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Station choice for Australian commuter rail lines: equilibrium and optimal fare design

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Abstract

We examine how park-and-ride commuters living along a rail line compete for seats when they travel to their workplace in Australian metropolitan areas. First, we prove that at user equilibrium in which each commuter minimizes her expected travel cost, there exists one station on the rail line at which some commuters could find a seat and the others have to stand; all of the commuters boarding at its upstream stations have seats and all of the commuters boarding at its downstream stations must stand in the train. We derive a solution algorithm for obtaining a user equilibrium, which involves solving an equation with only one variable. We demonstrate that more than one user equilibrium may exist. Second, we examine the system optimal station choice that assumes all of the commuters cooperate and minimizes their total travel cost. An analytical solution approach is proposed based on the structure of the problem. Third, we investigate the optimal train fare design that leads to the system optimal station choice. We prove that the optimal train fare satisfies: there exists a particular train station that has some seats and the train is full after this station. All of its upstream stations have the same fare, which is higher than or equal to the fare of this particular station; and all of its downstream stations have the same fare, which is lower than the fare of this particular station.

Key Words: Transportation; Rail; User equilibrium; Park-and-Ride; Seat availability

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1 Introduction

Commuter rail (suburban rail) is a passenger rail transport service that primarily operates between a city centre (central business district, or CBD) and the middle to outer suburbs. People using commuter rail services usually travel on a daily basis: from home to workplace in the morning and from workplace back home in the evening. In Australia, commuters use cars to get to train stations more than almost any others in the world (Department of Infrastructure, 2005) because of a high rate of car ownership and poor coverage of the broad urban areas by conventional public transport modes (e.g., buses and trams). Commuter rail forms a vital part of public transportation in major Australian cities. For instance, around 1 million people travel in the New South Wales commuter rail system every day, covering Sydney, New Castle, and Wollongong, amongst others; the daily ridership of the Melbourne railway system is 0.8 million (Commuter Rail in Australia, 2015).

Fig. 1 shows the South Coast Line of the New South Wales commuter rail system, where “Central” is the Central Railway Station of Sydney (Sydney CBD). People living along the south coast of New South Wales, mainly including cities and suburbs of Nowra (Bomaderry), Kiama, Albion Park, Dapto, Port Kembla, Wollongong, Thirroul, and Helensburgh, travel on this line to their workplace in Sydney CBD. The trip is long, for instance, it takes about 1 hour and 50 minutes from Nowra (Bomaderry) to Sydney CBD, and 1 hour 30 minutes from Wollongong to Sydney CBD. Therefore, the commuters have to wake up very early in order to arrive at their workplaces on time. As a result, they are still very sleepy when they get on the train and most people choose to sleep for half an hour to two in the train. This is in contrast to the evening trip from workplace back home during which most people play with their electronic devices such as iPad’s and smart phones. An important precondition for sleeping in trains in the morning is having a seat. Commuters could not sleep without a seat. Due to peak demand and limited train capacity, commuters who get on trains at downstream stations may not have a seat and hence could not sleep in trains. Some commuters therefore drive to upstream stations so that they could find a seat and then sleep during the trip. Our study hence aims to analyse how commuters choose stations to compete for seats when traveling to workplaces in the morning and the resulting implications for transport authorities.

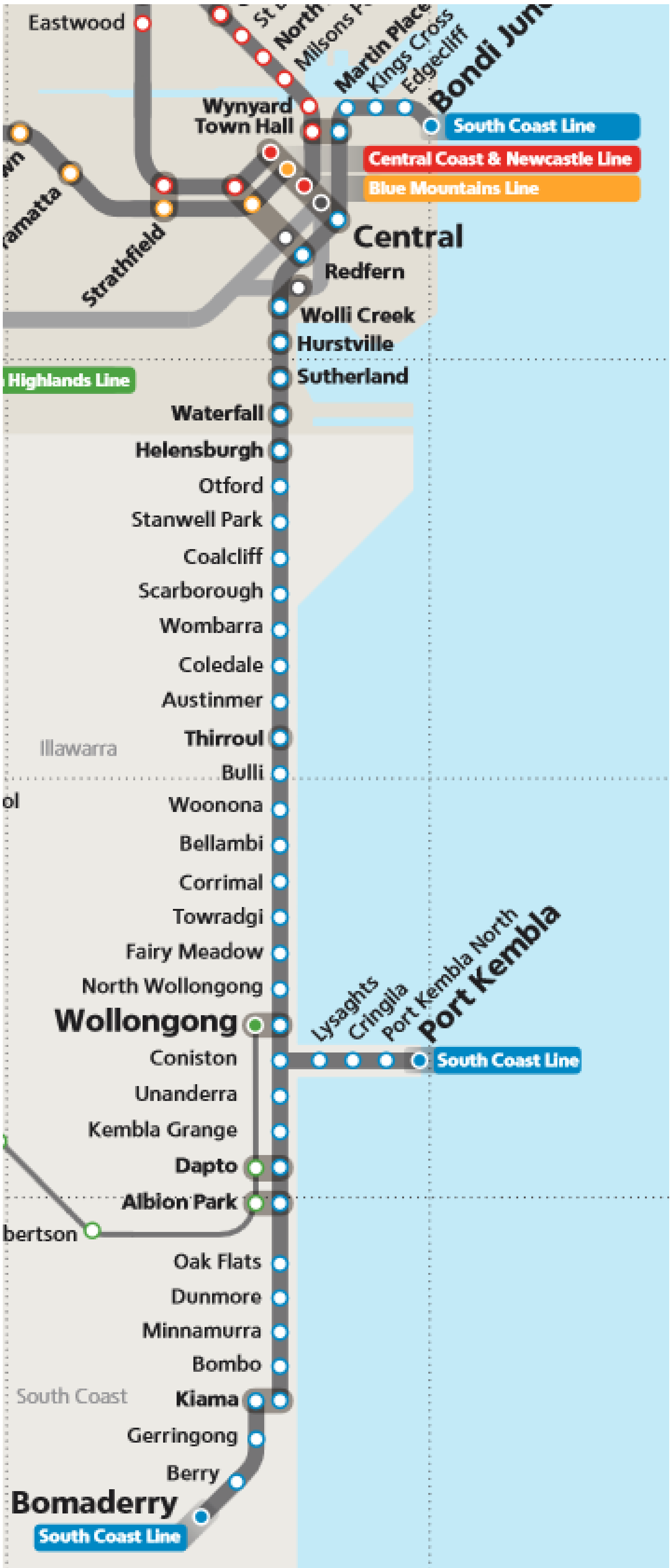

Fig. 1 South Coast Line (Source: Intercity Trains Network, 2015)

1.1 Literature review

Our study is related to the stream of works on park-and-ride (P&R). A large amount of research on P&R investigates the factors that affect the percentage of P&R commuters such as train/parking fare, availability of parking facilities at train stations and at workplaces, frequency of train services, and culture (Li et al., 2012; Duncan and Christensen, 2013; Habib et al., 2013; Mingardo, 2013; Liu and Meng, 2014). Once the factors are identified, effective policies for promoting the usage of P&R could be initiated. Some efforts are devoted to the location of P&R facilities (i.e., car parks) as well as the capacities of the facilities (Wang et al., 2004; Liu et al., 2009). Discrete choice models and transit assignment models are usually used to formulate transport mode split and commuters' route choice behaviour (Farhan and Murray, 2008; Aros-Vera et al., 2013). Our study differs from the above-mentioned works as we focus on one rail line along which all of the potential commuters park and ride as they all live far away from their workplaces. We further assume that parking slots are always available, which is the case for most train stations in remote suburbs of Australia. Moreover, in our model commuters choose stations based on the generalized cost that incorporates the availability of seats.

Another relevant category of research is transit assignment considering passenger congestion. An implicit approach is derived from road network modelling, for which strictly non-decreasing continuous disutility functions with respect to the number of commuters in trains are defined (Wong and Tong, 1999; Nuzzolo et al., 2001; Wu, et al., 2013). The main drawback of this approach is the approximation in assessing the disutility for boarding users at stops with respect to users already on board. In other words, the effect of congestion is the same for both standing passengers and sitting passengers. Another approach is to impose a strict vehicle capacity constraint and passengers are rejected if the vehicles are full (Poon et al., 2004; Hamdouch and Lawphongpanich, 2008). The third approach differentiates the discomfort level experienced by sitting and standing passengers (Tian et al., 2007; Sumalee et al., 2009; Hamdouch et al., 2011; Schmöcker et al., 2011; Leurent, 2012; Palma et al., 2015). Nevertheless, models in these studies generally consider only the public transport mode. In our research, we analyse a single line and develop analytical models to gain deeper insights into the problem. Moreover, we consider a park-and-ride system consisting of both private car mode and public transport, rather than just public transport.

