Noname manuscript No.
(will be inserted by the editor)

Notes on discrepancy in the pairwise comparisons method

Konrad Kułakowski

the date of receipt and acceptance should be inserted later

Abstract The pairwise comparisons method is a convenient tool used when the relative order among different concepts (alternatives) needs to be determined. One popular implementation of the method is based on solving an eigenvalue problem for the pairwise comparisons matrix. In such cases the ranking result the principal eigenvector of the pairwise comparison matrix is adopted, whilst the eigenvalue is used to determine the index of inconsistency. A lot of research has been devoted to the critical analysis of the eigenvalue based approach. One of them is the work of Bana e Costa and Vansninck [1]. In their work authors define the conditions of order preservation (COP) and show that even for a sufficiently consistent pairwise comparisons matrices, this condition can not be met. The present work defines a more precise criteria for determining when the COP is met. To formulate the criteria a discrepancy factor is used describing how far the input to the ranking procedure is from the ranking result.

1 Introduction

The origins of pairwise comparisons (herein abbreviated as PC) date back to the thirteenth century [5]. The contemporary form of the method owes to *Fechner* [7], *Thurstone* [23] and Saaty [18]. The latter proposed the Analytic Hierarchy Process *(AHP)* - a hierarchical, eigenvalue based extension to the *PC* theory, which provides useful methods for dealing with a large number of criteria. In its early stages the *PC* method was a voting method [5]. Later it was used in psychometrics and psychophysics [23]. Over time, it began to be used in decision theory [19], economics [17], and other fields. The utility of the method has been confirmed by numerous examples [24,15,22]. Despite its long existence it is still an interesting subject for researchers. Some of its aspects still raise vigorous discussions [6,2,1] and prompt researchers to enquire further into this area. Example of such exploration are the *Rough Set* approach [9], fuzzy *PC* relation handling [16,8,25], incomplete *PC* relation [3,12], non-numerical rankings [11], rankings with the reference set of alternatives [13, 14] and others. A more thorough discussion of the *PC* method can be found in [21,10].

2 Preliminaries

2.1 Pairwise comparisons method

Central to the *PC* method is a *PC* matrix $M = [m_{ij}]$, where $m_{ij} \in \mathbb{R}_+$ and $i, j \in \{1,\ldots,n\}$, that expresses a quantitative relation R over the finite set of concepts $C \overset{df}{=} \{c_i \in \mathscr{C} \wedge i \in \{1,\ldots,n\}\}$. The set $\mathscr{C}$ is a non empty universe of concepts and $R(c_i, c_j) = m_{ij}$, $R(c_j, c_i) = m_{ji}$. The values m_{ij} and m_{ji} are interpreted as the relative importance, value or quality indicators of concepts c_i and c_j, so that according to the best knowledge of experts $c_i = m_{ij} c_j$ should hold.

Definition 1 A matrix M is said to be reciprocal if $\forall i, j \in \{1,\ldots,n\} : m_{ij} = \frac{1}{m_{ji}}$ and M is said to be consistent if $\forall i, j, k \in \{1,\ldots,n\} : m_{ij} \cdot m_{jk} \cdot m_{ki} = 1$.

AGH University of Science and Technology, al. Mickiewicza 30, Kraków, Poland

Since the knowledge stored in the *PC* matrix usually comes from experts in the field of R, it may results in inaccuracy. In such a case it may be that there exists a certain triad of values m_{ij}, m_{jk}, m_{ki} from M for which $m_{ij} \cdot m_{jk} \cdot m_{ki} \neq 1$. In other words, different ways of estimating concept value may lead to different results. This observation gave rise to the concept of an inconsistency index describing how far the matrix M is inconsistent. There are a number of inconsistency indexes [4]. The most popular, proposed by Saaty [18], is defined below.

Definition 2 The eigenvalue based consistency index *(Saaty's Index)* of $n \times n$ reciprocal matrix M is equal to:

$$\mathscr{S}(M) = \frac{\lambda_{max} - n}{n-1} \tag{1}$$

where λ_{max} is the principal eigenvalue of M.

