An Analysis of Efficiency of Time-consistent Coordination Mechanisms in a Model of Supply Chain Management*

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Abstract

In this paper, we study an advertising dynamic game in supply chain management under the assumption that the agents differ in their time preference rates. We study two coordination mechanisms: the cost sharing program, where the retailer can get some reimbursement of the advertising cost from the manufacturer; and the vertical integration, where the two players aim to maximize the joint profit. We derive the time-consistent cooperative advertising strategies in each coordination setting, and we compare them with the non-cooperative case. Our results show that, the cost sharing program is Pareto superior to the non-cooperative setting, while vertical integration could be more preferred by the manufacturer and less preferred by the retailer if the initial goodwill level is sufficiently high. Besides, unlike previous results in the literature, we found that when the agents' discount rates are very different, joint profits could be lower under vertical integration than in the non-cooperative case, which yields an inefficient cooperation.

Keywords: OR in Marketing, Supply Chain Management, Advertising Coordination, Heterogeneous Discounting, Differential Games

1. Introduction

Research interest regarding the interaction between the members of a supply chain covers various topics including inventory management, production and pricing competition, quality improvement and advertising competition, among others. Advertising coordination, which is commonly believed to be beneficial to the channel, has been highlighted in recent years.

There exist different interpretations of what is understood by advertising coordination (see, e.g., Aust & Buscher, 2014; Jørgensen & Zaccour, 2014, for reviews on advertising coordination). One prevailing setting is the cost-sharing program, also called cooperative advertising/co-op advertising, which is a binding contract on the sharing of the advertising cost: the follower of the supply chain

*We are very grateful to the editor and the two anonymous reviewers for their valuable comments and constructive suggestions. The second and the third authors' research is supported by MINECO [ECO2013-48248-P, ECO2017-82227-P (AEI/FEDER,UE)].

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(typically, the retailer) can get some reimbursement of advertising cost from the leader (typically, the manufacturer). According to a report by Marketing-Land (2018), the annual cooperative advertising expenditure was estimated to be 70 billion dollars in the United States. This program has been empirically tested and quite intensively studied in static models (for example, see Berger, 1972; Bergen & John, 1997; Dutta et al., 1995; Nagler, 2006). Adopting the goodwill dynamics to model the carry-over effect of advertising, Jørgensen et al. (2000) introduce this cooperation scheme into inter-temporal setting. They consider the case where both manufacturer and retailer can (but are not forced to) implement two types of advertising with long-term and short-term effect, which contribute to the goodwill and market demand respectively. In line with the interest on different advertising effects, Jørgensen et al. (2003) argue that the negative influence of the retailer's promotion on goodwill should be studied. Jørgensen et al. (2001b) apply a more flexible functional form for the demand function, and introduce decreasing marginal returns to goodwill. The situation in which the retailer can launch a private-label product is studied in Karray & Zaccour (2005), whereas in De Giovanni (2011), the manufacturer has quality improvement as an additional operational tool. Other cases include a pre-launch advertising campaign with two customers segments to which the players' access is asymmetric (as in Buratto et al., 2007), mechanisms combining revenue sharing and advertising cost sharing (De Giovanni & Roselli, 2012), and the interaction between inventory management and cost-sharing program (De Giovanni et al., 2019).

Despite of the different elements incorporated in different models, what is clear is that when the cost-sharing program is implemented, i.e., the subsidy-rate is strictly positive (this mainly depends on the relationship between the margins of each member, with the exception of De Giovanni, 2011, where the necessary conditions are related to the effectiveness parameter), the retailer is induced, directly or indirectly (through higher goodwill level), to invest more in advertising, and the outcome is Pareto-improving.

Another common mechanism is vertical integration, also known as centralized coordination. In this setting, all members of the supply chain act in a coordinated way to maximize the joint profit. The vertical integration happens rather frequently, for instance, Amazon acquired Whole Foods Market partly for their private label products. Due to its higher total channel profit (in the case of equal discount rates), research interest has been mainly put on how to maintain the cooperation over time with the implementation of incentive strategies (Jørgensen et al., 2006; Jørgensen & Zaccour, 2003a). It is worth mentioning that this setting sometimes also serves as a benchmark in the literature of cost-sharing to make the comparison of channel efficiency (Buratto et al., 2007; Zhang et al., 2013).

Nevertheless, the vertical integration efficiency is based on an essential hypothesis that all agents share equal time preferences. However, asymmetry in time discounting may arise as the result of many different aspects, such as the distinct firm sizes, which could imply distinct financial costs or financial constraints. Besides, different firms conduct divergent economic activities, which are regulated by the corresponding (very often divergent) legislation. Moreover, the asymmetric power in the chain could be associated with different survival probabilities for the firms involved. It is known that when firms have different survival density functions to evaluate the expected utility, by assuming an exponential distribution, the survival probabilities are integrated into the discount rates. Hence, it appears appealing to generalize the time preferences setting via incorporating a possible asymmetry. Such asymmetry in time discounting implies that the joint time preferences are time-inconsistent and we face a trade-off between efficiency and time-consistency (Jackson & Yariv, 2015). If players cooperate by using time-consistent (subgame perfect) strategies, it can happen that joint profits are lower than the joint non-cooperative payoffs. This situation, that we call group inefficiency, may arise for the reason that the set of non-cooperative strategies is, in general, not included in the set of time-consistent strategies. In this paper we concentrate our attention on the time-consistency since they can be seen as more credible, in the sense that agents have no incentives to deviate from their decisions.

The objective of the paper is threefold. First, we extend previous supply chain models by considering that agents can differ in their time preferences, and analyzing how this asymmetry affects non-cooperative and cooperative outcomes. While different time preferences have been considered in other economic areas (e.g., in environmental resources models, De Paz et al., 2013; Breton & Keoula, 2014), to the best of our knowledge these issues have not been addressed in the framework of dynamic marketing models. Second, and as a consequence of the previous extension, we study situations when cooperation does not pay off by identifying cases of group inefficiency. Third, we adopt a more general demand function that integrates the private effect of goodwill, the synergistic effect of goodwill and retailer's advertising, and the decreasing marginal returns to goodwill. We compare different cooperation programs to see which cooperation mechanism can yield a better outcome, thus offering some implementation guidelines.

We analyze three scenarios. Scenario Nash (N) describes the situation where the two firms act simultaneously in the absence of cooperation. In scenario Stackelberg (S) a cost sharing program is applied: the manufacturer, acting as the leader in a Stackelberg game (this is a prevalent assumption in the literature and is consistent with the nature of many industries such as automobile, gasoline and so on), supports part of the retailer's advertising cost. Another cooperation mechanism, the vertical integration, is modeled in scenario joint maximization (J) where the two players of the supply chain act coordinately to maximize the joint profit.

