

The Packing While Traveling Problem

S. Polyakovskiy^{a,*}, F. Neumann^a

^a*School of Computer Science
The University of Adelaide
Adelaide, SA 5005, Australia.*

Abstract

This paper introduces the Packing While Traveling problem as a new non-linear knapsack problem. Given are a set of cities that have a set of items of distinct profits and weights and a vehicle that may collect the items when visiting all the cities in a fixed order. Each selected item contributes its profit, but produces a transportation cost relative to its weight. The problem asks to find a subset of the items such that the total gain is maximized. We investigate constrained and unconstrained versions of the problem and show that both are \mathcal{NP} -hard. We propose a pre-processing scheme that decreases the size of instances making them easier for computation. We provide lower and upper bounds based on mixed-integer programming (MIP) adopting the ideas of piecewise linear approximation. Furthermore, we introduce two exact approaches: one is based on MIP employing linearization technique, and another is a branch-infer-and-bound (BIB) hybrid approach that compounds the upper bound procedure with a constraint programming model strengthened with customized constraints. Our experimental results show the effectiveness of our exact and approximate solutions in terms of solution quality and computational time.

Keywords: Combinatorial optimization; non-linear knapsack problem; linearization technique; piecewise approximation; hybrid optimization.

1. Introduction

Generally, the traditional statements of routing problems studied in the operations research literature base the computation of transportation costs on a linear function. However, in real practice, it might be necessary to deal with costs that have a nonlinear nature. For example, the study on the factors affecting truck fuel economy published by GOODYEAR (2008) reveals that vehicle miles per gallon decreases as gross combination weight increases assuming speed

*Corresponding author

Email addresses: `sergey.polyakovskiy@adelaide.edu.au` (S. Polyakovskiy),
`frank.neumann@adelaide.edu.au` (F. Neumann)

is maintained constant. In other words, a heavily loaded truck will use much more fuel than a lightly loaded one, and this relation is not linear.

In recent years, the research on dependence of fuel consumption on different factors, like a travel velocity, a load’s weight, and vehicle’s technical specifications, in various Vehicle Routing Problems (VRP) has gained attention from the operations research community. Mainly, this interest is motivated by a wish to be more accurate with the evaluation of transportation costs, and therefore to stay closer to reality. Indeed, an advanced precision would immediately benefit to transportation efficiency measured by the classic petroleum-based costs and the novel greenhouse gas emission costs. Furthermore, the proper estimation of costs and its computational simplicity should evolve optimization approaches and enhance their performance. In VRP in general, and in the Green Vehicle Routing Problems (GVRP) that consider energy consumption in particular, given are a depot and a set of customers that are to be served by a set of vehicles collecting (or delivering) required items. While the set of items is fixed, the goal is to find a route for each vehicle such that the total size of assigned items does not exceed the vehicle’s capacity and the total transportation cost over all vehicles is minimized. We refer to the book of Toth & Vigo (2014) and the recent surveys of Laporte (2009) and Lin et al. (2014) for an extended overview on VRP and GVRP.

Oppositely to VRP and GVRP, we consider the situation of a single vehicle whose route is given, but items can be either collected or skipped. This situation gives rise to a problem that we designate as Packing While Traveling (PWT). In the PWT, the items are distributed among the cities. The vehicle visits all the cities in a specific order and collects the items of its choice. Each item has a profit and a weight, and the vehicle may collect any unless the total weight of chosen items exceeds the vehicle’s capacity. The vehicle travels between two cities with a velocity that depends on the weight of the items collected in all the previously attended cities. Being selected, an item contributes its profit to the overall reward. However, its weight slows down the vehicle. This leads to a transportation cost depended on a traveling time, and therefore has a negative impact on the reward. The problem asks to find a packing plan that maximizes the difference between the total profit of the selected items and their transportation cost. The PWT arises as a baseline problem in some practical applications. For example, a supplier having a single truck has to decide on goods to purchase going through a fixed route in order to maximize profitability of future sales. Specifically, there might exist only a single major route that a vehicle has to follow while any deviations from it in order to visit particular cities on the way may be negligible with respect to the length of the route. The importance of items may be variable and affected by a specific demand, therefore the profits of items can be altered accordingly from trip to trip. Obviously, PWT is a part of a larger picture as a potential subproblem in various VRP with non-linear costs. Indeed, the objective function of the PWT studied here is constructed similarly to one that may be based on the dependence between a fuel cost to drive a distance unit and a gross combination weight as provided in GOODYEAR (2008).

The PWT originates from the Traveling Thief Problem (TTP) introduced by Bonyadi et al. (2013). The TTP combines the classical Traveling Salesperson Problem (TSP) with the 0-1 Knapsack Problem (KP) and allows permutation of the order of the cities. The PWT uses the same cost function as the TTP, and the only difference is the assumption of a fixed route. In this sense, any approach to the PWT can also be applied to the TTP as a subroutine to solve its packing component. The TTP has been introduced and studied mainly by the evolutionary computation community during the last few years. A benchmark set based on the TSP and KP instances has been presented in Polyakovskiy et al. (2014). Approaches to handle the TTP include various meta-heuristics such as evolutionary algorithms, randomized local search and co-evolutionary approaches. Mei et al. (2015) solve the problem approximately with the two-stage memetic algorithm, which consists of a tour improvement stage and an item picking stage. For the former stage, the local search operators have been adopted different to those that are traditional for the TSP. The second stage is solved either by a constructive heuristic or by means of genetic programming. Mei et al. (2016) propose two meta-heuristics for TTP: one is the cooperative co-evolution approach that solves the sub-problems separately and transfers the information between them in each generation, and another is the memetic algorithm that solves TTP as a whole. Faulkner et al. (2015) provide multiple constructive heuristics where solutions are obtained by finding a near-optimal TSP tour by applying the Chained Lin-Kernighan heuristic (Applegate et al. (2003)) to the underlying TSP part first, and then selecting a subset of items heuristically. The results produced by the heuristics have been compared then to the same approach where items are selected now by the approximate MIP-based approach of Polyakovskiy & Neumann (2015). They conclude that, when giving the same time limit, constructive heuristics perform generally better since are able to check more TSP tours as the packing part can be solved much faster than the MIP-based approach does. Indeed, existing approaches solve the TTP by fixing one of the components, usually the TSP, and then tackling the KP. Lourenço et al. (2016) follow in a different direction and propose an evolutionary algorithm that addresses both sub-problems at the same time. Their experimental results show that solving the TTP as a whole creates conditions for discovering solutions with enhanced quality, and that fixing one of the components might compromise the overall results. Recently, the formulation of the TTP, which allows to skip cities and to visit one city by multiple thieves, has been investigated by Chand & Wagner (2016). To our best knowledge, no exact approach has been proposed for the TTP so far.

Vansteenwegen et al. (2011) give a review on the so-called orienteering problem that is somehow related to TTP. There a set of vertices is given, each with a score, and the goal is to determine a path, limited in length, that visits some vertices and maximizes the sum of the collected scores. Feillet et al. (2005) present a classification of traveling salesman problems with profits (TSPs with profits) and survey the existing literature on this field. TSPs with profits are a generalization of the TSP where it is not necessary to visit all vertices. A profit is associated with each vertex. The overall goal is the simultaneous optimization

of the collected profit and the travel costs. In this sense, TTP has some relation to the Prize Collecting TSP (Balas (1989)) where a decision is to be made on whether to visit a given city. In the Prize Collecting TSP, a city-dependent reward is obtained when a city is visited and a city-dependent penalty has to be paid for each non-visited city. In contrast to this, TTP requires that each given city is visited. Furthermore, each city has a set of available items with weights and profits and a decision has to be made on which items to pick. TTP also relates to the traveling salesman subtour problem studied by Westerlund et al. (2006), where given is an undirected graph with edge costs and both revenues and weights on the vertices, and the goal is to find a subtour that includes a depot vertex, satisfies a knapsack constraint on the vertex weights, and that minimizes edge costs minus vertex revenues along the subtour. Beham et al. (2015) discuss optimization strategies for integrated knapsack and traveling salesman problems and study the Lagrangian decomposition to the knapsack constrained profitable tour problem.

In substance, the PWT considers a trade-off between the profits of collected items and the transportation cost affected by their total weight. It represents a class of nonlinear knapsack problems. Knapsack problems belong to the core combinatorial optimization problems and have been frequently studied in the literature from the theoretical as well as experimental perspective (Garey & Johnson (1979); Martello & Toth (1990); Kellerer et al. (2004)). While the classical knapsack problem asks for maximization of a linear pseudo-Boolean function under a single linear constraint, different generalizations and variations have been investigated such as the multiple knapsack problem (Chekuri & Khanna (2005)) and multi-objective knapsack problems (Erlebach et al. (2001)). Furthermore, knapsack problems with nonlinear objective functions have been studied in the literature from different perspectives (Bretthauer & Shetty (2002)). Hochbaum (1995) considers the problem of maximizing a separable concave objective function subject to a packing constraint and provided an FPTAS. An exact approach for a nonlinear knapsack problem with a nonlinear term penalizing the excessive use of the knapsack capacity has been given by Elhedhli (2005).

Recently, Wu et al. (2016) have investigated the role of the rent rate in the PWT, which is an important parameter in combining the total profit of selected items and the associated transportation cost. Specifically, the rent rate is a constant defining how much one needs to pay per unit time of traveling by the vehicle. The product of the rent rate and the total traveling time constitutes the transportation cost. In the TTP, when the value of the rent rate is small, searching for an efficient solution of the knapsack component becomes prioritized. From another hand, when the value of the rent rate is large, the TSP part of the TTP starts to play a dominating role. In this paper, the rent rate is formally introduced in Section 2 along with the PWT's statement. The theoretical and experimental investigations of Wu et al. (2016) show how the values of the rent rate influence the difficulty of a given problem instance through the items that can be excluded by the pre-processing scheme presented in this research. Furthermore, their investigations show how to create instances that are hard to be solved by simple evolutionary algorithms. The preliminary ver-

sion of our study on the PWT has appeared in Polyakovskiy & Neumann (2015). Here, we significantly improve our earlier results. We introduce an upper bound technique that along with the enhanced pre-processing scheme allows to solve a larger range of instances to optimality and to dramatically decrease running times. Furthermore, we introduce a hybrid approach that combines constraint programming with the upper bound procedure. It is superior on many test instances and produces optimal results in a very short time.

The rest of the paper is organized as follows. We give the formal statement of the PWT in Section 2 and discuss its complexity in Section 3. Section 4 addresses sequencing constraints that are repeatedly used later on in our approaches. In Section 5, we provide a pre-processing scheme which allows to identify unprofitable and compulsory items, and therefore decrease the size of the PWT's instances. Section 6 explains lower and upper bound techniques. In Sections 7 and 8, we introduce our two exact approaches: one that is based on MIP, and a hybrid one that adopts a branch-infer-and-bound paradigm. Finally, we report on the results of our experimental investigations in Section 9 and finish with some conclusions.

2. Problem Statement

The Packing While Traveling problem can be formally stated as follows. Given is a route $N = (1, 2, \dots, n+1)$ as a sequence of $n+1$ unique cities and a set M of m items distributed among first n cities. Distance $d_i > 0$ between two consecutive cities $(i, i+1)$ is known, for any $1 \leq i \leq n$. Every city i contains a set of distinct items $M_i = \{e_{i1}, \dots, e_{im_i}\}$, $M = \cup_{i=1}^n M_i$. Each item $e_{ik} \in M$ has a positive integer profit p_{ik} and a weight w_{ik} , $1 \leq k \leq m_i$. There is a single vehicle that visits all the cities in the order of a route N . The vehicle may collect any item in any city unless the total weight of selected items exceeds its capacity W . Collecting an item e_{ik} leads to a profit contribution p_{ik} , but increases the transportation cost as the weight w_{ik} slows down the vehicle. The vehicle travels along $(i, i+1)$ with velocity $v_i \in [v_{\min}, v_{\max}]$ which depends on the weight of the items collected in the first i cities. When the vehicle is empty, it runs with its maximal velocity v_{\max} . And vice versa, it runs with minimal velocity $v_{\min} > 0$ when is completely full. The objective is to find a subset of M such that the difference between the profit of the selected items and the transportation cost is maximized.

To state the problem precisely, we give a nonlinear binary integer program formulation. Let a binary decision vector $x \in \{0, 1\}^m$ represent a solution of the problem such that $x_{ik} = 1$ iff e_{ik} is selected. Then the travel time $t_i = \frac{d_i}{v_i}$ along $(i, i+1)$ is the ratio of the distance d_i and the current velocity

$$v_i = v_{\max} - \nu \sum_{j=1}^i \sum_{k=1}^{m_j} w_{jk} x_{jk}$$

which is determined by the weight of the items collected in cities $1, \dots, i$. The value $\nu = \frac{v_{\max} - v_{\min}}{W}$ is constant and defined by the input parameters. The

velocity depends on the weight of the chosen items linearly. The overall transportation cost is given by the sum of the transportation costs along all the edges $(i, i+1)$, $1 \leq i \leq n$, multiplied by a given rent rate $R > 0$. In summary, the problem is given by the following nonlinear binary program (PWT^c):

$$\begin{aligned} & \max \sum_{i=1}^n \left(\sum_{k=1}^{m_i} p_{ik} x_{ik} - \frac{R d_i}{v_{max} - \nu \sum_{j=1}^i \sum_{k=1}^{m_j} w_{jk} x_{jk}} \right) \\ & \text{s.t. } \sum_{i=1}^n \sum_{k=1}^{m_i} w_{ik} x_{ik} \leq W \\ & \quad x_{ik} \in \{0, 1\}, \quad e_{ik} \in M \end{aligned} \tag{1}$$

Here, (1) is a non-monotone submodular function.

We also consider the unconstrained version PWT^u of PWT^c where $W \geq \sum_{e_{ik} \in M} w_{ik}$ such that every selection of items yields a feasible solution. Given a real value B , the decision variant of PWT^c and PWT^u has to answer the question whether the value of (1) is at least B .

3. Complexity of the Problem

In this section, we investigate the complexity of PWT^c and PWT^u. PWT^c is \mathcal{NP} -hard as it is a generalization of the classical \mathcal{NP} -hard 0-1 knapsack problem (Martello & Toth (1990)). In fact, assigning zero either to the rate R or to every distance value d_i in PWT^c, we obtain the KP. We demonstrate that in contrast to the KP the unconstrained version PWT^u of the problem remains \mathcal{NP} -hard. We show this by reducing the \mathcal{NP} -complete *subset sum problem* (SSP) to the decision variant of PWT^u which asks whether there is a solution with objective value at least B . The input for SSP is given by m positive integers $S = \{s_1, \dots, s_m\}$ and a positive integer Q . The question is whether there exists a vector $x \in \{0, 1\}^m$ such that $\sum_{k=1}^m s_k x_k = Q$.

Theorem 1. *PWT^u is \mathcal{NP} -hard.*

PROOF. We start with encoding the instance of SSP given by the set of integers S and the integer Q as the instance I of PWT^u having two cities. The first city contains all the m items while the second city is a destination point free of items. We set the distance between two cities as $d_1 = 1$, the capacity of the vehicle as $W = \sum_{k=1}^m s_k$, and set $p_{1k} = w_{1k} = s_k$, $1 \leq k \leq m$. Subsequently, we set $v_{max} = 2$ and $v_{min} = 1$ which implies $\nu = 1/W$ and define $R^* = W(2 - Q/W)^2$.