The result of the pairwise comparisons method is ranking - a function that assigns values to the concepts. Formally, it can be defined as follows.

Definition 3 The ranking function for C (the ranking of C) is a function $\mu : C \to \mathbb{R}_+$ that assigns to every concept from $C \subset \mathscr{C}$ a positive value from $\mathbb{R}_+$.

In other words, $\mu(c)$ represents the ranking value for $c \in C$. The μ function is usually written in the form of a vector of weights $\mu \overset{df}{=} \left[\mu(c_1), \ldots \mu(c_n)\right]^T$. One of the popular methods of obtaining the vector μ is to calculate the principal eigenvector μ_{max} of M (i.e. the vector associated with the principal eigenvalue of M) and rescale them so that the sum of its elements is 1, i.e.

$$\begin{aligned} \mu_{ev} &= \left[\frac{\mu_{max}(c_1)}{s_{ev}}, \ldots, \frac{\mu_{max}(c_n)}{s_{ev}}\right]^T, \text{ and} \\ s_{ev} &= \sum_{i=1}^{n} \mu_{max}(c_i) \end{aligned} \tag{2}$$

where μ_{ev} - the ranking function, μ_{max} - the principal eigenvector of M. Due to the *Perron-Frobenius* theorem [18] one exists, because a real square matrix with the positive entries has a unique largest real eigenvalue such that the associated eigenvector has strictly positive components.

2.2 Eigenvalue heuristics

According to the *PC* approach m_{il} (an entry of M) should express the relative value of $c_i \in C$ with respect to $c_l \in C$. Therefore one would expect that $\mu(c_i)/\mu(c_l) = m_{il}$, i.e. $\mu(c_i) = m_{il}\mu(c_l)$ or conversely $m_{li}\mu(c_i) = \mu(c_l)$. In particular, it is desirable that

$$m_{li}\mu(c_i) = \mu(c_l) = m_{lj}\mu(c_j) \tag{3}$$

for every $c_i, c_j, c_l \in C$. Unfortunately due to possible data inconsistency this may not be possible, i.e. it may be the case that $m_{li}\mu(c_i) \neq m_{lj}\mu(c_j)$. Therefore the question arises of what $\mu(c_l)$ should be? Since the values $m_{li}\mu(c_i)$ for $i = 1, \ldots, n$ can vary from each other the natural (and probably one of the most straightforward) proposal is to adopt its arithmetic mean as the desired candidate for $\mu(c_l)$. This leads to the equation:

$$m_{l1}\mu(c_1) + \ldots + m_{ln}\mu(c_n) = n \cdot \mu(c_l) \tag{4}$$

which expresses the wish that $\mu(c_l)$ should be a compromise between all its putative values. A natural question is whether it is possible to achieve such a compromise for every $l = 1, \ldots, n$. In other words, whether it is possible to solve the following equation system:

$$\begin{array}{c} m_{11}\mu(c_1) + \ldots + m_{1n}\mu(c_n) \;=\; n \cdot \mu(c_1) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ m_{n1}\mu(c_1) + \ldots + m_{nn}\mu(c_n) \;=\; n \cdot \mu(c_n) \end{array} \tag{5}$$

This leads to the question of the solution of the following matrix equation:

$$M\mu = n\mu \tag{6}$$

Of course the solution of the above equation is the eigenvector of M, whilst n is replaced by λ - M's eigenvalue.

$$M\mu = \lambda\mu \tag{7}$$

There might be many eigenvectors and eigenvalues of M. However, when M is positive, real and reciprocal it has at least one positive real eigenvalue associated with the positive and real eigenvector [18]. Let λ_{max} be the real, largest, positive eigenvalue of M and μ_{max} be the associated eigenvector. *AHP* adopts μ_{max} as the solution of (7).

