Constraint qualifications in convex vector semi-infinite optimization*

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Abstract

Convex vector (or multi-objective) semi-infinite optimization deals with the simultaneous minimization of finitely many convex scalar functions subject to infinitely many convex constraints. This paper provides characterizations of the weakly efficient, efficient and properly efficient points in terms of cones involving the data and Karush-Kuhn-Tucker conditions. The latter characterizations rely on different local and global constraint qualifications. The results in this paper generalize those obtained by the same authors on linear vector semi-infinite optimization problems [Eur J Oper Res 227 (2013) 12-21].

1 Introduction

We consider convex optimization problems of the form

$$P : \text{ "min" } f(x) = (f_1(x), ..., f_p(x)) \quad \text{s.t. } g_t(x) \leq 0, t \in T, \tag{1}$$

where $x \in \mathbb{R}^n$ (the *space of decisions*), $f(x) \in \mathbb{R}^p$ (the *objective space*), the index set T is a compact Hausdorf topological space, $f_i : \mathbb{R}^n \to \mathbb{R}$ is a convex function, $i = 1, \ldots, p$, g_t is convex for each $t \in T$, and the function $(t, x) \mapsto g_t(x)$ is continuous on $T \times \mathbb{R}^n$. The continuity of f is consequence of the assumptions on its components $f_1, ..., f_p$. The model (1) includes ordinary convex (scalar and vector) optimization problems just taking the discrete topology on the (finite) index set. Since the optimality theory for this class of problems has been thoroughly studied, we assume in the sequel that T is infinite. When $p \geq 2$, P is

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a convex vector semi-infinite optimization (SIO in brief) problem; otherwise, P is a convex scalar SIO problem. Replacing in (1) the space of decisions $\mathbb{R}^n$ by an infinite dimensional space (typically a locally convex Hausdorff topological vector space) one gets a convex (scalar or vector) infinite optimization (IO in short) problem.

We assume throughout the paper that $p \geq 2$ and the *feasible set* of P, denoted by X, is non-empty. Obviously, X is a closed convex set whereas its image by the vector-valued objective function $f(X) \subset \mathbb{R}^p$ is possibly non-convex and non-closed. The vector SIO problem P can be reformulated as a vector optimization problem with the single convex constraint function $\varphi(x) := \max_{t \in T} g_t(x)$, called *marginal function:*

$$P : \text{" min "} \quad f(x) = (f_1(x), ..., f_p(x)) \quad \text{s.t.} \quad \varphi(x) \leq 0.$$

Throughout the paper we use the following notation. Given $x, y \in \mathbb{R}^m$, we write $x \leqq y$ $(x < y)$ when $x_i \leq y_i$ $(x_i < y_i$, respectively) for all $i = 1, ..., m$. Moreover, we write $x \leq y$ when $x \leqq y$ and $x \neq y$.

An element $\overline{x} \in X$ is said to be *efficient* (*weakly efficient*) if there is no $\widehat{x} \in X$ such that $f(\widehat{x}) \leq f(\overline{x})$ $(f(\widehat{x}) < f(\overline{x})$, respectively). There are many notions of proper efficiency in the literature, as those introduced by Geoffrion, Benson, Borwein and Henig. Since P is convex, all these concepts are equivalent (see, e.g., [9]), so that we recall only Geoffrion's definition: a feasible point $\overline{x} \in X$ is said to be *properly efficient* if there exists $\rho > 0$ such that, for all $i = 1, ..., p$ and $\widehat{x} \in X$ satisfying $f_i(\widehat{x}) < f_i(\overline{x})$, there exists $j \in \{1, ..., p\}$ such that $f_j(\widehat{x}) > f_j(\overline{x})$ and $\dfrac{f_i(\overline{x}) - f_i(\widehat{x})}{f_j(\widehat{x}) - f_j(\overline{x})} \leq \rho$.

We denote by X_{pE}, X_E, and X_{wE} the sets of properly efficient points, efficient points, and weakly efficient points of P, respectively. Obviously, $X_{pE} \subset X_E \subset X_{wE}$, with $X = X_{wE}$ whenever one component of f is identically zero, and $X = X_{pE}$ in the trivial case that f is the null function. Moreover, it is known that $f(X_{pE})$ is dense in $f(X_E)$ ([17]; see also [9, Theorem 3.17]).

Given a (possibly non-convex) vector SIO problem

$$P : \text{" min "} \quad f(x) \quad \text{s.t.} \quad x \in X,$$

$\overline{x} \in X$ is said to be *locally (properly, weakly) efficient solution* of P if there exists a neighborhood $\mathcal{N}$ of $\overline{x}$ such that $\overline{x}$ is (properly, weakly) efficient solution of

$$P_{\mathcal{N}} : \text{" min "} \quad f(x) \quad \text{s.t.} \quad x \in X \cap \mathcal{N}.$$

Global and local concepts coincide in convex vector SIO thanks to the convexity of X and the componentwise convexity of f. For instance, if $\overline{x} \in X$ is not weakly efficient there exists $\widehat{x} \in X$ such that $f(\widehat{x}) < f(\overline{x})$; since $f(\lambda \widehat{x} + (1 - \lambda) \overline{x}) < f(\overline{x})$ for all $\lambda \in]0, 1[$, with $\lambda \widehat{x} + (1 - \lambda) \overline{x} \in X \cap \mathcal{N}$ for λ sufficiently small, $\overline{x}$ cannot be a locally weakly efficient solution of P. For this reason, in convex vector SIO, we can characterize the (proper, weak) efficiency on the basis of local information. The known tests for non-linear vector optimization classify

a given $x \in X$ as locally (properly, weakly) efficient solution or not through conditions involving subsets of the objective space $\mathbb{R}^p$ or suitable scalarizations of P (see, e.g., [2], [9]).

In this paper, on convex vector SIO, we give conditions for $\overline{x} \in X_{pE}$, $\overline{x} \in X_E$, and $\overline{x} \in X_{wE}$ which are expressed in terms of convex cones contained in the decision space $\mathbb{R}^n$ or in terms of the existence of Karush-Kuhn-Tucker (KKT in short) multipliers which can be computed from $\overline{x}$ and the data describing P.

As a general rule, to obtain a checkable necessary optimality condition for a given constrained optimization problem, one needs to assume some property of the constraint system called *constraint qualification* (CQ in short). We consider in this paper four CQs which extend those used in our previous paper [12] on linear vector SIO. The strongest one is the natural extension of the CQ introduced by M. Slater in a seminal work on scalar optimization published in 1950, which was adapted to linear scalar SIO by Charnes, Cooper and Kortanek in the 1960s. A weaker CQ for convex scalar SIO has been proposed in [23]. The locally Farkas-Minkowski CQ was first defined in [26] for linear scalar SIO, and then extended to convex scalar SIO in [13] and to convex scalar IO in [8]. CQs weaker than the locally Farkas-Minkowski one have been introduced in [23], for convex SIO problems, and in [22], for convex IO problems. The local Slater CQ, introduced in Section 3 of this paper, seems to be new while the extended Kuhn-Tucker CQ was introduced in [29] for convex IO as an extension of that used by H.W. Kuhn and A.W. Tucker in [19] for ordinary non-linear optimization problems. Section 1 of [22] reviews the state of the art on CQs in convex scalar optimization.

The stability of linear and non-linear vector SIO has been investigated since the last 1980s from different perspectives, e.g., the pseudo-Lipschitz property and the lower and upper semicontinuity of the efficient set mapping under different types of perturbations, well-posedness, and generic stability (see, e.g., [4], [5], [6], [10], [30], [31]), while the existing literature on optimality conditions for vector SIO and vector IO problems is surprisingly limited.

The main antecedent of this paper is [12], on linear vector SIO, which provides characterizations of the weakly efficient, efficient and properly efficient solutions in terms of cones involving the data and KKT conditions. In [3], on a class of vector SIO problems involving differentiable functions whose constraints satisfy certain invex-type conditions and are required to depend continuously on an index t ranging on some compact topological space T, KKT conditions for $\overline{x} \in X_{pE}$, $\overline{x} \in X_E$ and $\overline{x} \in X_{wE}$ are given. In [14], on non-convex differentiable vector SIO, the authors discuss constraint qualifications as well as necessary and sufficient conditions for locally weakly efficient points and present optimality conditions for properly efficient points in the senses of Geoffrion and of Kuhn and Tucker [19]. Finally, in [7], on non-smooth vector IO problems posed on Asplund spaces whose index set T has no topological structure, necessary conditions as well as sufficient conditions for weakly efficient solutions are obtained appealing to the machinery of non-smooth analysis and a certain CQ, for non-convex systems introduced in [4], which can be seen as an extension of the so-called basic CQs introduced in [21], for scalar IO problems posed in

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Banach spaces.

