Revenue Maximization in Service Systems with Heterogeneous Customers

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Abstract-In this paper, we consider revenue maximization problem for a two server system in the presence of heterogeneous customers. We assume that the customers differ in their cost for unit delay and this is modeled as a continuous random variable with a distribution F. We also assume that each server charges an admission price to each customer that decide to join its queue. We first consider the monopoly problem where both the servers belong to a single operator. The heterogeneity of the customer makes the analysis of the problem difficult. The difficulty lies in the inability to characterize the equilibrium queue arrival rates as a function of the admission prices. We provide an equivalent formulation with the queue arrival rates as the optimization variable simplifying the analysis for revenue rate maximization for the monopoly. We then consider the duopoly problem where each server competes with the other server to maximize its revenue rate. For the duopoly problem, the interest is to obtain the set of admission prices satisfying the Nash equilibrium conditions. While the problem is in general difficult to analyze, we consider the special case when the two servers are identical. For such a duopoly system, we obtain the necessary condition for existence of symmetric Nash equilibrium of the admission prices. The knowledge of the distribution Fcharacterizing the heterogeneity of the customers is necessary to solve the monopoly and the duopoly problem. However, for most practical scenarios, the functional form of F may not be known to the system operator and in such cases, the revenue maximizing prices cannot be determined. In the last part of the paper, we provide a simple method to estimate the distribution F by suitably varying the admission prices. We illustrate the method with some numerical examples.

I. INTRODUCTION

In many service systems, the quality of service received is characterized by the queueing delay that is experienced by the customers in the system. Examples of such service systems that can be modeled as queueing systems include road and transport systems, health-care systems, computer systems, call centers and communications systems. The customers that receive service in such systems are usually sensitive to the delay experienced in these system. Further, such customers have non-identical preferences to the delay experienced. It is often beneficial for the service system to account for these heterogeneous preferences in any optimization concerning the use of system resources. Many service systems have emerged that exploit the heterogeneous nature of customers and use it to their advantage. For example, airlines offer priority boarding queues for payment of an additional fee. In this paper, we consider the problem of exploiting the heterogeneous nature of customers for revenue maximization in parallel server systems. We model heterogeneity of customers by assuming that different customers have different cost for a unit delay.

We consider service systems that consist of two parallel, possibly heterogeneous servers where each server has an associated queue for the customers to wait. The scheduling discipline at each server is work conserving and does not discriminate between customers on the basis of their preference for delay. The servers charge an admission price to every customer joining its queue. We assume that the queues are not observable and only the expected delay as a function of the arrival rate is available. We also assume that the expected delay at any server is monotone increasing in the arrival rate of customers to that server. The customers that use the system are strategic and make an individually optimal queue-join decision. We assume that customers differ in their cost for unit delay which is characterized by a random variable with a continuous distribution denoted by F. For a customer, the cost at a server is the sum of the admission price and the delay cost at the server. We assume that customers cannot balk from the system without obtaining service and such traffic is commonly seen in cloud-computing, purchase of essential services etc.

In this paper, we consider the problem of revenue maximization in such a service system by suitably choosing the admission prices at two parallel servers. Depending on the objective of each of these servers, we consider two natural scenarios. In a monopoly, we assume that the two servers belong to the same operator. The objective here is to maximize the total revenue rate, i.e., the sum of the revenue rate from the two servers. In the second scenario, we assume that each server belongs to separate operators and each server has the objective of maximizing its individual revenue rate. This is an example of a duopoly where the service systems compete with one another to maximize their individual revenue rate.

Now consider the scenario of a monopoly market discussed above where the service system has two parallel servers. In the absence of balking, it is not difficult to see that a revenue maximizing strategy for the monopoly is to keep both the admission prices at infinity. This is because as customers cannot balk, they are required to choose one of the server for service. Therefore one has to consider a more meaningful model for the monopoly market. Towards this, we assume that the admission price at one of the server, say Server 2 is fixed a-priori. This dissuades the service provider from fixing the admission price at Server 1 to unreasonably high values. Our interest for this model is to characterize the revenue maximizing admission price at Server 1 for different examples of the delay functions at the queue and when customers differ in their delay cost.

Classical monopoly models have been well studied for the

case of single server queues. One of the first work to analyze such a model is Naor [1]. This model considers a single server queueing system where homogeneous customers obtain a reward after service completion. The queue is observable to arriving customers who choose to either join the queue or balk. For such a system, the revenue maximizing admission price was first obtained in [1]. Subsequently, there have been several works analyzing the revenue maximization problem for various models such as a multiserver queue [2], GI/M/1queue [3], customers with heterogeneous service valuations [4] and queue length dependent prices [5]. While the above models assume that the queue lengths are observable, Edelson and Hilderbrand [6] were the first to consider the revenue maximization problem for the case when queues are not observable. See [7], [8], [9], [10], [11] for some other single server revenue maximization models.

The key difference of our model with that of the literature discussed above is as follows. Firstly, in our model, customers are inelastic in their demand and hence balking is not allowed. Secondly, the customers have to obtain service at either of the two servers and the admission price at one of the server is fixed. Finally, the customers have heterogeneous preference for the delay experienced in the queue. This feature makes our model meaningful but also difficult to analyze. For such parallel server models, the structural properties for the equilibrium routing have been obtained recently [12], [13]. We use the structural property of the equilibrium routing to solve the the revenue maximization problem for the monopoly.

For the duopoly problem with two competing and identical servers, we assume that the objective for each server is to set an admission price that maximizes its revenue rate. We are interested in studying the existence of Nash Equilibrium prices that would be set by the two servers. The earliest work analyzing the duopoly model with heterogeneous customers was by Luski [14] and Levhari and Luski [15]. Both the models assume that the customers are allowed to balk. Luski [14] is interested in knowing whether the revenue maximizing prices set by the two service systems can be equal. It is observed that when the parameters of the model are such that the customers have no incentive to balk, the revenue maximizing prices set by two identical servers is equal. This is however not the case when some of the customers prefer to balk. In this case, the equilibrium revenue maximizing prices are not equal. Levhari and Luski [15] provide a numerical analysis for the problem introduced in Luski [14]. Armony and Haviv [16] analyze this problem for the case when the customers are from a finite number of classes and each class has a distinct cost for unit delay. A numerical analysis of the Nash equilibrium admission prices between the two competing servers is provided. Chen and Wan [17] consider the revenue maximization in a duopoly with a single customer class. The service system is modeled by M/M/1 queues and the customers are allowed to balk from the system. These assumptions on the system model allows them to obtain the sufficient conditions for the existence of Nash equilibrium. Similar conditions were found in Dube and Jain [18] who consider an N-player oligopoly with multiclass customers. The customer classes differ only in their arrival rates and have the same delay cost per unit time. A differentiated service model is considered by Dube and Jain [19] where each player now operates two types of services and each service is used by a dedicated class of customers. Again, the key result in [19] is to obtain the sufficient condition for the Nash equilibrium prices. Mandjes and Timmers [20] consider a duopoly model with two customer classes differing in their delay cost. The model assumes a finite number of customers and the utility of a queue is a decreasing function of the number of customers using this server. Given the prices at the servers, they provide an algorithm that determines the equilibrium number of customers of each class that is to be allocated to the two servers. While the existence and uniqueness of such a customer equilibrium is provided, the existence of Nash equilibrium prices is only conjectured. In [21], [22] the demand rate at different servers is modeled using specific functions (known as demand models in such literature) instead of being calculated from the (Wardrop) equilibrium conditions [23]. This assumptions make the analysis relatively simpler. Ayesta et. al. [24] consider the oligopoly pricing game for a single customer class and obtain the necessary and sufficient conditions on the Nash equilibrium prices when the queues have identical delay functions. A best-response algorithm is then provided to numerically obtain these Nash equilibrium prices.

Most of the monopoly and duopoly models described above, make simplifying assumptions on the customer classes to characterize the underlying Wardrop equilibrium [23]. Additional simplification of the analysis is obtained by considering convex and increasing delay functions at the queues. We do not make any of these assumptions in this paper. We utilize the structure of the Wardrop equilibrium that was characterized in [12], [13] to analyze the two problems. This structure on the equilibrium allows us to provide an equivalent revenue maximization formulation for both the monopoly and the duopoly that is simpler to analyze. For the duopoly problem we provide sufficient conditions on the symmetric Nash equilibrium prices when the competing servers are identical.

For most practical scenarios, the distribution function $F(\cdot)$ characterizing the delay cost for a customer may not be known to the service system. The revenue maximizing strategy on the other hand depend on the distribution $F(\cdot)$. Without any knowledge of $F(\cdot)$, it is not be possible to ascertain a revenue optimal admission price at the servers and in such cases, the service system is required estimate this distribution function. Towards the end of this paper, we shall provide a simple method to estimate this distribution $F(\cdot)$ by varying the admission prices and observing the change in the equilibrium traffic routing. The service system can then use this estimate to perform the necessary revenue maximization.

The rest of the paper is organized as follows. In the next section, we shall formalize the notations and provide some preliminaries. We then formulate the revenue maximization problems in Section III. In Section IV, we consider the monopoly problem for revenue optimization followed by the duopoly problem in Section V. Finally in Section VI, we illustrate a mechanism based on admission pricing to estimate the distribution function F.

II. PRELIMINARIES

We will first introduce the notations that will be used throughout this paper. In both the monopoly and the duopoly model, we assume that the system has two servers. Let c_j denote the admission price at Server j where j=1,2. The customers arrive according to a homogeneous Poisson process with rate λ and have a service requirement that is i.i.d with exponential distribution and unit mean. Let $D_j(\gamma_j)$ denote the delay function associated with queue j when the queue arrival rate is γ_j , where j=1,2. Note that $\gamma_1+\gamma_2=\lambda$. We assume that D_j is monotone increasing and continuously differentiable in the interior of its domain with a strictly positive derivative. Additionally we assume that the cost function at the two server satisfies the following two conditions (1) $D_1(0) < D_2(\lambda) < \infty$ and (2) $D_2(0) < D_1(\lambda) < \infty$.

We associate with each arriving customer a continuous random variable β that quantifies a customer's sensitivity to delay or congestion. We shall assume that the delay sensitivity β for a customer is a realization of the random variable β . The customer arrivals constitute a marked Poisson process of intensity $\lambda \times F$ on $\mathbb{R} \times \mathbb{R}_+$. Here F is an absolutely continuous cumulative distribution function supported on the interval [a,b] of positive reals. We additionally assume that $F(\cdot)$ is strictly increasing and hence $f(x) \neq 0$ for any $x \in [a,b]$ where $f(\cdot)$ is the corresponding density function.