The main results show that: (1) The cost sharing mechanism implies higher advertising efforts from both manufacture and retailer, and leads to a Pareto superior outcome in comparison with non-cooperative case. This is in accordance with most of the studies. (2) Contrary to the existing marketing literature, a centralized coordination does not necessarily grant higher joint payoffs compared to scenario N. Particularly, if the retailer has a much higher discount rate than the manufacturer, low initial goodwill level and low revenue sharing rate could give rise to group inefficiency. Whereas for the opposite case, when the manufacturer discounts future payoffs much more heavily, and the revenue sharing rule does not extensively favor the retailer, the vertical integration is inefficient, no matter how the initial goodwill level is.

The rest of the paper is organized as follows. In Section 2 we describe the differential game model. The determination of feedback/time-consistent equilibria follows in Section 3. In Section 4 we make a fully detailed comparison of the strategies and payoffs obtained in Section 3 among the three distinct scenarios. We also run some numerical simulations to throw light on the existence of group inefficiency. Finally, in Section 5 we present the concluding remarks and suggest some future studies.

2. Model Formulation

We consider a two-echelon supply chain model consisting of one manufacturer and one retailer, and where the mechanism works in the way that the manufacturer's advertising policies have more impact on goodwill and sales. In practice, the advertising activities of each member usually have different properties. The manufacturer's global advertising $A_M(t)$ is normally more general and nationwide, with the objective of creating and improving the brand image; it doesn't necessarily generate immediate sales. On the contrary, the retailer's advertising $A_R(t)$ (e.g., promotion, fliers, point-of-sale display, etc.) works more in a local scale and could typically influence directly on the consumer demand (Aust & Buscher, 2014). In this sense, the manufacturer is to some extent dependent on the local advertising, and this is an important reason justifying the usefulness of cooperation.

We depart from the goodwill model proposed by Nerlove & Arrow (1962), where the goodwill is considered as a stock with dynamics given by

$$\dot{G}(t) = k_m A_M(t) - \delta G(t), \quad G(0) = G_0 \geq 0 \ , \tag{1}$$

where k_m and δ are positive constants representing the effectiveness of the manufacturer's advertising and the depreciation rate, respectively. From expression (1), goodwill only increases through manufacture's advertising. In the literature, assumptions on how the retailer's advertising effort affects the brand image are mixed. Some papers consider a positive effect (see, e.g., Jørgensen & Zaccour, 2003b; De Giovanni, 2014; De Giovanni & Roselli, 2012; Jørgensen et al., 2000; Zhang et al., 2013), some others report a negative effect (the main idea is that consumers very often relate frequent promotions to poor quality, for more details, see Jørgensen et al., 2003), and finally some assume a null effect and capture all the retailer's advertising influence in the sales function (Karray & Zaccour, 2005; Jørgensen et al., 2001a, 2006).

One common way to model the market demand is to consider the sum of functions of the goodwill and the advertising rate, assuming that they influence the demand independently. One example comes from Jørgensen et al. (2003) (we refer to Jørgensen et al., 2001b; Karray & Zaccour, 2005; Zhang et al., 2013, for more variations), where

$$S(t) = \mu G(t) + \gamma A_R(t) \ .$$

Alternatively, the following structure proposed by Jørgensen et al. (2006) (for more examples, see Jørgensen et al., 2001a, 2000) assumes that they interact in a multiplicative way:

$$S(t) = \theta + \gamma A_R(t)\sqrt{G(t)}\ .$$

In this setting, what could be arguable is that, in the most extreme case when there is no advertising or no goodwill, the global effect is null.

In this paper, we combine the separate effect of goodwill and the joint effect, thus sales are given by

$$S(t) = \theta + \mu G(t) + \gamma A_R(t)\sqrt{G(t)}, \tag{2}$$

where θ, μ and γ are positive constants representing the baseline sales, the effectiveness of goodwill and the synergy, respectively. Expression (2) can be considered as an extension of the additive type model by moderating the retailer's advertising's effect, or as the extension of the multiplicative type model by adding the separate effect of goodwill. This specification captures the properties of some specific markets such as car, infant food, domestic appliances etc., where goodwill plays a determinant role in consumer buying decisions. It reflects moreover the limited influence of the retailer, in that her advertising only works as a booster to the demand.

As in many other studies, advertising cost functions are assumed to be convex and given by

$$C(A_M) = \frac{c_m}{2}A_M(t)^2, \quad C(A_R) = \frac{c_r}{2}A_R(t)^2,$$

where c_m and c_r are positive constant cost parameters.

Finally, we assume that agents' time preferences can differ. As mentioned before, time discount rate can be thought as an aggregation of a series of factors (firm size, legislation, survival probabilities and so on). It is natural to think that such aggregation could result differently in different agents.

Let ρ_m and ρ_r denote the discount rate of the manufacturer and the retailer; $\pi \in (0,1)$ the revenue sharing rate, which is given exogenously; and $\Phi(t)$ the cost sharing rate, which is the fraction of the advertising cost of the retailer that the manufacturer offers to support. The objective functional of the manufacturer is

$$J_M = \int_0^\infty e^{-\rho_m t}\left[\pi S(t) - \frac{c_m}{2}A_M(t)^2 - \frac{c_r}{2}\Phi(t)A_R(t)^2\right]dt, \tag{3}$$

and the objective functional of the retailer is

$$J_R = \int_0^\infty e^{-\rho_r t}\left[(1-\pi)S(t) - \frac{c_r}{2}(1-\Phi(t))A_R(t)^2\right]dt. \tag{4}$$

Equations (1), (2), (3) and (4) define a two-player differential game with one state variable $G(t) \geq 0$, and with the manufacturer controlling $A_M(t) \geq 0$ and the retailer controlling $A_R(t) \geq 0$. We conduct a different treatment for the control variable $\Phi(t)$ depending on the scenario. Specifically, in the non-cooperation setting (N), $\Phi(t) = 0$. In the cost sharing scenario (S), the manufacturer can decide freely the value of $\Phi(t)$ in the interval $[0,1]$. In the vertical integration scenario (J) we keep $\Phi(t)$ as a constant. In a standard cooperative game of joint maximization with equal discount rates, neither revenue sharing rate nor cost sharing rate have impact on the strategies or on the joint outcome, since they are ultimately side payments. However when agents discount future revenues/costs in a heterogeneous way, it may happen that their behaviors are influenced thus the dynamics of the state also evolve differently.

In the following sections the time argument is omitted for brevity unless obvious ambiguity arises.

3. Determination of Feedback Equilibria

In this section, we compute the feedback Nash equilibrium, stage-wise feedback Stackelberg equilibrium (for the definition and distinction of Stackelberg solution types, we refer to Long, 2010; Haurie et al., 2012; Basar et al., 2018) and the time-consistent cooperative solution for the scenarios described above.

For the computation of feedback equilibria, we initially restrict the manufacturer's strategy space to the set of linear strategies and make use of the Lemma 1. We prove later that there is only one solution in each scenario, which gives rise to constant strategies for the manufacturer (this property coincides with previous results in the literature, see, e.g., Jørgensen et al., 2003), and to linear value functions for both players. It is important to realize that the structure of the strategies for the retailer does not depend on the assumptions on the manufacturer's strategy space, as can be easily checked from the proofs of Propositions 1-3.