The convex vector SIO problems considered in this paper arise in a natural way in robust linear vector optimization. Indeed, consider an uncertain linear vector optimization problem

$$(LP) \quad \text{" min "} \ (c_1^\top x, \ldots, c_p^\top x) \ \text{s.t.} \ a_t^\top x \geq b_t, \ t \in T,$$

where T is a finite set, $c_i \in \mathcal{U}_i \subset \mathbb{R}^n$, $i = 1, \ldots, p$, and $(a_t, b_t) \in \mathcal{V}_t \subset \mathbb{R}^{n+1}$, $t \in T$. The *uncertainty sets* $\mathcal{U}_i, i = 1, \ldots, p$, are arbitrary non-empty sets while $\mathcal{V}_t, t \in T$, are non-empty compact sets. The *robust minmax counterpart* of (LP) (term coined in [15]) enforces feasibility for any possible scenario and assumes that the cost of any (robust) feasible decision will be the worst possible, i.e., the problem to be solved is

$$\text{" min "} \ \left(\max_{c_1 \in \mathcal{U}_1} c_1^\top x, \ldots, \max_{c_p \in \mathcal{U}_p} c_p^\top x \right) \ \text{s.t.} \ a_t^\top x \geq b_t, \forall (a_t, b_t) \in \mathcal{V}_t, t \in T. \quad (2)$$

Observe that (2) is as (1), just taking $f_i(x) = \max_{c_i \in \mathcal{U}_i} c_i^\top x$ (i.e., the support function of $\mathcal{U}_i$), $i = 1, \ldots, p$, and expressing the constraints either as $b - a^\top x \leq 0$ for all $(a, b) \in \bigcup_{t \in T} \mathcal{V}_t$ (a compact index set) or as $g_t(x) \leq 0$, with $g_t(x) = \max \left\{ b - a^\top x : (a, b) \in \bigcup_{t \in T} \mathcal{V}_t \right\}$ for all $t \in T$ (a finite index set equipped with the discrete topology).

This paper is organized as follows. Section 2 recalls basic concepts of convex analysis to be used later, applying some of them to characterize the subdifferential cone and its interior, and to describe the relationships between several types of "tangent" cones which are closely related with the negative polar of the active cone. Section 3 extends to convex vector SIO four out of six constraint qualifications introduced in [12] for linear vector SIO. The two exceptions, the Farkas-Minkowski and the local polyhedral constraint qualifications, have not been considered in this paper as they are too strong in the convex framework. For methodological reasons, we give simple direct proofs of the lemmas in Section 3 even though most of them could be also obtained via linearization. Finally, Section 4 provides different characterizations of the sets X_{pE}, X_E, and X_{wE} in terms of cones and KKT multipliers. Here the proofs are necessarily direct as the objective functions are not linear. These results are applied to the robust linear vector optimization problem (LP).

2 Preliminaries

We start this section by introducing the necessary notation and concepts. Given $Z \subset \mathbb{R}^n$, int Z, cl Z, and bd Z denote the *interior*, the *closure*, and the *boundary* of Z, respectively. The scalar product of $x, y \in \mathbb{R}^n$ is denoted by $x^\top y$, the Euclidean norm of x by $\|x\|$, the corresponding open ball centered at x and radius $\varepsilon > 0$ by $B(x, \varepsilon)$, and the zero vector by 0_n. We also denote by conv Z

the *convex hull* of Z, while $\operatorname{cone} Z := \mathbb{R}_+ \operatorname{conv} Z$ denotes the *convex conical hull* of $Z \cup \{0_n\}$. If Z is a convex cone, its *positive (negative) polar cone is* $Z^+ := \left\{d \in \mathbb{R}^n : z^\top d \geq 0 \ \forall z \in Z\right\}$ $\left(Z^- := \left\{d \in \mathbb{R}^n : z^\top d \leq 0 \ \forall z \in Z\right\}\right.$, respectively). A convex cone is said to be *pointed* whenever it does not contain lines. We use frequently in this paper the topological interior of polar cones.

We make three claims concerning $\operatorname{cone} Z$, where Z is an arbitrary non-empty set of $\mathbb{R}^n$:

First,

$$0_n \notin \operatorname{conv} Z \Leftrightarrow \{\operatorname{cone} Z \text{ is pointed and } 0_n \notin Z\}. \tag{3}$$

We shall prove that $0_n \in \operatorname{conv} Z$ if and only if $\operatorname{cone} Z$ contains lines or $0_n \in Z$.

If $\operatorname{cone} Z$ contains lines, there exists $u \in \mathbb{R}^n \setminus \{0_n\}$ such that $\pm u \in \operatorname{cone} Z$. Then, we can write $u = \sum_{i=1}^n \alpha_i z_i$ and $-u = \sum_{i=1}^n \beta_i z_i$, with $\alpha_1, ..., \alpha_n, \beta_1, ..., \beta_n \in \mathbb{R}_+$, $z_i \in Z$, $i = 1, \ldots, n$, so that

$$0_n = \left(\sum_{i=1}^n (\alpha_i + \beta_i)\right)^{-1} \sum_{i=1}^n (\alpha_i + \beta_i) z_i \in \operatorname{conv} Z.$$

Alternatively, if $0_n \in Z$, it is obvious that $0_n \in \operatorname{conv} Z$.

Conversely, if $0_n \in \operatorname{conv} Z$, there exist $\alpha_i \in \mathbb{R}_+$ and $z_i \in Z$, $i = 1, \ldots, n$, such that $\sum_{i=1}^n \alpha_i = 1$ and $\sum_{i=1}^n \alpha_i z_i = 0_n$. Let $j \in \{1, ..., n\}$ be such that $\alpha_j > 0$. If $z_j \neq 0_n$, putting $\alpha := \sum_{i \neq j} \alpha_i > 0$, it follows that $u := -\frac{\alpha_j}{\alpha} z_j = \sum_{i \neq j} \frac{\alpha_i}{\alpha} z_i \in \operatorname{cone} Z$, so that $\operatorname{cone} Z$ contains the line spanned by u. Now, suppose that $z_j = 0_n$. Then

$$\sum_{i \neq j} \alpha_i z_i = 0_n. \tag{4}$$

If $\alpha_i = 0$ for all $i \neq j$, then $0_n = z_j \in Z$. Otherwise, there exists $k \neq j$ such that $\alpha_k > 0$, in (4), and we repeat the argument above.

Second,

$$0_n \notin Z \Rightarrow \operatorname{int} (\operatorname{cone} Z)^+ \subset \left\{d \in \mathbb{R}^n : z^\top d > 0 \ \forall z \in Z\right\}. \tag{5}$$

In fact, assume that $0_n \notin Z$ and $d \in \operatorname{int} (\operatorname{cone} Z)^+$. Let $\varepsilon > 0$ be such that $\operatorname{cl} B(d, \varepsilon) \subset (\operatorname{cone} Z)^+$. Given $z \in Z$, $d - \varepsilon \frac{z}{\|z\|} \in (\operatorname{cone} Z)^+$ while $z \in \operatorname{cone} Z$, so that $z^\top \left(d - \varepsilon \frac{z}{\|z\|}\right) = z^\top d - \varepsilon \|z\| \geq 0$ and so $z^\top d \geq \varepsilon \|z\| > 0$. Obviously, if $Z \cap (-Z) \neq \emptyset$, then both members of the inclusion in (5) are empty.

Third,

$$Z \text{ compact} \Rightarrow \left\{d \in \mathbb{R}^n : z^\top d > 0 \ \forall z \in Z\right\} \subset \operatorname{int} (\operatorname{cone} Z)^+. \tag{6}$$

In fact, by assumption, there exists $\delta > 0$ such that $\|z\| \leq \delta$ for all $z \in Z$. Let $d \in \mathbb{R}^n$ be such that $z^\top d > 0$ for all $z \in Z$. By the compactness of Z,

$\varepsilon := \min_{z \in Z} z^\top d > 0$. Given $x \in \operatorname{cone} Z$, we can write $x = \sum_{i=1}^{n} \mu_i z_i$, with $\mu_i \geq 0$ and $z_i \in Z$, $i = 1, ..., n$. Then, given u such that $\|u\| \leq 1$, one has

$$x^\top \left(d + \frac{\varepsilon}{\delta} u\right) = \sum_{i=1}^{n} \mu_i z_i^\top \left(d + \frac{\varepsilon}{\delta} u\right) \geq \sum_{i=1}^{n} \mu_i \left(\varepsilon - \frac{\varepsilon}{\delta} |z^\top u|\right) \geq 0.$$

Thus, $B\left(d, \frac{\varepsilon}{\delta}\right) \subset (\operatorname{cone} Z)^+$ and $d \in \operatorname{int} (\operatorname{cone} Z)^+$. The inclusion in (6) becomes an equation between non-empty sets whenever Z is compact and $0_n \notin \operatorname{conv} Z$ (as $\operatorname{cone} Z$ turns out to be a pointed cone).

