# When is the condition of order preservation met? 

Konrad Kułakowski<br>AGH University of Science and Technology, Poland<br>Jiří Mazurek<br>Silesian University Opava, Czech Republic<br>Jaroslav Ramík<br>Silesian University Opava, Czech Republic<br>Michael Soltys<br>California State University Channel Islands, USA


#### Abstract

This article explores a relationship between inconsistency in the pairwise comparisons method and conditions of order preservation. A pairwise comparisons matrix with elements from an alo-group is investigated. This approach allows for a generalization of previous results. Sufficient conditions for order preservation based on the properties of elements of pairwise comparisons matrix are derived. A numerical example is presented.


Keywords: pairwise comparisons, alo-groups, EVM, GMM, COP, AHP

## 1. Introduction

The first documented use of comparisons by pairs dates back to the XIII century [7]. Later, the method was developed by Fechner [10], Thurstone [36] and Saaty [30]. Saaty proposed the seminal Analytic Hierarchy Process (AHP) extension to pairwise comparisons (henceforth abbreviated as $P C$ )

[^0]theory, which is a framework for dealing with a large number of criteria. At the beginning of the XX century the method was used in psychometrics and psychophysics [36]. Now it is considered part of decision theory [31]. Its utility has been confirmed in numerous examples [37, 15, 27, 35]. Despite the relative maturity of the area, it still invites further exploration. Examples of new exploration are the Rough Set approach [13], voting systems [9], fuzzy PC relation handling [28, 38, 29], incomplete $P C$ relation [4, 11], non-numerical rankings [19, 18], nonreciprocal $P C$ relation properties [12], rankings with the reference set of alternatives [23, 26], applications to software correctness and cybersecurity [33] and others. Further references: [34, 16, 22].

Although the $A H P$ is a popular method for multiple-criteria decision making, it is often criticized as in [6]. An important objection to AHP originates from Bana e Costa and Vansnick, [2], where the authors formulated a so called $C O P$ (conditions of order preservations) and proved that the priority deriving method followed in $A H P$ does not meet $C O P$. This phenomenon is not, however, an inherent problem of $A H P$ [24, 14]. Instead it is the result of inconsistency and the size of the differences between the alternatives. Further study of COP, the Eigenvalue Method (EVM) and inconsistency can be found in [25]. In particular the work brings a theorem showing dependency of COP and Koczkodaj inconsistency index [20] in the context of EVM.

Verifying whether for a certain set of pairwise comparisons COP is met requires ranking calculation. As $C O P$ was originally formulated in the context of $E V M$ the criticism caused by [2] was directed at $E V M$ and $A H P$. An attentive reader will notice, however, that neither $C O P$ as such, nor the notion of consistency understood as the cardinal transitivity [5, p. 158], depend on prioritization methods. Hence the question arises whether the relationship between $C O P$ and inconsistency is of a general nature and if so, how does this relationship look like?

The present work is an attempt to answer this vital question. In order to emphasize the general nature of the relationship between COP and inconsistency we decided to use pairwise comparisons based on ordered abelian groups. These general results are presented in Section 4. Although AHP was defined in the context of $E V M$ other methods of prioritization are becoming more and more popular. The Geometric Mean Method (GMM) may serve as an example [17]. Therefore, in Section 5 we redefine GMM in the context of alo-groups and show that meeting the $C O P$ criteria under this generalized GMM depends on the locally defined inconsistency. Similarly to [25] we adopt the inconsistency index [20].

## 2. Preliminaries, pairwise comparisons method

The input data for the $P C$ method is a $P C$ matrix $C=\left[c_{i j}\right], i, j \in$ $\{1, \ldots, n\}$, that expresses a weight function $R$ with domain the finite set of alternatives $A=\left\{a_{i} \in \mathscr{A} \mid i \in\{1, \ldots, n\}\right\}$. The set $\mathscr{A}$ is a non empty universe of alternatives and $R\left(a_{i}, a_{j}\right)=c_{i j}$. The values of comparisons $c_{i j}$ indicate the relative importance of alternatives $a_{i}$ with respect to $a_{j}$. Here, the elements $c_{i j}$ of $P C$ matrix $C=\left[c_{i j}\right]$ belong to $\mathcal{G}$, an alo-group which will be defined bellow.

An abelian group is a set, $G$, together with an operation $\odot($ read "odot") that combines any two elements $a, b \in G$ to form another element in $G$ denoted by $a \odot b$, see [3, 1]. The symbol $\odot$ is a general placeholder for some concretely given operation. $(G, \odot)$ satisfies the following requirements known as the abelian group axioms, particularly: commutativity, associativity, there exists an identity element $e \in G$ and for each element $a \in G$ there exists an element $a^{(-1)} \in G$ called the inverse element to $a$.