1.2 Objectives and contributions

We conduct an in-depth analysis of how P&R commuters compete for seats when they travel to their workplace in the morning. We use the words "commuters" and "passengers" interchangeably. In our setting, all of the commuters live along a rail line and are homogeneous in that they have the same unit travel cost by train with a seat and the same unit travel cost by train without a seat. Our major findings are threefold. First, we prove that at user equilibrium (UE) in which each commuter minimizes her expected travel cost, there exists one station on the line at which some commuters could find a seat and the others have to stand; all of the commuters boarding at its upstream stations have seats and all of the commuters boarding at its downstream stations must stand in the train. It is possible, in extreme cases, that some stations are not used. We derive a solution algorithm for obtaining a user equilibrium, which involves solving an equation in one unknown. We demonstrate that more than one user equilibrium may exist. Second, we examine the system optimal (SO) station choice that assumes all of the commuters cooperate and minimizes their total travel cost. An analytical solution approach is proposed based on the structure of the problem. Third, we investigate the optimal train fare design that leads to the system optimal station choice. We prove that the optimal train fare satisfies: there exists a particular train station that has some seats and the train is full after this station. All of its upstream stations have the same fare, which is higher than or equal to the fare of this particular station; and all of its downstream stations have the same fare, which is lower than the fare of this particular station.

The remainder of the paper is organized as follows. Section 2 describes the problem. Section 3 develops a station choice model under user equilibrium. Section 4 investigates system optimal assignment of commuters to train stations. Section 5 examines the optimal train fare structure that achieves system optimum. Section 6 concludes. For better readability, the list of symbols used is summarized below:

Indices

k, m, n: A train station;

x, y: A passenger (defined by her location);

Parameters

a, b_1, b_2: Auxiliary parameters;

c_1: Traveling cost per unit distance by train if the passenger has a seat;

c_2: Traveling cost per unit distance by train if the passenger stands in the train;

c_3: Driving cost per unit distance (generalized cost that consists of both the direct expenses, e.g., fuel cost of cars, and the value of the driving time);

$c(x, n, p_n)$: Expected total travel cost of passenger x boarding at station L_n when the probability of having a seat is p_n;

$c^a(x, n)$: Expected total travel cost of passenger x boarding at station L_n when there are enough seats (the superscript a means "available");

$c^f(x, n)$: Expected total travel cost of passenger x boarding at station L_n when there is no seat (the superscript f means "full");

$f(x)$: Passenger density at location $x \in [0, X]$;

L_n: Distance of station $n = 1, 2 \cdots N+1$ to the origin;

M: Total number of seats in the train;

$\hat{M}$: Total number of commuters;

N: Number of train stations for commuters to board;

$N+1$: The train station at the commuters' workplace;

$n^{\min}, n^{\min}+1, \cdots, n^{\max}$: Possible stations n satisfying $p_n > 0, p_{n+1} = 0$ at UE;

$n', n'-1$: Possible stations n satisfying $p_n > 0, p_{n+1} = 0$ at SO;

$[0, X]$: Range of commuters' home locations;

Decision variables

θ_n: Train fare for traveling from station n to the workplace.

$n^*(x)$: The optimal station chosen by passenger x.

p_n: Probability that a passenger boarding at station n can find a seat;

p_n^{SO}: Probability that a passenger boarding at station n can find a seat at SO;

y_n: Critical points for choosing stations: all of the passengers $x \in (y_{n-1}, y_n)$ choose station n;

y_n^{SO}: Critical points for choosing stations at SO: all of the passengers $x \in (y_{n-1}^{SO}, y_n^{SO})$ choose station n at SO;

z_n: Number of commuters boarding at station n;

2 Problem description

Consider a commuter rail line shown in Fig. 2. We consider one train as trains have fixed schedules and a commuter chooses the train based on her working hours. In particular, among all of the trains that will make the commuter on time for work, the commuter chooses the one that departs the latest. This setting is applicable in two situations. First, if the frequency of the train is very low, e.g., one train per hour, then commuters have little choice of departure time. Hence, commuters will not depart one hour in advance to seek a seat. Second, when there are frequent train services (e.g., every 15 minutes), there is usually a busy period (say, 6:30 to 8:00) during which the trains are crowded. If a commuter's ideal departure time is in the middle of this busy period, then she should not depart early because departing early does not help her to find a seat. Only if her ideal departure time is at the beginning of this busy period should she consider departing early to guarantee a seat. Our subsequent analysis is hence applicable to the above two situations.

The commuters live along the railway line and board the train at stations $n = 1, 2 \cdots N$ (e.g., the cities and suburbs along the south coast of New South Wales shown in Fig. 1), and travel to their workplace at station $N+1$ (e.g., Sydney CBD). We define the passenger who lives the farthest to the workplace as the origin 0. The distance of station $n = 1, 2 \cdots N+1$ to the origin is L_n. Without ambiguity, we use n or L_n to refer to station n, and use x or y to refer to a passenger located at x or y.

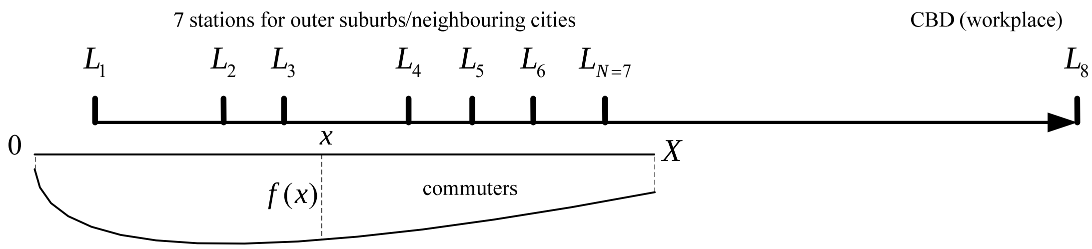

Fig. 2 An illustrative commuter rail line

The total number of seats in the train is M and the total number of passengers is $\hat{M}$, $0 < M < \hat{M}$. We assume that all of the passengers are able to get on the train (some passengers have to stand in the train). In reality, passengers may be rejected to take a full train (not enough space to stand) and have to wait for the next train to arrive. This is possible for some very busy cities (e.g., Beijing) and for some poor underdeveloped countries. In developed countries this is not the case as commuters require high-level of comfort (if some passengers are rejected, then the train must be extremely crowded). For example, our personal experience is that this almost never happens in Australia or Germany (with the exception of special events like world-class football games or Olympic games). For extremely busy subway systems in which passengers have to wait for a few trains before they can board as there are too many passengers waiting, we will have to model the station choice and departure time choice behaviour of passengers (Zhang et al., 2005, 2008; Yang et al., 2013; Liu et al., 2016).

We further assume that if there are more passengers boarding than the number of available seats at a station, all of the boarding passengers have the same chance of finding a seat. For instance, if $M = 400$, 370 passengers board the train at stations 1 to 4, and 60 passengers board the train at station 5, then each of these 60 passengers who compete for $400 - 370 = 30$ seats has a chance of 50% to find a seat.

A passenger x has to drive to a station n (n is to be determined), and then take the train to the workplace $N+1$. We define c_1 as the traveling cost per unit distance by train if the passenger has a seat, c_2 the traveling cost per unit distance by train if the passenger stands in the train, and c_3 the driving cost per unit distance, $c_1 < c_2 < c_3$. Note that here c_3 is the generalized cost that consists of both the direct expenses (e.g., fuel cost of cars) and the value of the driving time.

Our study investigates how the passengers choose the stations to board. We assume that each passenger minimizes her expected travel cost.

To simplify the notation as well as to focus on the main idea of the research, we assume that the commuters live along the railway line for a distance of X, $X \geq L_N$, and do not consider commuters living between X and L_{N+1}. Moreover, in our problem setting, there are actually more stations between X and L_{N+1}, denoted by station $N^{(1)}, N^{(2)} \cdots$, and we do not consider these stations. We do not consider these stations and commuters because commuters boarding at these stations never have a seat, provided that X satisfies two criteria: First, X should be large enough so that commuters living farther than X or even a little closer than X never have a seat, and the nearest upstream station at which passengers can find a seat is so far away such that it does not make sense for these commuters to drive to it to compete for a seat. Second, X should be much closer to station N than to the first station $N^{(1)}$, so that when neither N nor $N^{(1)}$ has a seat, commuter X will choose station N rather than $N^{(1)}$. Such an X is not difficult to identify in reality. We will see later that when stations $N^{(1)}, N^{(2)} \cdots$ never have a seat and commuters between X and L_{N+1} never have a seat, the UE station choice is the same for these commuters and at these stations as the SO station choice. Hence, they can be excluded from the model.

One may wonder, what if we cannot identify a suitable X, either because we do not have the practical knowledge about the commuter rail line or such an X does not exist. Under this circumstance, we can set $X := L_{N+1}$ and use the model proposed in the study with the following modifications: (i) we need to take into account all of the stations $1, 2 \cdots n \cdots N, N^{(1)}, N^{(2)} \cdots N+1$ rather than just $1, 2 \cdots N, N+1$; and (ii) we allow commuters to choose any of the stations including $N+1$ rather than allow commuters to board just at

$1,2\cdots N$; if a commuter chooses station $N+1$, that means the commuter drives to the workplace without taking the train.

Finally, we define the passenger density as $f(x)\geq 0$ at location $x\in[0,X]$. Therefore, the total number of passengers $\hat{M}=\int_0^X f(x)dx$.