We now recall the Wardrop equilibrium conditions [23], [13] that characterize the individually optimal choice of server made by the arriving customers. A customer with delay cost β entering the system must choose a queue j so as to minimize $c_i + \beta D_i(\gamma_i)$. Here γ_i is determined through the strategies of all customers. We assume that the quantities $\lambda_1, \lambda_2, D_i(\cdot), F(\cdot)$ and c_i , for j = 1, 2 is part of common knowledge. We also assume that the customers do not have access to current or past queue occupancies, or the history of arrival times. The strategy of a customer is restricted to choosing a server according to a fixed probability distribution and such joint strategies are represented by a stochastic kernel, denoted by K^W . We interpret $K^W(\beta, i)$ as the probability that a customer with delay sensitivity β chooses queue i at equilibrium. For the two server system, the equilibrium kernel K^W must satisfy the following Wardrop equilibrium conditions.

$$K^{W}(\beta, i) \ge 0 \text{ implies } c_i + \beta D_i(\gamma_i) \le c_{3-i} + \beta D_{3-i}(\gamma_{3-i}).$$
(1)

In words, this means that if customers with delay cost β choose Server i at equilibrium, then the expected cost for this customer at Server i must be at most the expected cost at Server 3-i for i=1,2. For a kernel K^W , note that the arrival rate of customers to Server j is given by

$$\gamma_j = \lambda \int_{\beta=a}^b K^W(\beta, j) dF(\beta).$$

We now provide the following theorem that is a restatement of Corollary 4 in [13]. This theorem characterizes the Wardrop equilibrium kernel for a system with two parallel servers.

Theorem 1: Define δ_i as the probability distribution that puts unit mass on i and suppose that the kernel K^W satis-

fies the Wardrop equilibrium condition. Then there exists a threshold β_1 with $\beta_1 \in [a, b]$ such that

• when $c_1 > c_2$ (resp. $c_1 < c_2$),

$$K^{W}(\beta,\cdot) = \begin{cases} \delta_{1} \text{ (resp. } \delta_{2}) & \text{for } \beta \in (\beta_{1},b], \\ \delta_{2} \text{ (resp. } \delta_{1}) & \text{for } \beta \in [a,\beta_{1}]. \end{cases}$$
 (2)

Further if $\beta_1 \in (a, b)$ then,

$$c_1 + \beta_1 D_1(\gamma_1) = c_2 + \beta_1 D_2(\gamma_2). \tag{3}$$

• When $c_1 = c_2$, K^W is not unique and any kernel K with $\gamma_1 = \gamma^+$ is a valid Wardrop equilibrium kernel K^W where $\gamma^+ := \{\gamma_1 : D_1(\gamma_1) = D_2(\gamma_2)\}$.

Refer Figures 1 and 2 for a representation of the Wardrop equilibrium kernel for the case when $c_1 > c_2$ and $c_1 < c_2$ respectively. Here $f(\cdot)$ denotes the underlying density function of the random variable β while the shaded region identifies the delay cost parameter of those customers that choose Server 1.

Proof: The first part is simply a restatement of Corollary 4 in [13] for the case $c_1 > c_2$ and the proof for $c_1 < c_2$ is along similar lines. We now prove the second part. Consider the case when $c_1 = c_2$ and recall the assumption that $D_1(0) < D_2(\lambda)$ and $D_2(0) < D_1(\lambda)$. K^W must be such that $D_1(\gamma_1) = D_2(\gamma_2)$. To see why this must be true, suppose that this is not true and let $D_1(\gamma_1) \neq D_2(\gamma_2)$. Customers from the queue with a higher delay cost will have an incentive to move to the queue with a lower delay cost. This implies that a K^W with $D_1(\gamma_1) \neq D_2(\gamma_2)$ is not at equilibrium. Recall the definition $\gamma^+ := \{ \gamma_1 : D_1(\gamma_1) = D_2(\gamma_2) \}$. Since, $D_1(0) < D_2(\lambda)$ and $D_2(0) < D_1(\lambda)$, we have $0 < \gamma^+ < \lambda$. Now for any kernel K satisfying $\gamma_1 = \gamma^+$, since $c_1 = c_2$, the cost for any customer at the two servers is equal. Hence there is no incentive for any customer to deviate from its choice of the server. The Wardrop equilibrium kernel K^W though not unique must however satisfy $\lambda \int_{\beta=a}^b K^W(\beta,j) dF(\beta) = \gamma^+$.

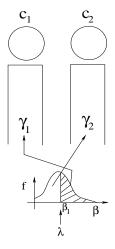
III. PROBLEM FORMULATIONS

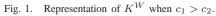
Having characterized the Wardrop equilibrium kernel K^W for a two server system, we will now formulate the revenue maximization problems for both the monopoly and the duopoly model. Let $R_j(c_j, \gamma_j) := c_j \gamma_j$ denote the revenue rate at server j when the arrival rate of customers due to the corresponding kernel K^W is γ_j for j=1,2. For the monopoly model, let $R_T(c_1, \gamma_1)$ denote the revenue rate for the monopoly service system. Since $\gamma_2 = \lambda - \gamma_1$, it suffices to express the revenue rate as a function of only γ_1 . We have

$$R_T(c_1, \gamma_1) := c_1 \gamma_1 + c_2 \gamma_2 = c_2 \lambda + (c_1 - c_2) \gamma_1.$$

Note from Theorem 1, that the argument γ_1 is determined by the kernel K^W which in turn depends on the admission prices c_1 and c_2 . This dependence will be made explicit by writing γ_1 as $\gamma_1(c_1,c_2)$ and the revenue optimization problem for the monopoly can now be stated as follows.

$$\max_{c_1} \qquad R_T(c_1,\gamma_1(c_1,c_2)) = c_2\lambda + (c_1-c_2)\gamma_1(c_1,c_2)$$
 subject to
$$0 \le c_1 \le c^1$$
 (P1)





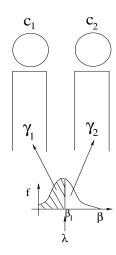


Fig. 2. Representation of K^W when $c_1 < c_2$.

where c^1 is an arbitrarily large value such that $\gamma_1(c^1,c_2)=0$. c^1 is a technical requirement to ensure a compact domain and one could also define $c^1:=\inf\{c:\gamma_1(c,c_2)=0\}$ in which case we have $\gamma_1(c_1,c_2)=0$ for any $c_1>c^1$. To be able to solve program P1 using standard optimization techniques, a closed form expression for $\gamma_1(c_1,c_2)$ would be convenient. When $c_1>c_2$, and $\beta_1\in(a,b)$, from Theorem 1 and the definition of γ_1 , it can be seen that

$$\gamma_1(c_1, c_2) = \lambda(1 - F(\beta_1))$$
 (4

where

$$\beta_1 = \{ \beta : c_1 + \beta D_1(\lambda (1 - F(\beta))) = c_2 + \beta D_2(\lambda F(\beta)) \}.$$

A similar condition follows when $c_1 < c_2$ and it can be seen that obtaining an explicit expression for $\gamma_1(c_1, c_2)$ is difficult. Note that we have not assumed any functional form for D_i and $F(\cdot)$ and for certain choice of these functions, a closed form expression for $\gamma_1(c_1, c_2)$ may not be possible. Without an analytic expression for $\gamma_1(c_1, c_2)$, it is difficult to solve the revenue maximization problem. Therefore we require an alternative approach to solve program P1. One possible alternative is to let the equilibrium γ_1 (the value of γ_1 at equilibrium) be the optimization variable and represent other variables of the system such as c_1, c_2, β_1 as a function of γ_1 . With slight abuse of notation, we will use $c_i(\gamma_i)$ to denote the admission price at Server j when the arrival rate to Server j at equilibrium is γ_j where j = 1, 2. Similarly, we shall use $\beta_1(\gamma_1)$ to represent the threshold β_1 corresponding to an equilibrium arrival rate of γ_1 to Server 1. Note that $c_1(\gamma_1)$ is also a function of c_2 . This is because the equilibrium γ_1 depends on the difference $(c_1 - c_2)$ and not on their individual values. This is clear from Theorem 1 (Eq. (3)). Therefore for a given c_2 and $\gamma_1 \in (0, \lambda)$ one can determine c_1 using Eq. (3). We have suppressed this dependence on c_2 to simplify notation. For the monopoly model $c_2(\gamma_2) = c_2$ as c_2 is assumed fixed. Thus the equivalent revenue optimization problem for the monopoly is as follows.

$$\max_{\gamma_1} R_T(c_1(\gamma_1), \gamma_1) = c_2 \lambda + (c_1(\gamma_1) - c_2) \gamma_1$$
subject to $0 \le \gamma_1 \le \gamma^1(c_2) \le \lambda$ (P2)

where $\gamma^1(c_2)$ determines the domain for the feasible values of γ_1 as a function of c_2 . An intuitive explanation for the quantity $\gamma^1(c_2)$ is as follows. Consider the case $c_1=c_2=0$. From Theorem 1, we have $\gamma_1=\gamma^+$ where $0<\gamma^+<\lambda$. Using the notation $\gamma_1(c_1,c_2)$, we have $\gamma_1(0,0)=\gamma^+$. For any $c_1>0$, $\gamma_1(c_1,0)<\gamma^+$ since the increase in the admission price at Server 1 makes the server more costly and decreases the resulting γ_1 . Clearly, for any $c_1\geq 0$ and $c_2=0$, $\gamma_1\notin (\gamma^+,\lambda]$ and $c_1(\gamma_1)$ in program P2 cannot be defined for $\gamma_1\in (\gamma^+,\lambda]$. Therefore when $c_2=0$, the domain for the optimization variable γ_1 should be restricted to $[0,\gamma^+]$. In general, for an arbitrary c_2 , the domain for γ_1 in program P2 is defined using $\gamma^1(c_2)$ and this will be characterized formally in Section IV.

Now consider the duopoly market with two competing servers charging admission prices c_1 and c_2 to their arriving customers. The objective of Server j is to choose an admission price c_j that maximizes its revenue rate R_j . For this duopoly, the revenue optimization problem for Server j is as follows.

$$\max_{c_j} \qquad R_j(c_j,\gamma_j) = c_j\gamma_j(c_j,c_{j^-})$$
 subject to $0 \le c_j \le c^j$ (P3) given c_{j^-}

where c_{j^-} represents the admission price at the server other than j, i.e., $c_{1^-} = c_2$ and $c_{2^-} = c_1$.