The *one-sided directional derivative* of a real-valued function $h : \mathbb{R}^n \to \mathbb{R}$ at $\overline{x} \in \mathbb{R}^n$ with respect to a vector $d \in \mathbb{R}^n$ is defined to be the limit

$$h'(\overline{x}; d) = \lim_{\varepsilon \downarrow 0} \frac{h(\overline{x} + \varepsilon d) - h(\overline{x})}{\varepsilon},$$

if it exists. If h is convex, then it is continuous, the directional derivative function at $\overline{x} \in \mathbb{R}^n$, $h'(\overline{x}; .)$, is a finite convex function too, and the *subdifferential*

$$\partial h(\overline{x}) := \left\{\xi \in \mathbb{R}^n : h(x) \geq h(\overline{x}) + \xi^\top (x - \overline{x}) \ \forall x \in \mathbb{R}^n\right\},$$

is a non-empty compact convex such that $h'(\overline{x}; d) = \max_{\xi \in \partial h(\overline{x})} \xi^\top d$ (see, e.g., [27, Theorems 23.1 and 23.4]).

Lemma 1 *Let $h : \mathbb{R}^n \to \mathbb{R}$ be a convex function and let $d \in \mathbb{R}^n$ be a descent direction of h at $\overline{x} \in \mathbb{R}^n$. Then, $h'(\overline{x}; d) < 0$.*

Proof: Since d is a descent direction of h at $\overline{x}$ there exists $\beta > 0$ such that

$$h(\overline{x} + \varepsilon d) - h(\overline{x}) < 0 \text{ for all } \varepsilon \in \,]0, \beta[\,. \tag{7}$$

Thus

$$h'(\overline{x}; d) = \lim_{\varepsilon \downarrow 0} \frac{h(\overline{x} + \varepsilon d) - h(\overline{x})}{\varepsilon} \leq 0.$$

On the other hand

$$h'(\overline{x}; d) = \max_{\xi \in \partial h(\overline{x})} \xi^\top d.$$

Now, taking into account that $0_n \notin \partial h(\overline{x})$ and $\partial h(\overline{x})$ is compact, if $h'(\overline{x}; d) = 0$ there would exists $\widetilde{\xi} \in \partial h(\overline{x})$, $\widetilde{\xi} \neq 0_n$ such that $\widetilde{\xi}^\top d = 0$. Thus, in particular, for $\widetilde{\xi}$ and $\overline{x} + \varepsilon d$ with $\varepsilon \in \,]0, \beta[$

$$h(\overline{x} + \varepsilon d) - h(\overline{x}) \geq \varepsilon \widetilde{\xi}^\top d = 0$$

which contradicts (7). $\square$

It is easy to prove that, under the assumptions on P,

$$X = \left\{x \in \mathbb{R}^n : \xi^\top x \leq \xi^\top y - g_t(y), \ \forall (t, y) \in T \times \mathbb{R}^n, \ \forall \xi \in \partial g_t(y)\right\}. \tag{8}$$

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From (8) and [13, Theorem 9.3] one gets that X is compact if and only if $\operatorname{cone}\left(\bigcup_{(t,y)\in T\times\mathbb{R}^n}\partial g_t\left(y\right)\right) = \mathbb{R}^n$. This condition (expressed in terms of the data), guarantees the compactness of $f\left(X\right)$. More information on X can be obtained from the linearization (8) of X under the constraint qualifications introduced in Section 3.

Two convex cones involving the data of the vector SIO problem P in (1) are basic in our approach: the convex conical hull of the subdifferentials at $\overline{x} \in X$ of the components of f,

$$G(\overline{x}) := \operatorname{cone}\left(\bigcup_{i=1}^{p}\partial f_i(\overline{x})\right),$$

that we shall call *subdifferential cone at* $\overline{x}$, and the *active cone* at $\overline{x} \in X$,

$$A\left(\overline{x}\right) := \operatorname{cone}\left(\bigcup_{t\in T(\overline{x})}\partial g_t(\overline{x})\right),$$

where $T\left(\overline{x}\right) := \{t \in T : g_t(\overline{x}) = 0\}$ is the *set of active indices* at $\overline{x}$.

We are interested in the negative polar of both cones, $G(\overline{x})^-$ and $A\left(\overline{x}\right)^-$, and their corresponding interiors, $\operatorname{int} G(\overline{x})^-$ and $\operatorname{int} A\left(\overline{x}\right)^-$.

Lemma 2 *Given $\overline{x} \in X$, the following statements hold:*

(i) $0_n \notin \operatorname{conv}\left(\bigcup_{i=1}^{p}\partial f_i(\overline{x})\right)$ *if and only if $G(\overline{x})$ is pointed and $0_n \notin \bigcup_{i=1}^{p}\partial f_i(\overline{x})$.*

(ii) $G(\overline{x})^- = \left\{d \in \mathbb{R}^n : f_i'(\overline{x}; d) \leq 0,\ i = 1,\ldots,p\right\}.$

(iii) If $0_n \notin \bigcup_{i=1}^{p}\partial f_i(\overline{x})$, *then*

$$\begin{aligned}\operatorname{int} G(\overline{x})^- &= \left\{d \in \mathbb{R}^n : \xi^\top d < 0\ \forall \xi \in \bigcup_{i=1}^{p}\partial f_i(\overline{x})\right\}\\&= \{d \in \mathbb{R}^n : f_i'(\overline{x}; d) < 0,\ i = 1,\ldots,p\},\end{aligned} \tag{9}$$

with $\operatorname{int} G(\overline{x})^- \neq \emptyset$ *whenever* $0_n \notin \operatorname{conv}\left(\bigcup_{i=1}^{p}\partial f_i(\overline{x})\right)$.

Proof: (i) It is straightforward consequence of (3), just taking $Z = \bigcup_{i=1}^{p}\partial f_i(\overline{x})$.

(ii) Since $f_i'(\overline{x}; d) = \max_{\xi\in\partial f_i(\overline{x})} \xi^\top d$, one has that $\xi^\top d \leq 0$ for all $\xi \in \partial f_i(\overline{x})$ if and only if $f_i'(\overline{x}; d) \leq 0,\ i = 1,\ldots,p$. So,

$$\begin{aligned}G(\overline{x})^- &= \left\{d \in \mathbb{R}^n : \xi^\top d \leq 0\ \forall \xi \in \bigcup_{i=1}^{p}\partial f_i(\overline{x})\right\}\\&= \{d \in \mathbb{R}^n : f_i'(\overline{x}; d) \leq 0,\ i = 1,\ldots,p\},\end{aligned} \tag{10}$$

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(iii) Again by the identity $f'_i(\overline{x}; d) = \max_{\xi \in \partial f_i(\overline{x})} \xi^\top d$, one has that $\xi^\top d < 0$ for all $\xi \in \partial f_i(\overline{x})$ if and only if $f'_i(\overline{x}; d) < 0$, $i = 1, \ldots, p$. Assuming that $0_n \notin \bigcup_{i=1}^{p} \partial f_i(\overline{x})$, from (5) and (6), one gets (9). The additional condition that $0_n \notin \mathrm{conv}\left(\bigcup_{i=1}^{p} \partial f_i(\overline{x})\right)$ guarantees, by (i), that $G(\overline{x})$ is pointed, which in turn implies that $\mathrm{int}\, G(\overline{x})^- \neq \emptyset$. $\qquad \square$

The KKT conditions will be obtained by analyzing the relationships between negative polar of the active cone $A(\overline{x})$ and four "tangent" cones at $\overline{x}$ defined as follows:

The *cone of feasible directions* at $\overline{x}$ is

$$D(X; \overline{x}) = \{d \in \mathbb{R}^n : \exists \mu > 0 \text{ such that } \overline{x} + \mu d \in X\}.$$

It is known that $D(X; \overline{x}) \subset A(\overline{x})^-$ ([13, Lemma 7.7]).

The *attainable cone* at $\overline{x}$, denoted by $A(X; \overline{x})$, is formed by those $d \in \mathbb{R}^n$ such that there exist $\tau > 0$ and a vector function $h \in \mathcal{C}^1\left([0, \tau[, \mathbb{R}^n\right)$ with $h(0) = \overline{x}$, $h'(0) = d$, and $h(s) \in X$ for all $s \in [0, \tau[$.

The *Bouligand tangent cone* at $\overline{x}$, denoted by $T(X; \overline{x})$, is formed by those $d \in \mathbb{R}^n$ such that there exist sequences $\{s^k\}_{k \in \mathbb{N}}$ and $\{d^k\}_{k \in \mathbb{N}}$ such that $s^k \downarrow 0$, $d^k \to d$ as $k \to \infty$ and $\overline{x} + s^k d^k \in X$ for all $k \in \mathbb{N}$. In that case, since $d^k \in D(X; \overline{x})$ for all $k \in \mathbb{N}$, $d \in \mathrm{cl}\, D(X; \overline{x})$.

The *interior tangent cone* at $\overline{x}$, denoted by $T^i(X; \overline{x})$, is formed by those $d \in \mathbb{R}^n$ such that there exist $\tau > 0$ and a neighborhood $\mathcal{N}$ of d such that $\overline{x} + s\mathcal{N} \subset X$ for all $s \in]0, \tau[$.