The inverse operation $\div$ to $\odot$ is defined for all $a, b \in G$ as follows:

$$
\begin{equation*}
a \div b=a \odot b^{(-1)} \tag{1}
\end{equation*}
$$

Note that the inverse operation is not necessarily associative.
An ordered triple $(G, \odot, \leq)$ is said to be an abelian linearly ordered group, alo-group for short, if $(G, \odot)$ is a group, $\leq$ is a linear order on $G$, and for all $a, b, c \in G$ :

$$
\begin{equation*}
a \leq b \text { implies } a \odot c \leq b \odot c . \tag{2}
\end{equation*}
$$

in other words, $\odot$ respects $\leq$.
If $\mathcal{G}=(G, \odot, \leq)$ is an alo-group, then $G$ is naturally equipped with the order topology induced by $\leq$ and $G \times G$ is equipped with the related product topology. We say that $\mathcal{G}$ is a continuous alo-group if $\odot$ is continuous on $G \times G$.

By definition, an alo-group $\mathcal{G}$ is a lattice ordered group. Hence, there exists $\max \{a, b\}$, for each pair $(a, b) \in G \times G$. Nevertheless, a nontrivial alogroup $\mathcal{G}=(G, \odot, \leq)$ has neither the greatest element nor the least element.

Because of the associative property, the operation $\odot$ can be extended by induction to $n$-ary operation.
$\mathcal{G}=(G, \odot, \leq)$ is divisible if for each positive integer $n$ and each $a \in G$ there exists the $(n)$-th root of $a$ denoted by $a^{(1 / n)}$, i.e., $\left(a^{(1 / n)}\right)^{(n)}=a$.

Let $\mathcal{G}=(G, \odot, \leq)$ be an alo-group. Then the function $\|\cdot\|: G \rightarrow G$ defined for each $a \in G$ by

$$
\begin{equation*}
\|a\|=\max \left\{a, a^{(-1)}\right\} \tag{3}
\end{equation*}
$$

is called a $\mathcal{G}$-norm.
The operation $d: G \times G \rightarrow G$ defined by $d(a, b)=\|a \div b\|$ for all $a, b$ $\in G$ is called a $\mathcal{G}$-distance. Next, we present the well known examples of alo-groups; for more details see also [1], or, [29].

Example 1. Additive alo-group $\mathcal{R}=(\mathbb{R},+, \leq)$ is a continuous alo-group with: $e=0, a^{(-1)}=-a, a^{(n)}=n \cdot a$.

Multiplicative alo-group $\mathcal{R}_{+}=\left(\mathbb{R}_{+}, \cdot, \leq\right)$ is a continuous alo-group with: $e=1, a^{(-1)}=a^{-1}=1 / a, a^{(n)}=a^{n}$. Here, by ' $\cdot$ ' we denote the usual operation of multiplication.

Fuzzy additive alo-group $\mathcal{R}_{a}=\left(\mathbb{R},+_{f}, \leq\right)$, see [29], is a continuous alogroup with:

$$
a+_{f} b=a+b-0.5, e=0.5, a^{(-1)}=1-a, a^{(n)}=n \cdot a-\frac{n-1}{2} .
$$

Fuzzy multiplicative alo-group $] \mathbf{0}, \mathbf{1}\left[\mathbf{m}=(] 0,1\left[, \bullet_{f}, \leq\right)\right.$, see [1] , is a continuous alo-group with:

$$
a \bullet_{f} b=\frac{a b}{a b+(1-a)(1-b)}, e=0.5, a^{(-1)}=1-a
$$

Remark 1. Usually, the $P C$ method is used with a multiplicative $P C$ matrix, i.e., with multiplicative alo-group, see e.g. [30, 24]. Then the relative importance of an alternative is multiplied with the relative importance of the other alternatives when considering a chain of alternatives. Now, our approach based on a more general concept applying alo-groups enables to extend the properties of the multiplicative system to the whole class of pairwise comparisons systems. The four instances listed in Example 1 show some useful non-trivial cases.

Now, we define two important properties of a $P C$ matrix.
Definition 1. A PC matrix $C=\left[c_{i j}\right], c_{i j} \in G$, is said to be $\odot$-reciprocal if

$$
\begin{equation*}
c_{i j} \odot c_{j i}=e \text { for all } i, j \in\{1, \ldots, n\} \tag{4}
\end{equation*}
$$

or, equivalently,

$$
\begin{equation*}
c_{j i}=c_{i j}^{(-1)} \text { for all } i, j \in\{1, \ldots, n\}, \tag{5}
\end{equation*}
$$

and it is said to be $\odot$-consistent if

$$
\begin{equation*}
c_{i j} \odot c_{j k} \odot c_{k i}=e \text { for all } i, j \in\{1, \ldots, n\} . \tag{6}
\end{equation*}
$$

Evidently, if $C$ is $\odot$-consistent, then it is also $\odot$-reciprocal, but not vice versa. Since the $P C$ matrix usually contains subjective evaluations provided by (human) experts, the information contained therein may be $\odot-$ inconsistent. That is, a triad of values $c_{i j}, c_{j k}, c_{k i}$ in $C$ may exist for which $c_{i j} \odot c_{j k} \odot c_{k i} \neq e$. In other words, different ways of estimating the value of a pair of alternatives may lead to different results. This fact leads to the concept of an $\odot$-inconsistency index describing the extent to which the matrix $C$ is $\odot$-inconsistent.