For the duopoly market, the aim is to obtain the Nash equilibrium set of admission prices to be charged at the two servers. We shall denote the Nash equilibrium prices by the tuple (c_1^*, c_2^*) . Using the notion of the best response function [25], (c_1^*, c_2^*) can be characterized as follows. Let $B_i(c_{i^-})$ denote the admission price at Server i that maximizes the server revenue R_i for a given value of c_{i^-} for i=1,2. Clearly, $B_i(c_{i^-})$ is the maximizer in program P3 and it is easy to see that

$$\begin{array}{lll} B_1(c_2) & := & \{c_1 \geq 0 : c_1 \gamma_1(c_1, c_2) \geq c_1' \gamma_1(c_1', c_2) \forall c_1' \geq 0\} \\ B_2(c_1) & := & \{c_2 \geq 0 : c_2 \gamma_2(c_1, c_2) \geq c_2' \gamma_2(c_1, c_2') \forall c_2' \geq 0\} \,. \end{array}$$

and

$$(c_1^*, c_2^*) = \{(c_1, c_2) : B_1(c_2) = c_1, B_2(c_1) = c_2\}.$$

However as argued earlier, the closed form expression for $\gamma_j(c_j,c_{j^-})$ is not easy to obtain. This makes it difficult to solve program P3 and obtain the best responses $B_i(c_{i^-})$ for i=1,2. As a result, obtaining (c_1^*,c_2^*) is in general not easy. As in the case of the monopoly program, to obtain (c_1^*,c_2^*) , we need to first reformulate program P3 by letting γ_j denote the optimizing variable. The corresponding optimization problem is as follows.

$$\begin{array}{ll} \max_{\gamma_{j}} & R_{j}(c_{j}(\gamma_{j}),\gamma_{j}) := c_{j}(\gamma_{j})\gamma_{j} \\ \text{subject to} & 0 \leq \gamma_{j} \leq \gamma^{j}(c_{j^{-}}) \leq \lambda \\ \text{given} & c_{j^{-}}. \end{array} \tag{P4}$$

 $c_j(\gamma_j)$ can be interpreted as the admission price at Server j that leads to the equilibrium arrival rate of γ_j when the other server charges c_{j^-} . Note again that $c_j(\gamma_j)$ will be a function of c_{j^-} but we do not make this explicit in the notation. To lighten notation, we will not make this dependence explicit. Now let $\gamma_1^*(c_2)$ denote the maximizer in program P4 for a given value of c_2 . Then the best response c_1 is in fact given by the function $c_1(\gamma_1^*(c_2))$. Therefore, once the function $c_1(\gamma_1)$ is characterized, the best response now denoted by $\hat{B}_1(c_2)$ satisfies $\hat{B}_1(c_2) = c_1(\gamma_1^*(c_2))$. We now have

$$(c_1^*, c_2^*) = \left\{ (c_1, c_2) : \hat{B}_1(c_2) = c_1, \hat{B}_2(c_1) = c_2 \right\}$$

where $\hat{B}_i(c_{i-}) = c_i(\gamma_i^*(c_{i-}))$ and as stated earlier, $\gamma_i^*(c_{i-})$ is the maximizer in program P4 for i=1,2. It is therefore clear that (c_1^*,c_2^*) can be obtained once we have characterized $c_1(\gamma_1)$. We shall analyze the program P4 in detail in Section V and explicitly characterize the functions $c_j(\gamma_j)$ for j=1,2 to be able to obtain (c_1^*,c_2^*) .

IV. MONOPOLY MARKET

In this section, we will analyze the monopoly program P2. To be able to solve program P2, we need to characterize $c_1(\gamma_1)$ for a fixed value of c_2 . This procedure is outlined below. From Eq. (3) of Theorem 1, we know that when $\beta_1 \in (a,b)$, (and hence $\gamma_1 \in (0,\lambda)$) we have

$$c_1 - c_2 = \beta_1 \left(D_2(\gamma_2) - D_1(\gamma_1) \right).$$

We will express the right hand side of the above equation as a function of γ_1 , i.e.,

$$g_1(\gamma_1) := \beta_1(\gamma_1) \left(D_2(\lambda - \gamma_1) - D_1(\gamma_1) \right)$$
 (5)

where $\beta_1(\gamma_1)$ represents the threshold β_1 for a kernel K^W that satisfies Theorem 1 and corresponds to an equilibrium arrival rate of γ_1 . Note that $g_1(\gamma_1)$ characterizes the difference (c_1-c_2) as a function of γ_1 . For a fixed c_2 and for a γ_1 satisfying $0 \le \gamma_1 \le \gamma^1(c_2) \le \lambda$ (the domain of γ_1 in program P2) we see that $c_1(\gamma_1) = c_2 + g_1(\gamma_1)$. We characterize $c_1(\gamma_1)$ in the following manner. We first characterize $\beta_1(\gamma_1)$ using Lemmas 1 and 2. Then in Lemma 3, we characterize $g_1(\gamma_1)$. For a fixed c_2 , we then obtain $\gamma^1(c_2)$ in Lemma 5 that determines the domain of $c(\gamma_1)$. To prove this lemma we need to characterize

the uniqueness of kernel K^W for a fixed difference $(c_1 - c_2)$. This is part of Lemma 4. Finally we characterize $c_1(\gamma_1)$ in Theorem 2 using $g_1(\gamma_1)$ and $\gamma^1(c_2)$.

Recall that we make minimal assumptions on the distribution $F(\cdot)$ and on the delay cost function $D_j(\cdot)$. For our numerical examples and also to illustrate the properties of the functions $\beta_1(\cdot), g_1(\cdot)$ and $c_1(\cdot)$, we consider the following examples for $F(\cdot)$ and $D_j(\cdot)$. The distribution $F(\cdot)$ is from one of the following;

- Uniform distribution over the range [a, b].
- Exponential distribution with mean τ .
- Gamma distribution with shape k and scale θ .

For the delay cost function, we shall assume one of the following.

- $D_j(\gamma_j) = \frac{\gamma_j}{\mu_j}$. This corresponds to the case of linear delay.
- $D_j(\gamma_j) = \frac{1}{\mu_j \gamma_j}$ and $\mu_j > \lambda$. This corresponds to M/M/1 type delay cost function.

The distribution and the delay cost functions outlined above are commonly used to model heterogeneous customers and congestion costs. (Refer [26], [12], [13], [14], [15])

We now begin with the following lemma that identifies the necessary and sufficient condition on the equilibrium γ_1 when either $c_1 \geq c_2$ or $c_1 < c_2$.

Lemma 1: $\gamma_1 \in [0, \gamma^+]$ iff $c_1 \geq c_2$ while $\gamma_1 \in (\gamma^+, \lambda]$ iff $c_1 < c_2$.

Proof: See Appendix for proof.

Refer Figure 3 and 4 for an illustration of the lemma.

Next, we express the threshold β_1 of Theorem 1 as a function of γ_1 . Recall from the theorem that K^W is characterized by β_1 when $c_1 \neq c_2$. We let $\beta_1(\gamma_1)$ to denote the value of the threshold β_1 (characterizing K^W) for a given γ_1 such that $\gamma_1 \neq \gamma^+$. We have the following lemma.

Lemma 2:

$$\beta_1(\gamma_1) = \begin{cases} F^{-1}\left(\frac{\lambda - \gamma_1}{\lambda}\right) & \text{for } 0 \le \gamma_1 < \gamma^+, \\ F^{-1}\left(\frac{\gamma_1}{\lambda}\right) & \text{for } \gamma^+ < \gamma_1 \le \lambda. \end{cases}$$
 (6)

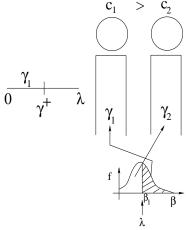
where F^{-1} represents the quantile function or the inverse function of the distribution F.

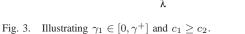
Note that $\beta_1(\gamma_1)$ is not defined in Lemma 2 when $\gamma_1=\gamma^+$. This is because the Wardrop kernel K^W with $\gamma_1=\gamma^+$ is not unique and need not be characterized by a single threshold. We shall however assume from now on that when $\gamma_1=\gamma^+$ (and hence $c_1=c_2$), the corresponding kernel K^W is also characterized by a single threshold β_1 . Hence for $c_1=c_2$, we have

$$K^{W}(\beta, 1) = \begin{cases} \delta_{1} & \text{for } \beta \in (\beta_{1}, b], \\ \delta_{2} & \text{for } \beta \in [a, \beta_{1}]. \end{cases}$$
 (7)

As a result, we define $\beta_1(\gamma^+) = F^{-1}\left(\frac{\lambda - \gamma^+}{\lambda}\right)$ and the modified $\beta_1(\gamma_1)$ is now as follows.

$$\beta_1(\gamma_1) = \begin{cases} F^{-1}\left(\frac{\lambda - \gamma_1}{\lambda}\right) & \text{for } 0 \le \gamma_1 \le \gamma^+, \\ F^{-1}\left(\frac{\gamma_1}{\lambda}\right) & \text{for } \gamma^+ < \gamma_1 \le \lambda. \end{cases}$$
(8)





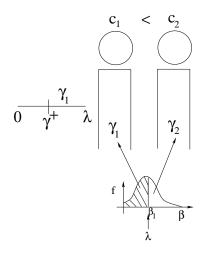


Fig. 4. Illustrating $\gamma_1 \in (\gamma^+, \lambda]$ and $c_1 < c_2$.

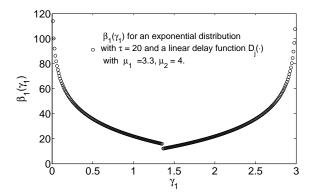


Fig. 5. Illustrating $\beta_1(\gamma_1)$ when the servers are not identical.

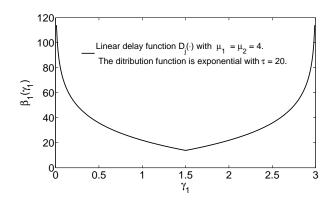


Fig. 6. Illustrating $\beta_1(\gamma_1)$ for the case of identical servers.

Refer Fig. 5 for a numerical evaluation of Eq. (8) for the case when $F(\cdot)$ is an exponential distribution with $\tau=20$. The delay functions are $D_j(\gamma_j)=\frac{\gamma_j}{\mu_j}$ where $\mu_1=3.3$ and $\mu_2=4$. Fig. 6 corresponds to the case when the two servers are identical, i.e., $\mu_1=\mu_2=4$.

Remark 1: Recall our assumption that $F(\cdot)$ is absolutely continuous and strictly increasing in its domain. Further, the support is [a,b] and hence $F(\cdot)$ is a bijective function whose inverse exists. In fact $F^{-1}(\cdot)$ is continuous and strictly increasing in its domain. Since $F^{-1}(\cdot)$ is continuous in its arguments, $\beta_1(\gamma_1)$ is continuous when $0 \le \gamma_1 < \gamma^+$ and $\gamma^+ < \gamma_1 \le \lambda$. However at $\gamma_1 = \gamma^+$, $\beta_1(\gamma_1)$ is in general not continuous (Refer Fig. 5). For the case when the servers are identical, i.e., $D_1(\gamma) = D_2(\gamma) = D(\gamma)$, we see from the definition of γ^+ that $\gamma^+ = \frac{\lambda}{2}$. For this case, it is easy to see that $\beta_1(\gamma_1)$ is continuous at $\gamma_1 = \gamma^+$, (but not differentiable). See Fig. 6.

Having obtained $\beta_1(\gamma_1)$, we shall now analyze $g_1(\gamma_1)$ that was defined in Eq. (5). $g_1(\gamma_1)$ will be used later to obtain $c_1(\gamma_1)$. We have the following lemma.