Lemma 3 *([2], [16], [20], [25], [27], [32]) Given $\overline{x} \in X$, the cones $T^i(Z; x)$, $D(Z; x)$, $A(Z; x)$, and $T(Z; x)$ are all convex and satisfy*

$$T^i(X; \overline{x}) = \mathrm{int}\, D(X; \overline{x}) \subset D(X; \overline{x}) \subset A(X; \overline{x}) = T(X; \overline{x}) = \mathrm{cl}\, D(X; \overline{x}). \quad (11)$$

Example 4 *Consider the closed convex set*

$$X = \left\{(x_1, x_2) \in \mathbb{R}^2 : t(x_1 - 2)^2 - t - x_2 \leq 0\ \forall t \in [0, 1]\right\}$$

and the point $\overline{x} = (1, 0) \in X$. Since

$$\varphi(x) = \begin{cases} -x_2, & 1 \leq x_1 \leq 3, \\ x_1^2 - 4x_1 + 3 - x_2, & \text{otherwise,} \end{cases}$$

one has

$$X = \left\{x \in \mathbb{R}^2 : \varphi(x) \leq 0\right\} = \left\{x \in \mathbb{R}^2 : x_2 \geq \max\left\{x_1^2 - 4x_1 + 3, 0\right\}\right\},$$

so that

$$D(X; \overline{x}) = \left\{d \in \mathbb{R}^2 : 2d_1 + d_2 > 0, d_2 \geq 0\right\} \cup \{0_2\},$$

$$A(X; \overline{x}) = T(X; \overline{x}) = \left\{d \in \mathbb{R}^2 : 2d_1 + d_2 \geq 0, d_2 \geq 0\right\} = \mathrm{cl}\, D(X; \overline{x}),$$

and

$$T^i(X; \overline{x}) = \left\{d \in \mathbb{R}^2 : 2d_1 + d_2 > 0, d_2 > 0\right\} = \text{int } D(X; \overline{x}).$$

Hence, the inclusions in (11) are strict. Observe that

$$A(\overline{x}) = \text{cone}\left\{(-2t, -1) : t \in [0, 1]\right\} = \text{cone}\left\{(0, -1), (-2, -1)\right\}$$

is the negative polar of any of the cones considered in Lemma 3.

Lemma 5 *Given $\overline{x} \in X$,*

$$A(\overline{x})^- = \{d \in \mathbb{R}^n : \varphi'(\overline{x}; d) \leq 0\}$$

and

$$\text{int } A(\overline{x})^- = \{d \in \mathbb{R}^n : \varphi'(\overline{x}; d) < 0\}$$

Proof: One has

$$
\begin{aligned}
\varphi'(\overline{x}; d) \quad &= \max_{t \in T(\overline{x})} \max_{\xi \in \partial g_t(\overline{x})} \xi^\top d \\
&= \max\left\{\xi^\top d : \xi \in \bigcup_{t \in T(\overline{x})} \partial g_t(\overline{x})\right\}.
\end{aligned}
\tag{12}
$$

From (12), we have $\varphi'(\overline{x}; d) \leq 0$ if and only if $\xi^\top d \leq 0$ for all $\xi \in \bigcup_{t \in T(\overline{x})} \partial g_t(\overline{x})$ if and only if $d \in A(\overline{x})^-$.

Similarly, by the compactness of $\bigcup_{t \in T(\overline{x})} \partial g_t(\overline{x})$, $\varphi'(\overline{x}; d) < 0$ if and only if $\xi^\top d < 0$ for all $\xi \in \bigcup_{t \in T(\overline{x})} \partial g_t(\overline{x})$ if and only if $d \in \text{int } A(\overline{x})^-$. $\quad\square$

3   Constraint qualifications

Next we introduce four constraint qualifications which are frequently encountered in the SIO literature or are inspired in classical constraint qualifications of non-linear optimization. When the constraint functions are affine, these constraint qualifications collapse to those introduced, under similar names, in [12] (for linear vector SIO). Even more, the below CQ hold for the convex system $\{g_t(x) \leq 0, t \in T\}$ if and only if the corresponding linear versions hold for the linear system in (8).

Definition 6 *We say that P satisfies the Slater constraint qualification (SCQ) if there is a Slater point x^0, i.e. $g_t(x^0) < 0$, for all $t \in T$.*

In other words, SCQ holds if and only if the marginal function φ takes a negative value at some point (observe that the marginal function of the linear system in (8) is also φ). By continuity of φ, SCQ implies that $\text{int } X \neq \emptyset$, but the converse is not true. It is worth noting that, in contrast with the other three CQs to be introduced next, SCQ is not associated with a given feasible solution.

Definition 7 *We say that the locally Farkas-Minkowski constraint qualification (LFMCQ) holds at $\overline{x} \in X$ if $A(\overline{x}) = D(X; \overline{x})^-$.*

Obviously, if LFMCQ holds at $\overline{x} \in X$, then $A(\overline{x})$ is closed. If $\overline{x} \in \text{int } X$, then $A(\overline{x}) = D(X; \overline{x})^- = \{0_n\}$. Therefore the LFMCQ should be investigated only at the boundary feasible points. Moreover, if this property holds and $\overline{x} \in \text{bd } X$, we have $A(\overline{x}) \neq \{0_n\}$, i.e. there are binding constraints at $\overline{x}$.

Lemma 8 *([13, Theorem 7.9]) SCQ implies LFMCQ at any feasible solution.*

Consequently, if P satisfies SCQ, then bd $X = \{x \in X : A(x) \neq \{0_n\}\}$. Moreover, geometric information on X in terms of the data can be obtained by combining (8) and [13, Theorem 5.9].

Definition 9 *We say that P satisfies the local Slater constraint qualification (LSCQ) at $\overline{x} \in X$ when either $T(\overline{x}) = \emptyset$ or there exists a vector $d \in \mathbb{R}^n$ satisfying*

$$d^\top \xi < 0 \text{ for all } \xi \in \bigcup_{t \in T(\overline{x})} \partial g_t(\overline{x}). \tag{13}$$

Proposition 10 *LSCQ holds at $\overline{x} \in X$ if and only if*

$$T(\overline{x}) \neq \emptyset \Rightarrow 0_n \notin \text{conv} \left(\bigcup_{t \in T(\overline{x})} \partial g_t(\overline{x}) \right).$$

Proof: We can assume that $T(\overline{x}) \neq \emptyset$. The direct statement is obvious while the converse statement is consequence of the assumptions on P. In fact, the continuity of $t \mapsto g_t(\overline{x})$ on the compact set T entails that $T(\overline{x})$ is a compact set, as well as $\bigcup_{t \in T(\overline{x})} \partial g_t(\overline{x})$ (see e.g. [18, Theorem 4.4.2]). Due to the compactness of conv $\left(\bigcup_{t \in T(\overline{x})} \partial g_t(\overline{x}) \right)$ and the separation theorem, the condition $0_n \notin \text{conv} \left(\bigcup_{t \in T(\overline{x})} \partial g_t(\overline{x}) \right)$ guarantees the fulfilment of LSCQ at $\overline{x}$. $\square$

Corollary 11 *If P satisfies LSCQ at $\overline{x} \in X$, then $A(\overline{x})$ is a pointed closed cone.*

Proof: We can assume $T(\overline{x}) \neq \emptyset$ (otherwise $A(\overline{x}) = \{0_n\}$ is closed). By Proposition 10, since conv $\left(\bigcup_{t \in T(\overline{x})} \partial g_t(\overline{x}) \right)$ is a compact convex set which does

not contain the origin, $A(\overline{x}) = \text{cone} \left(\bigcup_{t \in T(\overline{x})} \partial g_t(\overline{x}) \right)$ is a pointed closed cone.

$\square$

The following example shows that the converse statement of Corollary 11 does not hold.

Example 12 *Let $n = 2$ and $g_t(x) = (1-t)|x_1 - 1| + |x_2| - 1 + t$ for all $T = [0, 1]$. Then it is easy to see that*

$$X = \left\{ x \in \mathbb{R}^2 : g_t(x) \le 0, \ t = 0, 1 \right\} = [0, 2] \times \{0\}.$$

We have $T(0_2) = [0, 1]$, with

$$\partial g_t(0_2) = \text{conv} \ \{(t-1, -1), (t-1, 1)\} = \{t-1\} \times [-1, 1], t \in [0, 1].$$

Thus, $\bigcup_{t \in T(0_2)} \partial g_t(0_2) = [-1, 0] \times [-1, 1]$ and $A(0_2) = \text{cone} \left(\bigcup_{t \in T(0_2)} \partial g_t(0_2) \right) = \mathbb{R}_- \times \mathbb{R}$ is closed. Finally, as $\pm(0, 1) \in \partial g_0(0_2)$, LSCQ fails at 0_2.

Definition 13 *We say that P satisfies the extended Kuhn-Tucker CQ (EK-TCQ) at $\overline{x} \in X$ when*

$$\{d \in \mathbb{R}^n : \varphi'(\overline{x}; d) \le 0\} \subset A(X; \overline{x}). \tag{14}$$

As a consequence of the Ioffe and Tihkomirov's theorem on the subdifferential of the supremum function (see e.g. [32, Theorem 2.4.18] and [16, Proposition 6.3]), when $\overline{x} \in \text{bd} X$, it holds that

$$\varphi'(\overline{x}; d) = \max_{t \in T(\overline{x})} g_t'(\overline{x}; d). \tag{15}$$

The next lemma provides a useful approximations of the tangent cones to X at $\overline{x} \in \text{bd} X$ in terms of the directional derivative function $\varphi'(\overline{x}; \cdot)$.