2.3 Local discrepancy

In his seminal work [18, p. 238] Saaty proved the following equality:

$$\lambda_{max} - 1 = \sum_{i=1, i\neq j}^{n} m_{ji} \frac{\mu(c_i)}{\mu(c_j)} \tag{8}$$

Thus,

$$\lambda_{max} - n = \left(\sum_{i=1, i\neq j}^{n} m_{ji} \frac{\mu(c_i)}{\mu(c_j)} \right) - (n-1) \tag{9}$$

which leads to the equation describing the Saaty's inconsistency index (Def. 2):

$$\mathscr{S}(M) = \left(\frac{1}{n-1} \sum_{i=1, i\neq j}^{n} m_{ji} \frac{\mu(c_i)}{\mu(c_j)} \right) - 1 \tag{10}$$

Following Saaty [18, p. 238] let us denote:

$$\epsilon(i,j) \overset{df}{=} m_{ji} \frac{\mu(c_i)}{\mu(c_j)} = \frac{1}{m_{ij}} \frac{\mu(c_i)}{\mu(c_j)} \tag{11}$$

If the ranking μ_{max} were ideal, i.e. if each expert judgment perfectly corresponded to the ranking results, then every m_{ij} would equal the ratio $\mu(c_i)/\mu(c_j)$. In such a case every $\epsilon(i,j)$ would equal to 1. Otherwise, when the ranking is imperfect the values m_{ij} and $\mu(c_i)/\mu(c_j)$ may vary. In other words $\epsilon(i,j)$ describes the discrepancy[1] between the particular expert judgment m_{ij} and the ranking results $\mu(c_i)/\mu(c_j)$. The relationship between $\epsilon(i,j)$ and $\mathscr{S}(M)$ (adopting μ_{max} - the eigenvector of M as the ranking function) could be written as follows:

$$\mathscr{S}(M) = \frac{1}{(n-1)} \sum_{i=1, i\neq j}^{n} \left(\epsilon(i,j) - 1 \right) \tag{12}$$

In other words the given value of the inconsistency index $\mathscr{S}(M)$ guarantees that the arithmetic mean of the difference between assessment accuracy determinants and one, i.e. $\epsilon(i,j) - 1$ equals $\mathscr{S}(M)$.

2.4 Conditions of Order Preservation

In [1] *Bana e Costa* and *Vansnick* formulate two postulates (conditions of order preservation) as regards the meaning of an eigenvalue based ranking result. The first one, ordinal, *the preservation of order preference condition (POP)* claims that the ranking result in relation to the given pair of concepts (c_i, c_j) should not break with the expert judgement. In other words for pair of concepts $c_1, c_2 \in C$ such that c_1 dominates c_2 i.e. $m_{1,2} > 1$ should hold that:

$$\mu(c_1) > \mu(c_2) \tag{13}$$

The second one, cardinal, *the preservation of order of intensity of preference condition* (*POIP),* stipulates that if c_1 dominates c_2, more than c_3 dominates c_4 (for $c_1, \ldots, c_4 \in C$), i.e. if additionally
$m_{3,4} > 1$ and $m_{1,2} > m_{3,4}$ then also

$$\frac{\mu(c_1)}{\mu(c_2)} > \frac{\mu(c_3)}{\mu(c_4)} \tag{14}$$

Despite the fact that the both conditions of order preservation have been formulated in the context of eigenvalue based approach it is important to note that, in principle, they remain valid in the context of any priority deriving method. None of the two conditions does not require μ be a rescaled eigenvector of M. Moreover, meeting the *POP* and *POIP* conditions seem to be natural for any μ.

[1] In [18] the value $\epsilon(i,l)$ is referred as error.

3 The ranking discrepancy

It is easy to see that $\epsilon(i,j) = {}^{1}/_{\epsilon(j,i)}$. For example if some $\epsilon(i,j) = 2$ then $\epsilon(j,i) = 0.5$. In fact both of these values carry the same information, which is: the ranking result for the pair (c_i, c_j) differs twice from the expert judgement. I.e. one concept got 100% better score than they should. The usefulness of the $\epsilon(i,j)$ parameter has been recognized by *Saaty*. For instance in [20, p. 203] the matrix $[\epsilon(i,j)]$ is used to determine which expert judgments need to be improved in order to reduce inconsistency of M.