Lemma 1. *If the manufacturer and retailer strategies are given by $A_M = A_m G + B_m$ and $A_R = A_r\sqrt{G}$, then the corresponding value functions are quadratic in the goodwill for the manufacturer and linear in the goodwill for the retailer, i.e., $V_M = \lambda_M^2/2 + \alpha_M G + \beta_M$ and $V_R = \alpha_R G + \beta_R$, with λ_M, α_M, β_M, α_R and β_R constant numbers.*

Proof. See the Appendix. □

3.1. Determination of the Feedback Nash Equilibrium

In this non-cooperative scenario, the manufacturer and the retailer decide simultaneously their strategies and there is no cost sharing, $\Phi = 0$. With the absence of cost sharing program, the manufacturer cannot influence the retailer's decisions due to our model structure. Accordingly, the following Nash equilibrium coincides with the Stackelberg equilibrium without cost sharing program (for a rather detailed discussion on such coincidence, we refer to Rubio, 2006). Using a superscript N to denote "Nash", Proposition 1 characterizes the equilibrium.

Proposition 1. *The feedback Nash equilibrium is given by the pair of strategies*

$$A_M^{\mathrm{N}} = \frac{k_m}{c_m}\alpha_M^{\mathrm{N}}\ , \tag{5}$$

$$A_R^{\mathrm{N}} = \frac{(1-\pi)\gamma}{c_r}\sqrt{G} \ , \tag{6}$$

and the corresponding value functions are given by

$$V_M^{\mathrm{N}} = \alpha_M^{\mathrm{N}} G + \frac{k_m^2}{2c_m\rho_m}(\alpha_M^{\mathrm{N}})^2 + \frac{\pi\theta}{\rho_m} \ , \tag{7}$$

$$V_R^{\mathrm{N}} = \alpha_R^{\mathrm{N}} G + \frac{k_m^2}{c_m\rho_r}\alpha_M^{\mathrm{N}}\alpha_R^{\mathrm{N}} + \frac{(1-\pi)\theta}{\rho_r} \ , \tag{8}$$

where

$$\alpha_M^{\mathrm{N}} = \frac{c_r\pi\mu + \pi(1-\pi)\gamma^2}{c_r(\rho_m+\delta)} \ ,$$

$$\alpha_R^{\mathrm{N}} = \frac{2c_r(1-\pi)\mu + (1-\pi)^2\gamma^2}{2c_r(\rho_r+\delta)} \ .$$

Proof. See the Appendix. □

As usual, the manufacturer's advertising strategy is determined in the way that the advertising's marginal cost is equal to its marginal revenue:

$$c_m A_M^{\mathrm{N}} = \frac{\pi k_m}{(\rho_m+\delta)}\left[\mu + \frac{(1-\pi)\gamma^2}{c_r}\right] \ .$$

From (5) we can see that the manufacturer's policy is proportional to the ratio of effectiveness parameter to cost parameter, k_m/c_m, which can represent the efficiency of this investment, and is decreasing in δ, the goodwill depreciation rate. Next we study the sensitivity of A_M^{N} with respect to π. From

$$\frac{\partial A_M^{\mathrm{N}}}{\partial \pi} = \frac{k_m(c_r\mu + \gamma^2 - 2\gamma^2\pi)}{c_m c_r(\rho_m+\delta)} \ ,$$

it holds that $\partial A_M^{\mathrm{N}}/\partial\pi > 0$ for all $\pi \in (0,1)$ when $c_r\mu \geq \gamma^2$. This implies that the manufacturer is motivated to invest more when her corresponding revenue sharing rate π is higher, as expected. However, when γ, the effectiveness parameter of the synergy $A_R^{\mathrm{N}}\sqrt{G}$, is sufficiently high, such that $c_r\mu < \gamma^2$, A_M^{N} increases in π in the interval $(0,\tilde{\pi})$, and decreases in π in the interval $(\tilde{\pi},1)$ with $\tilde{\pi} = (c_r\mu+\gamma^2)/(2\gamma^2) > 1/2$. The same happens if c_r and/or μ, the retailer's advertising cost parameter and the effectiveness of goodwill, are sufficiently small. We can understand this behavior via two aspects. Firstly, both large γ and small c_r imply high efficiency of the retailer's advertising, and μ partly measures the efficiency of the manufacturer's advertising. Secondly, large π leads to low local advertising effort, and the manufacturer's advertising is somehow constrained by the retailer's level due to the synergistic effect of goodwill and promotion. As a result, in the situation where the retailer has highly efficient marketing tool but takes little part of the revenue (and low A_R^{N} follows), the manufacturer reduces her investment as π increases. This result is new in the literature and can provide some managerial insights to practitioners. It is also worth mentioning that when the manufacturer is more impatient (larger ρ_m), she invests less. In our model, the manufacturer has no direct influence on the revenue. In addition, an impatient agent discounts heavily the future rewards, which makes the immediate loss prioritized in the decision making.

Regarding the retailer's advertising strategy, it is increasing in the goodwill with a decreasing marginal effect. This is intuitive. Recall how the retailer's advertising works jointly with the goodwill level in (2): on the one hand, its effect gets strengthened by the goodwill level, thus a response of increase results; on the other hand, such reinforcement effect is decreasing and, as a consequence, the retailer adjusts her increasing speed. Besides, A_R^{N} is proportional to γ/c_r, the ratio of effectiveness parameter to cost parameter. A higher revenue sharing rate corresponding to her $(1-\pi)$ implies a higher advertising effort.

3.2. Determination of the Feedback Stackelberg Equilibrium under Cost Sharing

In this scenario, a cost sharing program is applied. The retailer, which is the follower (as it is usual in most of the studies and in many industries like automobile and gasoline; this is also a natural assumption derived from the limited influence of the retailer on the sales in our setting), can get some reimbursement of the advertising cost from the leader - the manufacturer. As seen in scenario N, the retailer slows down the increment in advertising as the goodwill level goes higher. By supporting part of the retailer's cost, the manufacturer can reach a more desirable sales level. Letting the superscript S refer to "Stackelberg", the following proposition describes the equilibrium.