Lemma 14 *Let $\overline{x} \in \text{bd} X$. Then,*

$$\{d \in \mathbb{R}^n : \varphi'(\overline{x}, d) < 0\} \subset T^i(X; \overline{x}) \subset T(X; \overline{x}) \subset \{d \in \mathbb{R}^n : \varphi'(\overline{x}, d) \le 0\}.$$

Proof: We first show that $\{d \in \mathbb{R}^n : \varphi'(\overline{x}, d) < 0\} \subset T^i(X; \overline{x})$. Let $\overline{d} \in \mathbb{R}^n$ be such that $\varphi'(\overline{x}, \overline{d}) < 0$. Then, there exists $\tau > 0$ such that $\varphi(\overline{x} + s\overline{d}) < 0$ for all $s \in]0, \tau[$. By continuity of φ, there exist $t_0 > 0$ and an open neighborhood $\mathcal{N}$ of $\overline{d}$ such that $\varphi(\overline{x} + td) < 0$ for all $t \in]0, t_0[$ and all $d \in \mathcal{N}$. That is,

$$\overline{d} \in T^i(\{x \in \mathbb{R}^n : \varphi(x) < 0\}, \overline{x}) \subset T^i(\{x \in \mathbb{R}^n : \varphi(x) \le 0\}, \overline{x}) = T^i(X; \overline{x}).$$

The inclusion $T^i(X; \overline{x}) \subset T(X; \overline{x})$ is well-known (see, e.g. in [28]).

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Finally, consider $\overline{d} \in T(X; \overline{x})$. Then, there exist sequences $\{s^k\}_{k \in \mathbb{N}}$ and $\{d^k\}_{k \in \mathbb{N}}$ such that $s^k \downarrow 0$, $d^k \to \overline{d}$ as $k \to \infty$ and $\varphi(\overline{x} + s^k d^k) \leq 0$ for all $k \in \mathbb{N}$. Since $\varphi(\overline{x}) = 0$, one has

$$\frac{\varphi(\overline{x} + s^k d^k)}{s^k} = \frac{\varphi(\overline{x} + s^k d^k) - \varphi(\overline{x})}{s^k} \leq 0 \ \forall k \in \mathbb{N}.$$

Now, taking limits as $k \to \infty$, we conclude that

$$\frac{\varphi(\overline{x} + s^k d^k)}{s^k} \to \varphi'(\overline{x}; \overline{d}) \leq 0. \qquad \square$$

Theorem 15 *The following statements are true:*
(i) SCQ implies LSCQ at any $\overline{x} \in X$.
(ii) If LSCQ holds at $\overline{x} \in X$ and $T(\overline{x})$ is a set of isolated points of T, then SCQ holds.
(iii) If LSCQ holds at $\overline{x} \in X$, then LFMCQ holds at $\overline{x}$.
(iv) If LSCQ holds at $\overline{x} \in X$, then EKTCQ holds at $\overline{x}$.

Proof: (i) Let x^0 be a Slater point and $\overline{x} \in X$. Let $d := x^0 - \overline{x}$ and $\xi \in \partial g_t(\overline{x})$ for some $t \in T(\overline{x})$. Since

$$0 > g_t(x^0) \geq g_t(\overline{x}) + \xi^\top (x^0 - \overline{x}) = d^\top \xi,$$

d satisfies (13).

(ii) We shall prove that, under the assumption (equivalent to assert that $T(\overline{x})$ is finite and $T \setminus T(\overline{x})$ is compact), there exists a Slater point in the half-line emanating from $\overline{x}$ in some direction d satisfying (13). In fact, given $t \in T(\overline{x})$, as $g'_t(\overline{x}; d) = \max_{\xi \in \partial g_t(\overline{x})} \xi^\top d < 0$, there exists $\varepsilon_t > 0$ with $g_t(\overline{x} + \lambda d) < 0$ for any $\lambda \in]0, \varepsilon_t[$. On the other hand, by continuity of the function $\max_{t \in T \setminus T(\overline{x})} g_t$, there exists a neighborhood $\mathcal{N}$ of $\overline{x}$ where $\max_{t \in T \setminus T(\overline{x})} g_t$ is negative. Taking a sufficiently small $\lambda_0 > 0$, we get $g_t(\overline{x} + \lambda_0 d) < 0$ for all $t \in T(\overline{x})$ and $\overline{x} + \lambda_0 d \in \mathcal{N}$. So, $\overline{x} + \lambda_0 d$ is a Slater point.

(iii) By a well-known result (see, e.g., [24, Proposition 5]), if $\varphi'(\overline{x}; d) < 0$, then $d \in D(X; \overline{x})$. This, combined with Lemma 5, yields

$$\text{int } A(\overline{x})^- \subset D(X; \overline{x}), \tag{16}$$

where $\text{int } A(\overline{x})^- \neq \emptyset$ by Corollary 11, as the negative polar of a pointed closed convex cone contains interior points. Taking negative polars in both members of (16) one gets, by the Farkas lemma for cones,

$$D(X; \overline{x})^- \subset \left(\text{int } A(\overline{x})^-\right)^- = \left(A(\overline{x})^-\right)^- = \text{cl } A(\overline{x}) = A(\overline{x}).$$

We now prove the reverse inclusion by contradiction. Suppose that there exists $\xi \in A(\overline{x}) \setminus D(X; \overline{x})^-$. Then there exists $d \in D(X; \overline{x})$ such that $\xi^\top d > 0$,

with $\xi = \sum_{i=1}^{m} \xi_i$, $\xi_i \in \partial g_{t_i}(\overline{x})$, $t_i \in T(\overline{x})$, $i = 1, ..., m$. Let $i_0 \in \{1, ..., m\}$ such that $\xi_{i_0}^{\top} d > 0$. This means that for any $\varepsilon > 0$, we have

$$g_{t_{i_0}}(\overline{x} + \varepsilon d) = g_{t_{i_0}}(\overline{x} + \varepsilon d) - g_{t_{i_0}}(\overline{x}) \geq \varepsilon \xi_{i_0}^{\top} d > 0,$$

so that $d \notin D(X; \overline{x})$ (contradiction). Thus, $A(\overline{x}) = D(X; \overline{x})^{-}$.

(iv) Let $\overline{x} \in \operatorname{int} X$. If $\overline{x} \in \operatorname{int} X$, then $A(X; \overline{x}) = \mathbb{R}^n$ and (14) holds trivially. Thus we can assume without loss of generality (w.l.o.g. in short) that $\overline{x} \in \operatorname{bd} X$.

Let $d \in \mathbb{R}^n$ satisfy (13). By (15), we have, from (12),

$$\varphi'(\overline{x}; d) = \max \left\{ \xi^{\top} d : \xi \in \bigcup_{t \in T(\overline{x})} \partial g_t(\overline{x}) \right\} < 0.$$

Since $\{d \in \mathbb{R}^n : \varphi'(\overline{x}; d) < 0\} \neq \emptyset$ and $\varphi'(\overline{x}; \cdot)$ is a finite-valued convex function ([27, Theorem 23.4]), we get

$$\operatorname{cl}\{d \in \mathbb{R}^n : \varphi'(\overline{x}; d) < 0\} = \{d \in \mathbb{R}^n : \varphi'(\overline{x}; d) \leq 0\}.$$

Then, by Lemma 14 and Lemma 3,

$$\{d \in \mathbb{R}^n : \varphi'(\overline{x}; d) \leq 0\} = T(X; \overline{x}) = A(X; \overline{x}),$$

and so EKTCQ holds at $\overline{x}$. $\qquad\square$

Observe that the assumptions on $T(\overline{x})$ and $T \setminus T(\overline{x})$ in Theorem 15(ii) are not superfluous (see [12, Example 4]) and imply the non-connectedness of T. In the particular case that T is finite, SCQ and LSCQ are equivalent. Notice also that Lemma 8 follows straighforwardly from statements (i) and (iii) of Theorem 15.

The next example shows that LFMCQ does not imply LSCQ (consequently does not imply SCQ).

Example 16 *Let $n = 2$ and $g_t(x) = \|x\| - t$ for all $t \in T = [0, 1]$. It follows that $X = \{0_2\}$ and $T(0_2) = \{0\}$. We have $D(X, 0_2)^{+} = \mathbb{R}^2$, $\partial g_0(0_2) = \operatorname{cl} B(0_2, 1)$, $A(\overline{x}) = \operatorname{cone} \partial g_0(0_2) = \mathbb{R}^2 = D(X, 0_2)^{-}$ but (13) fails. So, LFMCQ holds at 0_2 while LSCQ fails.*

Example 17 *([12, Example 25]) Consider the linear vector SIO problem*

$$P: \quad "\min" \quad f(x) = (x_1 - x_2, -x_2)$$
$$s.t. \quad tx_1 + \left(\sqrt{1 - (t-1)^2}\right) x_2 \leq 0, \ t \in [0, 2],$$

whose feasible set is $X = \mathbb{R}^2_{-}$. It follows that EKTCQ holds at 0_2 (see [12, Example 25]) and $A(0_2) = (\mathbb{R}_{++} \times \mathbb{R}_{+}) \cup \{0_2\}$ is non-closed. Thus, EKTCQ does not imply LFMCQ.