3 Station choice model at user equilibrium

In this section we assume that the train fares for the travel between any station $n=1,2\cdots N$ to station $N+1$ are all the same and hence do not affect the choice of stations to board. This assumption will be extended in Section 3.2.3. We identify properties of the station choice problem in section 3.1, based on which we formulate the station choice model in section 3.2 and prove that more than one equilibrium may exist. Section 3.3 conducts a case study to show the key findings.

3.1 Fundamental properties

Suppose that z_n passengers board the train at station n (z_n is to be determined), then the probability that a passenger boarding at station n can find a seat is:

$$p_n=\begin{cases}1, & \text{if } \sum_{m=1}^{n} z_m \leq M \\ 0, & \text{if } \sum_{m=1}^{n-1} z_m \geq M \\ \dfrac{M-\sum_{m=1}^{n-1} z_m}{z_n}, & \text{otherwise.}\end{cases} \tag{1}$$

Hence, the probabilities p_n are functions of $(z_1,z_2\cdots z_n)$. To simplify the notation, we use p_n instead of $p_n(z_1,z_2\cdots z_n)$ in the sequel.

Proposition 1: The probability of having a seat at each station $n=1,2\cdots N$ satisfies:

$$1\geq p_1\geq p_2\geq\cdots\geq p_N\geq 0, \tag{2}$$

$$p_N < 1, \tag{3}$$

and at most at one station the probability is strictly greater than 0 and strictly less than 1. Moreover, there exists exactly one $\bar{n} \in \{1, 2 \cdots N\}$, such that $p_{\bar{n}} > 0$ and $p_{\bar{n}+1} = 0$, and such that all of the commuters who board at its upstream stations can find a seat and no commuters boarding at its downstream stations can find a seat:

$$p_k = \begin{cases} 1, k = 1, 2 \cdots \bar{n} - 1, \\ 0, k = \bar{n} + 1, \bar{n} + 2 \cdots N. \end{cases} \tag{4}$$

Proposition 1 is self-explanatory. The expected travel cost of passenger x boarding at station L_n is

$$c(x, n, p_n) = c_3 \mid x - L_n \mid + [p_n c_1 + (1 - p_n) c_2](L_{N+1} - L_n). \tag{5}$$

At equilibrium, passenger x chooses station $n^*(x)$ with the lowest expected travel cost:

$$n^*(x) \in \arg \min_{n=1,2\cdots N} c(x, n, p_n). \tag{6}$$

3.2 Station choice model

Lemma 1: If $x \le L_n$, then passenger x will not use stations $k = n+1, n+2 \cdots N$.

Proof: For $k = n+1, n+2 \cdots N$, we have

$$\begin{aligned} & c(x, n, p_n) = c_3 \mid x - L_n \mid + [p_n c_1 + (1 - p_n) c_2](L_{N+1} - L_n) \\ & = c_3 (L_n - x) + [p_n c_1 + (1 - p_n) c_2](L_{N+1} - L_n) \\ & = c_3 [L_k - x - (L_k - L_n)] + [p_n c_1 + (1 - p_n) c_2][L_{N+1} - L_k + (L_k - L_n)] \\ & = \{c_3 (L_k - x) + [p_n c_1 + (1 - p_n) c_2](L_{N+1} - L_k)\} - \{c_3 (L_k - L_n) + [p_n c_1 + (1 - p_n) c_2](L_k - L_n)\} \\ & \le \{c_3 \mid L_k - x \mid + [p_k c_1 + (1 - p_k) c_2](L_{N+1} - L_k)\} + (L_k - L_n)[p_n c_1 + (1 - p_n) c_2 - c_3] \\ & = c(x, k, p_k) + (L_k - L_n)[p_n c_1 + (1 - p_n) c_2 - c_3] \\ & < c(x, k, p_k). \end{aligned} \tag{7}$$

The "≤" holds because of Eq. (2), and the "<" holds because $c_3 > c_1$ and $c_3 > c_2$.□

Similar to Lemma 1, we have

Lemma 2: If $x \ge L_n$ and $p_n = 1$, then passenger x will not board at stations $m = 1, 2 \cdots n-1$.□

Proposition 1, Eq. (6), and Lemmas 1 and 2 imply that the choice of stations by passengers has the structure shown in Fig. 3, i.e., $n^*(x_1) \le n^*(x_2)$ if $x_1 < x_2$; in words, a passenger x_2 who lives closer to the workplace than another passenger x_1 will not choose a station farther to the workplace than the one chosen by x_1. To simplify the notation, we define y_n as the critical points for choosing stations, $n = 1, 2 \cdots N$, that is, all of the passengers $x \in [0, y_1)$ choose station 1, all of the passengers $x \in (y_1, y_2)$ choose station 2, etc. Note that we do not need to worry about the difference between (y_1, y_2) and $[y_1, y_2]$ as we consider continuously distributed passengers on the interval between 0 and X. Defining $y_0 = 0$, we have

$$0 = y_0 \le y_1 \le y_2 \le \cdots \le y_N = X . \tag{8}$$

The number of passengers boarding at station n can be expressed as

$$z_n = \int_{y_{n-1}}^{y_n} f(x)dx . \tag{9}$$

Hence, once we know y_n, all of the information, including the number of passengers boarding the train at each station and the probability that a passenger boarding at each station could find a seat, i.e., Eq. (1), can be calculated.

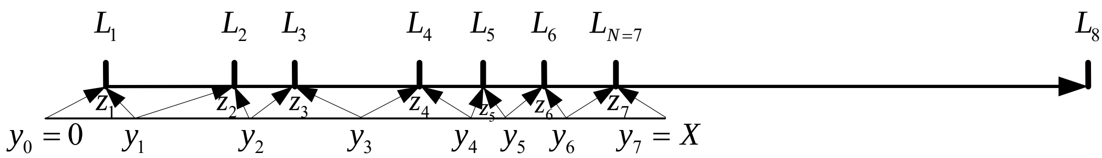

Fig. 3 Station choice by passengers

3.2.1 Cases in which some stations are not used at equilibrium

It should be noted that it is possible, at least in theory, to have $y_{n-1} = y_n$ in Eq. (8), which means station n is not used by any passenger. For instance, stations 4 and 5 are very close on the line shown in Fig. 4. The probability $p_4 \in (0,1)$, meaning that no passenger at station 5 can find a seat. Therefore, a passenger between station 4 and station 6 will either board at station 4 (to compete for a seat) or at station 6 (to reduce the driving distance). This insight has practical significance for commuter rail planners. Take the South Coast Line show in Fig. 1 as an example. Most people board the train at the stations of Wollongong and North Wollongong because the population of the city of Wollongong is much larger than the other cities/suburbs. The distance between the two stations is only 3.1 km. If the train capacity were not large enough for all of the passengers boarding at the station of North Wollongong, the station of Wollongong, and all of their upstream stations, it would be possible that all of the passengers living in the city of Wollongong would board the train at the station of Wollongong to compete for a seat and nobody would use the station of North Wollongong.

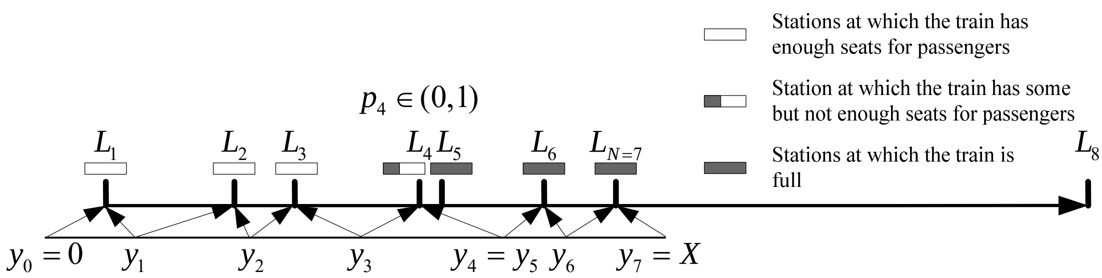

Fig. 4 A commuter rail line with station 5 not used

3.2.2 Station choice equations at equilibrium

Now we can state the formula regarding the station choice at equilibrium:

Theorem 1: Suppose that we know the station n satisfying $p_n > 0, p_{n+1} = 0$, that is, station n is the last station that has some available seats. Suppose further we know the station $k \geq n+1$ that is the first station used by passengers after n in view of the case shown in Fig. 4. In other words, stations $n+1, n+2\cdots k-1$ are not used. Then at equilibrium, p_n and $y_m, m=1,2\cdots N$ satisfy

$$y_0 = 0 \tag{10}$$

$$y_m = L_m + \frac{(c_3 - c_1)(L_{m+1} - L_m)}{2c_3}, m = 1,2,3\cdots n-2 \tag{11}$$

$$y_{n-1} = L_{n-1} + \frac{(c_3 - c_1)(L_n - L_{n-1}) + (1-p_n)(c_2 - c_1)(L_{N+1} - L_n)}{2c_3} \le L_n \tag{12}$$

$$\begin{aligned} & y_n = y_{n+1} = \cdots = y_{k-1} \\ & = L_n + \frac{\{c_3 - [p_n c_1 + (1-p_n)c_2]\}(L_k - L_n) + p_n(c_2 - c_1)(L_{N+1} - L_k)}{2c_3} \end{aligned} \tag{13}$$

$$L_{k-1} + \frac{(c_3 - c_2)(L_k - L_{k-1})}{2c_3} \le y_n = y_{n+1} = \cdots = y_{k-1} \le \min\left(y_k, L_k\right) \tag{14}$$

$$y_m = L_m + \frac{(c_3 - c_2)(L_{m+1} - L_m)}{2c_3}, m = k, k+1\cdots N-1 \tag{15}$$

$$y_N = X \tag{16}$$

$$p_n = \frac{M - \int_{y_0}^{y_{n-1}} f(x)dx}{\int_{y_{n-1}}^{y_n} f(x)dx} \tag{17}$$

$$0 < p_n \le 1 \tag{18}$$

Eqs. (11)–(13) and (15) follow from Eq. (6). Let us take Eq. (11) as an example. The definition of y_m implies that commuter y_m is indifferent between station m and station $m+1$. Since both stations have available seats, the travel cost of boarding at station m is $c(y_m, m, 1) = c_3(y_m - L_m) + c_1(L_{N+1} - L_m)$ and the travel cost of boarding at station $m+1$ is $c(y_m, m+1, 1) = c_3(y_m - L_{m+1}) + c_1(L_{N+1} - L_{m+1})$. As $c(y_m, m, 1) = c(y_m, m+1, 1)$, we have $y_m = L_m + (c_3 - c_1)(L_{m+1} - L_m)/(2c_3)$.

Eq. (11) shows that when there is always a seat at very upstream stations, the station choice depends on the relative cost of seating in a train c_1 and driving c_3. In this case, when c_3 is much larger than c_1, reducing the driving distance is the first priority and y_m lies in the middle of L_m and L_{m+1}. When c_3 is just slightly larger than c_1, hardly anyone will drive to the upstream station. This equation also shows that if the driving distance to the downstream station is shorter than that to the upstream one and both stations have enough seats, the passenger will always choose the downstream station.

Eq. (12) demonstrates that when p_n is larger, station L_n is more attractive and hence y_{n-1} is smaller. It also implies when L_{N+1} is larger, meaning that a seat is more desirable, more people will drive to the upstream station L_{n-1} as indicated by a larger y_{n-1}. Comparing Eq. (12) with Eq. (11), we see that the former has an extra term $(1-p_n)(c_2-c_1)(L_{N+1}-L_n)/(2c_3)$. Because of this term, it is possible that a passenger living closer to L_n than L_{n-1} chooses station L_{n-1} to ensure the availability of a seat. Such a scenario will not happen if both L_{n-1} and L_n always have seats, as shown in Eq. (11).

Eq. (13) implies that when p_n is larger, station L_n is more attractive; when L_{N+1} is larger, the advantage of using station L_n rather than downstream stations is more evident because a seat is more desirable for longer trips. In both cases, y_n is larger.

Eq. (14) is based on the definition of k, ensuring that stations $n+1, n+2 \cdots k-1$ will not be used. If Eq. (14) is violated, then the assumption that station $k \geq n+1$ is the first station used by passengers after n is incorrect.

Eq. (15) is similar to Eq. (13): when there is always no seat at downstream stations, the choice of station only depends on the relevant costs c_2 and c_3 and the distances to the upstream and downstream stations.

Eq. (16) comes from the problem definition. Eq. (17) is implied by Eqs. (1) and (9). Eq. (18) is based on the definition of n. If Eq. (18) is violated, then the assumption that station n is the last station that may have a seat is incorrect.

3.2.3 Extensions of the station choice model

The station choice model consisting of Eqs. (10)–(18), although based on simplifying assumptions, can be slightly revised to handle a number of practical issues.

First, if we consider the commuters who live very close to a station and hence walk to the station (note that in this case there is no station choice), we can simply add the number of commuters who walk to station n to the denominator $\int_{y_{n-1}}^{y_n} f(x)dx$ of Eq. (17), and deduct the total number of commuters who walk to stations $1,2\cdots n-1$ from the numerator of Eq. (17).

Second, if we consider that commuters may need to arrive early because of the uncertain time required for finding a parking place, then we can add a fixed extra cost term, denoted by C_n for station n, which is the buffer time multiplied by the value of time. If C_n is the same for all stations, then the model does not need to be changed. Otherwise, in Eqs. (11) and (15) y_m should increase by $(C_m - C_{m+1})/(2c_3)$ (note y_m decreases if $C_m < C_{m+1}$); in Eq. (12) y_{n-1} should increase by $(C_{n-1} - C_n)/(2c_3)$; in Eq. (13) y_n should increase by $(C_n - C_k)/(2c_3)$; and in Eq. (14) the first term should increase by $(C_{k-1} - C_k)/(2c_3)$.

Third, if the train fare is to be included, then we can add a fixed train fare θ_n for boarding at station n and revise the model similar to the above buffer time cost. If there is parking fee at a train station, or if a schedule delay cost at each station is to be included, the model can be revised accordingly.

3.2.4 Algorithm for finding the station choice at equilibrium

To find the station choice at equilibrium for a commuter rail line based on Theorem 1, we can enumerate all of the possible values of n from $n^{\min} = 1$ to $n^{\max} = \arg\min_{n=1,2\cdots N}\left\{\int_0^{L_n} f(x)dx \geq M\right\}$, then enumerate all of the possible values of k from

$n+1$ to N, then calculate p_n and $y_m, m=1,2\cdots N$ by Eqs. (11)–(13) and (15)–(17), and finally check whether Eqs. (14) and (18) hold. The difficult part is obtaining p_n, y_{n-1} and y_n based on Eqs. (12), (13) and (17). To this end, we rewrite Eqs. (12) and (13) as

$$y_{n-1} = -ap_n + b_1 \tag{19}$$

$$y_n = ap_n + b_2 \tag{20}$$

in which

$$a := \frac{(c_2 - c_1)(L_{N+1} - L_n)}{2c_3} \tag{21}$$

$$b_1 := L_{n-1} + \frac{(c_3 - c_1)(L_n - L_{n-1}) + (c_2 - c_1)(L_{N+1} - L_n)}{2c_3} \tag{22}$$

$$b_2 := L_n + \frac{(c_3 - c_2)(L_k - L_n)}{2c_3} \tag{23}$$

Eqs. (19) and (20) imply that the change in p_n will lead to the same amount of change in y_{n-1} and y_n, albeit in opposite directions. Substituting Eqs. (19) and (20) into Eq. (17),

$$p_n = \frac{M - \int_{y_0}^{-ap_n+b_1} f(x)dx}{\int_{-ap_n+b_1}^{ap_n+b_2} f(x)dx} \tag{24}$$

Since Eq. (24) has only one unknown p_n with limited support (0,1] and the integrand $f(x)$ is bounded in real applications, given a tolerance $\varepsilon > 0$, we can discretize the domain (0,1] and use enumeration methods to find all of the ε-approximate solutions of p_n and then derive $y_m, m=1,2\cdots N$.

3.2.5 Possibility of multiple equilibria

Evidently, Eq. (24) may admit more than one solution even for very simple density functions such as $f(x):=1$. This means that there are multiple equilibria. In fact, since the functional form of $f(x)$ is arbitrary as long as $f(x) \geq 0$, it is possible to construct an example with any number of equilibria by ensuring that Eq. (24) has the required number of solutions in $(0,1]$.

We look at a basic example shown in Fig. 5 to demonstrate multiple equilibria. Passengers are uniformly distributed between 0 and $X = 3.5$ with density $f(x) = 1$, $0 \leq x \leq X$. There are four stations: $L_1 = 0$, $L_2 = 2$, $L_3 = 3.25$, $L_4 = 12$. The total number of seats in the train is $M = 127/64$. The cost parameters are $c_1 = 0$, $c_2 = 0.2$, $c_3 = 1$.

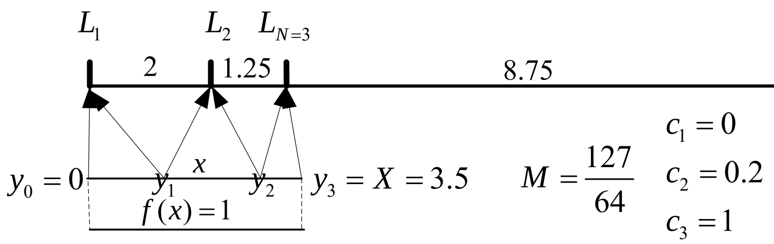

Fig. 5 A case with two equilibria

We can find that station 2 satisfies $p_2 \in (0,1)$. Therefore, $n = 2$ and $k = 3$. We calculate that

$$y_1 = -p_2 + 2, \tag{25}$$

$$y_2 = p_2 + 2.5. \tag{26}$$

Hence

$$p_2 = \frac{M - \int_{y_0}^{-p_2+2} f(x)dx}{\int_{-p_2+2}^{p_2+2.5} f(x)dx} = \frac{\frac{127}{64} - \int_0^{-p_2+2} 1dx}{\int_{-p_2+2}^{p_2+2.5} 1dx} \tag{27}$$

which is a quadratic equation in one unknown:

$$4(p_2)^2 - p_2 + \frac{1}{32} = 0 \tag{28}$$

It has two solutions:

$$p_2^{(1)} = \frac{1-\sqrt{0.5}}{8}, p_2^{(2)} = \frac{1+\sqrt{0.5}}{8}.$$

Therefore, this problem has two equilibria.