Consider the nonlinear function $f_{R^*}: [0, W] \rightarrow \mathbb{R}$ defined as

$$f_{R^*}(w) = w - \frac{R^*}{2 - w/W}.$$

f_{R^*} defined on the interval $[0, W]$ is a continuous concave function that reaches its unique maximum in the point $w^* = W \cdot (2 - \sqrt{R^*/W}) = Q$, i.e. $f_{R^*}(w) < f_{R^*}(w^*)$ for $w \in [0, W]$ and $w \neq w^*$. Then $f_{R^*}(Q)$ is the maximum value for f_{R^*} when being restricted to integer input, too. Therefore, we set $B = f_{R^*}(Q)$ and the objective function for PWT^u is given by

$$g_{R^*}(x) = \sum_{k=1}^m p_{1k}x_k - \frac{R^*}{2 - \frac{1}{W} \sum_{k=1}^m w_{1k}x_k}.$$

There exists an $x \in \{0, 1\}^m$ such that $g_{R^*}(x) \geq B = f_{R^*}(Q) = 2(Q - W)$ iff $\sum_{k=1}^m s_k x_k = \sum_{k=1}^m w_{1k}x_k = \sum_{k=1}^m p_{1k}x_k = Q$. Therefore, the instance of SSP has answer YES iff the optimal solution of the PWT^u instance I has objective value at least $B = f_{R^*}(Q)$. Obviously, the reduction can be carried out in polynomial time which completes the proof. \square

4. Sequencing Constraints

In this section, we derive a set of constraints that speed up the reasoning to be done within our algorithms. Specifically, the constraints establish priority among the items positioned in the same or different cities of the route.

The first portion of the constraints results from the fact that item e_{il} in city i should not be selected prior to another item e_{ik} , $1 \leq l, k \leq m_i$ and $l \neq k$, positioned in the same city when the condition $(p_{il} < p_{ik}) \wedge (w_{il} \geq w_{ik})$ holds. This constraint (SC^i) has the form of

$$x_{il} \leq x_{ik}, l \neq k, e_{il}, e_{ik} \in M_i : (p_{il} < p_{ik}) \wedge (w_{il} \geq w_{ik}).$$

Let Δ_l^{ji} denote a lower bound on the cost of transportation of item e_{jl} from city j to succeeding city i computed as

$$\Delta_l^{ji} = R \sum_{a=j}^{i-1} d_a \left(\frac{1}{v_{max} - \nu \left(w_{jl} + \sum_{b=1}^a w_b^c \right)} - \frac{1}{v_{max} - \nu \sum_{b=1}^a w_b^c} \right),$$

where w_b^c is the total weight of the compulsory items collected in city b . Compulsory items must be a part of an optimal solution (see Section 5 for details). Specifically, Δ_l^{ji} is based on the difference between traveling with all the compulsory items and item e_{jl} and traveling with all the compulsory items only. In this case, no other items are picked up and the vehicle runs with the maximal feasible velocity that it may achieve on each of the edges. Then another set of constraints uses the fact that item e_{jl} in city j should not be selected prior to item e_{ik} in city i such that the condition $(p_{jl} - \Delta_l^{ji} < p_{ik}) \wedge (w_{jl} \geq w_{ik})$ holds.

This constraint (SC^{ji}) takes the form of

$$x_{jl} \leq x_{ik}, j < i, e_{jl} \in M_j, e_{ik} \in M_i : (p_{jl} - \Delta_l^{ji} < p_{ik}) \wedge (w_{jl} \geq w_{ik}).$$

Similarly, let $\bar{\Delta}_l^{ji}$ denote an upper bound on the cost of transportation of item e_{jl} from city j to succeeding city i computed as

$$\bar{\Delta}_l^{ji} = R \sum_{a=j}^{i-1} d_a \left(\frac{1}{v_{max} - \nu \cdot \min\left(\sum_{b=1}^a w_b^{max}, W\right)} - \frac{1}{v_{max} - \nu \left(\min\left(\sum_{b=1}^a w_b^{max}, W\right) - w_{jl}\right)} \right),$$

where w_b^{max} is the total weight of all the items existing in city b . Specifically, $\bar{\Delta}_l^{ji}$ represents the difference between traveling having the vehicle loaded as much as possible and doing so but leaving a free space just for item e_{jl} . In this case, as many items as possible are picked up and the vehicle travels with the least feasible velocity that it may achieve on each of the edges. Then one more set of constraints arises from the fact that item e_{ik} in city i should not be selected prior to item e_{jl} in city j such that $(p_{jl} - \bar{\Delta}_l^{ji} > p_{ik}) \wedge (w_{jl} \leq w_{ik})$ holds. This constraint (SCⁱⁱⁱ) has the following form:

$$x_{jl} \geq x_{ik}, \quad j < i, \quad e_{jl} \in M_j, \quad e_{ik} \in M_i : (p_{jl} - \bar{\Delta}_l^{ji} > p_{ik}) \wedge (w_{jl} \leq w_{ik}).$$

5. Pre-processing

In this section, we introduce a pre-processing scheme to identify items of a given instance I that can be either directly included or discarded. Excluding such items from solution process can significantly speed up algorithms. We distinguish between two kinds of items that are identified in the pre-processing: *compulsory* and *unprofitable* items. We call an item *compulsory* if its inclusion in any feasible solution increases the objective function value, and call an item *unprofitable* if it does not do that. Therefore, an optimal solution must contain all compulsory items while all unprofitable items must be discarded. In order to identify *compulsory* and *unprofitable* items, we consider the total transportation cost that a set of items produces.

Definition 1 (Total Transportation Cost). Let $O \subseteq M$ be a subset of items. We define the total transportation cost along route N when the items of O are selected as

$$t_O = R \cdot \sum_{i=1}^n \frac{d_i}{v_{max} - \nu \sum_{j=1}^i \sum_{e_{jk} \in O_j} w_{jk}},$$

where $O_j = M_j \cap O$, $1 \leq j \leq n$, is the subset of O selected in city j .

Based on the given instance I , we can identify unprofitable items for PWT^c according to the following proposition.

Proposition 1 (Unprofitable Item, PWT^c Case). Let I be an arbitrary instance of PWT^c. If $p_{ik} \leq R(t_{\{e_{ik}\}} - t_{\emptyset})$, then e_{ik} is an unprofitable item.

PROOF. We assume that $p_{ik} \leq R(t_{\{e_{ik}\}} - t_\emptyset)$ holds. Let $M^* \subseteq M \setminus \{e_{ik}\}$ denote an arbitrary subset of items excluding e_{ik} such that $w_{ik} + \sum_{e_{jl} \in M^*} w_{jl} \leq W$ holds. We consider $t_{M^* \cup \{e_{ik}\}}$ and t_{M^*} . Since the velocity depends linearly on the weight of collected items and the travel time $t_i = d_i/v_i$ along $(i, i+1)$ depends inversely proportional on the velocity v_i , the inequality $(t_{\{e_{ik}\}} - t_\emptyset) \leq (t_{M^* \cup \{e_{ik}\}} - t_{M^*})$ holds. Therefore, $p_{ik} \leq R(t_{M^* \cup \{e_{ik}\}} - t_{M^*})$ holds for any $M^* \subseteq M \setminus \{e_{ik}\}$ that completes the proof. \square

Proposition 1 helps to determine whether the profit p_{ik} of item e_{ik} is large enough to cover the least incremental transportation cost it incurs when selected in the packing plan x . In this case, the least incremental transportation cost results from accepting the selection of e_{ik} as only selected item in x versus accepting empty x as a solution. It is important to note that Proposition 1 can reduce PWT^c problem to PWT^u by excluding items so that the sum of the weights of all remaining items does not exceed the weight bound W . In this case, we can further refine the set of items by searching for those ones that must be a part of any solution of PWT^c . We identify compulsory items for the unconstrained case according to the following proposition.

Proposition 2 (Compulsory Item, PWT^u Case). *Let I be an arbitrary instance of PWT^u . If $p_{ik} > R(t_M - t_{M \setminus \{e_{ik}\}})$, then e_{ik} is a compulsory item.*

PROOF. We work under the assumption that $p_{ik} > R(t_M - t_{M \setminus \{e_{ik}\}})$ holds. In the case of PWT^u , all the existing items can fit into the vehicle at once and all subsets $O \subseteq M$ are feasible. Let $M^* \subseteq M \setminus \{e_{ik}\}$ be an arbitrary subset of items excluding e_{ik} , and consider $t_{M \setminus M^*}$ and $t_{M \setminus M^* \setminus \{e_{ik}\}}$, respectively. Since the velocity depends linearly on the weight of collected items and the travel time $t_i = d_i/v_i$ along $(i, i+1)$ depends inversely proportional on the velocity v_i , we have $(t_M - t_{M \setminus \{e_{ik}\}}) \geq (t_{M \setminus M^*} - t_{M \setminus M^* \setminus \{e_{ik}\}})$. This implies that $p_{ik} > R(t_{M \setminus M^*} - t_{M \setminus M^* \setminus \{e_{ik}\}})$ holds for any subset $M \setminus M^*$ of items which completes the proof. \square

For the unconstrained variant PWT^u , Proposition 2 is valid to determine whether item e_{ik} is able to cover by its p_{ik} the largest possible incremental transportation cost it may generate when has been selected in x . Here, the largest possible incremental transportation cost is computed with respect to the worst case scenario when all the possible items are selected along with e_{ik} , and therefore the vehicle has the maximal possible load and the least velocity, versus accepting all the items but e_{ik} . In the unconstrained case, having compulsory items included according to Proposition 2, we may identify some more unprofitable items. Indeed, compulsory items contribute to the collected weight and therefore limit the potential positive contribution of other items. As a result, some of the items may become unprofitable after a number of compulsory items has been detected. We find unprofitable items for PWT^u with respect to the following proposition.

Proposition 3 (Unprofitable Item, PWT^u Case). *Let I be an arbitrary instance of PWT^u and M^c be the set of all compulsory items. If $p_{ik} \leq R(t_{M^c \cup \{e_{ik}\}} - t_{M^c})$, then e_{ik} is an unprofitable item.*

PROOF. We assume that $p_{ik} \leq R(t_{M^c \cup \{e_{ik}\}} - t_{M^c})$ holds. Let $M^* \subseteq M \setminus \{M^c \cup \{e_{ik}\}\}$ be an arbitrary subset of M that does not include any item of $M^c \cup \{e_{ik}\}$ and consider $t_{M^c \cup M^*}$ and $t_{M^c \cup M^* \cup \{e_{ik}\}}$. Since the velocity depends linearly on the weight of collected items and the travel time $t_i = d_i/v_i$ along $(i, i+1)$ depends inversely proportional on the velocity v_i , we have $(t_{M^c \cup \{e_{ik}\}} - t_{M^c}) \leq (t_{M^c \cup M^* \cup \{e_{ik}\}} - t_{M^c \cup M^*})$. Hence, we have $p_{ik} \leq R(t_{M^c \cup M^* \cup \{e_{ik}\}} - t_{M^c \cup M^*})$ for any $M^* \subseteq M \setminus \{M^c \cup \{e_{ik}\}\}$ which completes the proof. \square

Proposition 3 determines for PWT^u whether the profit p_{ik} of item e_{ik} is large enough to cover the least incremental transportation cost resulted from its selection along with all known compulsory items. Specifically, in Proposition 3 the list incremental transportation cost follows from accepting the selection of e_{ik} along with the set of compulsory items M^c in x versus accepting just the selection of M^c as a solution. One can see that Proposition 3 is a special case of Proposition 1 with only the difference that it has t_\emptyset replaced by t_{M^c} .

It takes only a linear time to check any instance of PWT^c for unprofitable items with respect to Proposition 1. In fact, each item e_{ik} can be checked in a constant time if the total length of the path from city i to city $n+1$ is known. When dealing with PWT^u, Propositions 2 and 3 can be applied iteratively to the remaining set of items until no compulsory or unprofitable item is found. The running time of all the rounds of the search is bounded by $\mathcal{O}(nm^2)$. Our preliminary investigation has shown that it is rather time-consuming to solve large and even moderate-sized unconstrained instances due to the time spent on computing the incremental transportation cost for each of the items separately as the Propositions 2 and 3 advise. Indeed, we cannot perform the pre-processing step reasonably fast with respect to the time limits we apply in Section 9. Obviously, slow pre-processing can easily stultify all benefits of its use. To manage this, we use the reasoning similar to one that the sequencing constraints adopt in Section 4. Specifically, we deduce whether item e_{ik} is compulsory or unprofitable from the answer concerning item e_{jl} for which it has been already obtained. Algorithm 1 sketches the pseudocode of the enhanced algorithm, which runs in $\mathcal{O}(m^3)$, but operates up to two orders of magnitude faster in practice and allows to handle the largest instances of the test suite (see Section 9 for details).

The pre-processing algorithm works as follows. The loop (6-25) searches for compulsory items and the loop (30-53) searches for unprofitable ones. Once either no compulsory or no unprofitable item has been found within the corresponding loop, the algorithm terminates (cf. lines 26 and 54). We use two Boolean variables μ_{ik}^u and μ_{ik}^c that take value *true* to mark item e_{ik} as unprofitable and compulsory, respectively. Both values are initialized as $\mu_{ik}^u = \mu_{ik}^c = \text{false}$. Subsequently, variable $\bar{w}_i^{max} = \sum_{b=1}^i w_b^{max}$ computes the maxi-

mal possible weight of the items that can be collected in city i and in all the preceding cities. Similarly, variable $\bar{w}_i^c = \sum_{b=1}^i w_b^c$ computes the total weight of compulsory items existing in city i and in all the preceding cities. We use \bar{w}_i^{max} and \bar{w}_i^c to calculate, respectively, the largest possible incremental cost c_{max} and the minimal possible incremental cost c_{min} for each of the items in the loops of our algorithm (cf. lines 8-25 and lines 32-53). Furthermore, to make reasoning on the properties of item e_{ik} with respect to the known properties of item e_{jl} , we introduce the dummy profit p'_{jl} of e_{jl} . Specifically, when item e_{jl} has been shown to be either compulsory or not, or either unprofitable or not, p'_{jl} defines how large or small profit p_{ik} must be with respect to computed c_{max} or c_{min} to let e_{ik} have the same property as e_{jl} (cf. lines 12, 15, 36, and 39). For example, when item e_{ik} is proved to be compulsory independently of any another item (cf. line 25), its p'_{ik} is set to c_{max} (cf. line 24). This means that e_{ik} would be compulsory even if its profit was less than p_{ik} , but mainly greater than c_{max} . Therefore, to become a compulsory item as p_{ik} is, another item, say $e_{i'k'}$ in city $i' : i' \leq i$, should have a weight that is smaller or equal to w_{ik} and its profit $p_{i'k'}$ minus the corresponding largest possible incremental cost must be strictly larger than p'_{ik} (cf. line 15). Similarly, when e_{ik} is proved to be not a compulsory item independently of any other item, its p'_{ik} is also set to c_{max} (cf. line 24). This is because e_{ik} would not be compulsory even if its profit was larger than p_{ik} , but mainly less or equal to c_{max} . Therefore, to stay as not a compulsory item as p_{ik} is, another item, say $e_{i'k'}$ in city $i' : i' \leq i$, should have a weight that is greater or equal to w_{ik} and its profit $p_{i'k'}$ minus the corresponding least possible incremental cost must be at most p'_{ik} (cf. line 12). The same reasoning is to be done for the case when e_{ik} is proved as a compulsory item according to already known compulsory item e_{jl} . Here, p'_{ik} is set to $p'_{jl} + c_{max}$ where c_{max} plays a role of the largest cost of transportation of e_{ik} from city i to succeeding city j (cf. line 17). In such a way, when considering other items in the future iterations, they are compared to item e_{jl} through the current item e_{ik} since e_{ik} implicitly points to e_{jl} using the assigned dummy profit p'_{ik} . The same reasoning is valid for proving e_{ik} to be not a compulsory item according to already known non-compulsory item e_{jl} where p'_{ik} is set to $p'_{jl} + c_{min}$ (cf. line 13). In a similar way, we proceed with deduction of unprofitable items in the loop (30-53). Utilizing the dummy profits of items significantly strengthen deductions by relating the items to one of the items' group rapidly. Before applying our approaches given in Section 6, Section 7, and 8, we remove all unprofitable and compulsory items from the set M using these pre-processing steps.