## 3. Priority deriving methods

There are a number of inconsistency indexes associated with deriving PC rankings, including the Eigenvector Method [30], the Least Squares Method, the Chi Squares Method [5], Koczkodaj's distance based inconsistency index ${ }^{1}$ [20], the Geometric Mean Method (GMM) and others. The three most prominent methods are described below.

The result of the pairwise comparisons method is a ranking - a mapping that assigns values to the concepts. Formally, it can be defined as the following function.

Definition 2. The ranking function for $A$ (the ranking of $A$ ) is a function $w: A \rightarrow \mathbb{R}_{+}$that assigns to every alternative from $A \subset \mathscr{A}$ a positive value from $\mathbb{R}_{+}$.

In other words, $w(a)$ represents the ranking value for $a \in A$. The $w$ function is usually written in the form of a vector of weights, i.e., $w \stackrel{d f}{=}$ $\left[w\left(a_{1}\right), \ldots w\left(a_{n}\right)\right]^{T}$ and is called the priority vector.

Now, for the moment, we consider the usual multiplicative alo-group $\mathcal{R}_{+}=\left(\mathbb{R}_{+}, \cdot, \leq\right)$. The eigenvalue based consistency index $C I(C)$ called Saaty's index of $n \times n$ reciprocal matrix $C=\left[c_{i j}\right]$ is defined as:

$$
\begin{equation*}
C I(C)=\frac{\lambda_{\max }-n}{n-1} \tag{7}
\end{equation*}
$$

where $\lambda_{\text {max }}$ is the principal eigenvalue of $C$.

[^1]The value $\lambda_{\max } \geq n$ and $\lambda_{\max }=n$ only if $C$ is consistent [32]. Vector $w$ is determined as the rescaled principal eigenvector of $C$. Thus, assuming that $C w_{\max }=\lambda_{\max } w_{\max }$ the priority vector $w$ is

$$
w=\gamma\left[w_{\max }\left(a_{1}\right), \ldots, w_{\max }\left(a_{n}\right)\right]^{T}
$$

where $\gamma$ is a scaling factor. Usually it is assumed that $\gamma=\left(\sum_{i=1}^{n} w_{\max }\left(a_{i}\right)\right)^{-1}$. This method is called the Eigenvector Method (EVM).

Here, we consider a space of evaluations; an alo-group with the only one binary operation, particularly, $\odot=\cdot$. Therefore, the Saaty's index cannot be defined, because two group operations in (77), e.g. •, +, are necessary. In what follows, we shall not deal with the Eigenvector Method.

Koczkodaj's inconsistency index $K I$ of $n \times n$ and $(n>2)$ reciprocal matrix $C=\left[c_{i j}\right]$ is defined as

$$
\begin{equation*}
K I(C)=\max _{i, j, k \in\{1, \ldots, n\}}\left\{1-\min \left\{\frac{c_{i j}}{c_{i k} c_{k j}}, \frac{c_{i k} c_{k j}}{c_{i j}}\right\}\right\} . \tag{8}
\end{equation*}
$$

Similarly, as we consider here a space alo-group with one binary operation, particularly, $\odot=\cdot$, the Koczkodaj's inconsistency index cannot be defined, as two group operations (8), e.g. •, $+/-$, are necessary. That is why we shall not deal with the Koczkodaj's inconsistency index. Later on (Theorem 3), however, we shall derive a relationship between Koczkodaj's inconsistency index and the generalized inconsistency index (which will be also defined later) in the multiplicative alo-group $\mathcal{R}_{+}=\left(\mathbb{R}_{+}, \cdot, \leq\right)$ together with the additional field operation + , see Example 2,

One of the most important, and still gaining in importance, methods of deriving priorities from pairwise comparisons has been proposed by Crawford [8]. According to this approach, referred in the literature as geometric mean method (GMM) the weight of $i$-th alternative is given by the geometric mean of the $i$-th row of $C$. Thus, the priority vector is given as

$$
\begin{equation*}
w=\gamma\left[\left(\prod_{r=1}^{n} c_{1 r}\right)^{\frac{1}{n}}, \ldots,\left(\prod_{r=1}^{n} c_{n r}\right)^{\frac{1}{n}}\right]^{T} \tag{9}
\end{equation*}
$$

where $\gamma$ is a scaling factor. As previously, $\gamma=\left(\sum_{i=1}^{n} w_{\max }\left(a_{i}\right)\right)^{-1}$. Following [30], we obtain the following definition.