3.3 Case study

We conduct a case study based on the South Coast Line in Fig. 1. We consider $N = 17$ stations that represent the 17 stations from Wollongong to Helensburgh (both inclusive). The layout is shown in Fig. 6. For simplicity, we assume that the distance between two adjacent stations is 2km, and the distance from the 17th station to the destination is 32km. $X = 32$. The density of commuters is $f(x) = 10$, $0 \le x \le 32$. The total number of seats in the train is $M = 100$. The cost parameters are $c_1 = 0.5$, $c_2 = 0.7$, $c_3 = 1.5$.

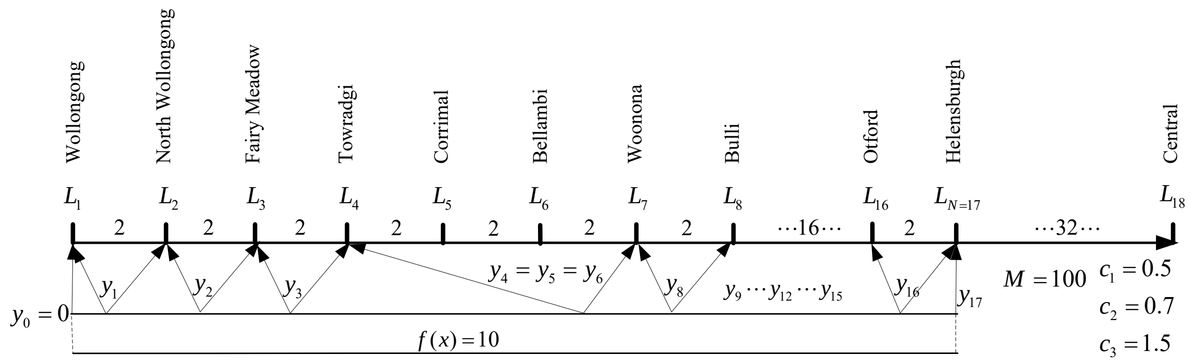

Fig. 6 Case study based on the South Coast Line of Australia

We calculate that $n = 4$, $k = 7$, and $p_n \approx 0.85$. The travel cost of each commuter x is shown in Fig. 7. The travel cost for commuter y_0 is 32. $y_3 \approx 5.27$ and commuter y_3 has the same travel cost 31.90 no matter whether she uses station 3 or station 4. Stations 5 and 6 are not used. $y_4 = y_5 = y_6 \approx 10.87$ and commuter y_4 has the same travel cost 38.10 no matter whether she uses station 4 or station 7.

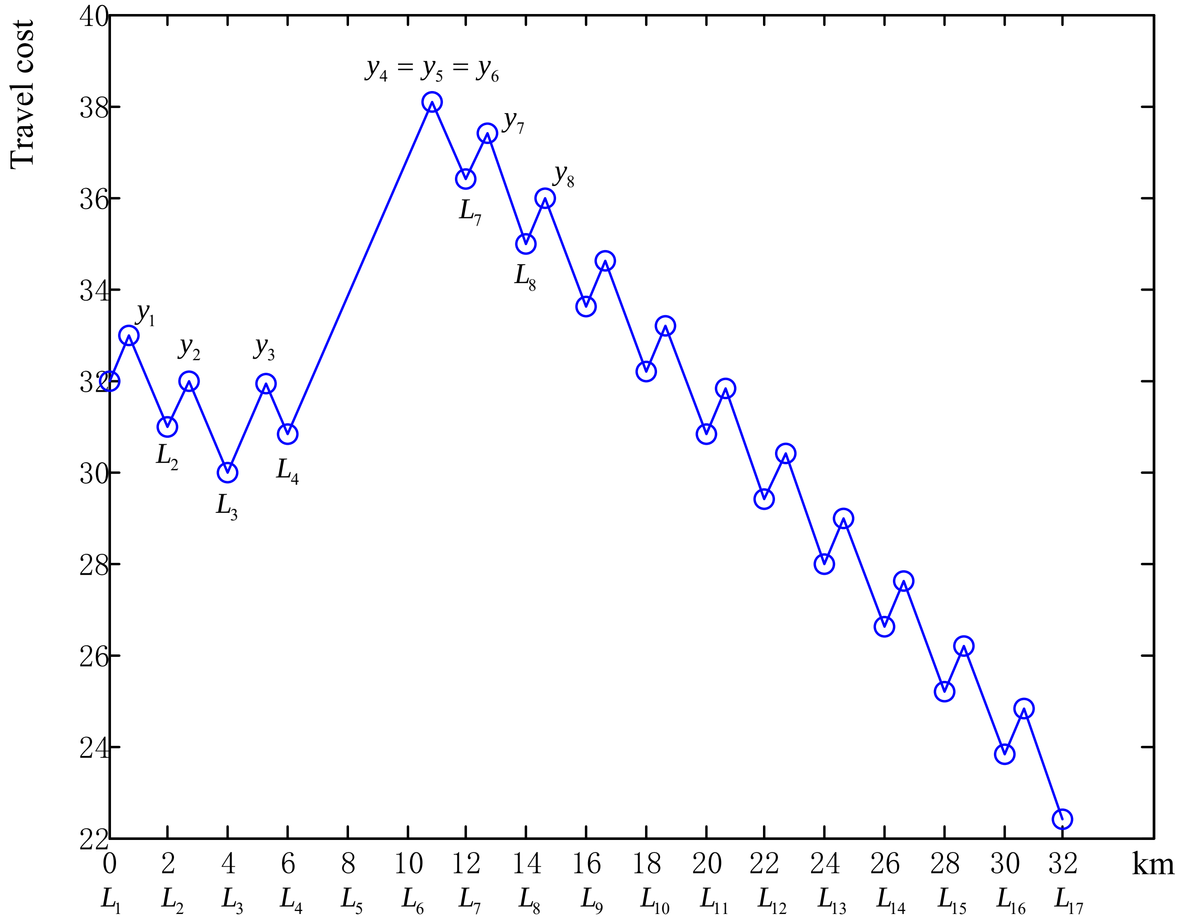

Fig. 7 Travel cost of each passenger at equilibrium

In addition, we see very interesting and similar "zigzag" travel cost patterns for commuters at very upstream locations who can always find a seat (e.g., commuters on $[0, L_3]$) and whose at very downstream locations who never have a seat (e.g., commuters on $[L_7, L_{17}]$). The similarities of the travel cost of the two clusters of commuters are (i) a commuter living at a station has a lower travel cost than her neighbouring commuters who live between two stations, for instance, commuter L_2 has a lower cost than commuters y_1 and y_2, and commuter L_8 has a lower cost than commuters y_7 and y_8; and (ii) the overall pattern of each cluster is that the travel cost decreases at locations closer to the workplace, for example, commuter L_1 has a lower cost than commuter L_2, commuter y_1 has a lower cost than commuter y_2, commuter L_7 has a lower cost than commuter L_8, commuter y_7 has a lower cost than commuter y_8. The difference of the travel cost of the two clusters of commuters is, for example, y_2 is farther away from L_2 than the distance from y_7 to L_7; in other words, commuter y_2 would drive farther upstream than y_7 because the travel cost in train with a seat from L_2 to L_3 is smaller than that without a seat from L_7 to L_8.

Based on the case study, we find that in extreme cases, it is possible that all of commuters drive to station 1 to compete for a seat. Note that the commuter located at the middle of L_1 and L_{N+1} will not drive to station 1 because the cost for the commuter to use station 1 is

$$\begin{aligned} & c_3(L_1 + L_{N+1})/2 + (p_1c_1 + (1-p_1)c_2)(L_{N+1} - L_1) \\ & > c_3(L_1 + L_{N+1})/2 \end{aligned}$$

and the cost of driving to the workplace L_{N+1} is $c_3(L_1 + L_{N+1})/2$. Therefore, we slightly change the case study by setting $L_{N+1} = 80$ instead of 64 and setting $f(x) = 3.5$, $0 \le x \le 32$, and investigate the relation between c_1, c_2, and c_3, that makes all of the commuters drive to station 1. To this end, we have $p_1 = 100/112 \approx 0.89$ and all of the commuters drive to station 1 only if the commuter located at L_N drives to station 1, i.e., the travel cost for commuter L_N to use station 1 is not greater than that of station L_N. We thus have

$$c_3(L_N - L_1) + (p_1c_1 + (1-p_1)c_2)(L_{N+1} - L_1) \le c_2(L_{N+1} - L_N). \tag{29}$$

We can see that all commuters tend to drive to station 1 if c_3 is not too much larger than c_1 and c_2 (to make sure that the first term on the left-hand side of Eq. (29) is small), c_1 is much smaller than c_2 (to make sure that the second term on the left-hand side of Eq. (29) is small), and p_1 is large (i.e., not too many people take the train, to make sure that the second term on the left-hand side of Eq. (29) is small), among other conditions. In particular, if $c_1 = 0.1$,

$c_2 = 1.4$, and $c_3 = 1.5$ in this example, then all of commuters drive to station 1 to compete for a seat.