The following example shows that the converse statement of Theorem 15(iv) does not hold.

Example 18 *Consider Example 12, for which LSCQ is not fulfilled at 0_2. We have*

$$
\begin{aligned}
\varphi'(0_2; d) \leq 0 \quad &\Leftrightarrow g_t'(0_2, d) \leq 0 \ \forall t \in [0, 1] \\
&\Leftrightarrow (t-1)d_1 + d_2 \leq 0 \ \text{and} \ (t-1)d_1 - d_2 \leq 0 \ \forall t \in [0, 1] \\
&\Leftrightarrow d_1 \geq 0, d_2 = 0.
\end{aligned}
$$

Let $d \in \mathbb{R}^2$ be such that $\varphi'(0_2; d) \leq 0$. Then the vector function $h(s) := (sd_1, sd_2) = (sd_1, 0) \in X$ for all $s \in [0, 2/d_1]$, satisfies $h(0) = 0_2$ and $h'(0) = d$. That is, $d \in A(X; 0_2)$. So, EKTCQ holds at 0_2.

Lemma 19 *Let $\overline{x} \in \operatorname{bd} X$. Then, P satisfies EKTCQ at $\overline{x}$ if and only if*

$$
T(X; \overline{x}) = \{d \in \mathbb{R}^n : \varphi'(\overline{x}, d) \leq 0\}. \tag{17}
$$

Proof: Assume that P satisfies EKTCQ at $\overline{x}$, i.e.,

$$
\{d \in \mathbb{R}^n : \varphi'(\overline{x}, d) \leq 0\} \subset A(X; \overline{x}) = T(X; \overline{x}). \tag{18}
$$

By (18) and Lemma 14, (17) holds. The converse statement is trivial. $\qquad \square$

The four constraint qualifications introduced above fail in the next example.

Example 20 *Consider the following set*

$$
X = \left\{ x \in \mathbb{R}^2 : g_t(x_1, x_2) = tx_1^2 - t(1-t) + (1-t)\left(x_2^2 + x_2\right) \leq 0, \ t \in [0, 1] \right\}.
$$

Since

$$
g_0(x_1, x_2) = x_2^2 + x_2 \leq 0 \Rightarrow x_2 \in [-1, 0]
$$

and

$$
g_1(x_1, x_2) = x_1^2 \leq 0 \Rightarrow x_1 = 0,
$$

one gets $X = \{0\} \times [-1, 0]$. As $T(0_2) = \{0, 1\}$, $0_2 \in \bigcup\limits_{t \in T(0_2)} \partial g_t(0_2) = \{(0, 0), (0, 1)\}$ and so LSCQ fails at 0_2. We also have $A(0_2) = \operatorname{cone}\{(0, 1)\}$ and $D(X, 0_2) = \operatorname{cone}\{(0, -1)\}$, so that $A(0_2) \subsetneq -D(X, 0_2)^+$, which implies the failure of LFMCQ. Since $g_0'(0_2; d) = d_2$ and $g_1'(0_2; d) = 0$, $\{d \in \mathbb{R}^2 : \varphi'(0_2; d) \leq 0\} = \mathbb{R} \times \mathbb{R}_-$. On the other hand $A(X; 0_2) = \{0\} \times \mathbb{R}_-$. So, EKTCQ fails to hold at 0_2.

4   Cone conditions

It is well-known (see, e.g. [9, Theorem 3.21 and Corollary 3.23]) that, if T is finite, f_i, $i = 1, \ldots, p$ and g_t, $t \in T$, are convex differentiable functions, $\overline{x} \in X$, and $0_n \notin \operatorname{conv}\{\nabla f_1(\overline{x}), \ldots, \nabla f_p(\overline{x})\}$, then $\overline{x}$ is a weakly efficient solution of the (ordinary) convex vector optimization problem P if and only if there exist

$t^j \in T(\overline{x})$, $j = 1, \ldots, q$, as well as non-negative scalars $\lambda_1, \ldots, \lambda_p, \mu_1, \ldots, \mu_q$ satisfying

$$\sum_{i=1}^{p} \lambda_i \nabla f_i(\overline{x}) = \sum_{j=1}^{q} \mu_j \nabla g_{t^j}(\overline{x}) \neq 0_n. \tag{19}$$

In geometric terms, the KKT condition (19) asserts that $G(\overline{x}) \cap A(\overline{x}) \neq \{0_n\}$ (it is sufficient to take $q \leq n$ by Carathéodory's theorem applied to the convex cone $A(\overline{x})$). In this section we give similar conditions for convex vector SIO problems and different types of efficiency.

The characterizations of efficient and weakly efficient solutions in this section extend similar results on linear vector SIO in [12] to convex vector SIO. We start with two sufficient conditions for $\overline{x} \in X$ to be efficient and weakly efficient solution independently of the constraints under assumptions which already appeared in Lemma 2.

Proposition 21 *If $\overline{x} \in X$, then the following statements are true:*
(i) If $0_n \in \text{conv}\left(\bigcup\limits_{i=1}^{p} \partial f_i(\overline{x})\right)$, then $\overline{x} \in X_{wE}$.

(ii) if $0_n \in \bigcup\limits_{i=1}^{p} \partial f_i(\overline{x})$ and the components of f are strictly convex, then $\overline{x} \in X_E$.

Proof: (i) Let $0_n = \sum\limits_{i=1}^{p} \alpha_i \xi_i$, with $\alpha_i \geq 0$ and $\xi_i \in \partial f_i(\overline{x})$, $i = 1, \ldots, p$, and $\sum\limits_{i=1}^{p} \alpha_i = 1$. Let $\alpha_j > 0$, $j \in \{1, \ldots, p\}$. Then, $\xi_j = -\sum\limits_{i \neq j} \frac{\alpha_i}{\alpha_j} \xi_i$. We can assume w.l.o.g. the existence of $i \neq j$ such that $\alpha_i > 0$ (otherwise, $\xi_j = 0_n$ and $\overline{x} \in X_{wE}$ because it is a minimizer of f_j on $\mathbb{R}^n$, and so on X). If we suppose that there exists $\widehat{x} \in X$ such that $f(\widehat{x}) < f(\overline{x})$, for every $i = 1, \ldots, p$ and $\xi_i \in \partial f_i(\overline{x})$, then

$$0 > f_i(\widehat{x}) - f_i(\overline{x}) \geq \xi_i^\top (\widehat{x} - \overline{x}).$$

Thus,

$$0 > \xi_j^\top (\widehat{x} - \overline{x}) = -\sum_{i \neq j} \frac{\alpha_i}{\alpha_j} \xi_i^\top (\widehat{x} - \overline{x}) > 0,$$

which is a contradiction whereby $\overline{x} \in X_{wE}$.

(ii) If $0_n \in \bigcup\limits_{i=1}^{p} \partial f_i(\overline{x})$ and all the objective functions are strictly convex, then $\overline{x}$ is the unique minimizer of at least one of the objective functions on $\mathbb{R}^n$ (and so on X). Then, there is no $\widehat{x} \in X$ such that $f(\widehat{x}) \leq f(\overline{x})$, i.e., $\overline{x} \in X_E$. $\qquad\square$

Theorem 22 *Let $\overline{x} \in X$ be such that $0_n \notin \bigcup\limits_{i=1}^{p} \partial f_i(\overline{x})$. Then, the following statements hold:*
(i) $\overline{x} \in X_E$ if and only if

$$D(X; \overline{x}) \cap G(\overline{x})^- \subset \{d \in \mathbb{R}^n : f_i'(\overline{x}; d) = 0, i = 1, \ldots, p\}.$$

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(ii) $\overline{x} \in X_{wE}$ *if and only if* $D(X; \overline{x}) \cap \operatorname{int} G(\overline{x})^{-} = \emptyset$.

Proof: (i) Denote $F(\overline{x}) := \{d \in \mathbb{R}^n : f_i'(\overline{x}; d) = 0, i = 1, \ldots, p\}$.

Suppose that $\overline{x} \in X_E$. Let $d \in D(X; \overline{x}) \cap G(\overline{x})^{-}$. Then, by Lemma 2(ii),

$$f_i'(\overline{x}; d) \leq 0, \text{ for all } i = 1, \ldots, p. \tag{20}$$

Since $d \in D(X; \overline{x})$ and $\overline{x} \in X_E$, if there exists $i_0 \in \{1, \ldots, p\}$ such that $f_{i_0}'(\overline{x}; d) < 0$, d would be a feasible descent direction of f_{i_0}' at $\overline{x}$ and there would exist $j_0 \in \{1, \ldots, p\}$, $j_0 \neq i_0$, such that d is a feasible ascent direction of f_{j_0} at $\overline{x}$. Then, by Lemma 1, $f_{j_0}'(\overline{x}; d) > 0$, which contradicts (20). Thus $f_i'(\overline{x}; d) = 0$, for all $i = 1, \ldots, p$, i.e., $d \in F(\overline{x})$.