Definition 3. Let $C=\left[c_{i j}\right]$ be a reciprocal PC matrix. For each pair $i, j \in$ $\{1, \ldots, n\}$, and a priority vector $w=\left[w\left(a_{1}\right), \ldots, w\left(a_{n}\right)\right]^{T}, a$ local error index $\epsilon(i, j, w)$ is given as

$$
\begin{equation*}
\epsilon(i, j, w) \stackrel{d f}{=} c_{i j} \odot w\left(a_{j}\right) \div w\left(a_{i}\right) \tag{10}
\end{equation*}
$$

and similarly (as in [24]) let us define

$$
\begin{equation*}
\mathscr{E}(i, j, w) \stackrel{d f}{=} \max \left\{\epsilon(i, j, w),(\epsilon(i, j, w))^{(-1)}\right\} \tag{11}
\end{equation*}
$$

The global error index $\mathscr{E}(C, w)$ for the $P C$ matrix $C$ and a priority vector $w=\left(w_{1}, \ldots, w_{n}\right)$, is the maximal value of $\mathscr{E}(i, j, w)$, i.e.,

$$
\begin{equation*}
\mathscr{E}(C, w) \stackrel{d f}{=} \max _{i, j \in\{1, \ldots, n\}} \mathscr{E}(i, j, w) \tag{12}
\end{equation*}
$$

Now, let us derive the following properties of $\mathscr{E}(C, w)$.
Lemma 1. Let $C=\left[c_{i j}\right]$ be a reciprocal PC matrix and $w=\left[w\left(a_{1}\right), \ldots, w\left(a_{n}\right)\right]^{T}$ be a priority vector. Then

$$
\begin{equation*}
\mathscr{E}(C, w) \geq e \tag{13}
\end{equation*}
$$

moreover, if

$$
\mathscr{E}(C, w))=e
$$

then $C$ is $\odot$-consistent.
Proof. Either $\epsilon(i, j, w) \geq e$, or, $\epsilon(i, j, w) \leq e$, then $\epsilon(i, j, w))^{(-1)} \geq e$. Hence,

$$
\mathscr{E}(i, j, w)=\max \left\{\epsilon(i, j, w),(\epsilon(i, j, w))^{(-1)}\right\} \geq e
$$

By (12) we obtain

$$
\mathscr{E}(C, w) \geq e
$$

Moreover, let $\mathscr{E}(C, w)=e$. Then by (11), (12) for all $i, j \in\{1, \ldots, n\}$, it holds

$$
\epsilon(i, j, w)=c_{i j} \odot w\left(a_{j}\right) \div w\left(a_{i}\right)=e,
$$

hence, equivalently,

$$
c_{i j}=w\left(a_{i}\right) \div w\left(a_{j}\right)
$$

Then we obtain

$$
c_{i j} \odot c_{j k} \odot c_{k i}=w\left(a_{i}\right) \div w\left(a_{j}\right) \odot w\left(a_{j}\right) \div w\left(a_{k}\right) \odot w\left(a_{k}\right) \div w\left(a_{i}\right)=e
$$

hence by (6), $C$ is $\odot$-consistent.

Remark 2. The global error index $\mathscr{E}(C, w)$ depends not only on the elements $c_{i j}$ of $P C$ matrix $C$, but also on the priority vector $w$. It is, however, always greater or equal to the identity element $e \in G$. If the global error index of $C$ is equal to $e$ then $P C$ matrix $C$ is $\odot$-consistent. Later, in Lemma 2, we will show that if $P C$ matrix $C$ is $\odot$-consistent, then there exists a priority vector $w$ such that $\mathscr{E}(C, w)=e$ holds.

## 4. Condition of Order Preservation

In [2] Bana e Costa and Vansnick formulate two conditions of order preservations. Here, we formulate these conditions in a more general setting, i.e., for alo-groups. The first, the preservation of order preference condition $(P O P)$, claims that the ranking result in relation to the given pair of alternatives $\left(a_{i}, a_{j}\right)$ should not break with the expert judgment, that is, if for a pair of alternatives $a_{i}, a_{j} \in \mathscr{A}$ such that $a_{i}$ dominates $a_{j}\left(c_{i j}>e\right)$ then:

$$
\begin{equation*}
w\left(a_{i}\right)>w\left(a_{j}\right), \text { or, equivalently } w\left(a_{i}\right) \div w\left(a_{j}\right)>e \tag{14}
\end{equation*}
$$

Here, $w\left(a_{k}\right), k=1,2, \ldots, n$, are individual weights of a priority vector $w$.
The second one the preservation of order of intensity of preference condition (POIP), claims that if $a_{i}$ dominates $a_{j}$ more than $a_{k}$ dominates $a_{l}$ $\left(a_{i}, a_{j}, a_{k}, a_{l} \in \mathscr{A}\right)$, i.e., if $c_{i j}>e, c_{k l}>e$ and $c_{i j}>c_{k l}$ then

$$
\begin{equation*}
w\left(a_{i}\right) \div w\left(a_{j}\right)>w\left(a_{k}\right) \div w\left(a_{l}\right) \tag{15}
\end{equation*}
$$

We show that POP and POIP condition is satisfied if the PC matrix is $\odot-$ consistent. We start with the well known necessary and sufficient condition for a PC matrix to be $\odot$-consistent, see also [29].