3.1 Local discrepancy

It turns out that $\epsilon(i,j)$ can also be used to formulate sufficient conditions for which both *COP* postulates (Def. 2.4) hold. For this purpose, let us define the local discrepancy $\mathscr{E}(i,j)$ as:

$$\mathscr{E}(i,j) \overset{df}{=} \max\{\epsilon(i,j) - 1, {}^{1}/_{\epsilon(i,j)} - 1\} \tag{15}$$

The value $\mathscr{E}(i,j)$ reflects local differences between ranking results and given expert judgements. Information that for certain $\widehat{i}, \widehat{j}$ the value $\mathscr{E}(\widehat{i}, \widehat{j}) = 0.8$ means that the discrepancy between the expert judgment $m_{\widehat{i}\widehat{j}}$ and the ranking results $\mu(c_{\widehat{i}})$ and $\mu(c_{\widehat{j}})$ reach 80%. Similarly as the matrix $[\epsilon(i,j)]$, also the local discrepancy matrix $[\mathscr{E}(i,j)]$ may help to discover where the highest discrepancy is, hence, where the expert judgement (or the ranking function) could be improved.

3.2 Global discrepancy

In order to reduce (to limit) the local discrepancies it is reasonable to introduce the concept of the global ranking discrepancy.

Definition 4 Let the global ranking discrepancy for the pairwise comparisons matrix M, and the ranking μ, be the maximal value of $\mathscr{E}(i,j)$ for $i,j = 1,\ldots,n$, i.e.

$$\mathscr{D}(M,\mu) \overset{df}{=} \max_{i,j=1,\ldots,n} \mathscr{E}(i,j) \tag{16}$$

Thus, a certain value of the global ranking discrepancy $\mathscr{D}(M,\mu) \leq \delta$ provides a guarantee that the maximal discrepancy between a single assessment of an expert and the comparison of corresponding results will not be greater than δ. The ranking discrepancy $\mathscr{D}(M,\mu)$ translates directly into the inconsistency $\mathscr{S}(M)$. The relationship can be expressed as the following theorem.

Theorem 1 *For every pairwise comparisons matrix M and the eigenvector based ranking μ_{max} holds that:*

$$\mathscr{D}(M,\mu_{max}) \leq \delta \Rightarrow \mathscr{S}(M) \leq \delta \tag{17}$$

Proof Since $\mathscr{D}(M,\mu) \leq \delta$, thus according to the definition 4, every $\mathscr{E}(i,j) \leq \delta$ for $i,j = 1,\ldots,n$. Thus, due to definition of $\mathscr{E}$ (see 15), holds that $\epsilon(i,j) - 1 \leq \delta$ for every $i,j \in \{1,\ldots,n\}$. In particular for any $j \in \{1,\ldots,n\}$ it is true that:

$$\sum_{i=1, i\neq j}^{n} \left(\epsilon(i,j) - 1\right) \leq (n-1)\delta \tag{18}$$

hence

$$\frac{1}{(n-1)} \sum_{i=1, i\neq l}^{n} \left(\epsilon(i,j) - 1\right) \leq \delta \tag{19}$$

which, in the light of (12) satisfies the assertion $\mathscr{S}(M) \leq \delta$.

Hence, besides the fact that the global ranking discrepancy $\mathscr{D}(M,\mu_{max})$ detects and limits the worst case discrepancy between a single expert judgement and the ranking result, it also provides a guarantee in the original sense proposed by *Saaty* [19]. Therefore, wherever the inconsistency index $\mathscr{S}(M)$ has so far been used, $\mathscr{D}(M,\mu_{max})$ might be used instead. Provided of course, that $\mathscr{D}(M,\mu_{max})$ is sufficiently small. In return, in addition to the requirements of the level of inconsistency, the users receive a guarantee of even discrepancy distribution.

4 The ranking discrepancy and the conditions of order preservation

Similarly as *POP* and *POIP* (Sec. 2.4), the global ranking discrepancy is derived from eigenvalue based approach but it does not depend on it. Thus, the definition (Def. 4) remains valid for any priority deriving method and any μ. Moreover, the value $\mathscr{D}(M,\mu)$ remains in the immediate connection with *POP* and *POIP*. This relationship could be expressed in the form of the following two assertions.