Proposition 2. *The feedback Stackelberg equilibrium is given by the strategies*

$$A_M^{\mathrm{S}} = \begin{cases} \frac{k_m}{c_m}\alpha_M^{\mathrm{S}} & \text{if} \quad \frac{1}{3} < \pi < 1 \ , \\ \frac{k_m}{c_m}\alpha_M^{\mathrm{N}} & \text{if} \quad 0 < \pi \leq \frac{1}{3} \ , \end{cases} \tag{9}$$

$$\Phi^{\mathrm{S}} = \begin{cases} \frac{3\pi-1}{\pi+1} & \text{if} \quad \frac{1}{3} < \pi < 1 \ , \\ 0 & \text{if} \quad 0 < \pi \leq \frac{1}{3} \ , \end{cases} \tag{10}$$

$$A_R^{\mathrm{S}} = \begin{cases} \frac{(1+\pi)\gamma}{2c_r}\sqrt{G} & \text{if} \quad \frac{1}{3} < \pi < 1 \ , \\ \frac{(1-\pi)\gamma}{c_r}\sqrt{G} & \text{if} \quad 0 < \pi \leq \frac{1}{3} \ , \end{cases} \tag{11}$$

and the corresponding value functions are given by

$$V_M^{\mathrm{S}} = \begin{cases} \alpha_M^{\mathrm{S}}G + \frac{k_m^2}{2c_m\rho_m}(\alpha_M^{\mathrm{S}})^2 + \frac{\pi\theta}{\rho_m} & \text{if} \quad \frac{1}{3} < \pi < 1 \ , \\ \alpha_M^{\mathrm{N}}G + \frac{k_m^2}{2c_m\rho_m}(\alpha_M^{\mathrm{N}})^2 + \frac{\pi\theta}{\rho_m} & \text{if} \quad 0 < \pi \leq \frac{1}{3} \ , \end{cases} \tag{12}$$

$$V_R^{\mathrm{S}} = \begin{cases} \alpha_R^{\mathrm{S}}G + \frac{k_m^2}{c_m\rho_r}\alpha_M^{\mathrm{S}}\alpha_R^{\mathrm{S}} + \frac{(1-\pi)\theta}{\rho_r} & \text{if} \quad \frac{1}{3} < \pi < 1 \ , \\ \alpha_R^{\mathrm{N}}G + \frac{k_m^2}{c_m\rho_r}\alpha_M^{\mathrm{N}}\alpha_R^{\mathrm{N}} + \frac{(1-\pi)\theta}{\rho_r} & \text{if} \quad 0 < \pi \leq \frac{1}{3} \ , \end{cases} \tag{13}$$

where α_M^{N} and α_R^{N} are defined in Proposition 1 and

$$\alpha_M^{\mathrm{S}} = \frac{8c_r\pi\mu + (1+\pi)^2\gamma^2}{8c_r(\rho_m+\delta)} \ ,$$

$$\alpha_R^{\mathrm{S}} = \frac{4c_r(1-\pi)\mu + (1-\pi^2)\gamma^2}{4c_r(\rho_r+\delta)} \ .$$

Proof. See the Appendix. □

The manufacturer offers to support part of the retailer's advertising cost only in the case where her revenue sharing rate is sufficiently high. Otherwise, it is not profitable to pay an additional cost and the best she could do is to compete with the retailer, thus resulting the same outcome as that of in scenario Nash. This finding coincides with Jørgensen et al. (2000) and Jørgensen et al. (2003).

Now let us focus on the case of $1/3 < \pi < 1$, when a cost sharing program is conducted.

We can see some common properties between (5) and (9). Specifically, in both scenarios N and S, the manufacturer responses in a similar way to the efficiency ratio k_m/c_m, goodwill depreciation rate δ, and time preference ρ_m. However, unlike Scenario N, where a lower global advertising budget might be associated with a higher revenue sharing rate because of (relative) inefficiency and limited local advertising, here the larger proportion of revenue the manufacturer takes, the more resources she would spend in global advertising ($\partial A_M^{\mathrm{S}}/\partial\pi > 0$). With an active cost sharing program, the local advertising is greater than that in competition setting (we present a more detailed comparison in Section 4.1). This allows the manufacturer to invest more in advertisement, independently of how efficient the follower's marketing tool is.

With respect to the retailer's advertising, as in the competition scenario N, it is proportional to γ/c_r and increases in the goodwill with a decreasing marginal effect. What may be surprising at first sight is that A_R^{S} (equation (11)) is increasing in π, the manufacturer's revenue sharing rate. However, note that the higher fraction the manufacturer gets from the revenue, the higher percentage she would pay to the retailer ($\partial\Phi^{\mathrm{S}}/\partial\pi > 0$). Indeed, the actual cost the retailer is paying is decreasing in the manufacturer's revenue sharing rate $\big(\partial[A_R^{\mathrm{S}}(1-\Phi^{\mathrm{S}})]/\partial\pi < 0\big)$.

3.3. Determination of the Time-Consistent Cooperative Solution under Vertical Integration

In the vertical integration scenario, the manufacturer and the retailer form a coalition to maximize the sum of their individual payoffs defined in (3) and (4). Using the superscript J to represent "Joint maximization", we are facing a problem with the objective functional:

$$\begin{aligned} J^{\mathrm{J}} =& J_M + J_R \\ =& \int_0^\infty e^{-\rho_m t}\Big\{\pi S + e^{-(\rho_r-\rho_m)t}(1-\pi)S - \frac{c_m}{2}(A_M)^2 \\ &-\frac{c_r}{2}\Big[\Phi(A_R)^2 + e^{-(\rho_r-\rho_m)t}(1-\Phi)(A_R)^2\Big]\Big\}\,dt. \end{aligned} \tag{14}$$

Notice that when $\rho_m \neq \rho_r$, terms depending on π and Φ don't vanish, unlike the standard case. When agents are cooperating, π and Φ are side payments which can be negotiated in order to sustain the cooperation. However, when they discount them heterogeneously, as we will show later, these side payments do have impacts on their behaviors and the outcome.

As illustrated in De Paz et al. (2013), the aggregated time preferences given by (14) are time-inconsistent. Similar to hyperbolic discounting, time-consistent solutions can be defined for this kind of problems. Applying the approach proposed in the paper mentioned above, the time-consistent cooperative solution is characterized by the following proposition.

Proposition 3. *The time-consistent cooperative advertising strategies are determined by*

$$A_M^{\mathrm{J}} = \begin{cases} \frac{k_m}{c_m}(\alpha_M^{\mathrm{J}} + \alpha_R^{\mathrm{J}}) & \text{if } \alpha_M^{\mathrm{J}} + \alpha_R^{\mathrm{J}} \geq 0 \,, \\ 0 & \text{if } \alpha_M^{\mathrm{J}} + \alpha_R^{\mathrm{J}} < 0 \,, \end{cases} \tag{15}$$

$$A_R^{\mathrm{J}} = \frac{\gamma}{c_r}\sqrt{G} \,, \tag{16}$$

and the corresponding value functions are given by

$$V_M^{\mathrm{J}} = \alpha_M^{\mathrm{J}} G + \begin{cases} \frac{k_m^2}{2c_m\rho_m}((\alpha_M^{\mathrm{J}})^2 - (\alpha_R^{\mathrm{J}})^2) + \frac{\pi\theta}{\rho_m} & \text{if } \alpha_M^{\mathrm{J}} + \alpha_R^{\mathrm{J}} \geq 0 \,, \\ \frac{\pi\theta}{\rho_m} & \text{if } \alpha_M^{\mathrm{J}} + \alpha_R^{\mathrm{J}} < 0 \,, \end{cases} \tag{17}$$