Lemma 2. Let $C=\left[c_{i j}\right]$ be an $\odot$-reciprocal PC matrix. Then $C$ is $\odot$ consistent if and only if there exists a priority vector $w=\left[w\left(a_{1}\right), \ldots, w\left(a_{n}\right)\right]^{T}$ such that for all $i, j \in\{1, \ldots, n\}$

$$
\begin{equation*}
w\left(a_{i}\right) \div w\left(a_{j}\right)=c_{i j} \tag{16}
\end{equation*}
$$

Proof. Suppose that $C=\left[c_{i j}\right]$ is $\odot$-consistent, then by Definition 1

$$
c_{i j} \odot c_{j k} \odot c_{k i}=e \text { for all } i, j, k \in\{1, \ldots, n\}
$$

or, equivalently,

$$
c_{i j} \odot c_{j k}=c_{i k} .
$$

Let $w=\left(w_{1}, \ldots, w_{n}\right)$ be given by

$$
\begin{equation*}
w\left(a_{i}\right)=\delta \odot\left(\bigodot_{r=1}^{n} c_{i r}\right)^{\left(\frac{1}{n}\right)}, i \in\{1,2, \ldots, n\} \tag{17}
\end{equation*}
$$

where $\delta$ is a scaling factor, $\delta=\left(\bigodot_{r=1}^{n} c_{1 r}\right)^{\left(-\frac{1}{n}\right)} \odot \ldots \odot\left(\bigodot_{r=1}^{n} c_{n r}\right)^{\left(-\frac{1}{n}\right)}$. Then we obtain by consistency condition $c_{i r} \odot c_{r j}=c_{i j}$

$$
\begin{gathered}
w\left(a_{i}\right) \div w\left(a_{j}\right)=\delta \odot\left(\bigodot_{r=1}^{n} c_{i r}\right)^{\left(\frac{1}{n}\right)} \odot\left(\delta \bigodot_{r=1}^{n} c_{j r}\right)^{\left(-\frac{1}{n}\right)}= \\
=\left(\bigodot_{r=1}^{n}\left(c_{i r} \odot c_{r j}\right)\right)^{\left(\frac{1}{n}\right)}=\left(c_{i j}^{(n)}\right)^{\left(\frac{1}{n}\right)}=c_{i j}
\end{gathered}
$$

hence, (16) is satisfied. On the other hand, let condition (16) be satisfied. Then for each $i, j, k \in\{1, \ldots, n\}$ we obtain

$$
\begin{aligned}
& c_{i j} \odot c_{j k} \odot c_{k i}=\left(w\left(a_{i}\right) \div w\left(a_{j}\right)\right) \odot\left(w\left(a_{j}\right) \div w\left(a_{k}\right)\right) \odot\left(w\left(a_{k}\right) \div w\left(a_{i}\right)\right)= \\
& \quad=w\left(a_{i}\right) \odot w\left(a_{j}\right)^{(-1)} \odot w\left(a_{j}\right) \odot w\left(a_{k}\right)^{(-1)} \odot w\left(a_{k}\right) \odot w\left(a_{i}\right)^{(-1)}=e
\end{aligned}
$$

hence, $C$ is $\odot$-consistent.
Theorem 1. Let $C=\left[c_{i j}\right]$ be an $\odot$-reciprocal PC matrix, and let $w=$ $\left[w\left(a_{1}\right), \ldots, w\left(a_{n}\right)\right]^{T}$ be a priority vector, let $i, j, k, l \in\{1, \ldots, n\}$. If $C$ is $\odot$-consistent then condition POP is satisfied, i.e. $c_{i j}>e$ implies $w_{i}>w_{j}$. Moreover, if $c_{i j}>c_{k l}$, then condition POIP is also satisfied, i.e., $c_{i j}>c_{k l}$ implies $w_{i} \div w_{j}>w_{k} \div w_{l}$.

Proof. Suppose that $C=\left[c_{i j}\right]$ is $\odot$-consistent. If for some $i, j \in\{1, \ldots, n\}$ we have $c_{i j}>e$, then by (16) in Lemma 2 we have

$$
c_{i j}=w\left(a_{i}\right) \div w\left(a_{j}\right)>e,
$$

which is equivalent to $w\left(a_{i}\right)>w\left(a_{j}\right)$ and condition $P O P$ is satisfied. Moreover, by Lemma 2, it holds that $c_{i j}>c_{k l}$ if and only if

$$
w\left(a_{i}\right) \div w\left(a_{j}\right)>w\left(a_{k}\right) \div w\left(a_{l}\right),
$$

hence, (14) is satisfied.

Lemma 3. Let $C=\left[c_{i j}\right]$ be an $\odot$-reciprocal PC matrix and $w=\left(w_{1}, \ldots, w_{n}\right)$ be a priority vector. Then for all $i, j \in\{1, \ldots, n\}$

$$
\begin{equation*}
\mathscr{E}(C, w)^{(-1)} \odot w\left(a_{i}\right) \div w\left(a_{j}\right) \leq c_{i j} \leq \mathscr{E}(C, w) \odot w\left(a_{i}\right) \div w\left(a_{j}\right) \tag{18}
\end{equation*}
$$

Proof. By (11), (12) we obtain

$$
\begin{equation*}
\mathscr{E}(C, w) \geq \max \left\{\epsilon(i, j, w),(\epsilon(i, j, w))^{(-1)}\right\} \geq \epsilon(i, j, w)=c_{i j} \odot w\left(a_{j}\right) \div w\left(a_{i}\right) \tag{19}
\end{equation*}
$$