Lemma 3: For $0 \le \gamma_1 \le \lambda$, $g_1(\gamma_1)$ is continuous and monotonic decreasing in γ_1 . Further, $g_1(\gamma^+) = 0$.

Proof: See Appendix for proof. See Fig. 7 for a numerical evaluation of $g_1(\gamma_1)$ when $F(\cdot)$ is

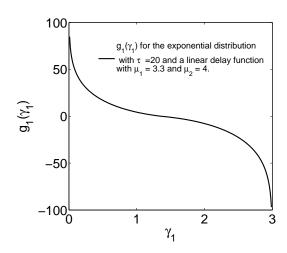


Fig. 7. Illustrating $g_1(\gamma_1)$ when the servers are not identical.

an exponential distribution with $\tau=20$ and when the servers have a linear delay with $\mu_1=3.3$ and $\mu_2=4$.

To determine $c_1(\gamma_1)$, we also need to identify the domain over which it can be defined. As argued earlier, this domain is determined by $\gamma^1(c_2)$ which is characterized in Lemma 5. Before stating Lemma 5, we shall first characterize the uniqueness of β_1 , and hence the kernel K^W , when $c_1 \neq c_2$. While Theorem 1, guarantees existence of a β_1 characterizing kernel K^W , it does not guarantee the uniqueness of β_1 and hence the uniqueness of the kernel K^W . This result will be used in the proof of Lemma 5.

Lemma 4: For a given $\Delta := (c_1 - c_2)$, the threshold β_1 characterizing the kernel K^W in Theorem 1 is as follows.

$$\beta_1 = \begin{cases} b & \text{if } \Delta \ge g_1(0) \text{ or } \Delta \le g_1(\lambda), \\ \beta_1(\hat{\gamma}) & \text{if } g_1(\lambda) < \Delta < g_1(0) \end{cases}$$
(9)

where $\hat{\gamma}$ satisfies $\Delta = g_1(\hat{\gamma})$. For a fixed Δ , the equilibrium γ_1 and the corresponding β_1 is unique and this implies the uniqueness of K^W .

Proof: See Appendix for proof.

We now characterize $\gamma^1(c_2)$ in the following lemma.

Lemma 5:

$$\gamma^{1}(c_{2}) = \begin{cases} \lambda & \text{for } c_{2} \geq -g_{1}(\lambda), \\ \gamma : g_{1}(\gamma) = -c_{2} & \text{for } c_{2} < -g_{1}(\lambda) \end{cases} (10)$$

In words, when $c_2 < -g_1(\lambda)$ we have $\gamma^1(c_2) = \{\gamma: g_1(\gamma) = -c_2\}$ and for any $c_1 \geq 0$, the equilibrium $\gamma_1 \notin (\gamma^1(c_2), \lambda]$. However when $c_2 \geq -g_1(\lambda)$, we have $\gamma^1(c_2) = \lambda$ in which case for suitable choices of $c_1, \gamma_1 \in [0, \lambda]$.

Proof: See Appendix for proof.

The above lemma also implies that, if $c_2 \geq -g_1(\lambda)$, then for any $c_1 \in (0, c_2 + g_1(\lambda))$ the equilibrium γ_1 satisfies $\gamma_1 = \lambda$. On the other hand, if the parameters of the system are such that $c_2 + g_1(\lambda) < 0$, then for any set of admission prices c_1 at Server 1, we have $\gamma_1 < \gamma^1(c_2)$.

Remark 2: From Lemma 5, when $c_2 < -g_1(\lambda)$, we have $\gamma^1(c_2) = \gamma$ where $g_1(\gamma) = -c_2$. From Lemma 3, we know that $g_1(\gamma) < 0$ for $\gamma > \gamma^+$. Hence when $c_2 \ge 0$, we have $\gamma^1(c_2) \ge \gamma^+$ with strict equality when $c_2 = 0$.

Finally, we have the following theorem to express c_1 as a function of γ_1 , denoted by $c_1(\gamma_1)$.

Theorem 2: $c_1(\gamma_1) = c_2 + g_1(\gamma_1)$ for $0 < \gamma_1 < \gamma^1(c_2) \le \lambda$. For $\gamma_1 = 0$, $c_1(0)$ must be at least equal to $c_2 + g_1(0)$, i.e., $c_1(0) \ge c_2 + g_1(0)$. Similarly when $\gamma^1(c_2) = \lambda$, we have $c_1(\lambda) \le c_2 + g_1(\lambda)$.

Proof: First consider a fixed γ_1 satisfying $\gamma_1 \in (0, \gamma^1(c_2))$ for a fixed c_2 . $(\gamma^1(c_2))$ was characterized in Lemma 5.) The corresponding threshold β_1 is determined by Eq. (8) and hence we have $\beta_1 \in (a,b)$ for $\gamma_1 \in (0,\gamma^1(c_2))$. Recall that Lemma 4 relates the threshold β_1 with Δ . Since $\beta_1 < b$, from Lemma 4, Δ must satisfy $\Delta = g_1(\gamma_1)$. Therefore for a fixed c_2 , the admission price $c_1(\gamma_1)$ resulting in the arrival rate of γ_1 at Server 1 is given by

$$c_1(\gamma_1) = c_2 + g_1(\gamma_1).$$

For the case $\gamma_1=0$, from Eq. (8), we have $\beta_1=b$. From Lemma 4, this implies that $\Delta \geq g_1(0)$. From the definition of Δ , we have $c_1(0) \geq c_2 + g_1(0)$. Similarly when $\gamma_1=\lambda$, from Eq. (8), we have $\beta_1=b$. From Lemma 4, this implies $\Delta \leq g_1(\lambda)$ and hence $c_1(\lambda) \leq c_2 + g_1(\lambda)$. This completes the proof.

In the above theorem, $c_1(0)$ and $c_1(\lambda)$ are not uniquely defined and can take values that satisfy $c_1(0) \ge c_2 + g_1(0)$ and

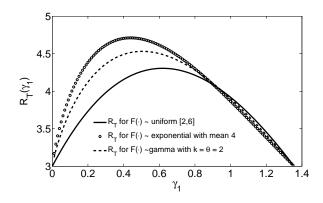


Fig. 8. R_T as a function of γ_1 when $D_j(\gamma_j) = \frac{\gamma_j}{\mu_j}$

 $c_1(\lambda) \leq c_2 + g_1(\lambda)$ respectively. As convention, we henceforth define $c_1(0) = c_2 + g_1(0)$ and $c_1(\lambda) = c_2 + g_1(\lambda)$. Further note that the domain for $c_1(\cdot)$ is $0 \leq \gamma_1 \leq \gamma^1(c_2)$ and for $\gamma^1(c_2) < \gamma_1 < \lambda$, $c_1(\gamma_1)$ is undefined. The function $c_1(\gamma_1)$ for $0 \leq \gamma_1 \leq \gamma^1(c_2) \leq \lambda$ can now be expressed as follows.

$$c_{1}(\gamma_{1}) = \begin{cases} c_{2} + g_{1}(\gamma_{1}) & \text{for } 0 < \gamma_{1} \leq \gamma^{1}(c_{2}) < \lambda, \\ c_{2} + g_{1}(0) & \text{for } \gamma_{1} = 0, \\ c_{2} + g_{1}(\lambda) & \text{for } \gamma_{1} = \gamma^{1}(c_{2}) = \lambda, \end{cases}$$
(11)

Now recall the revenue maximization problem P2. Define γ_1^* as the optimizer for this program with the revenue maximizing admission price given by $c_1(\gamma_1^*)$. Since $R_T(c_1(\gamma_1),\gamma_1)=c_2\lambda+(c_1(\gamma_1)-c_2)\gamma_1,\ \gamma_1^*$ must be such that $c_1(\gamma_1^*)>c_2$. From Eq. (11), this implies that $g_1(\gamma_1^*)>0$. From Lemma 3 we have $g_1(\gamma_1)>0$ for $\gamma_1\in(0,\gamma^+)$ and this implies that $\gamma_1^*\in(0,\gamma^+)$. The term $c_2\lambda$ in $R_T(c_1(\gamma_1),\gamma_1)$ is a constant and hence we have the following equivalent program for the revenue maximization problem.

$$\max_{\gamma_1} \qquad g_1(\gamma_1)\gamma_1$$
subject to $0 \le \gamma_1 \le \gamma^+$ (P4)

where $g_1(\gamma_1)$ is given by Eq. (5).

Note from Lemma 3 that $g_1(\cdot)$ is a continuous function of its domain. Program P4 involves maximizing a continuous function over a compact set and hence a maximizer γ_1^* exists. The original monopoly program P1 has been significantly simplified to the equivalent program P4. Since $g_1(\gamma_1)$ is strictly decreasing (and hence quasi-convex), $g_1(\gamma_1)\gamma_1$ is in fact a product of two quasi-convex functions. (However product of quasi-convex functions need not be quasi-convex function). One can now use standard non-linear optimization techniques to obtain γ_1^* . To further understand Program P4, we perform a numerical evaluation of $g_1(\gamma_1)\gamma_1$ under a combination of assumptions on the distribution functions F and the delay functions $D_i(\gamma_i)$ that were outlined earlier.

Example 1: In this example we shall assume that the $D_j(\gamma_j) = \frac{\gamma_j}{\mu_j}$ for j=1,2. We assume that $\mu_1=3.3$ and $\mu_2=4$. Further, the arrival rate $\lambda=3$ and we consider the following three examples for the distribution $F(\cdot)$. (1) F has a

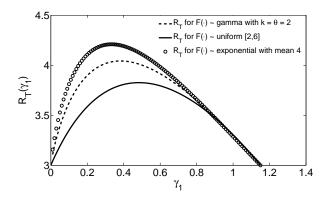


Fig. 9. R_T as a function of γ_1 when $D_j(\gamma_j) = \frac{1}{\mu_j - \gamma_j}$

uniform distribution with support on [2,6]. (2) F has an exponential distribution with mean $\tau=4$ and (3) F has a Gamma distribution with the scale k and shape θ parameters 2 and 2 respectively. Note that β with these three distributions have the same mean. We plot $R_T(c_1(\gamma_1),\gamma_1)=c_2\lambda+g_1(\gamma_1)\gamma_1$ as a function of γ_1 in Fig. 8 where we assume $c_2=1$. When F has the uniform distribution, $\gamma_1^*=0.62$. The optimal revenue rate $R_T(\gamma_1^*)=4.306$ while the admission price $c_1(\gamma_1^*)$ maximizing R_T is 3.106. The corresponding values for the exponential distribution are $\gamma_1^*=0.44$, $R_T(\gamma_1^*)=4.712$ and $c_1(\gamma_1^*)=4.89$ while the values for gamma distribution are $\gamma_1^*=0.51$, $R_T(\gamma_1^*)=4.532$ and $c_1(\gamma_1^*)=4$.