$$V_R^{\mathrm{J}} = \alpha_R^{\mathrm{J}} G + \begin{cases} \frac{k_m^2}{c_m\rho_r}(\alpha_M^{\mathrm{J}}\alpha_R^{\mathrm{J}} + (\alpha_R^{\mathrm{J}})^2) + \frac{(1-\pi)\theta}{\rho_r} & \text{if } \alpha_M^{\mathrm{J}} + \alpha_R^{\mathrm{J}} \geq 0 \,, \\ \frac{(1-\pi)\theta}{\rho_r} & \text{if } \alpha_M^{\mathrm{J}} + \alpha_R^{\mathrm{J}} < 0 \,, \end{cases} \tag{18}$$

where

$$\alpha_M^{\mathrm{J}} = \frac{2c_r\pi\mu + (2\pi - \Phi^{\mathrm{J}})\gamma^2}{2c_r(\rho_m + \delta)} \,,$$

$$\alpha_R^{\mathrm{J}} = \frac{2c_r(1-\pi)\mu + (1 + \Phi^{\mathrm{J}} - 2\pi)\gamma^2}{2c_r(\rho_r + \delta)} \,.$$

Proof. See the Appendix. □

We first look at the standard case, where both players share the same discount rate. In this case, the condition $\alpha_M^{\mathrm{J}} + \alpha_R^{\mathrm{J}} \geq 0$ is verified. We rewrite the non-zero manufacturer's strategy as

$$A_M^{\mathrm{J}} = \frac{k_m \left\{(\rho_m - \rho_r)\left[-2(\gamma^2 + c_r\mu)\pi + \gamma^2\Phi^{\mathrm{J}}\right] + (\rho_m + \delta)(\gamma^2 + 2c_r\mu)\right\}}{2c_m c_r(\rho_m + \delta)(\rho_r + \delta)} \,. \tag{19}$$

When $\rho_m = \rho_r = \rho$,

$$A_M^{\mathrm{J}} = \frac{k_m(\gamma^2 + 2c_r\mu)}{2c_m c_r(\rho + \delta)} \,,$$

and the payoffs of the grand coalition are

$$V_M^{\mathrm{J}} + V_R^{\mathrm{J}} = \frac{\gamma^2 + 2c_r\mu}{2c_r(\rho + \delta)}G + \frac{k_m^2}{2c_m\rho}\left[\frac{\gamma^2 + 2c_r\mu}{2c_r(\rho + \delta)}\right]^2 + \frac{\theta}{\rho} \,.$$

Note that none of them is affected by the revenue sharing rate nor the cost sharing rate. As we mentioned before, in the standard case, side payments given from π and Φ cancel in the coalition's objective functional. As a consequence, even though agents' individual payoffs vary with different values of side payments, their strategies and the joint payoffs are not affected.

However, when the two players discount future payoffs differently, the situation changes. In this setting with full cooperation among agents, giving rise to a vertical integration, we assume values of π and Φ^{J} as given, as the result, e.g., of a previous agreement (contract) among the manufacturer and the retailer. Note that in some extreme case (it is easy to check that $\alpha_M^{\mathrm{J}} + \alpha_R^{\mathrm{J}} \geq 0$ is fulfilled unless ρ_m is very different to ρ_r), the manufacturer would choose not to invest in advertising at all. When the parameters setting is in such a way that $A_M^{\mathrm{J}} > 0$, the manufacturer's advertising (19) is monotone in π and Φ^{J}, depending on the relation between ρ_m and ρ_r. From the shadow prices α_M^{J} and α_R^{J}, we can see that a higher revenue sharing rate and a lower cost sharing rate are beneficial to the manufacturer, but damaging to the retailer (and these effects decrease in their corresponding discount rate). Concretely, when $\rho_m > \rho_r$, the overall effect of higher π turns out to be unfavorable to the coalition, thus implying lower A_M^{J}. On the contrary, Φ^{J} and A_M^{J} move in the same direction due to the same reason. If $\rho_r > \rho_m$, the previous results are reversed. A more detailed analysis of the effects of π and Φ is provided in Section 4.3.

The property of being proportional to k_m/c_m also appears in this scenario, as in the other two settings. However, unlike (5) and (9), the effects of δ and ρ_m are not so straightforward. A_M^{J} (in the non-zero case) is increasing in ρ_m if $\Phi^{\mathrm{J}}/\pi > 2 + c_r\mu/\gamma^2$. Since the manufacturer has larger influence on the market, the above condition (implying a cost sharing rate more than twice the revenue sharing rate) seems unrealistic. Hence, it is more likely that A_M^{J} decreases in ρ_m. Another relevant difference is that the time preference of the retailer ρ_r also plays a role in determining the manufacturer's decision. We can observe that ρ_r and ρ_m have opposite effects on A_M^{J}.

The retailer's advertisement expenditure does not depend on the time preferences. Compared to the policies in the other two scenarios N and S, some similar properties remain: it is state dependent and proportional to γ/c_r. Nevertheless, it is not subject to the revenue sharing rate as in (6) and (11). Subsequently, if π and the goodwill level are sufficiently high, it can happen that the retailer is forced to invest too much but gets too little. In this situation, high goodwill level is harmful to the retailer, and this explains why the shadow price α_R^{J} can be even negative for large π, which is not so frequent in the literature.

It is hard to conclude the side-payments' impact on the joint payoffs but, as shown in (17) and (18), they do have their influence.

4. Analysis of the Results

4.1. Comparison of the Equilibrium Strategies and Payoffs

While seeking the time-consistent solutions in the joint maximization setting (Scenario J), we have kept Φ^{J} as an arbitrary constant to see how it affects the coalition's behaviors and payoffs. Here we choose, for Scenario J, a critical value: $\Phi^{\mathrm{J}} = 0$, as in the (feedback Nash) non-cooperative case. This natural choice implies that side payments, in the case of vertical integration, solely come via the revenue sharing rate π. A pairwise comparison between Scenario S and J with $\Phi^{\mathrm{J}} = \Phi^{\mathrm{S}}$ will be briefly represented in Remark 1.

We first compare the manufacturer's advertising strategy among three scenarios.