$\mathscr{E}(C, w) \geq \max \left\{\epsilon(i, j, w),\left(\epsilon(i, j, w)^{(-1)}\right\} \geq \epsilon(i, j, w)^{(-1)}=c_{j i} \odot w\left(a_{i}\right) \div w\left(a_{j}\right)\right.$,
hence, when multiplied both sides of (19) by $w\left(a_{i}\right) \div w\left(a_{j}\right)$, and both sides of (20) by $w\left(a_{j}\right) \div w\left(a_{i}\right)$, we get
$\mathscr{E}(C, w) \odot w\left(a_{i}\right) \div w\left(a_{j}\right) \geq c_{j i} \odot w\left(a_{i}\right) \div w\left(a_{j}\right) \odot w\left(a_{j}\right) \div w\left(a_{i}\right)=c_{i j} \odot e=c_{i j}$.
$\mathscr{E}(C, w)^{(-1)} \odot w\left(a_{i}\right) \div w\left(a_{j}\right) \leq c_{i j} \odot w\left(a_{j}\right) \div w\left(a_{i}\right) \odot w\left(a_{i}\right) \div w\left(a_{j}\right)=c_{i j} \odot e=c_{i j}$.
Combining (21) and (22) we obtain (18).
Now, we turn our attention to $\odot$-inconsistent $\odot$-reciprocal $P C$ matrix. The following theorem gives sufficient conditions for validity of $P O P$ and POIP.

Theorem 2. Let $C=\left[c_{i j}\right]$ be an $\odot$-reciprocal $P C$ matrix, and let $w=$ $\left[w\left(a_{1}\right), \ldots, w\left(a_{n}\right)\right]^{T}$ be a priority vector, let $i, j, k, l \in\{1, \ldots, n\}$. If

$$
c_{i j}>\mathscr{E}(C, w), c_{k l}>\mathscr{E}(C, w)
$$

and

$$
\begin{equation*}
c_{i j} \div c_{k l}>(\mathscr{E}(C, w))^{2}, \tag{23}
\end{equation*}
$$

then

$$
w\left(a_{i}\right)>w\left(a_{j}\right), w\left(a_{k}\right)>w\left(a_{l}\right)
$$

and

$$
w\left(a_{i}\right) \div w\left(a_{j}\right)>w\left(a_{k}\right) \div w\left(a_{l}\right),
$$

i.e., condition POP and also POIP is satisfied.

Proof. If for some $i, j \in\{1, \ldots, n\}$ we have $c_{i j}>\mathscr{E}(C, w)$, then by (18) in Lemma 3 and $c_{i j}>\mathscr{E}(C, w)$ we obtain

$$
\begin{equation*}
\mathscr{E}(C, w) \odot w\left(a_{i}\right) \div w\left(a_{j}\right) \geq c_{i j}>\mathscr{E}(C, w) \tag{24}
\end{equation*}
$$

which implies, when "multiplied" both sides of (2) by $\mathscr{E}(C, w)^{(-1)}$,

$$
w\left(a_{i}\right) \div w\left(a_{j}\right)>e
$$

i.e., $w_{i}>w_{j}$, hence condition POP is satisfied. Similarly, if for some $k, l \in$ $\{1, \ldots, n\}$ we have $c_{i j}>\mathscr{E}(C, w), c_{k l}>\mathscr{E}(w)$, then we obtain $w\left(a_{i}\right)>w\left(a_{j}\right)$ and $w\left(a_{k}\right)>w\left(a_{l}\right)$, i.e. POP condition. Moreover, by (18) in Lemma 3 we obtain

$$
\begin{gather*}
\mathscr{E}(C, w) \odot w\left(a_{i}\right) \div w\left(a_{j}\right) \geq c_{i j}  \tag{25}\\
\mathscr{E}(C, w) \odot w\left(a_{l}\right) \div w\left(a_{k}\right) \geq c_{l k}=c_{k l}^{(-1)} \tag{26}
\end{gather*}
$$

By " $\odot$-multiplying" left sides and right sides of (24) and (26), we obtain

$$
\begin{equation*}
(\mathscr{E}(C, w))^{2} \odot\left(w\left(a_{i}\right) \div w\left(a_{j}\right)\right) \odot\left(w\left(a_{k}\right) \div w\left(a_{l}\right)\right)^{(-1)} \geq c_{i j} \div c_{k l} . \tag{27}
\end{equation*}
$$

If we assume that $\left(w\left(a_{i}\right) \div w\left(a_{j}\right)\right) \odot\left(w\left(a_{k}\right) \div w\left(a_{l}\right)\right)^{(-1)} \leq e$, which is equivalent to $w\left(a_{i}\right) \div w\left(a_{j}\right) \leq w\left(a_{k}\right) \div w\left(a_{l}\right)$, then by (27) we obtain

$$
(\mathscr{E}(C, w))^{2} \geq c_{i j} \div c_{k l} .
$$

This result, however, is in contradiction with (23), hence, it must be $w\left(a_{i}\right) \div w\left(a_{j}\right)>w\left(a_{k}\right) \div w\left(a_{l}\right)$, therefore, condition POIP is satisfied.