Theorem 2 *For every pairwise comparisons matrix M expressing the quantitative relationships R between concepts $c_1,\ldots,c_n \in C$, and the ranking μ, the order preference condition is preserved i.e.*

$$m_{ij} > 1 \;\; \textit{implies} \;\; \mu(c_i) > \mu(c_j) \tag{20}$$

if wherever $\mathscr{D}(M,\mu) < \delta$ then also $m_{ij} \geq \delta + 1$.

Proof Since $\mathscr{D}(M,\mu) < \delta$, then according to the definition 4, every $\mathscr{E}(\widehat{\imath},\widehat{\jmath}) < \delta$ for $\widehat{\imath},\widehat{\jmath} = 1,\ldots,n$. In particular $\mathscr{E}(j,i) < \delta$, hence also $\epsilon(j,i) - 1 < \delta$. Therefore, due to the definition of ϵ (11) it is true that

$$\frac{1}{m_{ji}} \cdot \frac{\mu(c_j)}{\mu(c_i)} < \delta + 1 \tag{21}$$

hence

$$m_{ji} \frac{\mu(c_i)}{\mu(c_j)} > \frac{1}{\delta + 1} \tag{22}$$

and due to the reciprocity

$$\frac{\mu(c_i)}{\mu(c_j)} > \frac{m_{ij}}{\delta + 1} \tag{23}$$

Therefore the ratio $\mu(c_i)/\mu(c_j)$ is strictly greater than one if only $m_{ij}/\delta+1 \geq 1$. In other words the only requirement in addition to $\mathscr{D}(M,\mu) < \delta$ needed to meet the *POP* is $m_{ij} \geq \delta + 1$.

The above theorem easily translates into an algorithm that allows us to decide whether the pairwise comparison matrix M and the ranking μ are *POP*-safe, i.e. whether the *POP* condition will never be violated for this pair. Let us note that if we adopt a weak inequality as the upper bound of the ranking discrepancy index i.e. $\mathscr{D}(M,\mu) \leq \delta$, then to meet the *POP* the strong inequality $m_{ij} > \delta + 1$ is needed. Thus, assuming that $\delta = \mathscr{D}(M,\mu)$ is known, all the ratios greater than one i.e. $m_{ij} > 1$ need to be examined to determine whether they are also greater than $\delta + 1$. If so, M is *POP*-safe, which means that *POP* is not violated.

The relationship between *POIP* and $\mathscr{D}(M,\mu)$ also can be expressed in the form of assertion.

Theorem 3 *For every pairwise comparisons matrix M expressing the quantitative relationships R between concepts $c_1,\ldots,c_n \in C$, and the ranking μ, the order of intensity of preference condition is preserved i.e.*

$$m_{ij} > m_{kl} > 1 \;\; \textit{implies} \;\; \frac{\mu(c_i)}{\mu(c_j)} > \frac{\mu(c_k)}{\mu(c_l)} \tag{24}$$

if wherever $\mathscr{D}(M,\mu) < \delta$ then also $m_{ij}/m_{kl} \geq (\delta + 1)^2$

Proof Since $\mathscr{D}(M,\mu) < \delta$, then according to the definition 4, every $\mathscr{E}(p,q) < \delta$ for $p,q = 1,\ldots,n$. In particular $\mathscr{E}(j,i) < \delta$ and $\mathscr{E}(k,l) < \delta$, hence also $\epsilon(j,i) - 1 < \delta$ and $\epsilon(k,l) - 1 < \delta$. Thus, following the same reasoning as in Theorem 2 (21, 22 and 23) we obtain that

$$\frac{\mu(c_i)}{\mu(c_j)} > \frac{m_{ij}}{\delta + 1} \;\; \text{and} \;\; \frac{\mu(c_l)}{\mu(c_k)} > \frac{m_{lk}}{\delta + 1} \tag{25}$$

hence due to the reciprocity,

$$\frac{\mu(c_i)}{\mu(c_j)} > \frac{m_{ij}}{\delta + 1} \;\; \text{and} \;\; \frac{\mu(c_k)}{\mu(c_l)} < m_{kl}(\delta + 1) \tag{26}$$