4 System optimal station choice

In this section we examine the system optimal station choice to minimize the total expected travel cost of all of the $\hat{M}$ commuters assuming they cooperate.

No matter how commuters choose stations, Proposition 1 holds. That is, there exists a station $\bar{n} \in \{1, 2 \cdots N\}$ such that $p_{\bar{n}} > 0, p_{\bar{n}+1} = 0$. At station $\bar{n}$, some commuters can find a seat and the others (possibly nobody) cannot. Suppose that we have an SO station choice scheme with $p_{\bar{n}} > 0, p_{\bar{n}+1} = 0$. In the SO scheme, we can consider that some commuters who board at station $\bar{n}$ have a reserved seat and the others always have to stand. This does not change the total travel cost. We further consider that those who board at station stations $1, 2 \cdots \bar{n} - 1$ also have reserved seats. Now we can classify the commuters at SO into four categories: (i) those who board at stations $1, 2 \cdots \bar{n} - 1$ with a reserved seat, (ii) those who board at station $\bar{n}$ with a reserved seat, (iii) those who board at station $\bar{n}$ and stand, and (iv) those who board at stations $\bar{n}+1, \bar{n}+2 \cdots N$ and stand. Commuters in categories (i) and (ii) have reserved seats and commuters in categories (iii) and (iv) do not have seats and must stand. This "reserved seat" idea could considerably facilitate the analysis of the SO station choice scheme.

Lemma 3: At SO, the commuter located at L_k, $k = 1, 2 \cdots N$, will board at station L_k. This implies that no station is skipped at SO.

Proof: Consider a commuter located at L_k, $k = 1, 2 \cdots N$ and suppose that she does not board at station L_k. We prove that this is impossible at SO by contradiction. Using the "reserved seat" idea, the commuter L_k either has a reserved seat or not. If she has a reserved seat, then she should board at L_k because driving to upstream stations increases both the driving costs and travel cost by train, and driving to downstream stations leads to a higher driving costs increase than the decrease in travel cost by train. If she has no seat, then she should still board at L_k because driving to an upstream station increases both the driving costs and travel cost by train, and driving to downstream stations leads to a higher driving costs increase than the decrease in travel cost by train. Therefore, the commuter located at L_k, $k = 1, 2 \cdots N$, will board at station L_k at SO. □

Lemma 4: Similar to the argument in Lemma 3, at SO, the commuters located sufficiently close to station L_k, $k = 1, 2 \cdots N$, will board at station L_k. □

Theorem 2: At SO, we have

$$0 = y_0 \le L_1 < y_1 < L_2 < y_2 < \cdots < y_N = X \ . \tag{30}$$

Proof: At SO, a passenger x_2 who lives closer to the workplace than another passenger x_1 will not choose a station farther to the workplace than the one chosen by x_1, because otherwise using the "reserved seat" idea we can exchange the choice of stations of two commuters and exchange the two commuters' reservations of seats and thereby reduce the total travel cost. Hence, Eq. (8) holds at SO. Moreover, Lemma 4 implies that $L_k \in (y_{k-1}, y_k)$, $k = 1, 2 \cdots N$. The facts of Eq. (8) and $L_k \in (y_{k-1}, y_k)$ imply Eq. (30). □

The station n with $p_n > 0, p_{n+1} = 0$ satisfies $\int_0^{y_n} f(x)dx \ge M$ and $\int_0^{y_{n-1}} f(x)dx < M$. Since $L_{n-1} < y_{n-1} < L_n < y_n < L_{n+1}$, we have

$$\int_0^{L_{n+1}} f(x)dx > M \tag{31}$$

and

$$\int_0^{L_{n-1}} f(x)dx < M \tag{32}$$

Therefore, station n is either the station n' defined below:

$$n' := \arg\min \left\{ m \mid \int_0^{L_m} f(x)dx > M \right\} . \tag{33}$$

or station $n'-1$. We will prove later that at SO station n takes exact one value from n' and $n'-1$.

4.1 SO station choice model

Given station n satisfying $p_n > 0, p_{n+1} = 0$, the values of $y_m, m = 1,2,3\cdots n-2$ can be determined by Eq. (11), and $y_m, m = n+1, n+2\cdots N-1$ can be determined by Eq. (15), meaning that the total travel cost of the passengers on $[0, L_{n-1}]\cup[L_{n+1}, X]$ is fixed. Hence, we optimize y_{n-1} and y_n to minimize the total travel cost of the passengers on $[L_{n-1}, L_{n+1}]$, which consists of four parts: $[L_{n-1}, y_{n-1}]$, $(y_{n-1}, L_n]$, $(L_n, y_n]$, and $(y_{n+1}, L_{n+1}]$, as shown in Fig. 8. These four parts correspond to the four terms in the objective function of the model below.

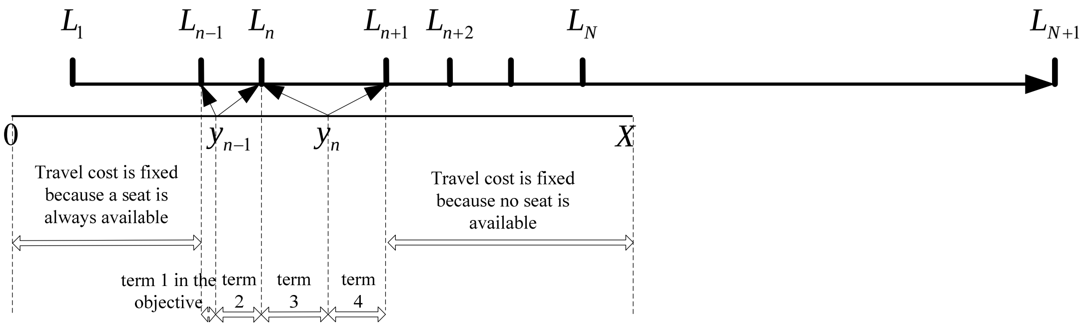

Fig. 8 Structure of the SO station choice

The model is:

$$[\text{SO}] \qquad \min_{y_{n-1}, y_n, p_n} \left\{ \begin{array}{l} \int_{L_{n-1}}^{y_{n-1}} \left[c_3(x - L_{n-1}) + c_1(L_{N+1} - L_{n-1})\right] f(x)dx \\ +\int_{y_{n-1}}^{L_n} \left\{c_3(L_n - x) + \left[p_n c_1 + (1-p_n)c_2\right](L_{N+1} - L_n)\right\} f(x)dx \\ +\int_{L_n}^{y_n} \left\{c_3(x - L_n) + \left[p_n c_1 + (1-p_n)c_2\right](L_{N+1} - L_n)\right\} f(x)dx \\ +\int_{y_n}^{L_{n+1}} \left[c_3(L_{n+1} - x) + c_2(L_{N+1} - L_{n+1})\right] f(x)dx \end{array} \right\} \tag{34}$$

subject to:

$$p_n = \frac{M - \int_{y_0}^{y_{n-1}} f(x)dx}{\int_{y_{n-1}}^{y_n} f(x)dx} \tag{35}$$

$$0 < p_n \le 1 \tag{36}$$

$$L_{n-1} < y_{n-1} < L_n < y_n < L_{n+1} \tag{37}$$

4.2 Structure of SO station choice model

The model (34) seems to be very difficult to solve because of the integrations in the objective function and constraints. We examine its properties and identify an analytical solution.

Define

$$c^f(x,n) := c_3 \,|\, x - L_n \,| + c_2(L_{N+1} - L_n) \tag{38}$$

and

$$c^a(x,n) := c_3 \,|\, x - L_n \,| + c_1(L_{N+1} - L_n) \tag{39}$$

where the superscript f means the train is full (has no available seats) before it arrives at station n, and a means the train has sufficient available seats at station n. Therefore, $c^f(x,n)$ is the total travel cost of passenger x boarding at station n without a seat, and $c^a(x,n)$ is the total travel cost of passenger x boarding at station n with a seat.

Theorem 3: Given station n satisfying $p_n > 0, p_{n+1} = 0$, if $p_n = 1$ at SO, then y_{n-1} satisfies $c^a(y_{n-1}, n-1) = c^a(y_{n-1}, n)$, y_n can be determined by

$$\int_{y_0}^{y_n} f(x)dx = M \tag{40}$$

and it has to satisfy

$$c^a(y_n, n) \le c^a(y_n, n+1) \text{ and } c^f(y_n, n) \ge c^f(y_n, n+1) . \tag{41}$$

Proof: Using the "reserved seat" idea, commuter y_{n-1} has a reserved seat and is indifferent between station $n-1$ and station n, and hence $c^a(y_{n-1}, n-1) = c^a(y_{n-1}, n)$. Eq. (40) holds because of the assumption that $p_n = 1$. Using the "reserved seat" idea, commuter y_n^- (i.e., the commuter located slightly closer to the origin 0 than y_n) has a reserved seat and she chooses to board at station n instead of $n+1$. This implies $c^a(y_n^-, n) \le c^a(y_n^-, n+1)$ and thereby $c^a(y_n, n) \le c^a(y_n, n+1)$ due to continuity of the cost function. Similarly, commuter y_n^+ (i.e., the commuter located slightly farther to the origin 0 than y_n) has no seat and she chooses to board at station $n+1$ instead of n. This implies $c^f(y_n^+, n) \ge c^f(y_n^+, n+1)$ and thereby $c^f(y_n, n) \ge c^f(y_n, n+1)$. □

Consider the example shown in Fig. 5 except that the total number of seats in the train is $M = 2.5$. Then at SO, $n = 2$, $p_n = 1$, $y_1^{SO} = 1$, $y_2^{SO} = 2.5$, and $y_3^{SO} = 3.5$.