Remark 3. Notice that in the previous lemmas and theorems, there is no special assumption concerning the method for generating the priority vector $w=\left[w\left(a_{1}\right), \ldots, w\left(a_{n}\right)\right]^{T}$. The priority vector, or, vector of weights $w$, may be an arbitrary positive vector with normalized elements. Specifically, in the case of multiplicative alo-group of positive real numbers $\mathcal{R}_{+}=\left(\mathbb{R}_{+}, \cdot, \leq\right)$ with some field operation + , we may use $E V M, G M M$ or any other priority vector generating method. In the following section we shall investigate a generalized version of GMM.

## 5. Generalized Geometric Mean Method (GGMM)

Following the Geometric Mean Method (GMM) we define the Generalized Geometric Mean Method (GGMM), where the weight of $i$-th alternative is given by the $\odot$-mean of the $i$-th row of $C=\left[c_{i j}\right]$.

Definition 4. Let $C=\left[c_{i j}\right]$ be a reciprocal PC matrix. Let for $i, j, k \in$ $\{1, \ldots, n\}$

$$
\begin{equation*}
e(i, j, k) \stackrel{d f}{=} c_{i j} \odot c_{j k} \odot c_{k i}, \tag{28}
\end{equation*}
$$

and similarly let us define

$$
\begin{equation*}
\eta(i, j, k) \stackrel{d f}{=} \max \left\{e(i, j, k),(e(i, j, k))^{(-1)}\right\} . \tag{29}
\end{equation*}
$$

The generalized inconsistency index of the PC matrix $C$ is defined as

$$
\begin{equation*}
G I(C) \stackrel{d f}{=} \max \{\eta(i, j, k) \mid i, j, k \in\{1, \ldots, n\}\} . \tag{30}
\end{equation*}
$$

Lemma 4. Let $C=\left[c_{i j}\right]$ be a reciprocal PC matrix, $w=\left(w_{1}, \ldots, w_{n}\right)$ be a priority vector defined as

$$
\begin{equation*}
w=\delta \odot\left[\left(\bigodot_{r=1}^{n} c_{1 r}\right)^{\left(\frac{1}{n}\right)}, \ldots,\left(\bigodot_{r=1}^{n} c_{n r}\right)^{\left(\frac{1}{n}\right)}\right]^{T} \tag{31}
\end{equation*}
$$

where $\delta$ is a scaling factor, $\delta=\left(\bigodot_{r=1}^{n} c_{1 r}\right)^{\left(-\frac{1}{n}\right)} \odot \ldots \odot\left(\bigodot_{r=1}^{n} c_{n r}\right)^{\left(-\frac{1}{n}\right)}$. The individual weights are given as

$$
\begin{equation*}
w_{i}=\delta \odot\left(\bigodot_{r=1}^{n} c_{i r}\right)^{\left(\frac{1}{n}\right)}, i \in\{1, \ldots, n\} \tag{32}
\end{equation*}
$$

Then the global error index of $C$ is always less or equal to the generalized inconsistency index, i.e.,

$$
\begin{equation*}
\mathscr{E}(C, w) \leq G I(C) \tag{33}
\end{equation*}
$$

Proof. Providing the use of GGMM we have

$$
c_{i j} \odot w\left(a_{j}\right) \div w\left(a_{i}\right)=c_{i j} \odot\left(\bigodot_{k=1}^{n} c_{j k} \div \bigodot_{k=1}^{n} c_{i k}\right)^{\left(\frac{1}{n}\right)}
$$

thus,
$c_{i j} \odot w\left(a_{j}\right) \div w\left(a_{i}\right)=\left(\bigodot_{k=1}^{n} c_{i j} \odot c_{j k} \div \bigodot_{k=1}^{n} c_{i k}\right)^{\left(\frac{1}{n}\right)}=\left(\bigodot_{k=1}^{n} c_{i j} \odot c_{j k} \odot c_{k i}\right)^{\left(\frac{1}{n}\right)}$
However, it holds that

$$
\begin{gather*}
\left(\bigodot_{k=1}^{n} c_{i j} \odot c_{j k} \odot c_{k i}\right)^{\left(\frac{1}{n}\right)} \leq \max _{k \in\{1, \ldots, n\}}\left\{c_{i j} \odot c_{j k} \odot c_{k i}\right\}= \\
\max _{k \in\{1, \ldots, n\}}\{\eta(i, j, k)\}, \tag{34}
\end{gather*}
$$

hence, $\mathscr{E}(C, w) \leq G I(C)$.
Now, we shall derive a relationship between Koczkodaj's inconsistency index (8) and the generalized inconsistency index (30) in the multiplicative alo-group $\mathcal{R}_{+}=\left(\mathbb{R}_{+}, \cdot, \leq\right)$ together with the additional field operation + . First, we modify Theorem 2 with respect to the above mentioned Koczko$d a j$ 's inconsistency index $K I(C)$ and the new generalized inconsistency index $G I(C)$, see [24, Corollary 1].