Proposition 4. *The manufacturer's strategies are related as follows:*

1. *If* $\rho_m < \rho_r$, $A_M^{\mathrm{N}} \leq A_M^{\mathrm{S}} < A_M^{\mathrm{J}}$ *for all* $\pi \in (0,1)$ *(with the first inequality strict for* $\pi \in (\frac{1}{3}, 1)$*);*
2. *If* $\rho_r < \rho_m < 2\rho_r + \delta$,
   - $A_M^{\mathrm{N}} \leq A_M^{\mathrm{S}} < A_M^{\mathrm{J}}$ *for all* $\pi \in (0, \pi^*)$ *(with the first inequality strict for* $\pi \in (\frac{1}{3}, 1)$*),*
   - $A_M^{\mathrm{N}} < A_M^{\mathrm{J}} < A_M^{\mathrm{S}}$ *for all* $\pi \in (\pi^*, 1)$*, where* $\pi^* \in (\frac{1}{3}, 1)$ *solves*

$$-\gamma^2(\rho_r+\delta)\pi^2 + \left[6\gamma^2(\rho_r+\delta) - 8(\gamma^2 + c_r\mu)(\rho_m+\delta)\right]\pi + 4(\gamma^2 + 2c_r\mu)(\rho_m+\delta) - \gamma^2(\rho_r+\delta) = 0\ ;$$

3. *If* $2\rho_r + \delta < \rho_m$,
   - $A_M^{\mathrm{N}} \leq A_M^{\mathrm{S}} < A_M^{\mathrm{J}}$ *for all* $\pi \in (0, \pi^*)$ *(with the first inequality strict for* $\pi \in (\frac{1}{3}, \pi^*)$*),*
   - $A_M^{\mathrm{N}} < A_M^{\mathrm{J}} < A_M^{\mathrm{S}}$ *for all* $\pi \in (\pi^*, \pi^{**})$,
   - $A_M^{\mathrm{J}} < A_M^{\mathrm{N}} < A_M^{\mathrm{S}}$ *for all* $\pi \in (\pi^{**}, 1)$*, where* $\pi^{**} \in (\frac{1}{3}, 1)$ *is the solution to*

$$-2\gamma^2(\rho_r+\delta)\pi^2 + 2(\gamma^2 + c_r\mu)(\rho_m+\delta)\pi - (\gamma^2 + 2c_r\mu)(\rho_m+\mu) = 0$$

   and $\pi^* < \pi^{**}$.

Proof. See the Appendix. □

The comparison between Nash and Stackelberg equilibria is clear. For $\pi > \frac{1}{3}$, the manufacturer's advertising is lower in Scenario N than in Scenario S. However, the nature of global advertising in the vertical integration (joint maximization) setting is more complex and highly depend on the agents' time preferences.

As a side note, large firms usually tend to have low discount rates, because they are very often associated with less financial constraints, lower potential crisis likelihood and higher survival probability. On the contrary, small firms are generally more eager for current payments due to the necessity of development[1]. If the manufacturer is more powerful and farsighted than the retailer, the global advertising would be the highest in vertical integration than in any other two cases. A similar result can arise if the manufacturer is slightly more myopic and the revenue sharing rate is small-intermediate. However, as discussed in the previous section with respect to (19), when $\rho_m > \rho_r$ holds, A_M^{J} is decreasing in π due to the joint shadow price. As a consequence, the manufacturer's advertisement expenditure in Scenario J would be between that in Scenario N and S for π considerably high ($\pi > \pi^*$). In the case where the manufacturer is much more shortsighted than the retailer ($\rho_m > 2\rho_r + \delta$), if π is sufficiently near to 1 ($\pi > \pi^{**}$), the global advertising in centralized coordination might be insufficient and become the minimum among all three scenarios.

We next compare the retailer's policies in all the three scenarios.

[1]Obviously this is not always true, time preferences also depend on the financial health of the company.

Proposition 5. *The retailer's strategies have the following properties (with the first inequality strict for $\pi \in (\frac{1}{3}, 1)$):*

$A_R^{\mathrm{N}} \leq A_R^{\mathrm{S}} < A_R^{\mathrm{J}}$.

Proof. It follows from (6), (11) and (16). □

For a given goodwill level, the retailer's advertising is the highest under vertical integration, and the lowest in competition setting.

Finally, we compare individual payoffs.

Proposition 6. *Equilibrium payoffs are related as follows (with strict inequality for $\pi \in (\frac{1}{3}, 1)$):*

1. $V_M^{\mathrm{N}}(G) \leq V_M^{\mathrm{S}}(G)$, $V_R^{\mathrm{N}}(G) \leq V_R^{\mathrm{S}}(G)$, *for all* $G \geq 0$.
2. *Shadow prices are ranked as* $\alpha_M^{\mathrm{N}} \leq \alpha_M^{\mathrm{S}} < \alpha_M^{\mathrm{J}}$ *and* $\alpha_R^{\mathrm{J}} < \alpha_R^{\mathrm{N}} \leq \alpha_R^{\mathrm{S}}$.

Proof. See the Appendix. □

Clearly, both manufacturer and retailer get better off in the cost-sharing setting than in the fully non-cooperative game. From Propositions 4 and 5, both firms invest more in the Stackelberg Scenario than in the Nash Scenario. Acting in this way together, they reach a larger market size. As for the vertical integration, we can conclude that the shadow price of goodwill (α_i^{J} with $i = M, R$ and $j =$N, S, J) in Scenario J is the greatest for the manufacturer and the smallest for the retailer than in the other two settings. Since the shadow price represents the increase of the payoff when increasing the initial goodwill by one unit, when the initial goodwill level is sufficiently high, vertical integration would be more preferred by the manufacturer and less preferred by the retailer compared to any other program.

Remark 1. *Although $\Phi^{\mathrm{J}} = 0$ seems to be the natural choice in the cooperative setting (agents share profits, not costs), there are situations in which it could make sense to consider $\Phi^{\mathrm{J}} = \Phi^{\mathrm{S}}$. Take, for instance, the case of a supply chain with an active cost sharing program ($\Phi^{\mathrm{S}} > 0$) that is looking at the feasibility of carrying out a vertical merger. It can be checked (see the Appendix) that, for $\Phi^{\mathrm{J}} = \Phi^{\mathrm{S}}$ and $\pi \in (\frac{1}{3}, 1)$, the manufacturer's strategies, the retailer's strategies and the shadow prices are ranked as:*

1. $A_M^{\mathrm{S}} < A_M^{\mathrm{J}}$.
2. $A_R^{\mathrm{S}} < A_R^{\mathrm{J}}$.
3. $\alpha_M^{\mathrm{S}} < \alpha_M^{\mathrm{J}}$ *and* $\alpha_R^{\mathrm{J}} < \alpha_R^{\mathrm{S}}$.

Proof. See the Appendix. □

4.2. Existence of Group Inefficiency

In this section we study the possible existence of group inefficiency in the vertical integration setting. By group inefficiency we mean a situation in which the joint payment if players cooperate is smaller than the sum of the individual payoffs of all members in the coalition when no cooperation happens. By construction, in the case of equal discount rates, group inefficiency cannot appear. However, if discount rates are heterogeneous, the restriction to the search of time-consistent solutions in a cooperative framework can have as a price the loss of group efficiency. This property was illustrated with a simple example in Marín-Solano (2015). Note that, although utilities are transferable, in the case of group inefficiency, no side payments exist such that all players can get, at least, what they obtain in the non-cooperative framework.

Since, from Proposition 6, payments for both players in Scenario S (where we are considering a partial cooperation via Φ^{S}) are higher than those in Scenario N, we center our analysis on the more demanding comparison between Scenario N and Scenario J. *A priori*, it is hard to conclude the overall effect of vertical integration: the manufacturer benefits more from the high initial goodwill level in Scenario J than in Scenario N, whereas the contrary happens to the retailer.

