components. European Journal of Operational Research 165, 786–793.

[6] da Costa Bueno, V.; do Carmo, I.M. (2007) Active redundancy allocation for a k-out-of-n:F system of dependent components. European Journal of Operational Research 176, 1041–1051.

[7] Di Crescenzo, A. (2007) A Parrondo paradox in reliability theory. The Mathematical Scientist 32, 17–22.

[8] Di Crescenzo, A.; Pellerey, F. (2011) Stochastic comparisons of series and parallel systems with randomized independent components. Operations Research Letters 39, 380–384.

[9] Durante, F.; Sempi, C. (2003) Copulae and Schur-concavity. International Mathematical Journal 3, 893–905.

[10] Eryilmaz, S. (2012) On the mean residual life of a k-out-of-n:G system with a single cold standby component. European Journal of Operational Research 222, 273-277.

[11] Hazra, N.K.; Nanda, A.K. (2014). Some results on series and parallel systems of randomized components. Operations Research Letters (to appear). Article first published online. DOI: http://dx.doi.org/10.1016/j.orl.2014.01.002.

[12] Jaworski, P., Durante, F., Härdle, W.K, Rychlik, T. (Eds.) (2010). Copula Theory and Its Applications, Copula Theory and Its Applications. Proceedings of the Workshop Held in Warsaw, 25-26 September 2009. Springer.

[13] Kuo, W.; Zhu, X. (2012) Importance Measures in Reliability, Risk, and Optimization: Principles and Applications. Wiley.

[14] Levitin, G.; Xing, L.; Dai, Y. (2014) Cold vs. hot standby mission operation cost minimization for 1-out-of-N systems. European Journal of Operational Research 234, 155–162.

[15] Li, X.; Ding, W. (2010) Optimal allocation of active redundancies to k-out-of-n systems with heterogeneous components. Journal of Applied Probability 47, 254–263.

[16] Marshall, A.W.; Olkin, I.; Arnold, B.C. (2011) Inequalities: theory of majorization and its applications. Second edition. Springer Series in Statistics. Springer, New York.

[17] McNeil, A.; Nešlehová, J. (2009) Multivariate Archimedean copulas, d-monotone functions and ℓ_1-norm symmetric distributions. The Annals of Statistics 37, 3059–3097.

[18] Misra, N.; Dhariyal, I.D.; Gupta, N. (2009) Optimal allocation of active spares in series systems and comparisons of component and system redundancies. Journal of Applied Probability 46, 19–34.

[19] Müller, S.; Scarsini, M. (2005) Archimedean copulae and positive dependence. Journal of Multivariate Analysis 93, 434–445.

[20] Navarro, J.; del Aguila, Y.; Sordo, M.A.; Suarez-Llorens, A. (2013) Stochastic ordering properties for systems with dependent identically distributed components. Applied Stochastic Models in Business and Industry 29, 264–278.

[21] Navarro, J.; del Aguila, Y.; Sordo, M.A.; Suarez-Llorens, A. (2013) Preservation of reliability classes under the formation of coherent systems. Applied Stochastic Models in Business and Industry (to appear). Article first published online. DOI: 10.1002/asmb.1985.

[22] Navarro, J.; Rubio, R. (2012) Comparisons of coherent systems with non-identically distributed components. Journal of Statistical Planning and Inference 142, 1310–1319.

[23] Navarro, J.; Rychlik, T. (2010) Comparisons and bounds for expected lifetimes of reliability systems. European Journal of Operational Research 207, 309–317.

[24] Navarro, J.; Spizzichino, F. (2010) Comparisons of series and parallel systems with components sharing the same copula. Applied Stochastic Models in Business and Industry 26, 775–791.

[25] Nelsen R.B. (2006) An introduction to copulas. Second edition. Springer Series in Statistics. Springer, New York.

[26] Shaked, M.; Shanthikumar, J.G. (2007) Stochastic orders. Springer Series in Statistics. Springer, New York.

[27] Zhang, Z. (2010) Ordering conditional general coherent systems with exchangeable components. Journal of Statistical Planning and Inference 140, 454–460.

[28] Zhao, P.; Chan, P.S.; Li, L.; Ng, H.K.T. (2013) Allocation of two redundancies in two-component series systems. Naval Research Logistics 60, 588–598.

[29] Zhao, P.; Chan, P.S.; Ng, H.K.T. (2012) Optimal allocation of redundancies in series systems. European Journal of Operational Research 220, 673–683.