6. Lower and Upper Bounds

In practice, approximation of nonlinear terms is an efficient way to deal with them. Although an approximate solution is likely to be different from an exact one, it might be close enough and obtainable in a reasonable computational time. In this section, we propose lower and upper bound techniques based on mixed-integer programming (MIP) adopting the ideas of piecewise linear approximation.

Algorithm 1: The Pre-processing Algorithm

```

1 initialize the indicator variables  $\mu_{ik}^u = \mu_{ik}^c = false$  for each item  $e_{ik} \in M$ ;
2 while true do
3   set  $flag \leftarrow false$ ;
4   for each city  $i$  from 1 to  $n$  do
5     calculate  $\bar{w}_i^{max}$ ;
6   for each city  $i$  from  $n$  to 1 do
7     for each item  $e_{ik} \in M_i : \neg(\mu_{ik}^u \vee \mu_{ik}^c)$  do
8        $c_{max} \leftarrow 0$ ;  $c_{min} \leftarrow 0$ ;
9       initialize  $flag' \leftarrow false$ ;
10      for each city  $j$  from  $i$  to  $n$  do
11        for each item  $e_{jl} \in M_j : \neg\mu_{jl}^u \wedge ((i \neq j) \vee (k > l))$  do
12          if  $(\neg\mu_{jl}^c) \wedge (w_{ik} \geq w_{jl}) \wedge (p_{ik} - c_{min} \leq p'_{jl})$  then
13             $p'_{ik} \leftarrow p'_{jl} + c_{min}$ ;
14             $flag' \leftarrow true$ ; break;
15          if  $\mu_{jl}^c \wedge (w_{ik} \leq w_{jl}) \wedge (p_{ik} - c_{max} > p'_{jl})$  then
16             $\mu_{ik}^c \leftarrow true$ ;
17             $p'_{ik} \leftarrow p'_{jl} + c_{max}$ ;
18             $flag \leftarrow true$ ;
19             $flag' \leftarrow true$ ; break;
20        if  $flag'$  then break;
21         $c_{min} \leftarrow c_{min} + Rd_j \left( \frac{1}{v_{max} - \nu(\bar{w}_j^c + w_{ik})} - \frac{1}{v_{max} - \nu\bar{w}_j^c} \right)$ ;
22         $c_{max} \leftarrow c_{max} + Rd_j \left( \frac{1}{v_{max} - \nu\bar{w}_j^{max}} - \frac{1}{v_{max} - \nu(\bar{w}_j^{max} - w_{ik})} \right)$ ;
23      if  $flag'$  then break;
24       $p'_{ik} \leftarrow c_{max}$ ;
25      if  $c_{max} < p_{ik}$  then  $\mu_{ik}^c \leftarrow true$ ;  $flag \leftarrow true$ ;
26  if  $\neg flag$  then break;
27  set  $flag \leftarrow false$ ;
28  for each city  $i$  from 1 to  $n$  do
29    calculate  $\bar{w}_i^c$ ;
30  for each city  $i$  from  $n$  to 1 do
31    for each item  $e_{ik} \in M_i : \neg(\mu_{ik}^u \vee \mu_{ik}^c)$  do
32       $c_{max} \leftarrow 0$ ;  $c_{min} \leftarrow 0$ ;
33      initialize  $flag' \leftarrow false$ ;
34      for each city  $j$  from  $i$  to  $n$  do
35        for each item  $e_{jl} \in M_j : \neg\mu_{jl}^c \wedge ((i \neq j) \vee (k > l))$  do
36          if  $(\neg\mu_{jl}^u) \wedge (w_{ik} \leq w_{jl}) \wedge (p_{ik} - c_{max} > p'_{jl})$  then
37             $p'_{ik} \leftarrow p'_{jl} + c_{max}$ ;
38             $flag' \leftarrow true$ ; break;
39          if  $\mu_{jl}^u \wedge (w_{ik} \geq w_{jl}) \wedge (p_{ik} - c_{min} \leq p'_{jl})$  then
40             $\mu_{ik}^u \leftarrow true$ ;
41             $p'_{ik} \leftarrow p'_{jl} + c_{min}$ ;
42             $flag \leftarrow true$ ;
43             $flag' \leftarrow true$ ; break;
44        if  $flag'$  then break;
45         $c_{min} \leftarrow c_{min} + Rd_j \left( \frac{1}{v_{max} - \nu(\bar{w}_j^c + w_{ik})} - \frac{1}{v_{max} - \nu\bar{w}_j^c} \right)$ ;
46        if  $c_{min} \geq p_{ik}$  then
47           $\mu_{ik}^u \leftarrow true$ ;
48           $p'_{ik} \leftarrow c_{min}$ ;
49           $flag \leftarrow true$ ;
50           $flag' \leftarrow true$ ; break;
51         $c_{max} \leftarrow c_{max} + Rd_j \left( \frac{1}{v_{max} - \nu\bar{w}_j^{max}} - \frac{1}{v_{max} - \nu(\bar{w}_j^{max} - w_{ik})} \right)$ ;
52      if  $flag'$  then break;
53       $p'_{ik} \leftarrow c_{min}$ ;
54  if  $\neg flag$  then break;

```

6.1. Lower Bound

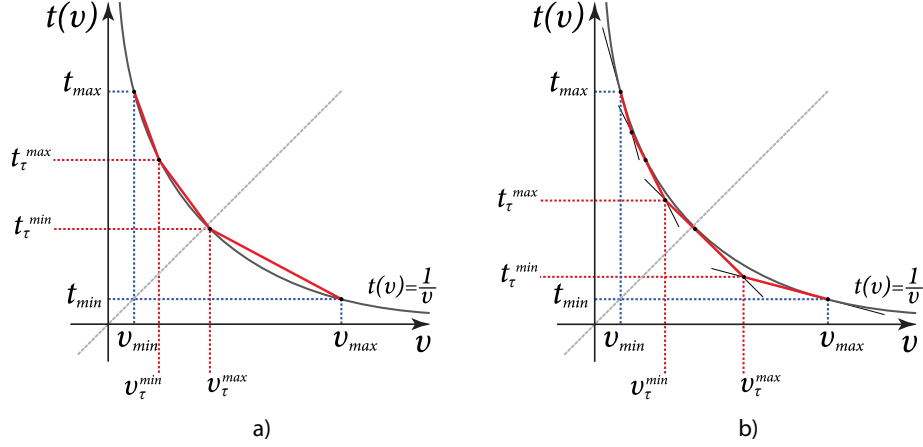


Figure 1: Piecewise linear approximation of $t(v) = 1/v$

Consider an arbitrary edge $(i, i + 1)$ and the traveling time $t'_i \in [t_{\min}, t_{\max}]$ per distance unit along it, for any $i = 1, \dots, n$. Here, $t_{\min} = 1/v_{\max}$ and $t_{\max} = 1/v_{\min}$ bound t'_i from below and from above, respectively. We partition the interval $[t_{\min}, t_{\max}]$ into λ equal-sized sub-intervals and determine thus a set $T = \{\tau_1, \dots, \tau_\lambda\}$ of straight line segments to approximate the curve of the function $t(v)$ as illustrated in Figure 1a. Each segment $\tau \in T$ is characterized by its minimal velocity v_τ^{\min} and its corresponding maximum traveling time per distance unit t_τ^{\max} , and by its maximum velocity v_τ^{\max} and its corresponding minimum traveling time per distance unit t_τ^{\min} . Specifically, $(v_\tau^{\min}, t_\tau^{\max})$ and $(v_\tau^{\max}, t_\tau^{\min})$ are the endpoints of segment τ referred to as breakpoints. We approximate t'_i by the linear combination of t_τ^{\min} and t_τ^{\max} if $v_i \in [v_\tau^{\min}, v_\tau^{\max}]$.

Our lower bound MIP-based model uses three types of variables in addition to the binary decision variable x_{ik} for each item $e_{ik} \in M$ from Section 2. Let w_i be a real variable equal to the total weight of selected items when traveling along the $(i, i + 1)$. Let p_i be a real variable equal to the difference of the total profit of selected items and their total transportation cost when delivering them to city $i + 1$. Let $T_i \subseteq T$, $1 \leq i \leq n$, denote a set of possible segments to which velocity v_i of the vehicle may relate, i.e. $T_i = \{\tau \in T : (v_\tau^{\min} \in [v_i^{\min}, v_i^{\max}]) \vee (v_\tau^{\max} \in [v_i^{\min}, v_i^{\max}])\}$, where $v_i^{\max} = v_{\max} - \nu \sum_{j=1}^i w_j^c$ is the maximal possible velocity that the vehicle can move along $(i, i + 1)$ when packing in all compulsory items only, and $v_i^{\min} = v_{\max} - \nu \cdot \min\left(\sum_{j=1}^i w_j^{\max}, W\right)$ the minimum possible velocity along $(i, i + 1)$ after having packed in all items available in cities $1, \dots, i$. Actually, we have $v_i \in [v_i^{\min}, v_i^{\max}]$. When $v_i \in [v_\tau^{\min}, v_\tau^{\max}]$ for $\tau \in T$, any point in between endpoints of τ is a weighted sum of them. Let B_i denote a set of all break-

points that the linear segments of T_i have. Then the value of the real variable $y_{ib} \in [0, 1]$ is a weight assigned to the breakpoint $b \in B_i$ associated with the pair of values (v_b, t_b) . When the linear combination $\sum_{b \in B_i} v_b y_{ib}$ under the constraint $\sum_{b \in B_i} y_{ib} = 1$ equals v_i , the linear combination $\sum_{b \in B_i} t_b y_{ib}$ overestimates t'_i . This underestimates the resulting profit minus the total transportation cost and gives a valid lower bound for PWT^c (and PWT^u) that can be obtained by solving the following linear mixed 0-1 program (LB^λ):

$$\max p^{LB}(x) = p_n \quad (2)$$

$$\text{s.t. } p_i = p_{i-1} + p_i^c + \sum_{e_{ik} \in M_i} p_{ik} x_{ik} - Rd_i \sum_{b \in B_i} t_b y_{ib}, \quad i = 1, \dots, n \quad (3)$$

$$w_i = w_{i-1} + w_i^c + \sum_{e_{ik} \in M_i} w_{ik} x_{ik}, \quad i = 1, \dots, n \quad (4)$$

$$\nu w_i + \sum_{b \in B_i} v_b y_{ib} = v_{max}, \quad i = 1, \dots, n \quad (5)$$

$$\sum_{b \in B_i} y_{ib} = 1, \quad i = 1, \dots, n \quad (6)$$

$$w_n \leq W \quad (7)$$

$$x_{ik} \in \{0, 1\}, \quad e_{ik} \in M \quad (8)$$

$$y_{ib} \in [0, 1], \quad i = 1, \dots, n, \quad b \in B_i \quad (9)$$

$$p_i \in \mathbb{R}, \quad i = 1, \dots, n \quad (10)$$

$$w_i \in \mathbb{R}_{\geq 0}, \quad i = 1, \dots, n \quad (11)$$

$$p_0 = w_0 = 0 \quad (12)$$

The value of λ in LB^λ sets its precision. Indeed, the precision of the lower bound may be increased at the cost of a running time as this also increases the number of segments, and thus raises the number of y -type variables to be involved. Equation (2) defines the objective function $p^{LB}(x)$ as p_n that is the difference of the total profit of selected items delivered to city $n + 1$ and their total transportation cost. Since the transportation cost is approximated in LB^λ , the actual objective value for PWT^c (and PWT^u) should be computed on the values of the decision variables of vector x . The resulting value then is also a valid lower bound. Equation (3) computes the difference p_i of the total profit of selected items and their total transportation cost when arriving at city $i + 1$ by summing up the value of p_{i-1} concerning $(i - 1, i)$, the profit of compulsory items p_i^c and the profit $\sum_{e_{ik} \in M_i} p_{ik} x_{ik}$ of items selected in city i , and subtracting the approximated transportation cost along $(i, i + 1)$. Equation (4) gives the weight w_i of the selected items when the vehicle departs city i by summing up w_{i-1} , the weight of compulsory items w_i^c and the weight $\sum_{e_{ik} \in M_i} w_{ik} x_{ik}$ of items selected in city i . Remind that we determine compulsory items according to the Proposition 2 of Section 5 when the problem is unconstrained. Equation (5) implicitly defines segment $\tau \in T_i$ to which the velocity of the vehicle v_i belongs and sets the weights for its endpoints. Equation (6) forces the total weight of the breakpoints of B_i be exactly 1. Equation (7) imposes the capacity constraint, and Eq. (8) declares x_{ik} as binary. Equation (9) states y_{ib} as a real

variable defined in $[0, 1]$. Equation (10) declares p_i as a real variable, while Eq. (11) defines w_i as a non-negative real. Finally, Equation (12) establishes the base cases for p_0 and w_0 . Obviously, one can relax the integrality imposed on the x -type variables that leads to a linear programming model. In fact, this generally worsens the lower bound value, but gives an advantage in running time.

6.2. Upper Bound

We now describe the upper bound technique that adopts the piecewise linear approximation proposed for the lower bound. This time, our goal is to underestimate the traveling time t'_i that the vehicle spends to pass a distance unit when traveling along the $(i, i + 1)$, for any $i = 1, \dots, n$. We utilize the same set of breakpoints B_i generated from the set of linear segments T_i . In each point $b \in B_i$, we draw a tangent to the curve of the function $t(v) = 1/v$ as depicted in Figure 1b. Subsequently, a new set of points \bar{B}_i is derived from the left and the rightmost points of B_i , and the points of intersection of each pair of neighboring tangents. This yields totally $|B_i| + 1$ points that produce a new set of $|B_i|$ linear segments \bar{T}_i resulted from connecting every two closest points in \bar{B}_i . Then a valid upper bound for PWT^c (and PWT^u) can be obtained via the model of LB^λ with only the difference that the set of breakpoints \bar{B}_i is used instead of B_i . We designate this altered model as UB^λ and the corresponding objective function as $p^{UB}(x)$. Again, one can manage precision of the upper bound adjusting the value of λ . Furthermore, the integrality imposed on the x -type variables may be relaxed to speed up computations at the price of the upper bound's quality.