Example 2: In this example, we assume that $D_j(\gamma_j) = \frac{1}{\mu_j - \gamma_j}$ where again $\mu_1 = 3.3$ and $\mu_2 = 4$. Note that $\lambda < \mu_j$ for j = 1, 2. The choice of $F(\cdot)$ is as in the previous example. A plot of $R_T(c_1(\gamma_1), \gamma_1)$ as a function of γ_1 is provided in Fig. 9. When F has the uniform distribution, $\gamma_1^* = 0.48$. The optimal revenue rate $R_T(\gamma_1^*) = 3.83$ while the admission price $c_1(\gamma_1^*)$ maximizing R_T is 2.72. The corresponding values for the exponential distribution are $\gamma_1^* = 0.33, R_T(\gamma_1^*) = 4.21$ and $c_1(\gamma_1^*) = 4.67$ while the values for gamma distribution are $\gamma_1^* = 0.38, R_T(\gamma_1^*) = 4.04$ and $c_1(\gamma_1^*) = 3.74$.

We conclude the analysis of the revenue maximization problem with the following observations made from the two examples given above.

- Firstly, we see that for the given examples of F, $R_T(c_1(\gamma_1), \gamma_1)$ is a unimodal function in γ_1 . For the three distribution functions, it can be shown that $F^{-1}(\cdot)$ is differentiable in its arguments. For such distribution functions with differentiable $F^{-1}(\cdot)$, this implies that $g_1(\gamma_1)$ and hence $R_T(c_1(\gamma_1), \gamma_1)$ is differentiable in γ_1 when $0 < \gamma_1 < \gamma^+$. From Rolle's Theorem (Theorem 10.2.7 [27]), this implies that there exists a $\gamma_1 \in (0, \gamma^+)$ such that $\frac{dR_T}{d\gamma_1} = 0$. A γ_1^* satisfying this equation is the revenue maximizing arrival rate to server 1. The admission price corresponding to this γ_1^* can now be obtained using Eq. (11).
- For each of the three distributions, note that we have $\mathsf{E}\left[\boldsymbol{\beta}\right]=4$. However, the Revenue rate R_T as a function of γ_1 is distinct in all the three cases. This implies that the revenue rate R_T depends on the higher moments of

- the distribution F and not just on its mean value.
- Finally, note that R_T depends on admission price through the addition factor of $c_2\lambda$. For different values of c_2 , the corresponding γ_1^* does not change. However it is easy to see from Eq. (11) that $c_1(\gamma_1)$ increases linearly in c_2 .

V. Duopoly

In this section, we shall consider program P4 for revenue maximization in the duopoly system. Much of the analysis in this section follows from that of the previous section. Let $\gamma_j, j=1,2$ denote the optimization variable and represent the admission prices at the respective servers as a function of the arrival rates. Towards this, we continue with the use of the notation $c_j(\gamma_j)$ for j=1,2. Note that while in the monopoly case, the admission price c_2 was considered fixed, in the duopoly of this section, it is the strategy for the second server and hence will not be a constant. The revenue function for Server j is given by

$$R_j(c_j(\gamma_j), \gamma_j) = c_j(\gamma_j)\gamma_j$$

where $c_j(\gamma_j)$ represents the admission price at Server j resulting in an equilibrium arrival rate of γ_j . As noted in the previous section, $c_j(\gamma_j)$ is a function of c_{j^-} , the admission price at the other server. For a fixed strategy c_2 at Server 2, from Eq. (11) the revenue function $R_1(c_1(\gamma_1), \gamma_1)$ can be redefined as

$$R_1(c_1(\gamma_1), \gamma_1) = (g_1(\gamma_1) + c_2) \gamma_1. \tag{12}$$

It can be argued as in the previous section that for a fixed c_1

$$R_2(c_2(\gamma_2), \gamma_2) = (g_2(\gamma_2) + c_1) \gamma_2 \tag{13}$$

where

$$g_2(\gamma_2) = \beta_1(\lambda - \gamma_2) \left(D_1(\lambda - \gamma_2) - D_2(\gamma_2) \right) \tag{14}$$

where from Eq. (8) $\beta_1(\lambda - \gamma_2)$ is as follows

$$\beta_1(\lambda - \gamma_2) = \begin{cases} F^{-1}\left(\frac{\gamma_2}{\lambda}\right) & \text{for } \lambda - \gamma^+ \le \gamma_2 \le \lambda, \\ F^{-1}\left(\frac{\lambda - \gamma_2}{\lambda}\right) & \text{for } 0 < \gamma_2 < \lambda - \gamma^+. \end{cases}$$

It is easy to see that $g_2(\gamma_2)$ is also continuous and strictly decreasing in γ_2 . Further, $g_2(\gamma_2) = 0$ when $\gamma_2 = \lambda - \gamma^+$. The revenue maximization problem for the duopoly is re-stated as follows.

$$\begin{aligned} \max_{\gamma_j} & R_j(\gamma_j) = \left(g_j(\gamma_j) + c_{j^-}\right)\gamma_j \\ \text{subject to} & 0 \leq \gamma_j \leq \gamma^j(c_{j^-}) \leq \lambda \\ \text{given} & c_{j^-}. \end{aligned} \tag{P7}$$

For a given c_{j^-} , recall that $\gamma_j^*(c_{j^-})$ denotes the maximizer of program P4 and hence of the above program. Also recall that $\hat{B}_j(c_{j^-})$ denotes the best response admission price at Server j in response to the admission price c_{j^-} at the other facility. Then the Nash equilibrium set of admission prices, denoted by (c_1^*, c_2^*) , is characterized as follows.

$$(c_1^*, c_2^*) = \left\{ (c_1, c_2) : \hat{B}_1(c_2) = c_1, \hat{B}_2(c_1) = c_2 \right\},$$
 (16)

where $\hat{B}_{j}(c_{j^{-}}) = g_{j}(\gamma_{i}^{*}(c_{j^{-}})) + c_{j^{-}}$ for j = 1, 2.

We begin the analysis for the duopoly problem by first identifying that $\gamma_j^*(c_{j^-})$ lies in the interior of the domain. We have the following lemma.

Lemma 6:
$$\gamma_j^*(c_{j^-}) \notin \{0, \gamma^j(c_{j^-})\}$$
.
Proof: See Appendix for proof.

For a given c_{j^-} since $\gamma_j^*(c_{j^-})$ lies in the interior of the domain, $\gamma_j^*(c_{j^-})$ satisfies $\frac{dR_j}{d\gamma_j}\bigg|_{\gamma_j=\gamma_j^*}=0$ and $\frac{d^2R_j}{d\gamma_j^2}\bigg|_{\gamma_j=\gamma_j^*}\leq 0$

Define $S_j(c_{j^-}):=\left\{\gamma_j:\frac{dR_j}{d\gamma_j}=0,\frac{d^2R_j}{d\gamma_j^2}\leq 0\right\}$. Then $\gamma_j^*(c_{j^-})$ is obtained as a solution to the following.

$$\gamma_j^*(c_{j^-}) = \underset{\gamma_j \in S_j(c_{j^-})}{\arg \max} \quad R_j(\gamma_j). \tag{P8}$$

From the above discussion, it should be clear that obtaining the closed form expression for (c_1^*, c_2^*) satisfying the simultaneous equations of (16) is, in general, not easy. Note that our analysis till now makes minimal assumptions on the distribution function F or on the delay function $D_i(\cdot)$. For certain choices of these functions, it may be difficult to obtain a closed form expression for $\gamma_i^*(c_{j-})$. The objective function R_i also need not be a concave function. In that case, a brute force search among all the local maxima points needs to be carried out to choose the right $\gamma_i^*(c_{i-})$. Instead of satisfying ourselves with some numerical examples, in the following subsection we shall analyze the Nash equilibrium under the restriction that the two servers are identical i.e., the average delay at any queue is the same for the same arrival rate. Under this setting, our interest is to characterize the symmetric Nash equilibrium such that $c_1^* = c_2^*$.

A. Characterizing a symmetric Nash equilibrium

In this section we shall characterize the necessary conditions for the existence of a symmetric Nash equilibrium, i.e., (c_1^*, c_2^*) where $c_1^* = c_2^* := c^*$. A natural scenario where such an equilibrium is possible is when the two servers have identical delay functions. In this section, we restrict to this case and assume that $D_j(\cdot) = D(\cdot)$ for j=1,2. As the service systems are identical in their delay characteristics, it is desirable to identify conditions for existence of a symmetric Nash equilibrium. We begin with the following definition. Define α_1,α_2 as follows.

$$\alpha_{1} = -\gamma^{+} \frac{dg_{1}(\gamma_{1})}{\gamma_{1}} \Big|_{\gamma_{1} = \gamma^{+}}$$

$$\alpha_{2} = -\gamma^{+} \frac{dg_{2}(\gamma_{2})}{\gamma_{2}} \Big|_{\gamma_{2} = \gamma^{+}}$$
(17)

Based on these definitions, we have the following lemma.

Lemma 7: $\alpha_1 = \alpha_2$.

Proof: From the definition of $g_1(\cdot)$ in Eq. (5) we have

$$\frac{dg_1(\gamma_1)}{\gamma_1} = \beta_1'(\gamma_1) \left(D_2(\lambda - \gamma_1) - D_1(\gamma_1) \right) + \beta_1(\gamma_1) \left(D_2'(\lambda - \gamma_1) - D_1'(\gamma_1) \right)$$

where the partial derivatives on the r.h.s. are w.r.t. γ_1 . Now from the definition of γ^+ and from Eq. (8) we have

$$\frac{dg_1(\gamma_1)}{\gamma_1}\bigg|_{\gamma_1=\gamma^+} = F^{-1}\left(\frac{\lambda-\gamma^+}{\lambda}\right)\left(D_2'(\lambda-\gamma^+) - D_1'(\gamma^+)\right).$$

Similarly, from the definition of $g_2(\cdot)$ we have

$$\frac{dg_{2}(\gamma_{2})}{\gamma_{2}} = \beta'_{1}(\lambda - \gamma_{2}) \left(D_{1}(\lambda - \gamma_{1}) - D_{2}(\gamma_{2}) \right) + \beta_{1}(\lambda - \gamma_{2}) \left(D'_{1}(\lambda - \gamma_{2}) - D'_{2}(\gamma_{2}) \right)$$

where the partial derivatives on the r.h.s are now w.r.t γ_2 . Note that since $\gamma_1=\lambda-\gamma_2$, we have $\frac{\partial D_1(\gamma_1)}{\partial \gamma_1}=-\frac{\partial D_1(\gamma_1)}{\partial \gamma_2}$. Further note that since the servers are identical, i.e., $D_j(\cdot)=D(\cdot)$ for j=1,2 from the definition of γ^+ we have $\gamma^+=\frac{\lambda}{2}$. From Eq. (15) and the fact that $\lambda-\gamma^+=\gamma^+$, we have

$$\frac{dg_2(\gamma_2)}{\gamma_2}\bigg|_{\gamma_2=\gamma^+} = F^{-1}\left(\frac{\lambda-\gamma^+}{\lambda}\right) \left(D_2'(\lambda-\gamma^+) - D_1'(\gamma^+)\right).$$

This proves that $\alpha_1 = \alpha_2$.