Hence, we proceed to check if the following relation holds:

$$
\begin{aligned}
&\left(V_M^{\mathrm{J}}(G)+V_R^{\mathrm{J}}(G)\right)-\left(V_M^{\mathrm{N}}(G)+V_R^{\mathrm{N}}(G)\right)\\
&=-\frac{\gamma^2\pi^2(\rho_m-2\rho_r-\delta)}{2c_r(\rho_m+\delta)(\rho_r+\delta)}G+\begin{cases}\Delta(\pi) & \text{if} \quad \alpha_M^{\mathrm{J}}+\alpha_R^{\mathrm{J}}\geq 0\\ -\frac{k_m^2}{c_m}\left[\frac{(\alpha_M^{\mathrm{N}})^2}{2\rho_m}+\frac{\alpha_M^{\mathrm{N}}\alpha_R^{\mathrm{N}}}{\rho_r}\right] & \text{if} \quad \alpha_M^{\mathrm{J}}+\alpha_R^{\mathrm{J}}<0\end{cases}\\
&<0\,,
\end{aligned}
\tag{20}
$$

where α_M^{N}, α_R^{N}, α_M^{J} and α_R^{J} are defined in Proposition 1 and 3, and

$$
\begin{aligned}
\Delta(\pi)=&\ \frac{k_m^2}{8c_mc_r^2\rho_m\rho_r(\rho_m+\delta)^2(\rho_r+\delta)^2}(a\pi^4+b\pi^3+c\pi^2+d\pi+e)\ \text{, with}\\
&a=4\gamma^4(\rho_m-\rho_r)(\rho_r+\delta)(\rho_m+\rho_r+\delta)\ ,\\
&b=-4\gamma^2(\gamma^2+c_r\mu)(\rho_r+\delta)\left[3\rho_m(\rho_m+\delta)-2\rho_r(\rho_r+\delta)\right]\ ,\\
&c=4(\rho_m+\delta)\left[c_r^2\mu^2(\rho_m+\delta)(2\rho_m-\rho_r)+\gamma^2(\gamma^2+2c_r\mu)(2\rho_m^2+3\rho_m\delta-\rho_r\delta)\right]\ ,\\
&d=-4(\gamma^2+c_r\mu)(\gamma^2+2c_r\mu)(2\rho_m-\rho_r)(\rho_m+\delta)^2\ ,\\
&e=(\gamma^2+2c_r\mu)^2(\rho_m+\delta)^2(2\rho_m-\rho_r)\ .
\end{aligned}
\tag{21}
$$

In the case of identical time preferences, $\rho_m=\rho_r=\rho$, given that the set of non-cooperative strategies is included in the set of time-consistent strategies, the total outcome of joint maximization is always equal or larger than that of Nash competition case. Nevertheless, when agents exhibit divergent discount rates, the aggregated time preferences become time-inconsistent and it could happen that (20) holds. Specifically, it becomes clear that, if $\rho_m>2\rho_r+\delta$, for any initial goodwill level, the coalition is inefficient when parameters are such that $\alpha_M^{\mathrm{J}}+\alpha_R^{\mathrm{J}}<0$, which implies zero manufacturer's advertising;

Table 1: Benchmark Parameter Setting

ρ_m	ρ_r	δ	c_m	c_r	k_m	μ	γ	θ
0.15	0.03	0.03	2	2	1	1	1	1

a sufficiently high initial goodwill level will also give rise to group inefficiency when $\alpha_M^{\mathrm{J}} + \alpha_R^{\mathrm{J}} \geq 0$, independently on the sign of $\Delta(\pi)$. Moreover, since $\Delta(0) < 0$ if $\rho_r > 2\rho_m$, group inefficiency can also happen when the initial goodwill level and the revenue sharing rate are sufficiently small.

We provide some numerical illustrations to throw light on the existence of group inefficiency. We confine our interest to the case when $\rho_m > 2\rho_r + \delta$ and $\alpha_M^{\mathrm{J}} + \alpha_R^{\mathrm{J}} \geq 0$. Accordingly, if $\Delta(\pi)$ in (20) takes a negative value, the cooperation is group inefficient no matter how the initial goodwill level is. Under this parameter setting, we have $\Delta(0) > 0$ and $\Delta(1) < 0$[2], which assures that there exists solution(s) of $\Delta(\pi) = 0$ for $\pi \in (0, 1)$. Due to the complexity of the quartic function $\Delta(\pi)$, it is difficult to prove formally the uniqueness of the solution in the interval $(0, 1)$. But after running many simulations, we found that there exists $\hat{\pi} \in (0, 1)$ such that $\Delta(\pi) > 0$ for all $\pi \in (0, \hat{\pi})$ and $\Delta(\pi) < 0$ for all $\pi \in (\hat{\pi}, 1)$.

In Table 1 we summarize the parameter values used as the benchmark case (corresponding to the solid line in all the figures). For simplicity the effectiveness parameters k_m, μ and γ are normalized to 1, and the cost parameters c_m, c_r are set to be 2. This benchmark sample is somehow symmetric, except for the agents' time preferences. Furthermore, by altering one single parameter value from the benchmark case, we have conducted some sensitivity analysis of how each parameter can affect the interval $(\hat{\pi}, 1)$ such that $\Delta(\pi) < 0$, which to some extent measures how likely an inefficient cooperation is to happen, as well as the group inefficiency level.

Some conclusions can be drawn directly from (20) and (21). Although the baseline sales θ, the manufacturer's advertising effectiveness k_m and cost parameter c_m determine the players' strategies and payoffs in both scenarios N and J, they do not affect the group inefficiency likelihood. However, for any given initial goodwill level, more effective (larger k_m) and/or less costly (smaller c_m) global advertising will imply a higher group inefficiency level.

Figures 1 to 6 represent the sensitivity analysis of ρ_m, ρ_r, δ, c_r, μ and γ, respectively[3]. As shown in all the figures, there exists group inefficiency for high levels of π. As explained previously, in the vertical integration setting, the retailer's advertising level is independent of the revenue sharing rate. A small participation on revenues may induce the retailer to earn less than what she spends and, as a result, she suffers a great loss of profit compared to the Nash setting. When the retailer loses so much that the improvement of the manufacturer's utility can not compensate, the group inefficiency arises. One may

[2]Proof: see the Appendix.

[3]Please notice that in order to show with more details the group inefficiency, we do not present the whole range of $\pi \in (0, 1)$. However, $\Delta(\pi)$ is strictly decreasing in the omitted interval.