Therefore, dividing the left inequality by the right inequality leads to the formula

$$\frac{\frac{\mu(c_i)}{\mu(c_j)}}{\frac{\mu(c_k)}{\mu(c_l)}} > \frac{\frac{m_{ij}}{\delta+1}}{m_{kl}(\delta + 1)} \tag{27}$$

Therefore, the ratio ${}^{\mu(c_i)/\mu(c_j)}/_{\mu(c_k)/\mu(c_l)}$ is greater than 1 if ${}^{m_{ij}/(\delta+1)}/_{m_{kl}(\delta+1)}$ is not smaller than 1. In other words the truth of the following inequality:

$$\frac{m_{ij}}{m_{kl}} \geq (\delta+1)^2 \tag{28}$$

implies that

$$\frac{\mu(c_i)}{\mu(c_j)} > \frac{\mu(c_k)}{\mu(c_l)} \tag{29}$$

which is the desired assertion.

Similar as before, to hold the above theorem it is enough for the weak inequality $\mathscr{D}(M,\mu) \leq \delta$ and the strong inequality ${}^{m_{ij}}/_{m_{kl}} > (\delta+1)^2$ to hold. Thus, for the practical verification of whether the *POIP* is violated, the condition ${}^{m_{ij}}/_{m_{kl}} > (\delta+1)^2$ needs to be examined for every pair m_{ij}, m_{kl} that meets the requirements of the theorem.

5 Numerical example

Let us consider a case of numerical judgment described in [1]. There are four concepts $c_1,\ldots,c_4$ for which the relative importance determined by a person J is given as the matrix M.

$$M = \begin{pmatrix} 1 & 2.5 & 4 & 9.5 \\ 0.4 & 1 & 3 & 6.5 \\ \frac{1}{4} & \frac{1}{3} & 1 & 5 \\ \frac{1}{9.5} & \frac{1}{6.5} & \frac{1}{5} & 1 \end{pmatrix} \tag{30}$$

The rescaled eigenvector corresponding to the maximal eigenvalue of M is given as:

$$\mu_{max} = [0.533,\ 0.287,\ 0.139,\ 0.041]^T \tag{31}$$

As already pointed in [1] *POIP* is not satisfied. In particular $m_{3,4} > m_{1,3}$ but $\mu_{max}(c_3)/\mu_{max}(c_4) < \mu_{max}(c_1)/\mu_{max}(c_3)$. The local discrepancy matrix $\mathscr{E} = [\mathscr{E}(i,j)]$ allows for identifying the most inconsistent entry in M. It is $m_{3,4}$, for which $\mathscr{E}(3,4) = 0.475$.

$$\mathscr{E} = \begin{pmatrix} 0 & 0.348 & 0.044 & 0.367 \\ 0.348 & 0 & 0.452 & 0.077 \\ 0.044 & 0.452 & 0 & 0.475 \\ 0.367 & 0.077 & 0.475 & 0 \end{pmatrix} \tag{32}$$

After re-evaluation by experts the value $m_{3,4}$ is set to 3. Re-creating the local discrepancy matrix for M where $m_{3,4} = 3$ and $m_{4,3} = {}^1/_3$ indicates that $m_{1,2}$ also needs expert attention. Re-evaluated $m_{1,2}$ is set to 1.5 and due to the reciprocity requirement $m_{2,1}$ is set to ${}^2/_3$. After adjusting four entries the matrix (30) takes the form:

$$M' = \begin{pmatrix} 1 & 1.5 & 4 & 9.5 \\ \frac{1}{1.5} & 1 & 3 & 6.5 \\ \frac{1}{4} & \frac{1}{3} & 1 & 3 \\ \frac{1}{9.5} & \frac{1}{6.5} & \frac{1}{3} & 1 \end{pmatrix} \tag{33}$$