We now have the following theorem, that characterizes the necessary condition for a symmetric Nash equilibrium.

Theorem 3: Let $c_1^* = c_2^*$ be a symmetric Nash equilibrium for the duopoly price competition. Then $c_1^* = c_2^* = \alpha_1$.

Proof: Recall that the Nash equilibrium is characterized as

$$(c_1^*, c_2^*) = \left\{ (c_1, c_2) : \hat{B}_1(c_2) = c_1, \hat{B}_2(c_1) = c_2 \right\}.$$

where $\hat{B}_j(c_{j^-})=g_j(\gamma_j^*(c_{j^-}))+c_{j^-}$ for j=1,2. This implies that $c_j^*=g_j(\gamma_j^*(c_{j^-}^*))+c_{j^-}^*$ and since $c_1^*=c_2^*$, we have $g_j(\gamma_j^*(c_{j^-}^*))=0$ for j=1,2. Now from Lemma 3 and symmetry of the servers, this implies that $\gamma_1^*(c_2^*)=\gamma^+$ and $\gamma_2^*(c_1^*)=\lambda-\gamma^+$. Since the servers are identical, we have $\gamma^+=\frac{\lambda}{2}$ and hence $\gamma_2^*(c_1^*)=\gamma^+$. Since $\gamma_j^*(c_{j^-})$ is also a solution to program P8, $\gamma_j^*(c_{j^-})\in S(c_{j^-})$. From the definition of $S(c_{j^-})$, this implies that $\frac{dR_j}{d\gamma_j}\Big|_{\gamma_j=\gamma^+}=0$. Further, this implies from the definition of $R_j(\gamma_j)$ that

$$\frac{dR_j}{d\gamma_j}\bigg|_{\gamma_j=\gamma^+} = \gamma^+ \frac{dg_j(\gamma_j)}{\gamma_j}\bigg|_{\gamma_j=\gamma^+} + g_j(\gamma^+) + c_{j^-}^* (18)$$

$$= 0.$$

We have $g_j(\gamma^+)=0$ and hence from the definition of α_j for j=1,2 we have $c_{j^-}^*=\alpha_j$. From Lemma 7, we have $\alpha_1=\alpha_2$ and hence $c_1^*=c_2^*=\alpha_1$. This completes the proof.

Note that the above theorem only provides a necessary condition for the Nash equilibrium pair and we shall soon see that in fact this condition is not sufficient. We shall now provide a few examples illustrating the occurrence of symmetric Nash equilibria.

Example 3: In this example, we assume that $D_j(\gamma_j)=\frac{\gamma_j}{\mu_j}$ for j=1,2. Let $\mu_1=\mu_2=4$ while the arrival rate is $\lambda=3$. We suppose that the distribution $F(\cdot)$ has a uniform distribution with support of [a,b]. We plot $R_1(\gamma_1)$ and $c_1(\gamma_1)$ as a function of γ_1 in Fig.10. The aim of this example is to check whether $(c_1^*,c_2^*)=(\alpha_1,\alpha_1)$ is a symmetric Nash equilibrium. For the set of parameters of this example we have

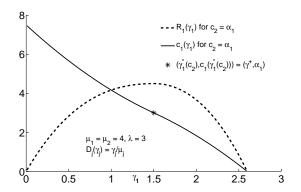


Fig. 10. R_1 and $c_1(\gamma_1)$ when $D_j(\gamma_j)=\frac{\gamma_j}{\mu_j}$ and $F(\cdot)$ is Uniform over [2,6]

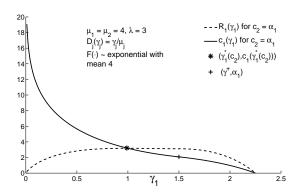


Fig. 11. R_1 and $c_1(\gamma_1)$ when $D_j(\gamma_j)=\frac{1}{\mu_j-\gamma_j}$ and $F(\cdot)$ is exponential with $\tau=4$

 $\gamma^+ = 1.5$ and since

$$\alpha_1 = -\gamma^+ \frac{dg_1(\gamma_1)}{\gamma_1} \bigg|_{\gamma_1 = \gamma^+}$$

we have $\alpha_1=3$. We now set $c_2=\alpha_1=3$. Clearly, for a symmetric Nash equilibrium $(c_1^*,c_2^*)=(\alpha_1,\alpha_1),\,\gamma_1^*(c_2)=\gamma^+=1.5$ must hold. It is easy to see from Fig. 10 that $R_1(\gamma_1)$ is indeed maximized when $\gamma_1=\gamma^+$ implying that $\gamma_1^*(c_2)=\gamma^+$. Further it can be verified that $(c_1(\gamma_1^*(c_2)))\alpha_1$. Clearly, $(c_1^*,c_2^*)=(\alpha_1,\alpha_1)$ is a symmetric Nash equilibrium for this example.

Example 4: With the help of this example, we will illustrate that the necessary conditions stated in the previous theorem need not be sufficient. We shall once again assume that $D_j(\gamma_j) = \frac{\gamma_j}{\mu_j}$ where $\mu_1 = \mu_2 = 4$. As for the choice of $F(\cdot)$, we consider an exponential distribution with $\tau = 4$. A plot of $R_1(\gamma_1)$ and $c_1(\gamma_1)$ as a function of γ_1 is provided in Fig. 11. For this example we start by setting $c_2 = \alpha_1$. However we observe that the best response $\gamma_1^*(c_2) \neq \gamma^+$ and hence $c_1(\gamma_1^*(c_2)) \neq c_2$. Both these points $\gamma_1^*(c_2), c_1(\gamma_1^*(c_2))$ and (γ^+, α_1) are represented in Fig. 11. Clearly, $(\alpha_1, \alpha_1) \neq (c_1^*, c_2^*)$ and therefore the sufficiency conditions differ from the necessary ones.

VI. ESTIMATING THE DISTRIBUTION F

Recall that F denotes the distribution function for the delay sensitivity of the arriving customers. The knowledge of F is necessary to determine the equilibrium kernel K^W introduced in Theorem 1. Further, the kernel K^W must be known for the revenue maximization problems seen in this paper. However in most practical situations, the distribution function F may not be known and due to the unobservable nature of the queues it may not be possible to even elicit such information from the arriving systems. In such situations the only alternative may be to estimate this distribution function. One possible method to do so is to vary the admission prices at the servers and then measure the change in the arrival rate of customers to the different server and then use the Wardrop equilibrium conditions to estimate F. In this section, we shall describe a simple procedure to estimate the underlying continuous distribution function F. Our proposed method is well suited for a monopoly system when the single service provider has access to both the admission prices. In this section, we also consider the case when β is a discrete random variable. In this case, the customers are divided into finite number of classes differing in their values of β . The aim is to identify the value of β for the different classes along with the Poisson arrival rates λ_i for the classes. Refer [13], [16], [20] for some examples of service systems where such discrete customer classes are considered.

Throughout this section, we shall make the following assumptions. We shall assume that the two servers are modeled as M/M/1 queues with service rates μ_1 and μ_2 and admission prices c_1 and c_2 respectively. With this assumption, we have $D_j(\gamma_j) = \frac{1}{\mu_j - \gamma_j}$. It goes without saying that our analysis will also hold for any delay cost $D_i(\cdot)$ that is monotonic and strictly increasing in its arguments. We assume that once the admission prices c_1 and c_2 at the servers are announced and that the Wardrop equilibrium is achieved, each server j will accurately determine or measure the equilibrium arrival rate γ_j and the mean delay cost $D_j(\gamma_j)$ for j=1,2. Hence the measured values γ_j and $D_j(\gamma_j)$ and the the corresponding quantities at the Wardrop equilibrium will be assumed to be the same. We also assume that the total arrival rate of customers to the system denoted by λ is known a priori and that $c_1 > c_2$, i.e., the admission price at the first server is higher than the second. Note that since the distribution $F(\cdot)$ is unknown, the functions $\beta_1(\cdot)$, $q_1(\cdot)$, $c_1(\cdot)$ also cannot be determined and used for our procedure.

We begin by estimating the distributions F that belongs to a parameterized family, say for example the exponential distribution. Let the parameter for the exponential distribution be denoted by α . When c_1 and c_2 at the two servers are fixed, the equilibrium γ_1 and γ_2 at the servers is measured immediately. We choose a c_1, c_2 such that $\gamma_j > 0$ for j = 1, 2. From this, the mean delay cost $D_j(\gamma_j)$ for j = 1, 2 is also calculated. Since all the quantities (except β_1) in Eq. (3) of Theorem (1) are known, the threshold β_1 can be determined as $\beta_1 = \frac{c_1 - c_2}{D_2(\gamma_2) - D_1(\gamma_1)}$. Now increase c_1 to c_1^1 where $c_1^1 = c_1 + \delta$ for $\delta > 0$. This decreases the equilibrium γ_1 to say γ_1^{δ} . Let β_1^{δ} denote the threshold when the arrival rate to Server 1

is γ_1^{δ} . Again, using the measurements of the arrival rates and the delay functions β_1^{δ} can be determined from Eq. (3). Since $\gamma_1^{\delta} < \gamma_1 < \gamma^+$, from Lemma 2, we know that $\beta_1(\gamma_1^{\delta}) > \beta_1(\gamma_1)$. This implies that $\beta_1^{\delta} > \beta_1$. Clearly, the ratio $\frac{\gamma_1 - \gamma_1^{\delta}}{\lambda}$ is the probability of an arriving customer with $\beta \in [\beta_1, \beta_1^{\delta}]$ and hence

$$\int_{\beta_1}^{\beta_1^{\delta}} \alpha e^{-x\alpha} dx = \frac{\gamma_1 - \gamma_1^{\delta}}{\lambda}.$$
 (19)

The only unknown quantity is the exponential parameter α which can now be obtained from the above equation.

Remark 3: Since the exponential distribution has a single parameter, the parameter could be obtained using only Eq. (19). For a parameterized distribution with k parameters, we need k simultaneous equations in terms of the underlying parameters. These can be obtained by following the procedure above for k different admission price $\{c_1^k\}$ at Server 1.

We will now describe a numerical method to obtain a piecewise constant approximation for the density function f that is not necessarily from a parameterized family of distribution functions. As an example, consider a random variable β supported on the range [0,4]. Suppose the distribution function is

$$P(\beta \le x) = F(x) = \frac{x^2}{16}.$$

The corresponding density function is denoted by f(x) is x/8 for $x \in [0,4]$. For this example assume that there are two M/M/1 servers with service rates $\mu_1=5$ and $\mu_2=5$, admission prices initially set to $c_1=c_2=5$ and the total arrival rate $\lambda=5$. As earlier, we assume that once the admission prices at the servers are announced, the Wardrop equilibrium is reached instantaneously and each servers can accurately determine the aggregate arrival rates and the mean delay per customer.