It should be noted that the SO result with $p_n = 1$ can only happen when $n = n'-1$ because otherwise Eq. (33) and Eq. (40) contradict with each other.

Theorem 4: Given station n satisfying $0 < p_n < 1$, at SO, y_{n-1} satisfies

$$c^a(y_{n-1}, n-1) = c^a(y_{n-1}, n) \tag{42}$$

and y_n satisfies

$$c^f(y_n, n) = c^f(y_n, n+1). \tag{43}$$

Proof: Using the "reserved seat" idea, commuter y_{n-1} has a reserved seat and she is indifferent between station $n-1$ and station n, and hence $c^a(y_{n-1}, n-1) = c^a(y_{n-1}, n)$. Some commuters on (y_{n-1}, y_n) have reserved seats and the others do not. We specify that there exists a $\bar{y}_n \in (y_{n-1}, y_n)$ such that all commuters on $(y_{n-1}, \bar{y}_n)$ have reserved seats and no commuter on $(\bar{y}_n, y_n)$ has a reserved seat. Of course, $\bar{y}_n$ can be determined by

$\int_{y_0}^{\bar{y}_n} f(x)dx = M$. Hence, commuter y_n has no seat and she is indifferent between station n and $n+1$. This implies $c^f(y_n, n) = c^f(y_n, n+1)$. □

Based on Theorem 3 and Theorem 4, we can classify all possible SO scenarios into three cases. Case I: $n = n'-1$, $p_n = 1$ and y_{n-1} and y_n , denoted by $y_{n'-2}(I)$ and $y_{n'-1}(I)$, respectively, can be calculated according to Theorem 3. Case II: $n = n'-1$, $p_n \in (0,1)$ and y_{n-1} and y_n, denoted by $y_{n'-2}(II)$ and $y_{n'-1}(II)$, respectively, can be calculated according to Theorem 4. Case III: $n = n'$, $p_n \in (0,1)$ and y_{n-1} and y_n, denoted by $y_{n'-1}(III)$ and $y_{n'}(III)$, respectively, can be calculated according to Theorem 4. Given a problem instance, we can analyze the three cases one by one. We have

Theorem 5: At SO exactly one of the above three cases holds.

Proof: We first prove Case (I) and Case (II) cannot be true simultaneously. (i) Because in Case (I) $p_{n'-1} = 1$ and in Case (II) $p_{n'-1} < 1$, we have

$$y_{n'-1}(II) > y_{n'-1}(I). \tag{44}$$

(ii) Eq. (41) implies

$$c^f(y_{n'-1}(I), n'-1) \geq c^f(y_{n'-1}(I), n'). \tag{45}$$

(iii) Eq. (43) implies

$$c^f(y_{n'-1}(II), n'-1) = c^f(y_{n'-1}(II), n'). \tag{46}$$

Points (i), (ii) and (iii) contradict with each other.

We then prove Case (I) and Case (III) cannot be true simultaneously. (i) We have $y_{n'-1}(III) < y_{n'-1}(I)$ because in Case (I) $p_{n'-1} = 1$ and in Case (III) $p_{n'} > 0$. (ii) Eq. (41) implies $c^a(y_{n'-1}(I), n'-1) \leq c^a(y_{n'-1}(I), n')$. (iii) Eq. (42) implies $c^a(y_{n'-1}(III), n'-1) = c^a(y_{n'-1}(III), n')$. Points (i), (ii) and (iii) contradict with each other.

We finally prove Case (II) and Case (III) cannot be true simultaneously. (i) We have $y_{n'-1}(III) < y_{n'-1}(II)$ because in Case (II) $p_{n'-1} > 0$ and in Case (III) $p_{n'} > 0$. (ii) Eq. (42)

implies $c^a(y_{n'-1}(III), n'-1) = c^a(y_{n'-1}(III), n')$. (iii) Eq. (43) implies $c^f(y_{n'-1}(II), n'-1) = c^f(y_{n'-1}(II), n')$, which implies $c^a(y_{n'-1}(II), n'-1) < c^a(y_{n'-1}(II), n')$ as $c_1 < c_2$. Points (i), (ii) and (iii) contradict with each other.□

We now summarize the algorithm to find the station choice at SO. First, there are two possible candidates of station n with $p_n > 0, p_{n+1} = 0$ based on Eq. (33). As at SO station n takes exact one value, we can investigate each candidate of station n and identify which one is SO. Given the station n, the values of y_{n-1} can be computed by Eqs. (42) and (43). As a result, all information about the SO can be obtained.

4.3 Case study (continued)

Consider the case in Section 3.3. Based on based on Eq. (33), we can calculate that $n' = 7$. When $n = n'$ ($p_n < 1$), Eq. (32) is violated. When $n = n'-1$ and $p_n = 1$, $y_n = L_6$ and the requirement $c^f(y_n, n) \geq c^f(y_n, n+1)$ in Eq. (41) is violated. Therefore, $n = n'-1$ and $p_n < 1$. Eqs. (42) and (43) imply $y_{n-1}(n') \approx 8.67$ and $y_n(n') \approx 10.53$. We then check $\int_{y_0}^{y_{n-1}(n')} f(x)dx \approx 87 < M$, and hence this is an SO solution with $p_n \approx 0.71$. Therefore, $y_5^{SO} = 8.67$ and $y_6^{SO} = 10.53$ at SO.

The travel cost of each commuter at SO, together with the travel cost at UE, is plotted in Fig. 9. Evidently at SO the travel cost curve is not continuous: there are two jumps at y_5^{SO} and y_6^{SO}. In particular, if the commuter y_5^{SO} boards at station 5, then she always has a seat and her travel cost is 29; if she boards at station 6, then she may not have a seat and her travel cost is 32.09; if the commuter y_6^{SO} boards at station 6, then she may have a seat and her travel cost is 30.89; if she boards at station 7, then she does not have a seat and her travel cost is 38.6. Comparing the SO travel cost for each passenger with the UE travel cost, we can see that those commuters living on (y_3^{SO}, y_6^{SO}) have their travel cost reduced at SO, the commuters who live on $[0, y_3^{SO}] \cup [y^6, L_N]$ have the same travel cost, and only a small proportion of commuters, i.e., those who live on (y_6^{SO}, y_6), experience a higher travel cost at SO.