7. Mixed-Integer Programming-Based Approach

Both PWT^c and PWT^u belong to the specific class of fractional binary programming problems for which several efficient reformulation techniques exist to handle nonlinear terms. We follow the approach of Li (1994) and Tawarmalani et al. (2002) to reformulate PWT^c (and PWT^u) as a linear mixed 0-1 program. It is applicable since the denominator of each fractional term in (1) is not equal to zero since $v_{min} > 0$. We start with introduction of auxiliary real-valued variables y_i , $i = 1, \dots, n$, such that $y_i = 1 / \left(v_{max} - \nu \sum_{j=1}^i \sum_{k=1}^{m_j} w_{jk} x_{jk} \right)$. The variables y_i express the travel time per distance unit along the edge $(i, i + 1)$. According to Li (1994), we can reformulate PWT^c as a mixed 0-1 quadratic program by replacing (1) with (13) and adding the set of constraints (14) and (15).

$$\max \sum_{i=1}^n \left(\sum_{k=1}^{m_i} p_{ik} x_{ik} - R d_i y_i \right) \quad (13)$$

$$\text{s.t. } v_{max} y_i + \nu \sum_{j=1}^i \sum_{k=1}^{m_j} w_{jk} x_{jk} y_i = 1, \quad i = 1, \dots, n \quad (14)$$

$$y_i \in \mathbb{R}_+, \quad i = 1, \dots, n \quad (15)$$

According to Tawarmalani et al. (2002), if $z = xy$ is a polynomial mixed 0-1 term where x is binary and y is a real-valued variable, then it can be linearized via the set of linear inequalities: (i) $z \leq Ux$; (ii) $z \geq Lx$; (iii) $z \leq y + L(x - 1)$; (iiii) $z \geq y + U(x - 1)$. Here, U and L are the upper and lower bounds on y , i.e. $L \leq y \leq U$. We can linearize the $x_{jk}y_i$ term in (14) by introducing a new real-valued variable $z_{jk}^i = x_{jk}y_i$ and new linear constraints. Let p_i^c and w_i^c denote the total profit and the total weight of the compulsory items in city i obtained with respect to Proposition 2. Similarly, let w_i^{max} be the total weight of the items (including all the compulsory items) in city i . Then variable y_i , $i = 1, \dots, n$, can be bounded from below by $L_i = 1 / (v_{max} - \nu \sum_{j=1}^i w_j^c)$ and from above by $U_i = 1 / (v_{max} - \nu \cdot \min(\sum_{j=1}^i w_j^{max}, W))$. In summary, we can formulate PWT^c (and PWT^u) as the following linear mixed 0-1 program (MIP ^{λ}):

$$\begin{aligned} \max p^{MIP}(x) &= \sum_{i=1}^n \left(p_i^c + \sum_{k=1}^{m_i} p_{ik} x_{ik} - R d_i y_i \right) \\ \text{s.t. } v_{max} y_i + \nu &\left(w_i^c + \sum_{j=1}^i \sum_{k=1}^{m_j} w_{jk} z_{jk}^i \right) = 1, \quad i = 1, \dots, n \\ z_{jk}^i &\leq U_i x_{jk}, \quad i, j = 1, \dots, n, \quad j \leq i, \quad e_{jk} \in M_j \\ z_{jk}^i &\geq L_i x_{jk}, \quad i, j = 1, \dots, n, \quad j \leq i, \quad e_{jk} \in M_j \\ z_{jk}^i &\geq y_i + U_i (x_{jk} - 1), \quad i, j = 1, \dots, n, \quad j \leq i, \quad e_{jk} \in M_j \\ z_{jk}^i &\leq y_i + L_i (x_{jk} - 1), \quad i, j = 1, \dots, n, \quad j \leq i, \quad e_{jk} \in M_j \\ \sum_{i=1}^n \sum_{k=1}^{m_i} w_{ik} x_{ik} &\leq W \end{aligned} \tag{16}$$

$$\begin{aligned} p^{LB}(x) &\leq p^{MIP}(x) \leq p^{UB}(x) \\ x_{ik} &\in \{0, 1\}, \quad e_{ik} \in M \\ z_{jk}^i &\in \mathbb{R}_+, \quad i, j = 1, \dots, n, \quad j \leq i, \quad e_{jk} \in M_j \\ y_i &\in \mathbb{R}_+, \quad i = 1, \dots, n \end{aligned} \tag{17}$$

A solution of LB ^{λ} can be used as a starting solution for MIP ^{λ} and can yield the lower bound value $p^{LB}(x)$. In its turn, UB ^{λ} can provide the upper bound value $p^{UB}(x)$. To set the value of λ to be used in both LB ^{λ} and UB ^{λ} , we specify its value through the notation MIP ^{λ} . To strengthen the relaxation of MIP ^{λ} , the sequencing constraints of Section 4 can be imposed as valid inequalities as has been earlier proposed in Polyakovskiy & Neumann (2015). However, our current investigations show that they are not beneficial anymore when the upper bound produced by UB ^{λ} is applied in Eq. 17. An effective set of inequalities in order to obtain tighter relaxations can be obtained from the reformulation-linearization technique (RLT) by Sherali & Adams (1999), which uses $3n$ additional inequalities for the capacity constraint (16). Specifically, multiplying (16) by y_l , $U_l - y_l$ and $y_l - L_l$, $l = 1, \dots, n$, we obtain the following inequalities:

$$\begin{aligned}
\sum_{i=1}^n \sum_{k=1}^{m_i} w_{ik} z_{ik}^l &\leq W y_l; \\
U_l \sum_{i=1}^n \sum_{k=1}^{m_i} w_{ik} x_{ik} - \sum_{i=1}^n \sum_{k=1}^{m_i} w_{ik} z_{ik}^l &\leq U_l W - W y_l; \\
\sum_{i=1}^n \sum_{k=1}^{m_i} w_{ik} z_{ik}^l - L_l \sum_{i=1}^n \sum_{k=1}^{m_i} w_{ik} x_{ik} &\leq W y_l - L_l W.
\end{aligned}$$

8. Branch-Infer-and-Bound Approach

Constraint programming (CP) has been shown to be a promising solution technique for various combinatorial optimization problems (Rossi et al. (2006, 2008)). It deals with a problem consisting of a set of variables $X = \{x_1, \dots, x_n\}$ and a finite set of constraints C given on the elements of X . Each variable $x_i \in X$ is associated with a domain D_i of available values. When the domain of every variable x_i is reduced to a singleton $\{v_i\}$, a values vector $v = (v_1, \dots, v_n)$ is obtained. A satisfiability problem asks for a decision vector $x = v$ such that all the constraints in C are satisfied simultaneously. A constraint optimization problem involves in addition an objective function $f(x)$ that is to be either maximized or minimized over the set of all feasible solutions. In CP, constraints are given in a declarative way, but are viewed individually as special-purpose procedures that operate on a solution space. Each procedure applies a filtering algorithm that eliminates those values from the domains of the involved variables which cannot be a part of any feasible solution with respect to that constraint. The restricted domains generated by the constraints are in effect elementary in-domain constraints that restrict a variable to a domain of possible values. They become a part of a constraint store. To link all the procedures together in order to solve a problem as a whole, the constraint store is passed on to the next constraint to be processed. In such a way, the results of one filtering procedure are propagated to the others. In general, filtering algorithms are called repeatedly to achieve a certain level of consistency. This is because achieving arc consistency for one constraint might make other constraints inconsistent. Specifically, constraint $c \in C$ involving variables x_i and x_j is said to be arc consistent with respect to x_i if for each value $v' \in D_i$ there is an allowed value of x_j . A constraint satisfaction problem is arc consistent iff every constraint $c \in C$ is arc consistent with respect to x_i as well as to x_j . Therefore, multiple runs of filtering are required for those constraints that share common variables of X . This process is called constraint propagation. CP aims to enumerate solutions with respect to the constraint store in order to find the best feasible solution. To cope with this, a search tree is used, and every variable x_i with domain D_i is examined in some node of the tree. If $D_i = \emptyset$, an infeasible solution is found. If $|D_i| > 1$, one can branch on x_i by partitioning D_i into smaller domains, each corresponding to a branch. The domains of the variables decrease as they are reduced via constraint propagation when one descends into the tree. In the

case of the constraint optimization problem, the search continues unless either the best solution is determined over those solutions where all the domains are singletons, or at least one of the domains is empty for every leaf node of the search tree. Certainly, the order in which the variables are instantiated and how the domains are partitioned matters for a running time.

Combining CP with the branch-and-cut method is a natural hybridization which results from the complementary strengths of both techniques. It gives rise to the so-called branch-infer-and-relax (BIR) approach presented by Bockmayr & Hooker (2005). The idea of BIR is to combine filtering and propagation used in CP with relaxation and cutting plane generation used in MIP. In each node of a search tree, constraint propagation creates a constraint store of in-domain constraints, while polyhedral relaxation creates a constraint store of inequalities. The two constraint stores can enrich each other, since reduced domains impose bounds on variables, and bounds on variables can reduce domains. The inequality relaxation is solved to obtain a bound on the optimal value, which prunes the search tree as in the branch-and-cut method.

Here, to solve PWT^c (and PWT^u), we adopt this idea and introduce a branch-infer-and-bound approach that compounds CP and the upper bound introduced in Section 6.2. Specifically, we substitute the relaxation used in BIR with a stand-alone upper bound procedure to be executed in each node of the search tree in order to prune some of its branches. In each node, we create a refined set of items $M' = M \setminus \cup_{e_{ik} \in M} e_{ik} : |D_{ik}| = 1$ and add the weight and profit of those items whose $D_{ik} = \{1\}$ to w_i^c and w_i^{max} in the model of $p^{UB}(x)$, respectively. In other words, we treat the items accepted by the search as compulsory. Finally, we apply UB^λ to x formed on M' and prune a branch if the resulting $p^{UB}(x)$ is smaller than the objective value of the best incumbent solution known.

Similarly to the previous MIP formulations, our CP model bases the search on binary decision vector x where variable x_{ik} takes the value of 1 to indicate that item $e_{ik} \in M$ is chosen. To speed up computations, it employs an auxiliary integer variable w_i that calculates the total weight of the items selected in city i and all the preceding cities when traveling along the edge $(i, i+1)$, for any $i = 1, \dots, n$. Again, w_i^c and p_i^c denote the total weight and the total profit of compulsory items collected in city i . Here, we assume that the both values come out of the pre-processing step. The model has the following objective function and constraints (BIB^λ):

$$\max p^{BIB}(x) = \sum_{i=1}^n \left(p_i^c + \sum_{k=1}^{m_i} p_{ik} x_{ik} - \frac{Rd_i}{v_{max} - \nu w_i} \right) \quad (18)$$

$$\text{s.t. } w_i = w_{i-1} + w_i^c + \sum_{e_{ik} \in M_i} w_{ik} x_{ik}, \quad i = 1, \dots, n \quad (19)$$

$$\sum_{i=1}^n \left(w_i^c + \sum_{k=1}^{m_i} w_{ik} x_{ik} \right) \leq W \quad (20)$$

$$x_{il} \leq x_{ik}, \quad i = 1, \dots, n, \quad e_{il}, e_{ik} \in M_i : (p_{il} < p_{ik}) \wedge (w_{il} \geq w_{ik}), \quad l \neq k, \quad (21)$$

$$x_{jl} \leq x_{ik}, \quad i = 1, \dots, n, \quad j < i, \quad e_{jl} \in M_j, \quad e_{ik} \in M_i : (p_{jl} - \Delta_l^{ji} < p_{ik}) \wedge (w_{jl} \geq w_{ik}) \quad (22)$$

$$x_{jl} \geq x_{ik}, \quad i = 1, \dots, n, \quad j < i, \quad e_{jl} \in M_j, \quad e_{ik} \in M_i : (p_{jl} - \overline{\Delta}_l^{ji} > p_{ik}) \wedge (w_{jl} \leq w_{ik}) \quad (23)$$

$$\text{sequencing}(x_{jl}, [w_1, \dots, w_n]), \quad j = 1, \dots, n,$$

$$e_{jl} \in M_j : (\exists e_{ik} \in M_i, j < i : p_{jl} - \overline{\Delta}_l^{ji} \leq p_{ik} \leq p_{jl} - \Delta_l^{ji}) \quad (24)$$

$$x_{ik} \in \{0, 1\}, \quad e_{ik} \in M \quad (25)$$

$$w_i \in \{0, \dots, W\}, \quad i = 1, \dots, n \quad (26)$$

$$w_0 = 0 \quad (27)$$

Function 18 represents the objective function of the problem. For each edge $(i, i+1)$, $i = 1, \dots, n$, it sums up the profits of items taken in city i minus the cost of transportation of all the items that have been placed to the vehicle in city i and all the cities prior to i . Equation (19) calculates the weight w_i of all the items taken in the cities $1, \dots, i$. Equation (20) is a capacity constraint. Equations (21), (22) and (23) impose the set of redundant sequencing constraints SC^i , SC^{ii} , and SC^{iii} of Section 4, respectively. This set may be rather small, and therefore might have a limited impact on inference of in-domain constraints during the search. Indeed, more constraints might be involved if w_b^c in Δ_l^{ji} of constraint SC^{ii} was larger and w_b^{max} in $\overline{\Delta}_l^{ji}$ of constraint SC^{iii} was smaller. The values of these two variables w_b^c and w_b^{max} can in fact be considered as initial lower and upper bounds on the weight of the items collected in city b . As one descends into the tree, items are either collected or rejected. Being picked up in some city, an item contributes its weight that increases the lower bound. Being rejected, it lowers the upper bound. We include those constraints into the pool that still may work out when the lower or upper bound on the weight reaches a certain level. Specifically, (24) adds a redundant customized constraint for each item e_{jl} that has at least one related item e_{ik} , $j < i$, such that their mutual sequencing depends on the weight of the items collected in cities $j, \dots, i-1$. Equations (25) and (26) define the domains of the variables. Finally, Equation (27) sets the base case $w_0 = 0$.

We assume a depth-first search strategy for traversing the binary search tree and instantiate variables in the order in which the cities appear in N . No order is given to the items within the same city. Therefore, at the moment when item e_{ik} is to be instantiated by the search, the domains of the variables associated with the items in cities $1, \dots, i-1$ and with those in city i that appear prior

to e_{ik} in the set M_i have been already reduced to singletons. Accordingly, the domain of variable w_i is reduced to a singleton once the decision variables of the items in city i have been all fixed to singletons.

The customized sequencing constraint **sequencing**($x_{jl}, [w_1, \dots, w_n]$) applies to variable x_{jl} for which there exists at least one item, say x_{ik} , such that $p_{jl} - \bar{\Delta}_l^{ji} \leq p_{ik} \leq p_{jl} - \Delta_l^{ji}$ holds and $j < i$. Algorithm 2 sketches the pseudocode of the corresponding filtering algorithm. We use a Boolean variable $\Theta_{e_{jl}e_{ik}}$, which takes value *true* to indicate the situation when selection of items e_{jl} and e_{ik} may be potentially sequenced. First, the algorithm initializes variables \bar{w}^{max} and \bar{w}^{min} that, respectively, represent the upper and lower bounds on the weight of collected items that the vehicle has in city i . It sets both variables to the sum of weights of items collected in the cities prior to city j , i.e. w_{j-1} , and adds the weight of item e_{jl} if its variable x_{jl} has been fixed to 1 (cf. line 1). Then the algorithm starts exploring the items positioned in the cities succeeding j . Each time the next city i is taken into consideration, it adds the weight of all the items existing in i to \bar{w}^{max} and the weight of all the compulsory items in i to \bar{w}^{min} (cf. line 5). In each city i , the algorithm examines the items that the city contains. When the corresponding variable x_{ik} of item e_{ik} is a singleton, the algorithm modifies the bounds on collected weight accordingly (cf. lines 10 and 11). Subsequently, if $\Theta_{e_{jl}e_{ik}}$ is *true*, it tries to establish a sequencing relation between e_{ik} and e_{jl} . If the condition in line 13 holds, it excludes 1 from domain D_{ik} , and therefore declines item e_{ik} since e_{jl} is assumed to be rejected by the search. At the same time, it lowers \bar{w}^{max} by the value of w_{ik} . On the other hand, if the condition in line 18 holds, it excludes 0 from domain D_{ik} , and therefore selects item e_{ik} because e_{jl} is assumed to be selected. In addition, it increases \bar{w}^{min} by the value of w_{ik} . If no relation has been established at that stage, the algorithm adds e_{ik} to the set of items M^* . Continuing to explore other items, it tries to prove each item of M^* to be unprofitable every time it finishes investigating city i (cf. line 31). Furthermore, having all the cities analyzed, the algorithm returns back to set M^* and tries to prove each of its items to be compulsory if the instance I at hands refers to PWT^u (cf. line 37).