Increase c_1 by $\delta>0$ and for the admission price vector $(c_1+\delta,c_2)$, measure the equilibrium arrival rates and the mean delay in the queues and calculate the corresponding threshold β_1 using Eq. (3). Repeat this for a finite number of times, each time increasing c_1 from its previous value by δ . This experiment is denoted in Table I.

c_1	c_2	γ_1	β_1
5.0	5	1.98	2.84
5.2	5	1.69	3.04
5.4	5	1.44	3.20
5.6	5	1.23	3.33
5.8	5	1.05	3.44
6.0	5	0.89	3.53
6.2	5	0.75	3.60
6.4	5	0.63	3.67
6.6	5	0.52	3.37
6.8	5	0.43	3.78

TABLE I

The table indicates the price vector (c_1,c_2) , the measured value of γ_1 and the threshold β obtained from Eq. (3).

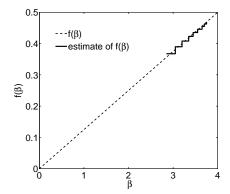


Fig. 12. Comparing the estimate of $f(\cdot)$ with the true density function.

Using the earlier notation, we observe from the table that as c_1 increases to, say $c_1 + \delta$, γ_1 decreases to γ_1^{δ} while the threshold β_1 increases (to β_1^{δ}). As earlier, we have

$$\int_{\beta_1}^{\beta_1^{\delta}} f(x)dx = \frac{\gamma_1 - \gamma_1^{\delta}}{\lambda}$$

where the density function f(x) is to be estimated. Assume for all $x \in (\beta_1, \beta_1^{\delta})$ that f(x) = z, where z is a constant. By assuming this, we are approximating the density function f(x) for $x \in (\beta_1, \beta_1^{\delta})$ by a horizontal line of magnitude z and thus approximating f(x) by a piecewise constant function. As $\delta \to 0$, the approximation should converge to the true density function. We now have

$$z = \frac{\gamma_1 - \gamma_1^{\delta}}{\lambda(\beta_1^{\delta} - \beta_1)}.$$
 (20)

The value of z for a fixed c_1 and $c_1 + \delta$ can be viewed as an estimate for the density function f(x) and obviously $z \to f(x)$ as $\delta \to 0$. These values of z for different values of c_1 are given in Table II.

c_1	$c_1 + \delta$	z
5	5.2	0.37
5.2	5.4	0.39
5.4	5.6	0.41
5.6	5.8	0.42
5.8	6.0	0.44
6.0	6.2	0.44
6.2	6.4	0.45
6.4	6.6	0.46
6.6	6.8	0.47

TABLE II

THE ESTIMATES *z* CAN BE OBTAINED FROM EQ. (20) FROM THE SUCCESSIVE CHANGES IN THE ADMISSION PRICES AND THE CORRESPONDING MEASUREMENTS OF THE ARRIVAL RATES.

A plot comparing the true density function and the estimate is given in Fig. 12. The plot shows that the estimate of the density function is reasonably accurate and for better estimation, one naturally required more of such measurement points.

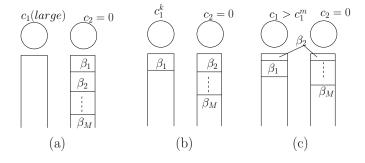


Fig. 13. Estimating the discrete distribution F

There is however a limitation to this method. Note that when $c_1=c_2$, the corresponding value of $\beta_1=2.84$. Any increase or decrease in either c_1 or c_2 cannot result in a β_1 such that $\beta_1<2.84$. This is because, for the underlying distribution we have from Eq. (8) that $\beta_1(\gamma^+)=2.84$ and for any $\gamma\in[0,\lambda]$ with $\gamma\neq\gamma^+$, we have $\beta_1(\gamma)>\beta_1(\gamma^+)$. As a result, the density function f(x) cannot be estimated for $x\leq 2.84$.

A. Estimating Discrete Distribution

We shall now consider the case where the distribution Fis a discrete distribution with M point masses. Thus, there are M customer classes and we will assume that for each Class i, the associated waiting cost β_i and the arrival rate λ_i are unknown. Further, $\beta_1 > \beta_2 \dots > \beta_M$. See [13] for the analysis of Wardrop equilibrium of such a model. We continue with the assumption that there are two servers each charging an admission price c_1 and c_2 . We begin by setting $c_2 = 0$ and c_1 to an arbitrarily large value such that $\gamma_1 = 0$ while $\gamma_2 = \lambda$. This is represented in part (a) of Fig. 13. It goes without saying that the necessary assumption is that $\mu_2 > \lambda$. Now start decreasing c_1 in steps of size δ and stop at the first instance when γ_1 increases to an arbitrarily small value ϵ . We use the notation c_1^j and γ_1^j to denote the admission price and the arrival rate at Server 1 when c_1 is decreased j times by δ , i.e., when $c_1^j = c_1 - j\delta$. $\gamma_1 = \epsilon$ implies that the most sensitive delay class β_1 must now be using Server 1 along with Server 2. Since the delay function at each queue can be measured, β_1 can be easily determined from the corresponding Wardrop condition

$$c_1^j + \beta_1 D_1(\gamma_1^j) = \beta_1 D_2(\gamma_2^j).$$

We will now determine λ_1 corresponding to this β_1 . Continue decreasing c_1 . The proportion of Class 1 customers using Server 1 keeps increasing till all Class 1 customers use only Server 1. When this happens, the corresponding Wardrop equilibrium condition for some k > j satisfies

$$c_1^k < \beta_1 \left(D_2(\gamma_2^k) - D_1(\gamma_1^k) \right)$$

and this is represented by part (b) in Fig. 13. For a Class 2 customer to start using Server 1, the Wardrop equilibrium condition is

$$c_1^m = \beta_2 \left(D_2(\gamma_2^m) - D_1(\gamma_1^m) \right)$$

where m > k. Further since m > k, we have

$$\beta_2 \left(D_2(\gamma_2^m) - D_1(\gamma_1^m) \right) < \beta_1 \left(D_2(\gamma_2^k) - D_1(\gamma_1^k) \right).$$

and hence for all l such that k < l < m, we have

$$\beta_2 \left(D_2(\gamma_2^m) - D_1(\gamma_1^m) \right) < c_1^l < \beta_1 \left(D_2(\gamma_2^k) - D_1(\gamma_1^k) \right).$$

This means that for any c_1^l satisfying $c_1^k < c_1^l < c_1^m$, γ_1^l and γ_2^l remain unchanged. Clearly in this case $\lambda_1 = \gamma_1^l$. Fig. 13, part (c) represents the fact that for any $c_1 > c_1^m$, Class 2 customers use both the servers at Wardrop equilibrium. Continue this process till all the λ_i , β_i as well as the number of customer classes M is determined. It should be noted that the accuracy of our method increases as $\delta \to 0$. A downside of a small δ is that the procedure may take a very long time to discover the system parameters.

VII. SUMMARY AND FUTURE WORK

In this paper, we have considered the problem of revenue maximization in parallel server systems. We specialize with the case of two servers and first assume the case when both the servers belong to the same service provider. The admission price at one of the server is required to be fixed and the service system can change the admission price at the other server to maximize its revenue. The Wardrop equilibrium when customers are heterogeneous and strategic has already been characterized in our earlier paper. We use this characterization to simplify the revenue maximization program to make it more amenable to analysis. The equivalent program is easy to interpret, analyze and provides more insight into the problem. While it is intuitive that for a fixed c_2 , the revenue maximizing c_1 should always be greater than c_2 , the program enables to characterize the revenue maximizing c_1^* as a function of c_2 .

In the second part of the paper, we consider the duopoly model where each server competes with the other one to maximize its revenue. This is a standard game-theoretic problem and the aim is to identify the Nash equilibrium set of prices. We see however that since the customers are heterogeneous, the first order necessary conditions are not easy to solve. Instead, we characterize this Nash equilibrium for a simplified case when the two servers are identical in their delay characteristics. In this case we are interested in the symmetric Nash equilibrium prices. We provide the necessary condition for this case and identify the Nash equilibrium prices for different distributions F and delay cost functions $D(\cdot)$.

In both these problems problems and also in the social welfare maximization problem of our previous paper, an important assumption is that the distribution function F is known. We relax this assumption in Section VI and provide a procedure to estimate this distribution. The proposed method is of course preliminary and assumes that one is allowed to change admission price any number of time to measure the change in the equilibrium arrival rate. Further, we have assumed that there is no cost to making such measurements. A more realistic method incorporating these practical limitations may make the problem more relevant and this is part of future work.

APPENDIX

Lemma 1

Proof: We first prove that $\gamma_1 \in [0, \gamma^+]$ implies $c_1 \geq c_2$. Recall the definition of γ^+ that

$$\gamma^+ = \{ \gamma_1 : D_1(\gamma_1) = D_2(\gamma_2) \}.$$

Since $D_j(\gamma_j)$ is monotonic and increasing in γ_j for j=1,2 and that $\gamma_2=\lambda-\gamma_1$ we have $D_1(\gamma_1)\leq D_2(\gamma_2)$ for $\gamma_1\in [0,\gamma^+]$. Now let $\gamma_1=0$. Since no customer uses Server 1 at equilibrium, this implies that $c_1+\beta D_1(0)>c_2+\beta D_2(\lambda)$ for all β . Since $D_1(0)< D_2(\lambda)$ (assumption) $c_1>c_2$ must be true.

When $\gamma_1 = \gamma^+$, we will show that $c_1 = c_2$. Suppose this is not true, i.e., $\gamma_1 = \gamma^+$ while $c_1 \neq c_2$. $\gamma_1 = \gamma^+$ implies $D_1(\gamma_1) = D_2(\gamma_2)$. As $c_1 \neq c_2$, customers have an incentive to move from the server with a higher admission price to the one with a lower price. This implies that $\gamma_1 = \gamma^+$ is not an equilibrium and this is a contradiction.

Now consider $\gamma_1 \in (0, \gamma^+)$ where $D_1(\gamma_1) < D_2(\gamma_2)$. From Theorem 1, $\gamma_1 \in (0, \gamma^+)$ implies $\beta_1 \in (a, b)$ and hence $c_1 + \beta_1 D_1(\gamma_1) = c_2 + \beta_1 D_2(\gamma_2)$. Since $D_1(\gamma_1) < D_2(\gamma_2)$ we have $c_1 \geq c_2$.

We now prove that if $c_1 \geq c_2$, then $\gamma_1 \in [0, \gamma^+]$. We first show that when $c_1 = c_2$, we have $\gamma_1 = \gamma^+$. Suppose that when $c_1 = c_2$, $\gamma_1 \neq \gamma^+$. From the definition of γ^+ we have $D_1(\gamma_1) \neq D_2(\gamma_2)$ and hence customers have an incentive to move from the server with higher expected delay to the one with lower expected delay. This implies that when $c_1 = c_2$, $\gamma_1 \neq \gamma^+$ is not an equilibrium.