Now assume that $\overline{x} \notin X_E$. Then, there exist $d \in D(X; \overline{x})$, $i_0 \in \{1, \ldots, p\}$, and $\beta > 0$ such that

$$f_i(\overline{x} + \varepsilon d) \leq f_i(\overline{x}), i = 1, \ldots, p, \tag{21}$$

with

$$f_{i_0}(\overline{x} + \varepsilon d) < f_{i_0}(\overline{x}) \tag{22}$$

for all $\varepsilon \in]0, \beta[$. From (21) it follows that $f_i'(\overline{x}; d) \leq 0$, $i = 1, \ldots, p$. Thus, again by Lemma 2(ii), $d \in D(X; \overline{x}) \cap G(\overline{x})^{-}$. But, from (22) we get $\varepsilon \xi_{i_0}^{\top} d \leq f_{i_0}(\overline{x} + \varepsilon d) - f_{i_0}(\overline{x}) < 0$ for all $\xi_{i_0} \in \partial f_{i_0}(\overline{x})$. That is, $f_{i_0}'(\overline{x}; d) = \max_{\xi_{i_0} \in \partial f_{i_0}(\overline{x})} \xi_{i_0}^{\top} d < 0$, so that $d \notin F(\overline{x})$.

(ii) If $0_n \in \operatorname{conv}\left(\bigcup_{i=1}^{p} \partial f_i(\overline{x})\right)$, then $\overline{x} \in X_{wE}$ by Proposition 21. So, we can assume that $0_n \notin \operatorname{conv}\left(\bigcup_{i=1}^{p} \partial f_i(\overline{x})\right)$, in which case $0_n \notin \bigcup_{i=1}^{p} \partial f_i(\overline{x})$. Let $d \in D(X; \overline{x}) \cap \operatorname{int} G(\overline{x})^{-}$. Then, again by Lemma 2(iii), $f_i'(\overline{x}; d) < 0$, $i = 1, \ldots, p$. That is, d is a feasible descent direction for f_i at $\overline{x}$, $i = 1, \ldots, p$. So, $\overline{x} \notin X_{wE}$.

Now, suppose that $\overline{x} \notin X_{wE}$, i.e. there exists $x \in X$ such that $f_i(x) < f_i(\overline{x})$, $i = 1, \ldots, p$. Let $d = x - \overline{x} \in D(X; \overline{x})$. Since d is a feasible descent direction for each f_i at $\overline{x}$, it follows that $f_i'(\overline{x}; d) = \max_{\xi_i \in \partial f_i(\overline{x})} \xi_i^{\top} d < 0$, again by Lemma 1. Thus, $d \in \operatorname{int} G(\overline{x})^{-}$ by Lemma 2(iii). Thus, $D(X; \overline{x}) \cap \operatorname{int} G(\overline{x})^{-} \neq \emptyset$, which completes the proof. $\square$

The following example shows that the assumption $0_n \notin \bigcup_{i=1}^{p} \partial f_i(\overline{x})$ in Theorem 22 is not superfluous.

Example 23 *Let $n = 2$, $p = 2$, $f_1(x) = x_1^2 + x_2^2$, $f_2(x) = x_1^2 - 2x_1 + x_2^2$, and $g_t(x) = x t_1^2 - x_2 + t - 1$ for all $t \in [0, 1]$. As g_t grows with t,*

$$X = \{x \in \mathbb{R}^2 : g_1(x) \leq 0\} = \{x \in \mathbb{R}^2 : x_2 \geq x_1^2\}.$$

Since f_1 and f_2 are strictly convex and $0_2 = \nabla f_1(0_2) \in \bigcup\limits_{i=1,2} \partial f_i(0_2)$, we have $0_2 \in X_E$. Now, take $d = (1,1)$. We have $D(X; 0_2) = \mathbb{R} \times \mathbb{R}_{++} \cup \{0_2\}$ and $G(0_2)^- = \mathbb{R} \times \mathbb{R}_+$, so that $d \in D(X; 0_2) \cap G(0_2)^-$, but $d \notin F(0_2)$ because $f_2'(0_2; d) = \nabla f_2(0_2)^\top d = -2 \neq 0$.

Corollary 24 *Given $\overline{x} \in X$, $\overline{x} \in X_{wE}$ if and only if*

$$\{d \in \mathbb{R}^n : f_i'(\overline{x}; d) < 0, \ i = 1, \dots, p\} \cap D(X; \overline{x}) = \emptyset. \tag{23}$$

Proof: If $0_n \in \bigcup\limits_{i=1}^{p} \partial f_i(\overline{x}; d)$, then $\overline{x} \in X_{wE}$ by Proposition 21 and there exists $i \in \{1, \dots, p\}$ such that $f_i'(\overline{x}; d) \geq 0$, so that (23) holds too. Otherwise, both statements are equivalent by Theorem 22(ii) and Lemma 2(iii). $\quad\square$

For the sake of brevity, given $\lambda = (\lambda_1, \dots, \lambda_p) \in \mathbb{R}^p$, we denote $\sum\limits_{i=1}^{p} \lambda_i f_i$ in matrix form as $\lambda^\top f$.

Theorem 25 *Given $\overline{x} \in X$, the following statements hold:*
(i) $\overline{x} \in X_{pE}$ if and only if $\partial\left(\lambda^\top f\right)(\overline{x}) \cap D(X; \overline{x})^+ \neq \emptyset$ for some $\lambda > 0_p$.
(ii) $\overline{x} \in X_{wE}$ if and only if $\partial\left(\lambda^\top f\right)(\overline{x}) \cap D(X; \overline{x})^+ \neq \emptyset$ for some $\lambda \geq 0_p$.

Proof: We associate with P the *parameterized (weighted) problem*

$$P(\lambda) : \min \ \left(\lambda^\top f\right)(x) = \sum\limits_{i=1}^{p} \lambda_i f_i(x) \ \text{ s.t. } x \in X, \tag{24}$$

where $\lambda \geq 0_p$ is the *weight vector* and $P(\lambda)$ is a convex SIO problem for each λ (we could aggregate $\sum\limits_{i=1}^{p} \lambda_i = 1$). By Theorem 27.4 of [27] it follows that $\overline{x}$ is an optimal solution of $P(\lambda)$ for some $\lambda \geq 0_p$ if and only if there exists $\xi \in \partial\left(\lambda^\top f\right)(\overline{x}) \cap D(X; \overline{x})^+$.

(i) According to the Geoffrion Theorem ([11], see also [9, Theorem 3.15]), $\overline{x}$ is an optimal solution of $P(\lambda)$ for some $\lambda > 0_p$ if and only if $\overline{x} \in X_{pE}$.

(ii) Similarly, by [9, Proposition 3.10], $\overline{x}$ is an optimal solution of $P(\lambda)$ for some $\lambda \geq 0_p$ if and only if $\overline{x} \in X_{wE}$. $\quad\square$

Observe that, given $\overline{x} \in X$, $0_n \in \text{conv}\left(\bigcup\limits_{i=1}^{p} \partial f_i(\overline{x})\right)$ entails that X_{wE}. We consider now the case where $0_n \notin \text{conv}\left(\bigcup\limits_{i=1}^{p} \partial f_i(\overline{x})\right)$.

Corollary 26 *Let $\overline{x} \in X$ be such that $0_n \notin \text{conv}\left(\bigcup\limits_{i=1}^{p} \partial f_i(\overline{x})\right)$. Then $\overline{x} \in X_{wE}$ if and only if $G(\overline{x}) \cap D(X, \overline{x})^+ \neq \{0_n\}$, in which case $\overline{x} \in \text{bd } X$.*

Proof: Let $0_n \notin \operatorname{conv}\left(\bigcup_{i=1}^{p} \partial f_i(\overline{x})\right)$. Then, $G(\overline{x}) \cap D(X,\overline{x})^+ \neq \{0_n\}$ if and only

if there exist $\lambda \geq 0_p$, $\xi_i \in \partial f_i(\overline{x})$, $i = 1, \ldots, p$ such that $\sum_{i=1}^{p} \lambda_i \xi_i \in D(X; \overline{x})^+$ if

and only if

$$\sum_{i=1}^{p} \lambda_i \partial f_i(\overline{x}) \cap D(X; \overline{x})^+ = \partial\left(\lambda^\top f(\overline{x})\right) \cap D(X; \overline{x})^+ \neq \{0_n\}$$

for some $\lambda \geq 0_p$. The first statement follows from Theorem 25(ii).