The rescaled principal eigenvector of M' is:

$$\mu'_{max} = [0.487,\ 0.338, 0.126,\ 0.048]^T \tag{34}$$

The local discrepancy matrix $\mathscr{E}' = [\mathscr{E}'(i,j)]$ calculated for M' and μ'_{max} shows that the global ranking discrepancy is 0.149.

$$\mathscr{E}' = \begin{pmatrix} 0 & 0.038 & 0.033 & 0.064 \\ 0.038 & 0 & 0.119 & 0.077 \\ 0.033 & 0.119 & 0 & 0.149 \\ 0.064 & 0.077 & 0.149 & 0 \end{pmatrix} \tag{35}$$

According to the Theorem 2 to meet the *POP* condition it is enough if

$$m'_{ij} > 1 \Rightarrow m'_{ij} > 1.149 \tag{36}$$

for every $i, j = 1, \ldots, 4$ and $M' = [m'_{ij}]$. Similarly, (Theorem 3) the *POIP* condition is satisfied if

$$m'_{ij} > m'_{kl} > 1 \Rightarrow m'_{ij}/m'_{kl} > (1+0.149)^2 \approx 1.32 \tag{37}$$

for every $i, j, k, l = 1, \ldots, 4$. It is easy to see that both (36) and (37) hold. Therefore, after the discrepancy reduction[2], there is a guarantee that the resulting pairwise comparisons matrix M' together with the ranking μ'_{max} satisfies *COP*.

6 Discussion and summary

In their work *Bana e Costa* and *Vansnick* [1] formulated two conditions whose fulfillment makes the ranking result indisputable. Therefore, in practice, meeting these two conditions may translate into a significant reduction in the number of appeals against the results of the ranking procedure. Hence, in addition to intangible benefits such as providing the ranking participants a sense of justice, meeting the *POP* and *POIP* conditions may contribute to the reduction of costs associated with the carrying out the evaluation procedure. The notion of global ranking discrepancy $\mathscr{D}(M,\mu)$ helps to fulfill the *Bana e Costa* and *Vansninck* postulate. The value $\mathscr{D}(M,\mu)$ directly translates to the requirements for the matrix M, so that the smaller $\mathscr{D}(M,\mu)$ the greater the chance that the *POP* and *POIP* conditions are met.

Although the global ranking discrepancy (Sec. 3) has been defined in the context of eigenvalue priority deriving method, it is not tied to it. In fact it could be useful for any pair of the *PC* matrix M and the ranking μ. The only, but crucial, assumption is that μ attempts to reflect the experts' judgments given as M. The conditions provided by Theorems (2) and (3) are sufficient, but they are not necessary. Thus, there may exist better estimates allowing to determine whether the *COP* are satisfied. The existence of such estimates remains as an open question.

This study addresses an important problem of discrepancies between expert judgments and ranking results that may appear in the pairwise comparison method. The notion of the global ranking discrepancy has been defined. Its relationship with the eigenvalue based inconsistency index and *POP* and *POIP* [1] postulates have been shown.

Acknowledgements I would like to thank Dr Jacek Szybowski and Prof. Antoni Ligęza for reading the first version of this work. Special thanks are due to Dan Swain for his editorial help. The research is supported by AGH University of Science and Technology, contract no.: 10.10.120.105.