9. Computational Experiments

In this section, we investigate the effectiveness of the proposed approaches by experimental studies. On the one hand, we assess the advantage of the pre-processing scheme in terms of quantity of discarded items and auxiliary decision variables. On the other hand, we evaluate our LB^λ , MIP^λ , and BIB^λ models in terms of solution quality and running time. The program code is implemented in JAVA using the IBM OPTIMIZATION STUDIO 12.6.2. To solve the mixed-integer programs LB^λ , UB^λ and MIP^λ , we use CPLEX with default settings. When running UB^λ , we increase CPLEX's precision by setting the relative tolerance on the gap between the best integer objective and the objective of the best node remaining to $1e-7$. To solve the constrained program within BIB^λ , we use CP OPTIMIZER switched to the depth-first search mode and set

Algorithm 2: The Filtering Algorithm of Constraint sequencing($x_{jl}, [w_1, \dots, w_n]$)

```

1 initialize  $\overline{w}^{max} \leftarrow w_{j-1} + w_{jl}x_{jl}$ ; initialize  $\overline{w}^{min} \leftarrow w_{j-1} + w_{jl}x_{jl}$ ;
2 initialize  $M^* \leftarrow \emptyset$ ;
3 set  $\overline{\Delta}_l^{jj} \leftarrow 0$ ;  $\Delta_l^{jj} \leftarrow 0$ ;
4 for each city  $i$  from  $j$  to  $n$  do
5    $\overline{w}^{max} \leftarrow \overline{w}^{max} + w_i^{max}$ ;  $\overline{w}^{min} \leftarrow \overline{w}^{min} + w_i^c$ ;
6   for each item  $e_{ik} \in M_i, e_{ik} \neq e_{jl}$  do
7     initialize  $flag \leftarrow true$ ;
8     if  $DomainSize(D_{ik}) \leq 1$  then
9        $flag \leftarrow false$ ;
10      if  $x_{ik} = 0$  then  $\overline{w}^{max} \leftarrow \overline{w}^{max} - w_{ik}$ ;
11      if  $x_{ik} = 1$  then  $\overline{w}^{min} \leftarrow \overline{w}^{min} + w_{ik}$ ;
12      if  $\Theta_{e_{jl}e_{ik}}$  then
13        if  $(w_{jl} \leq w_{ik}) \wedge (p_{jl} - \overline{\Delta}_l^{ji} > p_{ik}) \wedge (x_{jl} = 0)$  then
14          if  $DomainSize(D_{ik}) = 2$  then
15             $\overline{w}^{max} \leftarrow \overline{w}^{max} - w_{ik}$ ;
16             $flag \leftarrow false$ ;
17            RemoveValue( $D_{ik}, 1$ );
18          if  $(w_{jl} \geq w_{ik}) \wedge (p_{jl} - \Delta_l^{ji} < p_{ik}) \wedge (x_{jl} = 1)$  then
19            if  $DomainSize(D_{ik}) = 2$  then
20               $\overline{w}^{min} \leftarrow \overline{w}^{min} + w_{ik}$ ;
21               $flag \leftarrow false$ ;
22              RemoveValue( $D_{ik}, 0$ );
23      if  $flag$  then
24         $M^* \leftarrow M^* \cup \{e_{ik}\}$ ;
25        initialize  $\overline{\Delta}_k^{in+1} \leftarrow 0$ ;  $\Delta_k^{in+1} \leftarrow 0$ ;
26       $\overline{\Delta}_l^{ji+1} \leftarrow \overline{\Delta}_l^{ji} + Rd_i \left( \frac{1}{v_{max} - \nu \min(\overline{w}^{max}, W)} - \frac{1}{v_{max} - \nu \min(\overline{w}^{max}, W) - w_{ik}} \right)$ ;
27       $\Delta_l^{ji+1} \leftarrow \Delta_l^{ji} + Rd_j \left( \frac{1}{v_{max} - \nu \min(\overline{w}^{min}, W) + w_{ik}} - \frac{1}{v_{max} - \nu \min(\overline{w}^{min}, W)} \right)$ ;
28      for each item  $e_{ab} \in M^*$  do
29         $\overline{\Delta}_b^{an+1} \leftarrow \overline{\Delta}_b^{an+1} + Rd_i \left( \frac{1}{v_{max} - \nu \min(\overline{w}_j^{max}, W)} - \frac{1}{v_{max} - \nu \min(\overline{w}_j^{max}, W) - w_{ab}} \right)$ ;
30         $\Delta_b^{an+1} \leftarrow \Delta_b^{an+1} + Rd_j \left( \frac{1}{v_{max} - \nu \min(\overline{w}^{min}, W) + w_{ab}} - \frac{1}{v_{max} - \nu \min(\overline{w}^{min}, W)} \right)$ ;
31        if  $p_{ab} - \Delta_b^{an+1} \leq 0$  then
32          RemoveValue( $D_{ab}, 1$ );
33           $\overline{w}^{max} \leftarrow \overline{w}^{max} - w_{ab}$ ;
34           $M^* \leftarrow M^* \setminus \{e_{ab}\}$ ;
35 if  $ProblemType(I) \equiv PWT^u$  then
36   for each item  $e_{ab} \in M^*$  do
37     if  $p_{ab} - \overline{\Delta}_b^{an+1} > 0$  then RemoveValue( $D_{ab}, 0$ );

```

the relative tolerance gap to 0. Furthermore, we limit the parallel mode to only a single thread for CPLEX and CP OPTIMIZER to make them both comparable to each other when dealing with the small size instances in Section 9.1. We set the number of threads to the maximum number of cores available when investigating the large size instances in Section 9.2.

The test instances are adopted from the benchmark set B of Polyakovskiy et al. (2014). This benchmark set is constructed on TSP instances from TSPLIB introduced by Reinelt (1991) augmented by a set of items distributed among all the cities but the first one. We use the set of items available in each city and obtain the route from the corresponding TSP instance by running the Chained Lin-Kernighan heuristic proposed by Applegate et al. (2003). Given the permu-

tation $\pi = (\pi_1, \pi_2, \dots, \pi_n)$ of the cities computed by the Chained Lin-Kernighan heuristic, where π_1 is free of items, we use $N = (\pi_2, \pi_3, \dots, \pi_n, \pi_1)$ as the route for our problem. We consider the *uncorrelated* (uncorr), *uncorrelated with similar weights* (uncorr-s-w), and *bounded strongly correlated* (b-s-corr) types of items' generation, and set v_{min} and v_{max} to 0.1 and 1 as proposed for B .

9.1. Computational Experiments on the Set of Small Size Instances

Here, using a set of small instances, our goal is to evaluate the performance of our pre-processing scheme and the performance of the proposed approximate and exact approaches. Unlike experiments carried out in our earlier research (Polyakovskiy & Neumann (2015)), here we are able to find optimal solutions to all the small instances within the same time limit. Therefore, the main focus of our investigation with respect to the exact approaches is their running times rather than any qualitative performance measures.

We study three families of small size instances based on the TSP problems **eil51**, **eil76**, and **eil101** with 51, 76 and 101 cities, respectively. This series of experiments has been carried out on PC with 4 Gb RAM and a 3.06 GHz Dual Core processor. The results of the experiments are shown in Table 1. All the instances of a family have the same route N . We consider instances with 1, 5, and 10 items per city. The postfixes 1, 6 and 10 in the instances' names indicate the vehicle's capacity W . The greater the value of a postfix is, the larger W is given. Column 2 specifies the total number of items m . Ratio $\alpha = 100 \cdot (m - m') / m$ in Column 3 denotes a percentage of items discarded in a pre-processing step, where m' is the number of items left after pre-processing. Column *ver* identifies by “ u ” whether PWT^c has been reduced to PWT^u by pre-processing. Columns 5-7 report results for LB^λ with $\lambda = 100$. Specifically, column 5 gives ρ as a ratio between the lower bound obtained by LB^{100} and the optimum obtained by the branch-infer-and-bound approach. Column 6 contains the running time t of LB^{100} . Column 7 shows a rate β that is a percentage of auxiliary y -type variables used in practice by LB^λ . At most λn variables is required by LB^λ . Thus, β is computed as $\beta = 100 \cdot (\sum_{i=1}^n |B_i|) / (\lambda n)$. Column 8 presents the running time t for the MIP-based exact approach MIP^λ when λ is set to 1000. Therefore, both LB^λ and UB^λ incorporated into MIP^λ use $\lambda = 1000$ as well to compute initial lower and upper bounds. The time limit of 1 day has been given to MIP^λ in total, while LB^λ and UB^λ have got the time limit of 2 hours each. The running time of MIP^λ provided in the table includes the total time taken by LB^λ and UB^λ . For most of the instances except the instance “**uncorr-s-w.01**” of the family **eil101**, this time is found negligible. Column 9 reports ω as a relative gap in percents that compares the running time of MIP^λ to the smallest running time over all the algorithms studied in the experiment. In general, ω is to be computed as $\omega = 100 \cdot (t^{ALG} - t^{MIN}) / t^{MIN}$, where t^{ALG} is the running time of a particular algorithm and t^{MIN} is the minimum over the running times of the various configurations of MIP^λ and BIB^λ investigated here.

The rest columns of the table describe results for the branch-infer-and-bound approach BIB^λ with $\lambda \in \{500, 1000, 1500\}$. Furthermore, two cases of BIB^λ for

$\lambda = 1000$ have been studied. BIB^{1000} is exactly that one which is described in Section 8. $\text{BIB}_{no\ seq.}^{1000}$ is its copy that does not include the customized sequencing constraints to the model. In such way, we evaluate the impact of the constraint on the performance of BIB^λ . We employ LB^{100} , which we give 2 hours of running time limit, to provide BIB^λ with a lower bound to be used then to prune the search tree. Within BIB^{500} , BIB^{1000} , $\text{BIB}_{no\ seq.}^{1000}$, and BIB^{1500} , we run the variant of UB^λ where the integrality constraints on x -type decision variables are removed. This makes UB^λ a linear program and significantly speeds up computations at the very small cost of solution quality. Each of the columns t reports the total computational time for the corresponding BIB^λ and includes the time taken by LB^{100} . Similarly, each of the columns named ω does when reporting ω that compares the running time of BIB^λ to the smallest running time found over the studied MIP^λ and BIB^λ approaches. The least running time obtained for a particular instance is marked by bold and underlined in the entry t of the corresponding approach. The entries of the table marked by “-” indicate that optimal solutions have not been obtained within the given time limit.

9.1.1. Performance of the Pre-processing Scheme

Here, we evaluate the performance of our pre-processing scheme by calculating the percentage of discarded items and auxiliary decision variables. We aim to understand for which classes and types of the instances the pre-processing scheme works fine and which instances are hard to be reduced. Furthermore, we wonder how many of the instances of PWT^c become those of PWT^u .

The results of the experiments demonstrate efficiency of the pre-processing scheme. It is rather good with respect to the instances of uncorr type and removes on average 31.6% of their items. Concerning the uncorr-s-w type of the instances, it is able to exclude on average 18.5% of the items that they contain. Within these two categories, the instances with large W are rather liable to reduction to instances of PWT^u . Because W is large, they get more chances to loose enough items so that the total weight of rest items becomes less or equal to W . The pre-processing scheme does not work well for the b-s-corr type of the instances. No instance of this type has been reduced to PWT^u . Because the profit of an item approaches its weight, the pre-processing encounters a difficulty to find unprofitable items for this instance type. In general, the way of how profits and weights are generated is not an obstacle in itself for the pre-processing to be successful. There are other factors, like the value of rent rate R , the value of capacity W , and a distance to the destination from the city where an item is positioned, that hinder its application. For example, if a route is long enough, some items in the first cities can be shown to be unprofitable even in the case of their b-s-corr type of generation. Clearly, the fact that we cannot handle the instances of this type is a proper property of the benchmark suite B .

Obviously, discarding items within pre-processing reduces the number of auxiliary variables in LB^λ . The rate β demonstrates that in practice LB^{100}

uses a very reduced set of them. The average over all the entries is just 48.5%. Therefore, less than a half of all possible variables is used only. In general, β is significantly small when W is large, since latter results in a slower growth of diapason $[v_i^{\min}, v_i^{\max}]$ in LB^λ , for $i = 1, \dots, n$. In other words, the instances with large W require less number of auxiliary decision variables comparing to the instances where W is smaller.

9.1.2. Performance of the Approximate Approach

We now aim to evaluate the performance of LB^{100} concerning its running time and solution quality compared to optima. LB^{100} is particularly fast and its model is solved to optimality in a very short time for all the small size instances. Only one instance of the whole test suite causes a difficulty in terms of running time. The approximate approach looks very swift even with instances of the b-s-corr type and produces very good approximation for reasonably small $\lambda = 100$. The ratio ρ close to 1 points out that LB^{100} obtains approximately the same result as the optimal objective value is, but in a shorter time. Therefore, LB^λ gives an advanced trade-off in terms of computational time and solution's quality comparing to the exact approaches. The larger $\lambda = 1000$ has been tested in our earlier experiments (Polyakovskiy & Neumann (2015)). However, it results to a very limited improvement in the value of the total reward at the larger cost of running time, and more importantly at the larger cost of memory consumption. Indeed, as λ increases, the approach requires more memory as the number of auxiliary variables grows.

9.1.3. Performance of the Exact Approaches

In this part of the analysis, our goal is to evaluate the performance of the MIP^λ and BIB^λ approaches and determine which classes and types of the instances are hard to be solved by each of them. Since all the instances can be solved to optimality, we treat a time spent to achieve an optimal solution as hardness of an instance.

Comparing to the earlier results, we see now that the unconstrained instances of the problem are not to be easier to handle as it has been previously observed (Polyakovskiy et al. (2014); Polyakovskiy & Neumann (2015)). Now, our mixed-integer programming approach is able to solve all the small size instances to optimality as it is strengthened with the upper bound UB^λ . Specifically, MIP^λ takes much less time than the given limit and can be executed on an ordinary PC instead of the highly productive computational cluster that has been previously utilized.

The results show that the instances of the uncorr-s-w type are harder to solve for MIP^λ comparing to other types. This fact looks interesting. Comparing to the b-s-corr type, this type has around 20% of items per instance excluded by the pre-processing step and needs much less auxiliary variables, but takes more time to achieve an optimal solution. This also differs PWT from the classical 0-1 knapsack problem for which the b-s-corr type is shown to be the hardest one (Martello et al. (1999)). The b-s-corr type is not easier to solve than the

uncorr type in terms of running time, nor the latter is when compared to the former.

MIP^λ outperforms BIB^λ mainly on the set of b-s-corr type instances with the least capacity W . BIB^{500} is superior on the wide range of instances. It is highly effective for the set of uncorr and uncorr-s-w type instances with large capacity. Depending on the parameter λ , performance of BIB^λ can be further improved for some instances. Specifically, setting λ to larger values allows BIB^λ to solve b-s-corr type instances with many items and large capacity faster. The further increase of λ up to 1500, leads to the best result for some of those instances. However, this degrades performance of BIB^λ on other instances. Performance of BIB^λ with the values of λ less than 500 and greater than 1500 have also been investigated. However, using too small or too large values increases the running time of the approach. The same behavior has been observed for MIP^λ , for which the value of 1000 is the most promising one. In summary, we argue that selecting $\lambda = 1000$ represents a good balance when the classification of instances is unknown, e.g. a new problem instance is to be solved. BIB^{1000} works reasonably fast over all the instances. Although it loses against BIB^{500} on many instances, these loss are insignificant when compared to gains on some other instances, for example, on those ones prefixed by “b-s-corr_10”.

9.1.4. Impact of the Sequencing Constraints

Here, we are interested in understanding the impact of the sequencing constraints on the performance of the CP Search. We wonder how strong the constraint can be in pruning the search tree and for which types and classes of the instances it performs well.