Finally, if $\overline{x} \in \operatorname{int} X$, then $D(X; \overline{x})^+ = \{0_n\}$, so that $G(\overline{x}) \cap D(X,\overline{x})^+ = \{0_n\}$. Hence $\overline{x} \notin X_{wE}$ by the first statement. $\square$

5 KKT conditions

We are in a position to obtain KKT optimality conditions.

Theorem 27 *Given $\overline{x} \in X$, the following statements hold:*
(i) If there exists $\lambda \geq 0_p$ ($\lambda > 0_p$) satisfying

$$(KKT) \qquad \partial\left(\lambda^\top f\right)(\overline{x}) \cap (-A(\overline{x})) \neq \emptyset,$$

then $\overline{x} \in X_{wE}$ ($\overline{x} \in X_{pE}$, respectively).
(ii) If $\overline{x} \in X_{wE}$ ($\overline{x} \in X_{pE}$) satisfies LFMCQ, then there exist $\lambda \geq 0_p$ ($\lambda > 0_p$, respectively) such that (KKT) holds. If, additionally, $0_n \notin \operatorname{conv}\left(\bigcup_{i=1}^{p} \partial f_i(\overline{x})\right)$, then the following stronger condition holds:

$$\partial\left(\lambda^\top f\right)(\overline{x}) \cap (-A(\overline{x})) \neq \{0_n\}.$$

Proof: (i) Recall that $D(X; \overline{x}) \subset A(\overline{x})^-$. Taking positive polars we get

$$D(X; \overline{x})^+ \supset -A(\overline{x})^{++} = -\operatorname{cl} A(\overline{x}) \supset -A(\overline{x}),$$

so that (KKT) implies that $\partial\left(\lambda^\top f\right)(\overline{x}) \cap D(X; \overline{x})^+ \neq \emptyset$. The conclusion follows from Theorem 25.
(ii) We are assuming that $A(\overline{x}) = D(X; \overline{x})^- = -D(X; \overline{x})^+$. The first part is straightforward consequence of Theorem 25 while the second one follows from the argument of Corollary 26. $\square$

Lemma 28 *If $\overline{x} \in X_{wE}$, then*

$$\{d \in \mathbb{R}^n : f_i'(\overline{x}; d) < 0, \ i = 1, \ldots, p\} \cap T(X; \overline{x}) = \emptyset. \tag{25}$$

Proof: Assume the contrary, that is, there exists $\overline{d} \in T(X; \overline{x})$ satisfying

$$f_i'(\overline{x}; \overline{d}) < 0, \ i = 1, \ldots, p. \tag{26}$$

By $\overline{d} \in T(X; \overline{x})$, there exist sequences $\{s^k\}_{k \in \mathbb{N}}$ and $\{d^k\}_{k \in \mathbb{N}}$ such that $s^k \downarrow 0$, $d^k \to \overline{d}$ and $\overline{x} + s^k d^k \in X$ for all $k \in \mathbb{N}$. Since $\overline{x} \in X_{wE}$ there exists (perhaps after passing to a subsequence) an index $i_0 \in \{1, \ldots, p\}$ such that

$$f_{i_0}(\overline{x} + s^k d^k) \geq f_{i_0}(\overline{x}), \ k \in \mathbb{N}.$$

Since f_{i_0} is directional differentiable at $\overline{x}$ in the Hadamard sense (see e.g. [1, Proposition 2.126(v)(c)]), the latter inequalities provide

$$f_{i_0}'(\overline{x}; \overline{d}) = \lim_{k \to \infty} \frac{f_{i_0}(\overline{x} + s^k d^k) - f_{i_0}(\overline{x})}{s^k} \geq 0$$

which contradicts (26). $\qquad \square$

Observe that, since $D(X; \overline{x}) \subset T(X; \overline{x})$, the direct part of Corollary 24 is immediate consequence of Lemma 28.

We have shown in Theorem 27 that (KKT) is a necessary condition for weak efficiency under LFMCQ (and, by Lemma 8, also under SCQ). Finally, we prove that this necessary condition still holds under the remaining two CQ introduced in Section 3, namely, LSCQ and EKTCQ, together with the closedness of the active cone recall that LSCQ entails the latter property according to Corollary 11).

Theorem 29 *Let $0_n \notin \mathrm{conv} \left(\bigcup_{i=1}^p \partial f_i(\overline{x}) \right)$ and $\overline{x} \in X_{wE}$ satisfying one of the following conditions:*
(i) LSCQ;
(ii) EKTCQ and $A(\overline{x})$ is closed.
Then, there exists $\lambda \geq 0_p$ satisfying (KKT).

Proof: Since $\overline{x} \in X_{wE}$ and $0_n \notin \mathrm{conv} \left(\bigcup_{i=1}^p \partial f_i(\overline{x}) \right)$, $\overline{x} \in \mathrm{bd}\, X$ and

$$\left\{ d \in \mathbb{R}^n : \sum_{i=1}^p \lambda_i f_i'(\overline{x}; d) < 0, \text{ for all } \lambda \geq 0_p \right\} \cap T(X; \overline{x}) = \emptyset \tag{27}$$

by Corollary 26 and Lemma 28, respectively. Combining the formulas (15) and (27) with Theorem 15(iv) and Lemma 19, we conclude that there is no $d \in \mathbb{R}^n$ such that

$$\sum_{i=1}^p \lambda_i f_i'(\overline{x}; d) < 0 \text{ for all } \lambda \geq 0_p \tag{28}$$

and

$$g_t'(\overline{x}; d) \leq 0 \text{ for all } t \in T(\overline{x}). \tag{29}$$

(i) Assume that LSCQ is satisfied at $\overline{x}$. Now, (28)-(29) is equivalent to: there is no $d \in \mathbb{R}^n$ such that

$$\xi^\top d < 0 \text{ for all } \xi \in \partial\left(\lambda^\top f\right)(\overline{x}) \text{ for all } \lambda \geq 0_p. \tag{30}$$

and

$$\xi^\top d \leq 0 \text{ for all } \xi \in \partial g_t(\overline{x}) \text{ for all } t \in T(\overline{x}) \tag{31}$$

Since the homogeneous linear system formed by (31)-(30) is inconsistent, $\partial\left(\lambda^\top f\right)(\overline{x})$ is a compact convex set and $A(\overline{x}) := \mathrm{cone}\left(\bigcup\limits_{t \in T(\overline{x})} \partial g_t(\overline{x})\right)$ is closed (by Corollary 11), so that the Minkowski sum of both sets is closed, we can apply Motzkin's Theorem [13, Theorem 3.5] to conclude that

$$0_n \in \partial\left(\lambda^\top f\right)(\overline{x}) + A(\overline{x}).$$

(ii) The proof is the same, taking into account that now $A(\overline{x})$ is closed by assumption. $\square$

Example 17 shows that the closedness assumption in Theorem 29(ii) is not superfluous. Indeed, the unique solution of the system formed by the non-linear equations

$$\lambda_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \sum_{i=1}^{2} \mu_i \begin{pmatrix} -t^i \\ -\sqrt{1 - (t^i - 1)^2} \end{pmatrix}, t^1, t^2 \in [0, 2],$$

and the inequalities

$$\lambda_i \geq 0, \ \mu_i \geq 0, \ i = 1, 2, \tag{32}$$

is $\lambda_1 = \lambda_2 = \mu_1 = \mu_2 = 0$. Thus, (KKT) fails.

Example 30 *Consider the robust counterpart problem in (2) with* $\mathcal{U}_i = B(\overline{c}_i, \varepsilon) \subset \mathbb{R}^n$, $i = 1, ..., p$. *Then,* $f_i(x) = \max_{c_i \in \mathcal{U}_i} c_i^\top x = \overline{c}_i^\top x + \varepsilon \|x\|$, *with*

$$\partial f_i(x) = \begin{cases} \left\{\overline{c}_i + \varepsilon \frac{x}{\|x\|}\right\}, & \text{if } x \neq 0_n, \\ B(\overline{c}_i, \varepsilon), & \text{if } x = 0_n. \end{cases}$$

Let $X = \left\{x \in \mathbb{R}^n : p_k^\top x \geq q_k, k \in K\right\}$ *be the feasible set of (2) and let* $\overline{x} \in X$. *According to Theorem 15, Proposition 21, and Theorems 27 and 29, the following statements hold:*

(i) *If* $0_n \in \mathrm{conv}\left(\bigcup\limits_{i=1}^{p} \partial f_i(\overline{x})\right)$, *then* $\overline{x}$ *is a minmax robust weakly efficient solution.*

(ii) *Assume that* $0_n \notin \mathrm{conv}\left(\bigcup\limits_{i=1}^{p} \partial f_i(\overline{x})\right)$ *and either LSCQ or EKTCQ holds at* $\overline{x} \neq 0_n$. *Then,* $\overline{x}$ *is a weakly efficient solution of (2) if and only if there exists* $\lambda \geq 0_m$ *such that* $-\sum\limits_{i=1}^{p} \lambda_i \left(\overline{c}_i + \varepsilon \frac{\overline{x}}{\|\overline{x}\|}\right) \in A(\overline{x})$.

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