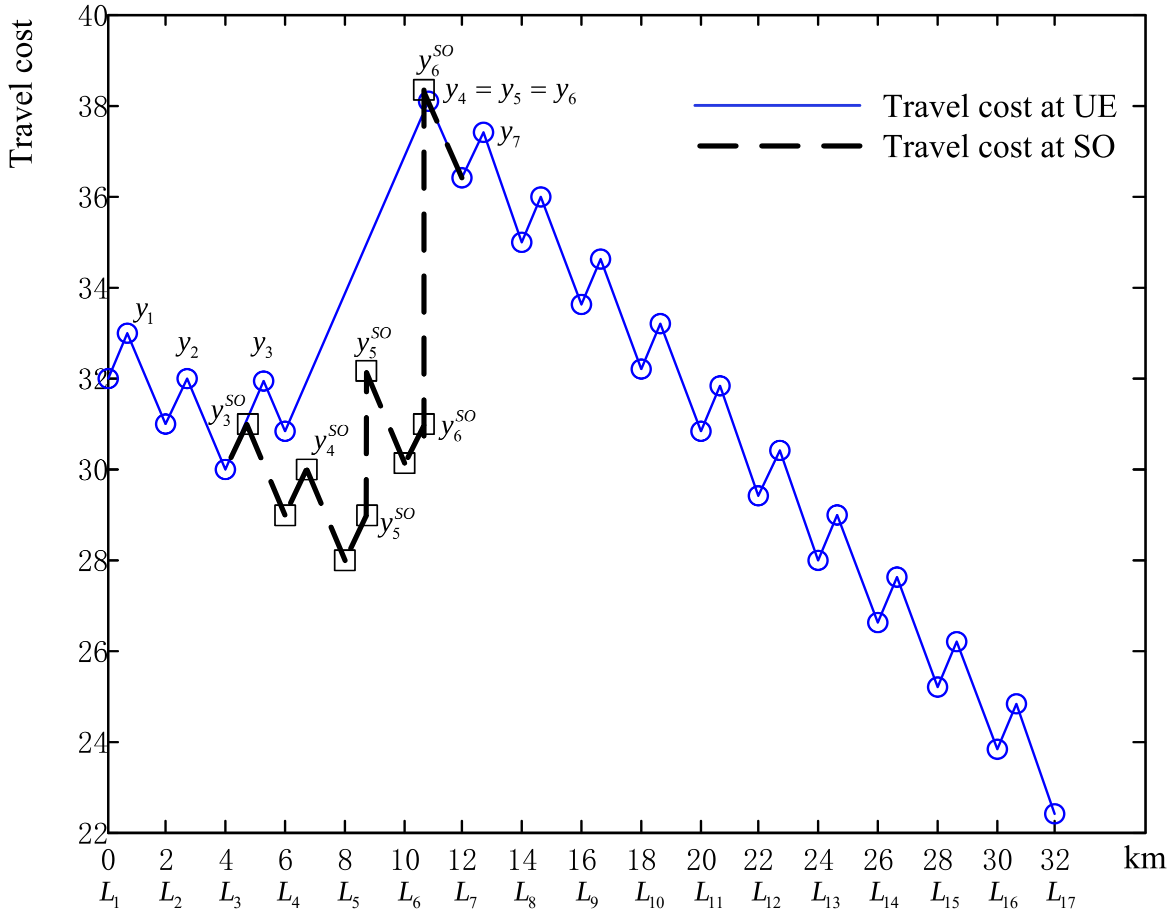

Fig. 9 Travel cost of each commuter at SO in comparison to UE

5 Optimal train fare design

The SO assignment is not sustainable when all of stations charge the same fare for the trips to the workplace because some commuters could switch to other stations to reduce their travel cost. We therefore examine the optimal train fares that could lead to SO assignment. When $p_n^{SO} < 1$, Eqs. (42) and (43) imply that

$$c(y_{n-1}^{SO}, n-1, p_{n-1}^{SO}) < c(y_{n-1}^{SO}, n, p_n^{SO}) \tag{47}$$

$$c(y_n^{SO}, n, p_n^{SO}) < c(y_n^{SO}, n+1, p_{n+1}^{SO}). \tag{48}$$

Theorem 6: Let θ_n be the train fare for traveling from station n to the workplace. The optimal fare design has the structure below: all of the stations $m = 1, 2, 3 \cdots n-1$ charge the same fare, station n charges a fare lower than station $n-1$ by

$$\theta_{n-1} - \theta_n = c(y_{n-1}^{SO}, n, p_n^{SO}) - c(y_{n-1}^{SO}, n-1, p_{n-1}^{SO}) \tag{49}$$

All of the stations $m = n+1, n+2 \cdots N$ charge the same fare that is lower than station n by

$$\theta_n - \theta_{n+1} = c(y_n^{SO}, n+1, p_{n+1}^{SO}) - c(y_n^{SO}, n, p_n^{SO}) . \tag{50}$$

The differences between fares for adjacent stations n and $n+1$ and adjacent stations n and $n-1$ are actually the "heights" of the "steps" in the SO travel cost curve in Fig. 9.

Theorem 6 tells us that the fare at station $n-1$ is higher than n, which is again higher than $n+1$. The intuition behind this result is that (i) station $n-1$ charges higher than n so that people between the two stations will not drive too far to the upstream station to secure a seat; and (ii) station n charges higher than $n+1$ so that people between the two stations will not drive too far to the upstream station to take the chance of finding a seat. Theorem 6 implies that the price should only decrease at the last station that may have a seat ($0 < p_n < 1$), and decrease again at the first station with no seat available. Note that in reality there could be equity considerations if many stations have the same fare.

When $p_n^{SO} = 1$, as there is always a seat at station n, "=" holds in Eq. (47) and $\theta_{n-1} = \theta_n$ in Eq. (49). Therefore, the train fare only needs to change once.

If the train fare is regulated and may not be able to reach the system optimal, then the park fee at train stations may be another complementary measure to be used to achieve the system optimal.

Consider the case study in Section 4.3. Assuming that θ_7 is given, then the optimal fares at stations 6 and 5 are, respectively,

$$\theta_6 = \theta_7 + \left[c(y_n^{SO}, n+1, p_{n+1}^{SO}) - c(y_n^{SO}, n, p_n^{SO}) \right] = \theta_7 + (38.6 - 30.89) = \theta_7 + 7.71 . \tag{51}$$

$$\theta_5 = \theta_6 + \left[c(y_{n-1}^{SO}, n, p_n^{SO}) - c(y_{n-1}^{SO}, n-1, p_{n-1}^{SO}) \right] = \theta_7 + 7.71 + (32.09 - 29) = \theta_7 + 10.8 . \tag{52}$$

Next, we analyze a few practical situations for determining the fares.

If a maximum fare $\theta^{\max}$, $\theta^{\max} \geq \theta_5 - \theta_7$, is imposed, the fares can be set as follows to maximize the total fare income:

$$\theta_5 = \theta^{\max}, \theta_6 = \theta_5 - 3.09, \theta_7 = \theta_6 - 7.71 \geq 0. \tag{53}$$

It is possible that a maximum fare is imposed on each travel zone. Suppose that station 5 is in zone 1 with a maximum fare $\theta_{[1]}^{\max}$, stations 6 and 7 are in zone 2 with a maximum fare $\theta_{[2]}^{\max}$. The fares can be set as follows to maximize the total fare income:

$$\theta_5 = \min\{\theta_{[1]}^{\max}, \theta_{[2]}^{\max} + 3.09\}, \theta_6 = \min\{\theta_{[1]}^{\max} - 3.09, \theta_{[2]}^{\max}\}, \theta_7 = \theta_6 - 7.71. \tag{54}$$

If the total fare income is to be minimized, then the fares are:

$$\theta_5 = 10.8, \theta_6 = 7.71, \theta_7 = 0. \tag{55}$$

If the total fare income is set at Θ, then the fares can be determined by:

$$\theta_5 = 3.09 + \theta_6, \theta_6 = 7.71 + \theta_7. \tag{56}$$

$$\theta_5 \int_0^{y_{n-1}^{SO}} f(x)dx + \theta_6 \int_{y_{n-1}^{SO}}^{y_n^{SO}} f(x)dx + \theta_7 \int_{y_n^{SO}}^{X} f(x)dx = \Theta. \tag{57}$$

6 Conclusions

We have conducted an in-depth analysis of how park-and-ride commuters compete for seats when they travel to their workplace in the morning. In our setting, all of the commuters live along a rail line and are homogeneous in that they have the same unit travel cost with a seat and the same unit travel cost without a seat. First, we proved that at user equilibrium in which each commuter minimizes her expected travel cost, there exists one station that has some seats; all of the commuters boarding at its upstream stations have seats and all of the commuters boarding at its downstream stations must stand in the train. It is possible, in extreme cases, that some stations are not used. We derived a solution algorithm for obtaining a user equilibrium, which involves solving an equation in one unknown. We demonstrated that more than one user equilibrium may exist. Second, we examined the system optimal station choice that assumes all of the commuters cooperate and minimizes their total travel cost. An analytical solution approach is proposed based on the structure of the problem. Third, we investigated the optimal train fare design that leads to the system optimal station choice. We proved that the optimal train fare satisfies: there exists a particular train station that has some seats and the train is full after this station. All of its upstream stations have the same fare, which is higher than or equal to the fare of this particular station; and all of its downstream stations have the same fare, which is lower than the fare of this particular station.

Two issues are noteworthy. First, the current train fare structure is mostly zone-based. The system-optimal fare structure studied in our paper is a variant of zone-based structure as one critical station on the boundary of two zones has a special fare: a few farthest stations have the highest fare (one zone), one critical station has a medium fare, and the other nearest stations have the lowest fare (the other zone). If a flat fare system is imposed, then the total cost for all commuters will not increase too much in general, as only passengers near the critical station may drive too much upstream to compete for a seat; however, since we study suburban train routes, the farthest passengers may be over 100 km away from the city center, while the nearest may be just e.g. 10 km; as a result, flat fare may be unfair for those who live very near the city center. Second, we have only considered the fare as the decision variable. We may also consider both the fare and the train frequency as the decision variables. In this case, if we assume that the train frequency is constant and the headway is so long that passengers will not choose trains but just take the one most suitable for their working hours, then the joint problem can be easily solved by enumerating the train frequency first, which determines the number of passengers to choose the train, and then the approach proposed by our study can be applied to evaluate the total travel cost; if the train frequency is not constant, then we can use dynamic programming to find the optimal timetable of the trains; if the frequency is very high and passengers choose both the train and the station, then this problem is much more complex and is thus a valuable future research topic.

There are a few extensions of our study. First, we have assumed that all of the commuters are homogeneous in that they have the same unit travel cost with a seat and the same unit travel cost without a seat. In reality, commuters are heterogeneous. For instance, some commuters may be used to waking up early. They thus do not need to sleep in trains and can play with their electronic devices even when they are standing. As a result, these commuters have a smaller unit travel cost difference with and without a seat than the commuters who are used to waking up late and are eager to have more sleep in trains. When heterogeneous commuters are considered, all of the train stations will be used at user equilibrium and the extreme case shown in Fig. 4 will not occur. Second, we have assumed all of the commuters live along a rail line, circumventing the complex routing problems in urban road networks. Since in the outer suburbs in Australia road congestion is not a problem and people drive to train stations very early in the morning (before the morning peak hours), we conjecture that incorporating the distribution of commuters in a realistic two-dimensional area and considering road networks will not change the major findings of the study.

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