Table 2 presents the details on the constraint programming search performed by BIB^λ on small size instances of the benchmark suite. Columns 1-3 specify the instance’s name, the total number of items m , and the version of the problem solved. Other columns of the table describe results for BIB^λ with $\lambda \in \{500, 1000, 1500\}$. Each of the sections shows the number of branches b totally explored and the number of fails f obtained by the CP solver during the search when the corresponding parameter value λ is in use. Furthermore, each of the columns “ UB^λ runs” reports the number of successful UB^λ runs, say r^s , that resulted in pruning of the search tree and the total number of runs, say r^t . The two values are separated by “|”. In the parentheses, it also gives a percentage of successful runs η computed as $\eta = 100 \cdot r^s / r^t$.

The sequencing constraints of BIB^λ look weak when dealing with the instances of the b-s-corr type. This can be observed from the results, which show that the number of UB^λ runs considerably decreases when λ increases. This means that BIB^λ needs to tighten the upper bound by increasing the value of λ in order to achieve a better performance for this type of the instances. In other words, BIB^λ relies more on a tight upper bound to prune the search tree rather than on the sequencing constraints. In contrast, considering other types of the instances, the growth of the number of runs is not that much for them. Moreover, the number of explored branches b and the number of fails f obtained

remain almost the same for different values of λ . This means that BIB^λ extensively rely on the sequencing constraints when solving the uncorr and uncorr-s-w types of the instances. When we turn off the sequencing constraints in the case of $\text{BIB}_{no\ seq.}^{1000}$, the approach explores more branches with respect to the uncorr and uncorr-s-w types and spends more time to do this, while the number of branches traversed and the running time stay almost the same for the b-s-corr type. The reason why the sequencing constraints are weak with respect to the b-s-corr type is the same as for the pre-processing scheme which also has low performance when dealing with this type.

9.2. Computational Experiments on the Set of Large Size Instances

The goal of our second experiment is to understand how fast LB^λ finds the approximate solutions of larger size instances and how efficient the pre-processing scheme is in this case. Solving large size instances turns out to be costly and requires computational capacity beyond that of an ordinary computer. Therefore, this series of experiments has been carried out on a computational cluster with 128 Gb RAM and 2.8 GHz 48-cores AMD Opteron processor. We use the same settings for the MIP solver as in our first experiment and set $\lambda = 100$ for LB^λ . We investigate two families of the largest size instances of benchmark suite B , namely those based on the TSP problems **pla33810** and **pla85900** with 33810 and 85900 cities, respectively. Table 3 reports the results. Columns 1-4 specify the instance's name, the total number of items m , the percentage of items discarded within the pre-processing step α , and the version of the problem solved. The rest of the table describes the results for LB^{100} and for its two special cases: $\text{LB}_{no\ pre-pr.}^{100}$ when LB^{100} is run without any pre-processing at all and $\text{LB}_{red.\ pre-pr.}^{100}$ when LB^{100} does not use the reasoning based on the sequencing constraints to accelerate deduction of compulsory and unprofitable items when a problem in hand is unconstrained (see Section 5 for details). Columns 5 and 10 provide the running time t of LB^{100} (including the time taken by pre-processing) and of $\text{LB}_{no\ pre-pr.}^{100}$, respectively, while columns 6 and 11 give details on β , which is the percentage of auxiliary y -type variables that they use. The way to calculate α and β is given in Section 9.1. Columns 8 and 12 report ω as a relative gap in percents that compares the running times of LB^{100} and $\text{LB}_{no\ pre-pr.}^{100}$. Specifically, ω is computed as $\omega = 100 \cdot (t^{ALG} - t^{MIN}) / t^{MIN}$, where t^{ALG} is the running time of LB^{100} or $\text{LB}_{no\ pre-pr.}^{100}$ and t^{MIN} is the minimum of their running times. Column t_p gives the pre-processing time taken by LB^{100} . Column γ shows a ratio between the number of y -type variables used by $\text{LB}_{no\ pre-pr.}^{100}$ and LB^{100} . Column ρ gives a ratio between the upper bound obtained by UB^λ with the same $\lambda = 100$ and the lower bound LB^{100} . We do not provide runtime and some other results for $\text{LB}_{red.\ pre-pr.}^{100}$ as it always performs worse than LB^{100} . For it, we only show the ratio η between the pre-processing time of $\text{LB}_{red.\ pre-pr.}^{100}$ and that one of LB^{100} to evaluate a speedup gained by utilizing the sequencing constraints for deduction of compulsory and unprofitable items in the case of PWT^u . Note that in $\text{LB}_{red.\ pre-pr.}^{100}$ the pre-processing

Table 1: Results of Computational Experiments on Small Size Instances

instance	m	α , %	ver	LB100			MIP1000			BIB500		BIB1000		BIB1000 _{no seq.}		BIB1500	
				ρ	t, sec	β , %	t, sec	ω , %	t, sec	ω , %	t, sec	ω , %	t, sec	ω , %	t, sec	ω , %	
instance family e1151																	
uncorr_01	50	42.0	c	1.00000	0.2	55.8	2.3	34.9	1.7	0.0	3.5	107.4	6.4	278.1	6.1	256.9	
uncorr_06	50	14.0	c	1.00000	0.2	39.2	3.2	255.5	0.9	0.0	1.7	82.8	3.2	252.2	2.6	184.7	
uncorr_10	50	12.0	u	1.00000	0.1	11.1	0.8	104.2	0.4	0.0	0.5	30.9	0.6	50.2	0.6	58.0	
uncorr-s-w-01	50	30.0	c	1.00000	0.3	77.4	2.9	84.3	1.6	0.0	3.1	94.1	7.6	381.5	4.8	205.0	
uncorr-s-w-06	50	24.0	c	1.00000	0.1	35.8	1.4	52.6	0.9	0.0	1.7	84.8	2.3	144.7	2.7	190.9	
uncorr-s-w-10	50	34.0	u	1.00000	0.1	13.2	1.6	379.2	0.3	0.0	0.4	9.6	0.5	36.1	0.4	28.1	
b-s-corr_01	50	4.0	c	1.00000	0.3	89.7	4.5	0.0	23.2	412.4	49.0	982.1	52.0	1049.0	77.8	1620.7	
b-s-corr_06	50	0.0	c	1.00000	0.2	53.4	2.5	0.0	2.9	14.0	6.4	152.1	6.3	146.0	11.0	333.1	
b-s-corr_10	50	0.0	c	1.00000	0.2	25.7	1.9	0.0	1.9	0.9	3.7	100.1	3.5	89.3	6.4	241.0	
uncorr_01	250	39.2	c	1.00000	0.3	65.5	10.4	127.5	4.6	0.0	9.1	97.6	21.5	368.7	14.7	220.7	
uncorr_06	250	16.4	c	1.00000	0.2	38.3	65.1	2353.2	2.7	0.0	4.2	57.0	7.3	175.2	6.1	131.7	
uncorr_10	250	54.4	u	1.00000	0.1	10.9	20.4	2512.1	0.8	0.0	1.0	28.1	1.8	131.3	1.2	60.0	
uncorr-s-w-01	250	20.8	c	1.00000	0.3	88.0	7.1	97.4	3.6	0.0	6.8	87.9	40.8	1026.9	10.4	187.1	
uncorr-s-w-06	250	14.0	c	1.00000	0.2	44.6	41.7	2520.9	1.6	0.0	2.4	50.0	5.8	262.0	3.5	118.8	
uncorr-s-w-10	250	19.2	u	0.99998	0.2	15.7	83.3	7761.7	1.1	0.0	1.3	19.2	2.3	120.0	1.5	42.7	
b-s-corr_01	250	0.0	c	1.00000	0.3	90.2	8.9	0.0	28.9	223.5	58.3	552.7	65.6	634.2	94.1	953.0	
b-s-corr_06	250	0.0	c	0.99997	0.2	55.8	20.9	0.0	27.3	31.0	53.7	157.3	55.4	165.7	57.4	175.0	
b-s-corr_10	250	0.0	c	1.00000	0.2	26.7	55.6	633.3	7.6	0.0	9.8	29.4	9.5	25.5	16.1	112.5	
uncorr_01	500	37.0	c	1.00000	0.3	67.7	12.9	12.0	11.5	0.0	22.1	91.4	68.5	493.1	35.8	210.4	
uncorr_06	500	15.2	c	0.99993	0.3	38.8	81.6	1369.8	5.6	0.0	8.3	49.2	17.8	221.4	12.0	116.6	
uncorr_10	500	51.4	u	1.00000	0.2	11.6	93.1	7093.6	1.3	0.0	1.5	19.4	3.4	160.2	1.9	43.4	
uncorr-s-w-01	500	20.2	c	1.00000	0.2	89.1	12.1	149.1	4.9	0.0	8.7	79.2	63.3	1200.5	13.1	168.9	
uncorr-s-w-06	500	15.2	c	0.99990	0.2	44.2	147.7	4854.8	3.0	0.0	3.9	32.4	9.9	231.5	5.3	78.5	
uncorr-s-w-10	500	18.6	u	1.00000	0.2	16.1	208.2	9788.2	2.1	0.0	2.3	11.1	4.6	118.6	2.6	21.5	
b-s-corr_01	500	0.0	c	0.99993	0.3	91.3	29.9	0.0	226.7	657.2	324.1	982.6	396.9	1225.8	535.0	1687.3	
b-s-corr_06	500	0.0	c	0.99995	0.3	55.4	71.3	0.0	316.7	344.1	149.0	109.0	161.5	126.4	235.7	230.4	
b-s-corr_10	500	0.0	c	1.00000	0.2	26.0	100.8	294.4	87.9	243.8	25.7	0.4	25.6	0.0	27.4	7.0	
instance family e1176																	
uncorr_01	75	26.7	c	1.00000	0.3	76.7	5.4	9.5	5.0	0.0	10.4	110.1	20.7	316.6	17.7	257.6	
uncorr_06	75	14.7	c	1.00000	0.3	33.9	9.8	433.3	1.8	0.0	3.3	80.5	4.3	134.7	5.2	183.2	
uncorr_10	75	48.0	u	1.00000	0.1	11.3	1.9	303.0	0.5	0.0	0.6	20.1	1.0	112.2	0.7	53.2	
uncorr-s-w-01	75	26.7	c	1.00000	0.4	78.2	4.9	2.3	4.8	0.0	9.6	99.0	30.4	529.9	15.6	224.3	
uncorr-s-w-06	75	17.3	c	1.00000	0.3	40.7	7.8	495.3	1.3	0.0	2.2	67.9	5.0	283.8	3.2	147.7	
uncorr-s-w-10	75	16.0	u	1.00000	0.2	16.6	10.8	1437.9	0.7	0.0	0.9	31.1	1.5	119.4	1.1	59.8	
b-s-corr_01	75	0.0	c	1.00000	0.4	93.5	6.2	0.0	215.6	3371.7	463.3	7362.1	510.3	8119.1	770.3	12305.6	
b-s-corr_06	75	0.0	c	1.00000	0.3	58.9	8.6	68.3	5.1	0.0	11.1	117.7	11.5	124.7	19.1	272.8	
b-s-corr_10	75	0.0	c	1.00000	0.3	25.5	6.7	15.5	5.8	0.0	9.8	68.7	10.1	73.8	11.1	90.4	
uncorr_01	375	38.1	c	1.00000	0.3	66.3	30.6	204.5	10.0	0.0	19.9	98.0	48.9	386.5	32.4	222.2	
uncorr_06	375	16.0	c	1.00000	0.2	37.0	162.0	2825.0	5.5	0.0	9.6	73.0	17.5	216.1	14.5	162.6	
uncorr_10	375	9.9	u	1.00000	0.2	11.8	105.4	7412.1	1.4	0.0	1.6	15.6	6.0	329.1	1.8	30.0	
uncorr-s-w-01	375	14.9	c	1.00000	1.0	89.7	26.4	205.8	8.6	0.0	15.3	77.1	347.2	3917.0	24.3	181.0	
uncorr-s-w-06	375	12.3	c	1.00000	0.3	46.8	165.9	4625.3	3.5	0.0	5.4	52.5	15.7	348.4	7.4	110.7	
uncorr-s-w-10	375	14.9	u	1.00000	0.2	17.0	230.9	10782.5	2.1	0.0	2.4	14.7	4.1	94.9	2.9	34.5	
b-s-corr_01	375	0.0	c	1.00000	0.3	94.1	24.4	0.0	51.2	110.4	100.2	311.6	116.8	379.6	154.4	533.9	
b-s-corr_06	375	0.0	c	1.00000	0.3	56.6	83.5	47.0	149.0	162.1	56.8	0.0	59.4	4.6	98.2	72.9	
b-s-corr_10	375	0.0	c	0.99998	0.3	27.5	181.4	579.8	92.7	247.3	26.7	0.0	27.6	3.5	37.8	41.8	
uncorr_01	750	32.5	c	1.00000	0.4	71.4	92.1	332.9	21.3	0.0	33.4	57.0	131.9	520.2	52.8	148.3	
uncorr_06	750	14.8	c	1.00000	0.2	39.0	429.8	2564.3	19.6	21.7	16.1	0.0	40.0	148.1	22.8	41.1	
uncorr_10	750	43.1	u	1.00000	0.2	13.0	306.7	8792.8	3.8	10.4	3.4	0.0	10.4	201.6	3.6	3.6	
uncorr-s-w-01	750	16.7	c	1.00000	0.5	88.7	117.0	950.9	11.1	0.0	19.2	72.0	255.2	2190.9	28.9	159.8	
uncorr-s-w-06	750	13.5	c	1.00000	0.2	45.7	472.0	7119.9	6.5	0.0	7.8	19.6	23.5	258.9	10.2	56.3	
uncorr-s-w-10	750	14.4	u	1.00000	0.3	17.0	823.7	17214.5	4.8	0.0	5.1	7.3	10.8	127.2	5.6	17.7	
b-s-corr_01	750	0.0	c	0.99999	0.3	93.7	161.4	0.0	183.8	13.8	287.4	78.0	388.3	140.5	442.4	174.0	
b-s-corr_06	750	0.0	c	1.00000	0.5	55.4	259.6	67.3	975.0	528.1	175.8	13.2	203.1	30.9	155.2	0.0	
b-s-corr_10	750	0.0	c	0.99999	0.2	25.9	281.7	106.7	20261.7	14767.4	176.7	29.7	185.2	35.9	136.3	0.0	
instance family e1101																	
uncorr_01	100	49.0	c	1.00000	0.4	60.7	7.4	144.7	3.0	0.0	6.0	98.5	12.9	325.4	10.3	237.6	
uncorr_06	100	16.0	c	0.99993	0.3	39.7	20.5	746.2	2.4	0.0	4.0	65.9	8.3	241.5	6.0	147.2	
uncorr_10	100	57.0	u	1.00000	0.2	10.0	4.8	524.7	0.8	0.0	1.0	28.5	1.8	136.2	1.2	63.4	
uncorr-s-w-01	100	25.0	c	1.00000	0.3	90.3	6.8	70.2	4.0	0.0	7.9	95.3	25.7	537.4	12.4	207.0	
uncorr-s-w-06	100	17.0	c	1.00000	0.4	41.9	17.2	1001.7	1.6	0.0	2.6	65.0	6.0	285.0	3.8	144.2	
uncorr-s-w-10	100	15.0	u	1.00000	0.2	17.2	39.6	3748.5	1.0	0.0	1.3	25.6	2.1	106.7	1.6	59.1	
b-s-corr_01	100	0.0	c	1.00000	0.5	94.4	12.7	0.0	60.2	373.0	126.5	893.6	135.5	964.3	209.2	1543.2	
b-s-corr_06	100	0.0	c	1.00000	0.4	56.2	11.3	8.5	10.4	0.0	22.9	119.6	22.9	119.1	40.1	284.5	
b-s-corr_10	100	0.0	c	0.99990	0.2	28.2	16.6	4.5	15.9	0.0	27.3	71.7	27.0	69.3	41.6	161.3	
uncorr_01	500	38.8	c	1.00000	0.4	65.9	31.6	120.2	14.4	0.0	28.0	94.8	76.3	430.7	43.8	204.8	
uncorr_06	500	14.4	c	1.00000	0.3	39.2	397.3	4678.5	8.3	0.0	13.7	64.2	26.3	216.9	19.6	135.7	
uncorr_10	500	51.4	u	1.00000	0.2	11.4	212.8	9707.2	2.2	0.0	2.5	16.5	5.6	157.1	3.0	38.8	
uncorr-s-w-01	500	20.4	c	1.00000	4.5	88.4	88.7	293.4	22.5	0.0	39.4	74.7	1924.5	8437.4	60.2	167.2	
uncorr-s-w-06	500	14.2	c	1.00000	0.4	44.8	365.2	6825.2	5.3	0.0	7.7	46.2	20.7	293.1	10.7	102.7	
uncorr-s-w-10	500	16.4	u	1.00000	0.2	16.3	525.4	16328.3	3.2	0.0	3.5	10.7					