Now let $c_1 > c_2$. From Theorem 1 we have either $\beta_1 = a$ or $\beta_1 = b$ or $\beta_1 \in (a,b)$. The case $\beta_1 = a$ corresponds to the case when all customers choose Server 2 at equilibrium and this cannot happen! This is because while $c_1 > c_2$, we have also assumed $D_1(\lambda) > D_2(0)$. K^W with $\beta_1 = a$ will be possible only if

$$c_1 - c_2 \le \beta(D_2(0) - D_1(\lambda))$$

for all $\beta \in [a,b]$. Now this is not possible as the left hand side is positive while the right hand side is negative. It is straightforward to see that when $\beta_1 = b$, we have $\gamma_1 = 0$ and hence $\gamma_1 \in [0,\gamma^+]$. When $\beta_1 \in (a,b)$ we have $c_1 + \beta_1 D_1(\gamma_1) = c_2 + \beta_1 D_2(\gamma_2)$. Again, since $c_1 > c_2$, we have $D_1(\gamma_1) \leq D_2(\gamma_2)$ and this requires $\gamma_1 \in (0,\gamma^+)$. The proof for $\gamma_1 \in (\gamma^+,\lambda]$ follows along similar lines and will not be provided. This completes the proof.

Lemma 2

Proof: From Lemma 1, $\gamma_1 \in [0, \gamma^+)$ implies that $c_1 > c_2$ while $\gamma_1 \in (\gamma^+, \lambda]$ implies $c_1 < c_2$. Now from Theorem 1, when $c_1 > c_2$, we have

$$\gamma_1 = \lambda \int_{\beta_1}^b 1 dF(\beta) = \lambda (1 - F(\beta_1)).$$

Similarly, when $c_1 < c_2$ we have

$$\gamma_1 = \lambda \int_0^{\beta_1} 1 dF(\beta) = \lambda(F(\beta_1)).$$

Now $\beta_1(\gamma_1)$ defined as the value of threshold β_1 when the equilibrium arrival rate to Server 1 is γ_1 can be represented as follows.

$$\beta_1(\gamma_1) = \begin{cases} \beta : \int_{\beta}^{b} \lambda dF(\beta) = \gamma_1 & \text{for } 0 \le \gamma_1 < \gamma^{+} \\ \beta : \int_{a}^{\beta} \lambda dF(\beta) = \gamma_1 & \text{for } \gamma^{+} < \gamma_1 < \lambda. \end{cases}$$
(21)

Now as seen earlier, $F(\cdot)$ is absolutely continuous and strictly increasing in its domain. Further, the support is [a,b] and hence $F(\cdot)$ is a bijective function whose inverse exists. In fact $F^{-1}(\cdot)$ is continuous and strictly increasing in its domain. The statement of the lemma now follows.

Lemma 3

Proof: Recall our assumption that $D_j(\gamma_j)$ is continuous and monotone increasing in γ_j where j=1,2. Since $\gamma_2=\lambda-\gamma_1$, $(D_2(\lambda-\gamma_1)-D_1(\gamma_1))$ is monotone decreasing in γ_1 for $0\leq \gamma_1\leq \lambda$. Recall Eq. (8) that determines $\beta_1(\gamma_1)$. For $0\leq \gamma_1<\gamma^+,\ \beta_1(\gamma_1)$ is continuous and strictly decreasing. The continuity follows from that of $F^{-1}(\cdot)$. Since $F^{-1}(\cdot)$ is strictly increasing in its arguments, $F^{-1}\left(\frac{\lambda-\gamma_1}{\lambda}\right)=\beta_1(\gamma_1)$ is decreasing in γ_1 . Clearly, $g_1(\gamma_1)$ is monotone decreasing when γ_1 is such that $0\leq \gamma_1<\gamma^+$.

When γ_1 is such that $\gamma^+ < \gamma_1 \leq \lambda$, from the definition of γ^+ , we have $(D_2(\lambda - \gamma_1) - D_1(\gamma_1)) < 0$. In this range of γ_1 , it can be seen from Eq. (8) that $\beta_1(\gamma_1)$ is continuous and increasing in γ_1 . This again implies that $g_1(\gamma_1)$ is continuous decreasing when γ_1 satisfies $\gamma^+ < \gamma_1 \leq \lambda$.

 $g_1(\gamma^+)=0$ follows from the definition of γ^+ where $D_1(\gamma^+)=D_2(\lambda-\gamma^+)$. The continuity at γ^+ is obvious from the fact that $g_1(\gamma^+)=0$ and $\lim_{\gamma_1\to\gamma^+}g_1(\gamma_1)=0$.

Lemma 4

Proof: Suppose $\Delta \geq g_1(0)$. From the definition of Δ and from Eq. (5), this implies that

$$c_1 - c_2 \ge b(D_2(\lambda) - D_1(0))$$

 $\ge \beta(D_2(\lambda) - D_1(0))$

for all $\beta \in [a,b]$. From the Wardrop equilibrium condition, this implies that $K^W(\beta,\cdot) = \delta_2$ for $\beta \in [a,b]$. This implies that $\gamma_1 = 0$ and from Eq. (2) we have $\beta_1 = b$. Similarly when, $\Delta \leq g_1(\lambda) < 0$ we have

$$c_1 - c_2 \le b(D_2(0) - D_1(\lambda))$$

 $\le \beta(D_2(0) - D_1(\lambda))$

where $\beta \in [a, b]$. Again, from the Wardrop equilibrium condition, this implies that $K^W(\beta, \cdot) = \delta_1$ for $\beta \in [a, b]$. Hence $\gamma_1 = \lambda$ and from Eq. (2), we have $\beta_1 = b$.

Now suppose $g_1(\lambda) < \Delta < g_1(0)$ where we know that $g_1(0) > 0$ and $g_1(\lambda) < 0$. From Lemma 3, we know that $g_1(\gamma_1)$ is monotonically decreasing in γ_1 . Therefore there exists a unique γ with $0 < \gamma < \lambda$ such that $\Delta = g_1(\gamma)$. This proves the uniqueness of γ_1 . To see how $\beta_1 = \beta_1(\gamma)$ note that $\Delta = g_1(\gamma)$ implies that

$$c_1 - c_2 = \beta_1(\gamma) \left(D_2(\lambda - \gamma) - D_1(\gamma) \right).$$

Now if $\gamma \leq \gamma^+$ we have $D_2(\lambda - \gamma) > D_1(\gamma)$. In this case,

$$c_1 - c_2 \le \beta \left(D_2(\lambda - \gamma) - D_1(\gamma) \right)$$

for $\beta \in [a, \beta_1(\gamma)]$. This means that $K^W(\beta, \cdot) = \delta_2$ for all $\beta \in [a, \beta_1(\gamma)]$. Similarly, we have

$$c_1 - c_2 \ge \beta \left(D_2(\lambda - \gamma) - D_1(\gamma) \right) \tag{22}$$

and $K^W(\beta,\cdot) = \delta_1$ when $\beta \in [\beta_1(\gamma), b]$. Similar arguments hold when $\gamma > \gamma^+$ and hence $\beta_1 = \beta_1(\gamma)$ when $g_1(0) < \Delta < g_1(\lambda)$.

From Theorem 1, K^W is characterized by β_1 and for a fixed Δ , β_1 is unique. This implies uniqueness of K^W . It is important to mention that K^W is unique when $\Delta = 0$ because of the assumptions made to ensure $\beta_1(\gamma_1)$ well defined at $\gamma_1 = \gamma^+$.

Lemma 5

Proof: Suppose c_2 satisfies $c_2 < -g_1(\lambda)$. Assume that $c_1 = 0$ so that we have $\Delta > g_1(\lambda)$. From Lemma 4 this implies that the equilibrium γ_1 satisfies $g_1(\gamma_1) = \Delta = -c_2$. Let us label this γ_1 as $\hat{\gamma}$. Now increase c_1 from $c_1 = 0$ by a small $\epsilon > 0$ such that there exists γ_1 that satisfies $\Delta = \epsilon - c_2 = g_1(\gamma_1)$. Now from the monotonicity of $g_1(\cdot)$ it is clear that the equilibrium γ_1 is decreasing as Δ increases. This implies that a higher Δ caused by increasing c_1 will only lead to a γ_1 satisfying $\gamma_1 < \hat{\gamma}$. Clearly, for any choice of $c_1 \geq 0$, we have $\gamma_1 \notin [\hat{\gamma}, \lambda]$ and hence for this case $\gamma^1(c_2) = \hat{\gamma}$.

Now suppose that $-c_2 \leq g_1(\lambda)$. When $c_1=0$, this implies $\Delta \leq g_1(\lambda)$ and from Lemma 4 this implies $\beta_1=b$ with the corresponding γ_1 satisfying $\gamma_1=\lambda$. As we increase c_1 , the equilibrium γ_1 decreases and hence γ_1 satisfies $\gamma_1\in[0,\lambda]$. The compact representation now follows.

Lemma 6

Proof: To reduce the notations, we represent $\gamma_i^*(c_{j^-})$ by γ_i^* in the proof of the lemma. We shall prove that $\gamma_1^* \notin \{0, \gamma^1(c_2)\}$ and the proof for $\gamma_2^* \notin \{0, \gamma^2(c_1)\}$ is along similar lines. Suppose $\gamma_1^* \in \{0, \lambda\}$. Then from the requirement that $\gamma_1^* = \lambda - \overline{\gamma_2^*}$, we have either (1) $\gamma_1^* = 0$ and $\gamma_2^* = \lambda$ or (2) $\gamma_1^* = \lambda$ and $\gamma_2^* = 0$. First consider the case when $\gamma_1^* = 0$ and $\gamma_2^* = \lambda$. This implies that $R_1(c_1(0), 0) = 0$ and hence the revenue made by Server 1 at equilibrium is zero. Further since this is an equilibrium, there is no incentive for the server to change the admission price and increase its revenue. We shall now show that this is not true. From Theorem 2, we know that for a given c_2 , the admission price at Server 1 must be at least $c_2 + g_1(0) > 0$. Now we know that setting $c_1 = c_2$ will result in $\gamma_1 = \gamma^+$. Now due to the assumption that (1) $D_1(0) < D_2(\lambda)$ and (2) $D_2(0) < D_1(\lambda)$, there exists an $\epsilon > 0$ such that setting $c_1 = c_2 + \epsilon$ will result in $\gamma_1 \in (0, \gamma^+)$. The revenue earned is non-zero and there is clearly an incentive to deviate from any value greater than $c_2 + g_1(0)$. This implies that $\gamma_1^* = 0$ and $\gamma_2^* = \lambda$ is not possible. The proof for $\gamma_1^* = \lambda$ and $\gamma_2^* = 0$ is along the same lines.

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