Theorem 3. Let $C=\left[c_{i j}\right]$ be an $\odot$-reciprocal PC matrix, $w=\left[w\left(a_{1}\right), \ldots, w\left(a_{n}\right)\right]^{T}$ be a priority vector generated by GGMM, i.e., (31) and (32). If

$$
c_{i j}>\frac{1}{1-K I(C)}, c_{k l}>\frac{1}{1-K I(C)}
$$

and

$$
\begin{equation*}
c_{i j} \div c_{k l}>\left(\frac{1}{1-K I(C)}\right)^{2} \tag{35}
\end{equation*}
$$

then

$$
w\left(a_{i}\right)>w\left(a_{j}\right), w\left(a_{k}\right)>w\left(a_{l}\right)
$$

and

$$
w\left(a_{i}\right) \div w\left(a_{j}\right)>w\left(a_{k}\right) \div w\left(a_{l}\right),
$$

i.e., condition POP and also condition POIP is satisfied.

Proof. Comparing (8) and (12), (30) we easily derive the relation between $K I(C)$ and $G I(C)$ as follows

$$
\begin{equation*}
G I(C)=\frac{1}{1-K I(C)} \tag{36}
\end{equation*}
$$

If for some $i, j, k, l \in\{1, \ldots, n\}$ we have $c_{i j}>G I(C)=\frac{1}{1-K I(C)}, c_{k l}>G I(C)$, then by Theorem 2 and Lemma 4 we obtain $w\left(a_{i}\right)>w\left(a_{j}\right)$ and $w\left(a_{k}\right)>$ $w\left(a_{l}\right)$.

Example 2. We consider an illustrating example of $4 \times 4 P C$ matrix $C$ in the usual multiplicative alo-group of positive real numbers $\mathcal{R}_{+}=\left(\mathbb{R}_{+}, \cdot, \leq\right)$, as follows:

$$
C=\left[\begin{array}{cccc}
1 & \frac{5}{2} & 3 & 5 \\
\frac{2}{5} & 1 & 2 & 4 \\
\frac{1}{3} & \frac{1}{2} & 1 & 3 \\
\frac{1}{5} & \frac{1}{4} & \frac{1}{3} & 1
\end{array}\right]
$$

The priority vector generated by $G G M M$ (in fact $G M M$ ) is:

$$
w=[0.494,0.2675,0.168,0.072]^{T},
$$

hence, the newly proposed generalized ' $\cdot$ '-inconsistency index $G I(C)=2.00$, $G I(C)^{2}=4.00$ and Koczkodaj's inconsistency index $K I(C)=0,5$. We conclude that if $c_{i j}>2.00$, then $w\left(a_{i}\right)>w\left(a_{j}\right)$. It is clear, that $P O P$ condition holds for all elements located above the main diagonal of $P C$ matrix $C$. Moreover, $c_{12}=2.50, c_{23}=2.00$ and $c_{12} / c_{23}=1.25$. We obtain $w_{1} / w_{2}=1.85$ and $w_{2} / w_{3}=1.59$, hence, $w_{1} / w_{2}>w_{2} / w_{3}$. Here, POIP condition is satisfied.

## 6. Discussion and summary

Is it possible to meet the $C O P$ criteria [2] when the ranking method is $E V M$ ? Would it be possible to use $G M M$ instead of $E V M$ while preserving $C O P$ ? Theorem 2 provides the evidence that as long as the result $c_{i j}$ of the direct comparison of the $i$-th and $j$-th alternative is large enough it is possible. In such a case the lower limit for $c_{i j}$ is given by the global error index $\mathscr{E}(C, w)$ defined for any priority vector $w$. Hence, the minimal value of $c_{i j}$ guaranting that $C O P$ is met depends on $w$, regardless of how $w$ is obtained.

An even more surprising conclusion comes from Theorem 3. Assuming that the weight vector was obtained using $G M M$ ( $G G M M$ ) the minimal value of $c_{i j}$-which guarantees compliance with the COP criteria-depends only on the inconsistency of the $P C$ matrix. In particular this means that by improving the consistency among the pairwise comparisons we are able to make the $P C$ matrix comply with $C O P$. Interestingly, a similar situation occurs in the case when the priority deriving method is $E V M$ [25]. This raises the question whether a similar property can be observed for any priority deriving method. This question remains unanswered today, however, it seem to be an interesting direction for further research.

By abstracting into an alo-group, we define GI a new generalized inconsistency index based only on the group operation $\odot$. We showed the relationship between the triad GI, KI (Koczkodaj's inconsistency index) and COP. The particulars of this relationship have been examined in the example at the end of Section 5

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[^0]:    Email addresses: konrad.kulakowski@agh.edu.pl (Konrad Kułakowski), mazurek@opf.slu.cz (Jiří Mazurek), ramik@opf.slu.cz (Jaroslav Ramík), michael.soltys@csuci.edu (Michael Soltys)

[^1]:    ${ }^{1}$ An alternate form of this definition can be found in [21].