Table 2: Details on the CP Search Performed by BIB^λ on Small Size Instances

instance	m	ver	BIB ⁵⁰⁰			BIB ¹⁰⁰⁰			BIB ¹⁰⁰⁰ _{no seq.}			BIB ¹⁵⁰⁰		
			b	f	UB ^λ runs	b	f	UB ^λ runs	b	f	UB ^λ runs	b	f	UB ^λ runs
instance family e1151														
uncorr.01	50	c	42	20	16 64 (25%)	42	20	16 64 (25%)	91	44	26 119 (22%)	42	20	16 64 (25%)
uncorr.06	50	c	34	17	17 65 (26%)	34	17	17 65 (26%)	72	35	31 101 (31%)	34	17	17 65 (26%)
uncorr.10	50	u	22	11	11 36 (31%)	22	11	11 36 (31%)	50	22	20 58 (35%)	22	11	11 36 (31%)
uncorr-s-w.01	50	c	46	21	14 84 (17%)	46	21	14 84 (17%)	163	79	41 199 (20%)	46	21	14 84 (17%)
uncorr-s-w.06	50	c	36	16	16 64 (25%)	36	16	16 64 (25%)	62	30	30 94 (32%)	36	16	16 64 (25%)
uncorr-s-w.10	50	u	8	4	4 33 (12%)	8	4	4 33 (12%)	26	12	12 54 (22%)	8	4	4 33 (12%)
b-s-corr.01	50	c	676	334	220 609 (36%)	676	334	220 609 (36%)	722	355	278 754 (37%)	676	334	220 609 (36%)
b-s-corr.06	50	c	100	50	50 139 (36%)	100	50	50 139 (36%)	104	51	61 169 (36%)	100	50	50 139 (36%)
b-s-corr.10	50	c	90	45	45 129 (35%)	90	45	45 129 (35%)	92	45	54 155 (35%)	90	45	45 129 (35%)
uncorr.01	250	c	110	52	50 254 (20%)	110	52	50 254 (20%)	293	138	97 472 (21%)	110	52	50 254 (20%)
uncorr.06	250	c	152	75	75 346 (22%)	130	64	64 324 (20%)	228	112	112 456 (24%)	130	64	64 324 (20%)
uncorr.10	250	u	36	18	18 145 (12%)	36	18	18 145 (12%)	151	74	73 264 (28%)	36	18	18 145 (12%)
uncorr-s-w.01	250	c	58	29	24 268 (9%)	58	29	24 268 (9%)	415	203	154 749 (21%)	58	29	24 268 (9%)
uncorr-s-w.06	250	c	48	23	23 255 (9%)	48	23	23 255 (9%)	175	86	85 415 (21%)	48	23	23 255 (9%)
uncorr-s-w.10	250	u	30	14	14 225 (6%)	30	14	14 225 (6%)	187	90	86 404 (21%)	30	14	14 225 (6%)
b-s-corr.01	250	c	600	296	265 1188 (22%)	600	296	265 1188 (22%)	684	338	366 1112 (33%)	600	296	265 1188 (22%)
b-s-corr.06	250	c	1148	572	567 1388 (41%)	662	331	331 885 (37%)	700	350	420 1104 (38%)	444	222	222 670 (33%)
b-s-corr.10	250	c	412	206	205 628 (33%)	232	116	116 452 (26%)	238	119	143 547 (26%)	232	116	116 452 (26%)
uncorr.01	500	c	1032	508	376 1080 (35%)	964	470	354 1034 (34%)	2292	1132	809 2482 (33%)	932	455	344 1014 (34%)
uncorr.06	500	c	332	164	163 755 (22%)	266	131	131 683 (19%)	610	301	300 1104 (27%)	266	131	131 683 (19%)
uncorr.10	500	u	36	18	18 285 (6%)	36	18	18 285 (6%)	269	131	128 548 (23%)	36	18	18 285 (6%)
uncorr-s-w.01	500	c	66	33	28 474 (6%)	66	33	28 474 (6%)	466	224	167 1160 (14%)	66	33	28 474 (6%)
uncorr-s-w.06	500	c	96	46	46 511 (9%)	94	45	45 509 (9%)	386	191	191 911 (21%)	94	45	45 509 (9%)
uncorr-s-w.10	500	u	38	17	16 442 (4%)	36	16	16 440 (4%)	382	187	187 872 (21%)	36	18	18 440 (4%)
b-s-corr.01	500	c	8318	4154	4061 9992 (41%)	4486	2238	2148 6128 (35%)	5310	2639	3034 7044 (43%)	4486	2238	2148 6128 (35%)
b-s-corr.06	500	c	21858	10918	10771 22543 (48%)	2804	1396	1391 3305 (42%)	3212	1601	1915 4433 (43%)	1960	976	976 2453 (40%)
b-s-corr.10	500	c	6402	3199	3167 6844 (46%)	628	314	313 1102 (28%)	650	324	388 1346 (29%)	354	177	177 830 (21%)
instance family e1176														
uncorr.01	75	c	96	46	44 138 (32%)	96	46	44 138 (32%)	199	97	65 254 (25%)	96	46	44 138 (32%)
uncorr.06	75	c	56	26	26 107 (24%)	56	26	26 107 (24%)	91	44	44 149 (30%)	56	26	26 107 (24%)
uncorr.10	75	u	10	5	5 45 (11%)	10	5	5 45 (11%)	58	28	28 91 (30%)	10	5	5 45 (11%)
uncorr-s-w.01	75	c	110	54	33 170 (19%)	110	54	33 170 (19%)	338	167	86 433 (20%)	110	54	33 170 (19%)
uncorr-s-w.06	75	c	38	19	17 96 (18%)	38	19	17 96 (18%)	94	46	42 151 (28%)	38	19	17 96 (18%)
uncorr-s-w.10	75	u	20	10	10 74 (14%)	20	10	10 74 (14%)	65	29	28 121 (23%)	20	10	10 74 (14%)
b-s-corr.01	75	c	6218	3100	2684 6446 (42%)	6218	3100	2684 6446 (42%)	6856	3419	3592 8249 (44%)	6210	3096	2680 6437 (42%)
b-s-corr.06	75	c	110	55	55 178 (31%)	110	55	55 178 (31%)	128	63	76 233 (32%)	110	55	55 178 (31%)
b-s-corr.10	75	c	318	159	157 358 (44%)	224	112	112 272 (41%)	246	123	148 353 (42%)	174	87	87 226 (38%)
uncorr.01	375	c	212	106	102 438 (23%)	212	106	102 438 (23%)	473	236	178 785 (23%)	212	106	102 438 (23%)
uncorr.06	375	c	160	79	79 461 (17%)	160	79	79 461 (17%)	302	148	146 653 (22%)	160	79	79 461 (17%)
uncorr.10	375	u	46	22	21 230 (9%)	46	22	21 230 (9%)	545	268	260 718 (36%)	46	23	23 230 (10%)
uncorr-s-w.01	375	c	200	99	96 560 (17%)	186	92	89 545 (16%)	3624	1806	1386 4938 (28%)	186	92	89 545 (16%)
uncorr-s-w.06	375	c	78	38	37 437 (8%)	78	38	37 437 (8%)	266	127	126 661 (19%)	78	38	37 437 (8%)
uncorr-s-w.10	375	u	32	14	14 347 (4%)	30	15	15 345 (4%)	156	77	76 528 (14%)	30	15	15 345 (4%)
b-s-corr.01	375	c	568	278	227 1901 (12%)	568	278	227 1901 (12%)	662	325	318 1252 (25%)	568	278	227 1901 (12%)
b-s-corr.06	375	c	3360	1679	1667 3708 (45%)	414	207	207 771 (27%)	454	227	272 973 (28%)	414	207	207 771 (27%)
b-s-corr.10	375	c	5232	2610	2572 5543 (46%)	586	290	289 935 (31%)	668	331	396 1217 (33%)	472	233	233 822 (28%)
uncorr.01	750	c	488	243	238 991 (24%)	352	175	170 851 (20%)	1094	545	419 1924 (22%)	352	175	170 851 (20%)
uncorr.06	750	c	1254	624	610 1891 (32%)	348	172	172 970 (18%)	869	430	427 1610 (27%)	348	172	172 970 (18%)
uncorr.10	750	u	174	84	71 604 (12%)	92	46	46 510 (9%)	626	307	301 1091 (28%)	58	29	29 475 (6%)
uncorr-s-w.01	750	c	152	75	71 838 (8%)	150	74	70 836 (8%)	1668	824	630 3122 (20%)	150	74	70 836 (8%)
uncorr-s-w.06	750	c	136	65	62 803 (8%)	74	36	35 736 (5%)	384	191	187 1200 (16%)	74	36	35 736 (5%)
uncorr-s-w.10	750	u	28	13	13 677 (2%)	22	11	11 667 (2%)	550	274	271 1342 (20%)	22	11	11 667 (2%)
b-s-corr.01	750	c	2576	1282	1152 6124 (19%)	1754	871	744 5281 (14%)	2324	1150	1170 3725 (31%)	1754	871	744 5281 (14%)
b-s-corr.06	750	c	39336	19666	19490 39884 (49%)	1866	932	930 2597 (36%)	2158	1077	1290 3470 (37%)	686	343	343 1416 (24%)
b-s-corr.10	750	c	1664464	832213	819235 1652903 (50%)	4906	2450	2435 5635 (43%)	5288	2640	3150 7241 (44%)	1890	943	940 2615 (36%)
instance family e1101														
uncorr.01	100	c	48	24	24 97 (25%)	48	24	24 97 (25%)	91	46	36 151 (24%)	48	24	24 97 (25%)
uncorr.06	100	c	90	42	42 161 (26%)	90	42	42 161 (26%)	151	72	72 228 (32%)	90	42	42 161 (26%)
uncorr.10	100	u	26	13	13 62 (21%)	26	13	13 62 (21%)	96	47	46 127 (36%)	24	12	12 60 (20%)
uncorr-s-w.01	100	c	44	21	12 121 (10%)	44	21	12 121 (10%)	149	73	34 269 (13%)	44	21	12 121 (10%)
uncorr-s-w.06	100	c	20	9	9 97 (9%)	20	9	9 97 (9%)	74	35	32 161 (20%)	20	9	9 97 (9%)
uncorr-s-w.10	100	u	22	11	11 105 (10%)	22	11	11 105 (10%)	67	31	30 158 (19%)	22	11	11 105 (10%)
b-s-corr.01	100	c	720	360	296 870 (34%)	720	360	296 870 (34%)	756	377	376 1015 (37%)	720	360	296 870 (34%)
b-s-corr.06	100	c	144	72	72 237 (30%)	144	72	72 237 (30%)	152	76	91 294 (31%)	144	72	72 237 (30%)
b-s-corr.10	100	c	582	291	289 658 (44%)	460	230	230 544 (42%)	466	233	280 660 (42%)	426	213	213 510 (42%)
uncorr.01	500	c	200	100	98 494 (20%)	200	100	98 494 (20%)	485	242	199 872 (23%)	200	100	98 494 (20%)
uncorr.06	500	c	196	98	98 606 (16%)	196	98	98 606 (16%)	360	176	176 840 (21%)	196	98	98 606 (16%)
uncorr.10	500	u	48	22	20 282 (7%)	44	22	21 278 (8%)	242	119	107 486 (22%)	44	22	22 278 (8%)
uncorr-s-w.01	500	c	626	308	253 1098 (23%)	612	301	247 1084 (23%)	27178	13577	10687 35473 (30%)	612	301	247 1084 (23%)
uncorr-s-w.06	500	c	68	33	33 488 (7%)	68	33	33 488 (7%)	300	148	144 815 (18%)	68	33	33 488 (7%)
uncorr-s-w.10	500	u	28	14	14 445 (3%)	28	14	14 445 (3%)	202	101	100 704 (14%)	28	14	14 445 (3%)
b-s-corr.01	500	c	6140	3065	2870 9999 (29%)	3732	1862	1702 7038 (24%)	4416	2203	2432 5966 (41%)	3706	1849	1689 7012 (24%)
b-s-corr.														

scheme decides whether an item is compulsory or unprofitable or none of these cases independently of other items whose properties are already known.

The results show that the pre-processing step is important and leads to great speeding-up. The pre-processing scheme excludes on average 27.7% of items per instance of the uncorr type and 13.2% of items per instance of the uncorr-s-w type. The instances of these two types are vulnerable for reduction when the problem is PWT^u and when capacities are large. Furthermore, they are processed relatively fast as require much less number of auxiliary variables. Pre-processing allows LB^{100} to accelerate computations for the uncorr and uncorr-s-w types of the instances by 328% and 421% on average, respectively. The average value of ratio γ is 1.5 for the uncorr type against 1.2 for the uncorr-s-w type. Interestingly, despite on the larger portion of auxiliary y -type variables excluded for the uncorr type, its speeding-up indicator ω is less than that one of the uncorr-s-w type. Pre-processing is rather costly when applied to the unconstrained instances. However, the reasoning based on the sequencing constraints significantly improves the situation. This is shown by the values of η , which indicate that the time taken by pre-processing can be reduced up to ~ 400 times when comparing LB^{100} to $LB_{red. \text{ pre-pr.}}^{100}$. Otherwise, the running time of $LB_{red. \text{ pre-pr.}}^{100}$ often dominates one of $LB_{no \text{ pre-pr.}}^{100}$ that leads to avoiding the pre-processing stage when dealing with PWT^u .

In general, LB^{100} proves its ability to master large instances in a reasonable time. It needs less than ~ 30 minutes to find an approximate solution to any instance of family `pla33810`. Almost all the instances of family `pla85900` can be solved approximately within 1 hour; it takes no longer than ~ 4 hours for any of them. The quality of approximate solutions is outstanding as is confirmed by the very small values of ratio ρ .

10. Conclusion

We have introduced a new non-linear knapsack problem where items to be selected are subject to the total reward that a vehicle obtains by summing up the profits of chosen items and the subtracting costs resulted from their transportation along a fixed route. We have shown that both the constrained and unconstrained versions of the problem are \mathcal{NP} -hard. Our proposed pre-processing scheme can significantly decrease the size of instances making them easier for computation. The experimental results show that small size instances can be solved to optimality in a reasonable time by any of the two proposed exact approaches. Larger instances can be efficiently handled by our approximate approach producing near-optimal solutions.

As a future work, this problem has several natural generalizations. The first evident generalization is for sure the traveling thief problem where the sequence of cities may be changed. This variant asks for the mutual solution of the traveling salesman and knapsack problems. Another interesting situation takes place when cities may be skipped because are of no worth, for example any item stored there has in fact low or negative contribution to the total reward.