References

1. C. A. Bana e Costa and J. Vansnick. A critical analysis of the eigenvalue method used to derive priorities in AHP. *European Journal of Operational Research*, 187(3):1422–1428, June 2008.
2. J. Barzilai and B. Golany. AHP rank reversal, normalization and aggregation rules. *INFOR - Information Systems and Operational Research*, 32(2):57–64, 1994.
3. S. Bozóki, J. Fülöp, and L. Rónyai. On optimal completion of incomplete pairwise comparison matrices. *Mathematical and Computer Modelling*, 52(1–2):318 – 333, 2010.
4. S. Bozóki and T. Rapcsak. On Saaty's and Koczkodaj's inconsistencies of pairwise comparison matrices. *Journal of Global Optimization*, 42(2):157–175, 2008.
5. J. M. Colomer. Ramon Llull: from 'Ars electionis' to social choice theory. *Social Choice and Welfare*, 40(2):317–328, October 2011.
6. J. S. Dyer. Remarks on the analytic hierarchy process. *Management Science*, 36(3):249–258, 1990.
7. G. T. Fechner. *Elements of psychophysics*, volume 1. Holt, Rinehart and Winston, New York, 1966.
8. M Fedrizzi and M Brunelli. On the priority vector associated with a reciprocal relation and a pairwise comparison matrix. *Journal of Soft Computing*, 14(6):639–645, 2010.
9. S. Greco, B. Matarazzo, and R. Słowiński. Dominance-based rough set approach on pairwise comparison tables to decision involving multiple decision makers. In *Rough Sets and Knowledge Technology*, volume 6954 of *LNCS*. Springer, 2011.
10. A. Ishizaka and A. Labib. Analytic hierarchy process and expert choice: Benefits and limitations. *OR Insight*, 22(4):201–220, 2009.
11. R. Janicki and Y. Zhai. On a pairwise comparison-based consistent non-numerical ranking. *Logic Journal of the IGPL*, 20(4):667–676, 2012.
12. W. W. Koczkodaj, M. W. Herman, and M. Orlowski. Managing Null Entries in Pairwise Comparisons. *Knowledge and Information Systems*, 1(1):119–125, 1999.
13. K. Kułakowski. A heuristic rating estimation algorithm for the pairwise comparisons method. *Central European Journal of Operations Research*, pages 1–17, 2013.
14. K. Kułakowski. Heuristic Rating Estimation Approach to The Pairwise Comparisons Method. *CoRR, (accepted for publication in Fundamenta Informaticae)*, abs/1309.0386, 2013.
15. M. J. Liberatore and R. L. Nydick. The analytic hierarchy process in medical and health care decision making: A literature review. *European Journal of Operational Research*, 189(1):14–14, August 2008.
16. L. Mikhailov. Deriving priorities from fuzzy pairwise comparison judgements. *Fuzzy Sets and Systems*, 134(3):365–385, March 2003.
17. G. L. Peterson and T. C. Brown. Economic valuation by the method of paired comparison, with emphasis on evaluation of the transitivity axiom. *Land Economics*, pages 240–261, 1998.
18. T. L. Saaty. A scaling method for priorities in hierarchical structures. *Journal of Mathematical Psychology*, 15(3):234 – 281, 1977.

[2] and inconsistency reduction. Note that $\mathscr{S}(M) = 0.04$ whilst $\mathscr{S}(M') = 0.003$.

19. T. L. Saaty. The analytic hierarchy and analytic network processes for the measurement of intangible criteria and for decision-making. In *Multiple Criteria Decision Analysis: State of the Art Surveys*, volume 78 of *International Series in Operations Research and Management Science*, pages 345–405. Springer New York, 2005.
20. T. L. Saaty. On the Measurement of Intangibles. A Principal Eigenvector Approach to Relative Measurement Derived from Paired Comparisons. *Notices of the American Mathematical Society*, 60(02):192, February 2013.
21. J. E. Smith and D. Von Winterfeldt. Anniversary article: decision analysis in management science. *Management Science*, 50(5):561–574, 2004.
22. N. Subramanian and R. Ramanathan. A review of applications of Analytic Hierarchy Process in operations management. *International Journal of Production Economics*, 138(2):215–241, August 2012.
23. L. L. Thurstone. A law of comparative judgment, reprint of an original work published in 1927. *Psychological Review*, 101:266–270, 1994.
24. O. S. Vaidya and S. Kumar. Analytic hierarchy process: An overview of applications. *European Journal of Operational Research*, 169(1):1–29, February 2006.
25. K. K. F. Yuen. Fuzzy cognitive network process: Comparison with fuzzy analytic hierarchy process in new product development strategy. *Fuzzy Systems, IEEE Transactions on*, PP(99):1–1, 2013.