Table 3: Results of Computational Experiments on Large Size Instances

instance	m	α , %	ver	LB ¹⁰⁰				LB ¹⁰⁰ _{no pre-pr.}				LB ¹⁰⁰ _{red. pre-pr.}	
				t, sec	β , %	t_p	ω , %	ρ	t, sec	β , %	ω , %	γ	η
instance family pla33810													
uncorr_01	33809	29.0	c	515	78.7	0	0	1.001	3358	93.6	551	1.19	1
uncorr_06	33809	12.8	c	342	42.8	0	0	1.001	2083	56.3	509	1.32	1
uncorr_10	33809	35.9	u	52	15.0	13	0	1.001	334	26.4	543	1.76	10
uncorr-s-w_01	33809	19.3	c	435	89.5	0	0	1.001	1411	93.5	225	1.04	1
uncorr-s-w_06	33809	11.2	c	607	47.7	0	0	1.001	2795	56.2	361	1.18	1
uncorr-s-w_10	33809	8.7	c	25	18.2	0	0	1.001	251	26.3	902	1.44	1
b-s-corr_01	33809	0.0	c	447	93.6	0	0	1.001	463	93.6	4	1.00	1
b-s-corr_06	33809	0.0	c	566	56.3	0	0	1.001	610	56.3	8	1.00	1
b-s-corr_10	33809	0.0	c	563	26.5	0	0	1.001	603	26.5	7	1.00	1
uncorr_01	169045	30.6	c	587	76.6	0	0	1.001	3050	93.6	419	1.22	1
uncorr_06	169045	12.8	c	1204	42.7	0	0	1.001	2299	56.2	91	1.32	1
uncorr_10	169045	35.8	u	157	14.7	94	36	1.001	115	26.4	0	1.79	8
uncorr-s-w_01	169045	15.2	c	348	90.5	0	0	1.001	1561	93.5	348	1.03	1
uncorr-s-w_06	169045	11.7	c	590	47.2	0	0	1.001	2141	56.2	263	1.19	1
uncorr-s-w_10	169045	9.0	c	549	18.0	0	0	1.001	6245	26.3	1037	1.46	1
b-s-corr_01	169045	0.0	c	1384	93.7	0	0	1.001	1410	93.7	2	1.00	1
b-s-corr_06	169045	0.0	c	438	56.4	0	0	1.002	442	56.4	1	1.00	1
b-s-corr_10	169045	0.0	c	718	26.3	0	0	1.002	814	26.3	13	1.00	1
uncorr_01	338090	31.6	c	1589	75.4	0	0	1.001	9315	93.5	486	1.24	1
uncorr_06	338090	12.8	c	1023	42.6	0	0	1.001	1645	56.2	61	1.32	1
uncorr_10	338090	35.9	u	947	14.7	238	0	1.001	1224	26.3	29	1.79	6
uncorr-s-w_01	338090	15.2	c	1209	90.5	0	0	1.001	5125	93.5	324	1.03	1
uncorr-s-w_06	338090	11.9	c	966	47.1	0	0	1.001	2033	56.2	111	1.19	1
uncorr-s-w_10	338090	9.0	c	1156	18.0	0	0	1.001	3621	26.3	213	1.46	1
b-s-corr_01	338090	0.0	c	829	93.6	0	0	1.001	857	93.6	3	1.00	1
b-s-corr_06	338090	0.0	c	873	56.3	0	0	1.002	947	56.3	8	1.00	1
b-s-corr_10	338090	0.0	c	1095	26.3	0	0	1.002	1176	26.3	7	1.00	1
instance family pla85900													
uncorr_01	85899	32.4	c	2514	73.8	0	0	1.002	10614	93.5	322	1.27	1
uncorr_06	85899	13.5	c	3028	41.6	0	0	1.002	40283	56.1	1230	1.35	1
uncorr_10	85899	40.8	u	213	13.7	59	0	1.002	323	26.3	52	1.92	28
uncorr-s-w_01	85899	16.4	c	1683	88.6	0	0	1.002	9566	93.5	468	1.06	3
uncorr-s-w_06	85899	12.3	c	1985	46.6	0	0	1.002	13408	56.2	575	1.20	1
uncorr-s-w_10	85899	13.6	u	158	17.2	3	0	1.002	1165	26.3	637	1.53	393
b-s-corr_01	85899	0.0	c	3967	93.6	0	2	1.002	3903	93.6	0	1.00	1
b-s-corr_06	85899	0.0	c	1570	56.3	0	0	1.002	1575	56.3	0	1.00	1
b-s-corr_10	85899	0.0	c	3491	26.4	0	0	1.002	3627	26.4	4	1.00	1
uncorr_01	429495	32.5	c	3436	73.6	0	0	1.002	27833	93.5	710	1.27	1
uncorr_06	429495	13.6	c	5495	41.4	0	0	1.002	47092	56.2	757	1.36	1
uncorr_10	429495	40.4	u	1471	13.8	936	58	1.001	934	26.3	0	1.90	18
uncorr-s-w_01	429495	16.3	c	2380	90.2	0	0	1.002	20790	93.5	773	1.04	1
uncorr-s-w_06	429495	12.8	c	4508	46.3	0	0	1.002	10825	56.2	141	1.21	1
uncorr-s-w_10	429495	13.2	u	545	17.4	23	0	1.001	1311	26.3	141	1.51	256
b-s-corr_01	429495	0.0	c	3105	93.6	0	0	1.002	3145	93.6	1	1.00	1
b-s-corr_06	429495	0.0	c	5483	56.2	0	0	1.002	5457	56.2	0	1.00	1
b-s-corr_10	429495	0.0	c	4751	26.3	0	0	1.002	4872	26.3	3	1.00	1
uncorr_01	858990	33.2	c	6904	72.6	0	0	1.002	11982	93.5	74	1.29	1
uncorr_06	858990	13.6	c	5510	41.4	0	0	1.002	7860	56.2	43	1.36	1
uncorr_10	858990	40.6	u	3650	13.9	1544	0	1.001	4627	26.3	27	1.90	16
uncorr-s-w_01	858990	16.4	c	4859	90.2	0	0	1.002	43064	92.9	786	1.03	1
uncorr-s-w_06	858990	12.7	c	7095	46.3	0	0	1.002	15904	56.2	124	1.21	1
uncorr-s-w_10	858990	13.2	u	5058	17.4	68	0	1.001	12868	26.3	154	1.51	183
b-s-corr_01	858990	0.0	c	5474	93.5	0	0	1.002	5498	93.5	0	1.00	1
b-s-corr_06	858990	0.0	c	12566	56.2	0	0	1.002	12681	56.2	1	1.00	1
b-s-corr_10	858990	0.0	c	13806	26.4	0	0	1.002	13981	26.4	1	1.00	1

Finally, the possibility to pickup and deliver the items is for certain one another challenging problem. The outcomes of our research can further be adopted to solve routing problems with nonlinear cost functions, for example those when such a measure as gallon per vehicle mile versus load is used.

11. Acknowledgements

We want to thank the referees for their valuable suggestions which helped to improve the paper. This research has been supported through ARC Discovery Project DP130104395.

12. References

References

- Applegate, D., Cook, W. J., & Rohe, A. (2003). Chained lin-kernighan for large traveling salesman problems. *INFORMS Journal on Computing*, 15, 82–92.
- Balas, E. (1989). The prize collecting traveling salesman problem. *Networks*, 19, 621–636.
- Beham, A., Fechter, J., Kommenda, M., Wagner, S., Winkler, S. M., & Affenzeller, M. (2015). Optimization strategies for integrated knapsack and traveling salesman problems. In R. Moreno-Díaz, F. Pichler, & A. Quesada-Arencibia (Eds.), *Computer Aided Systems Theory – EUROCAST 2015: 15th International Conference, Las Palmas de Gran Canaria, Spain, February 8-13, 2015, Revised Selected Papers* (pp. 359–366). Cham: Springer International Publishing. doi:10.1007/978-3-319-27340-2_45.
- Bockmayr, A., & Hooker, J. N. (2005). Constraint programming. In G. N. K. Aardal, & R. Weismantel (Eds.), *Discrete Optimization* (pp. 559 – 600). Elsevier volume 12 of *Handbooks in Operations Research and Management Science*. doi:10.1016/S0927-0507(05)12010-6.
- Bonyadi, M. R., Michalewicz, Z., & Barone, L. (2013). The travelling thief problem: The first step in the transition from theoretical problems to realistic problems. In *Proceedings of the IEEE Congress on Evolutionary Computation, CEC 2013, Cancun, Mexico, June 20-23, 2013* (pp. 1037–1044). IEEE. doi:10.1109/CEC.2013.6557681.
- Bretthauer, K. M., & Shetty, B. (2002). The nonlinear knapsack problem - algorithms and applications. *European Journal of Operational Research*, 138, 459–472.
- Chand, S., & Wagner, M. (2016). Fast heuristics for the multiple traveling thieves problem. In *Proceedings of the 2016 Annual Conference on Genetic and Evolutionary Computation GECCO '16*. New York, NY, USA: ACM. doi:10.1145/2908812.2908841.

- Chekuri, C., & Khanna, S. (2005). A polynomial time approximation scheme for the multiple knapsack problem. *SIAM J. Comput.*, *35*, 713–728.
- Elhedhli, S. (2005). Exact solution of a class of nonlinear knapsack problems. *Oper. Res. Lett.*, *33*, 615–624.
- Erlebach, T., Kellerer, H., & Pferschy, U. (2001). Approximating multi-objective knapsack problems. In F. K. H. A. Dehne, J.-R. Sack, & R. Tamassia (Eds.), *WADS* (pp. 210–221). Springer volume 2125 of *Lecture Notes in Computer Science*.
- Faulkner, H., Polyakovskiy, S., Schultz, T., & Wagner, M. (2015). Approximate approaches to the traveling thief problem. In *Proceedings of the 2015 Annual Conference on Genetic and Evolutionary Computation GECCO '15* (pp. 385–392). New York, NY, USA: ACM. doi:10.1145/2739480.2754716.
- Fillet, D., Dejax, P., & Gendreau, M. (2005). Traveling salesman problems with profits. *Transportation Science*, *39*, 188–205. doi:10.1287/trsc.1030.0079.
- Garey, M., & Johnson, D. (1979). *Computers and Intractability: A Guide to the Theory of NP-Completeness*. W. H. Freeman.
- GOODYEAR (2008). Factors Affecting Truck Fuel Economy. <http://www.goodyeartrucktires.com/pdf/resources/publications/FactorsAffectingTruckFuelEconomy.pdf>.
- Hochbaum, D. S. (1995). A nonlinear knapsack problem. *Oper. Res. Lett.*, *17*, 103–110.
- Kellerer, H., Pferschy, U., & Pisinger, D. (2004). *Knapsack Problems*. Springer, Berlin, Germany.
- Laporte, G. (2009). Fifty years of vehicle routing. *Transportation Science*, *43*, 408–416. doi:10.1287/trsc.1090.0301.
- Li, H.-L. (1994). A global approach for general 0-1 fractional programming. *European Journal of Operational Research*, *73*, 590 – 596. doi:10.1016/0377-2217(94)90257-7.
- Lin, C., Choy, K., Ho, G., Chung, S., & Lam, H. (2014). Survey of green vehicle routing problem: Past and future trends. *Expert Systems with Applications*, *41*, 1118 – 1138. doi:10.1016/j.eswa.2013.07.107.
- Lourenço, N., Pereira, F. B., & Costa, E. (2016). An evolutionary approach to the full optimization of the traveling thief problem. In F. Chicano, B. Hu, & P. García-Sánchez (Eds.), *Evolutionary Computation in Combinatorial Optimization: 16th European Conference, EvoCOP 2016, Porto, Portugal, March 30 - April 1, 2016, Proceedings* (pp. 34–45). Cham: Springer International Publishing. doi:10.1007/978-3-319-30698-8_3.

- Martello, S., Pisinger, D., & Toth, P. (1999). Dynamic programming and strong bounds for the 0-1 knapsack problem. *Manage. Sci.*, *45*, 414–424. doi:10.1287/mnsc.45.3.414.
- Martello, S., & Toth, P. (1990). *Knapsack Problems: Algorithms and Computer Implementations*. John Wiley & Sons.
- Mei, Y., Li, X., Salim, F., & Yao, X. (2015). Heuristic evolution with genetic programming for traveling thief problem. In *2015 IEEE Congress on Evolutionary Computation (CEC)* (pp. 2753–2760). doi:10.1109/CEC.2015.7257230.
- Mei, Y., Li, X., & Yao, X. (2016). On investigation of interdependence between sub-problems of the travelling thief problem. *Soft Computing*, *20*, 157–172. doi:10.1007/s00500-014-1487-2.
- Polyakovskiy, S., Bonyadi, M. R., Wagner, M., Michalewicz, Z., & Neumann, F. (2014). A comprehensive benchmark set and heuristics for the traveling thief problem. In *Proceedings of the 2014 Annual Conference on Genetic and Evolutionary Computation GECCO '14* (pp. 477–484). New York, NY, USA: ACM.
- Polyakovskiy, S., & Neumann, F. (2015). Packing while traveling: Mixed integer programming for a class of nonlinear knapsack problems. In L. Michel (Ed.), *Integration of AI and OR Techniques in Constraint Programming* (pp. 332–346). Springer International Publishing volume 9075 of *Lecture Notes in Computer Science*. doi:10.1007/978-3-319-18008-3_23.
- Reinelt, G. (1991). TSPLIB - A Traveling Salesman Problem Library. *ORSA Journal on Computing*, *3*, 376–384. doi:10.1287/ijoc.3.4.376.
- Rossi, F., van Beek, P., & Walsh, T. (2008). Chapter 4 constraint programming. In V. L. Frank van Harmelen, & B. Porter (Eds.), *Handbook of Knowledge Representation* (pp. 181 – 211). Elsevier volume 3 of *Foundations of Artificial Intelligence*. doi:10.1016/S1574-6526(07)03004-0.
- Rossi, F., Beek, P. v., & Walsh, T. (2006). *Handbook of Constraint Programming (Foundations of Artificial Intelligence)*. New York, NY, USA: Elsevier Science Inc.
- Sherali, H., & Adams, W. (1999). *A Reformulation Linearization Technique for Solving Discrete and Continuous Nonconvex Problems*. J Kluwer Academic Publishing, Boston, MA.
- Tawarmalani, M., Ahmed, S., & Sahinidis, N. (2002). Global optimization of 0-1 hyperbolic programs. *Journal of Global Optimization*, *24*, 385–416. doi:10.1023/A:1021279918708.
- Toth, P., & Vigo, D. (2014). *Vehicle Routing*. Society for Industrial and Applied Mathematics. doi:10.1137/1.9781611973594.

- Vansteenwegen, P., Souffriau, W., & Oudheusden, D. V. (2011). The orienteering problem: A survey. *European Journal of Operational Research*, 209, 1 – 10. doi:<http://10.1016/j.ejor.2010.03.045>.
- Westerlund, A., Göthe-Lundgren, M., & Larsson, T. (2006). A stabilized column generation scheme for the traveling salesman subtour problem. *Discrete Applied Mathematics*, 154, 2212 – 2238. doi:10.1016/j.dam.2005.04.012. International Symposium on Combinatorial Optimization CO'02.
- Wu, J., Polyakovskiy, S., & Neumann, F. (2016). On the impact of the renting rate for the unconstrained nonlinear knapsack problem. In *Proceedings of the 2016 Annual Conference on Genetic and Evolutionary Computation GECCO '16* (pp. 413–419). New York, NY, USA: ACM. doi:10.1145/2908812.2908862.