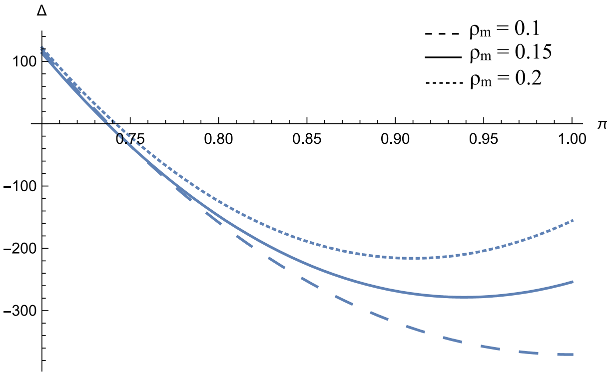

Figure 1: Group Inefficiency (ρ_m)

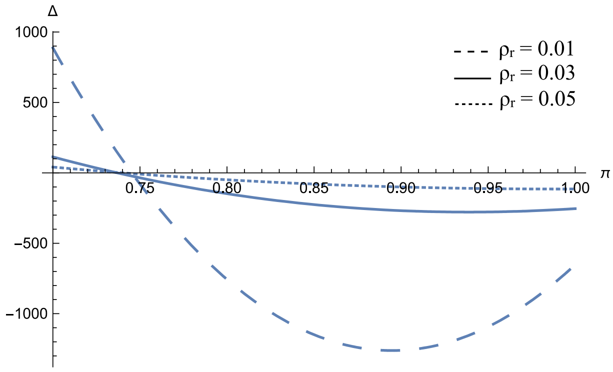

Figure 2: Group Inefficiency (ρ_r)

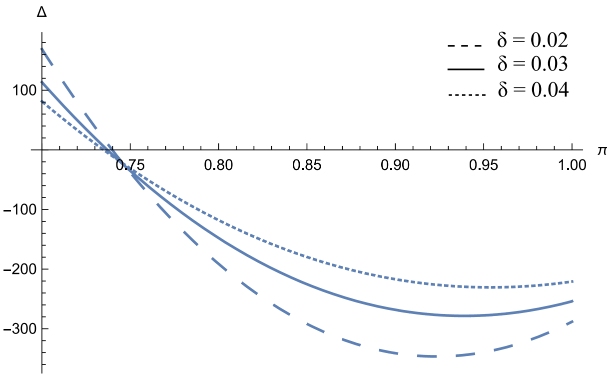

Figure 3: Group Inefficiency (δ)

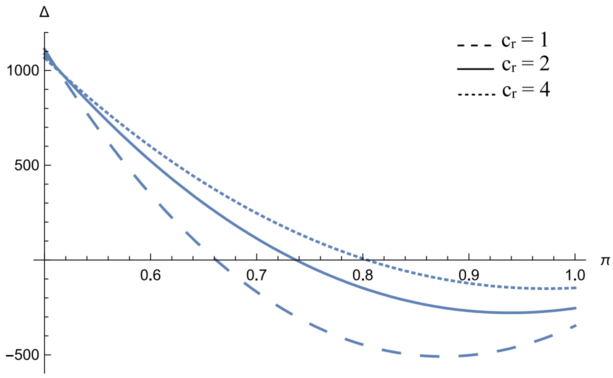

Figure 4: Group Inefficiency (c_r)

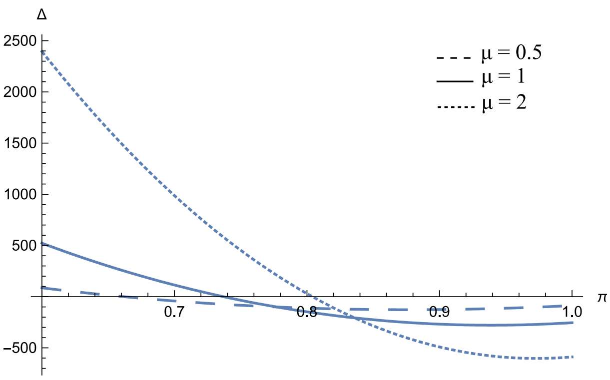

Figure 5: Group Inefficiency (μ)

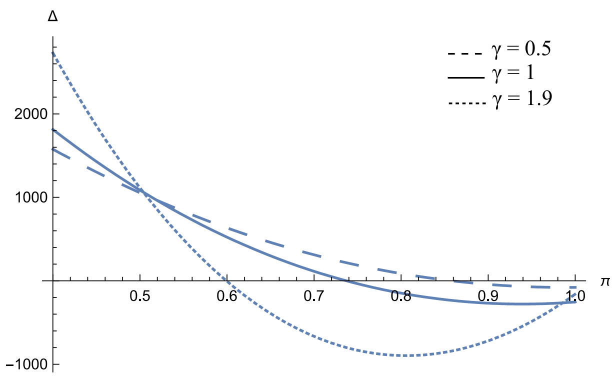

Figure 6: Group Inefficiency (γ)

Table 2: Values of $\hat{\pi}$ and π^{**} under Different Parameters Setting

	Figure 1			Figure 2		
	$\rho_m = 0.1$	$\rho_m = 0.15$	$\rho_m = 0.2$	$\rho_r = 0.01$	$\rho_r = 0.03$	$\rho_r = 0.05$
$\hat{\pi}$	0.737676	0.736966	0.741752	0.74525	0.736966	0.740058
π^{**}	0.981557	0.929286	0.904469	0.892313	0.929286	0.973828
	Figure 3			Figure 4		
	$\delta = 0.02$	$\delta = 0.03$	$\delta = 0.04$	$c_r = 1$	$c_r = 2$	$c_r = 4$
$\hat{\pi}$	0.739746	0.736966	0.734898	0.661669	0.736966	0.80407
π^{**}	0.915505	0.929286	0.942401	0.87868	0.929286	0.961652
	Figure 5			Figure 6		
	$\mu = 0.5$	$\mu = 1$	$\mu = 2$	$\gamma = 0.5$	$\gamma = 1$	$\gamma = 1.9$
$\hat{\pi}$	0.661669	0.736966	0.80407	0.857932	0.736966	0.599279
π^{**}	0.87868	0.929286	0.961652	0.980016	0.929286	0.823833

notice that joint payoffs are much higher when π is extremely small. This phenomenon derives from the fact that the manufacturer's advertising is decreasing in π for $\rho_m > \rho_r$. A smaller value of π would imply a higher national advertising level and this is beneficial to the whole supply chain. However, in our model setting, as the manufacturer has larger influence on the market, extremely small π would be less realistic.

Table 2 gives the parameter values we used for sensitivity analysis, and the corresponding $\hat{\pi}$ and π^{**}[4]. It is clear that in all of the cases, $\hat{\pi} < \pi^{**}$. For a revenue sharing rule $\hat{\pi} < \pi < \pi^{**}$, both manufacturer and retailer exert higher advertising effort in vertical integration than in Nash, however, this does not yield a higher joint payoffs, as one may expect. For $\pi > \pi^{**}$, we have $A_M^{\mathrm{J}} < A_M^{\mathrm{N}}$, implying a slower goodwill accumulation, and the group inefficiency follows. Moreover, smaller c_r and μ, greater δ and γ would induce a larger interval $(\hat{\pi}, 1)$[5]. As a result, group inefficiency is more likely to happen when the retailer's advertising is less costly, the synergistic effect of $A_R\sqrt{G}$ is stronger, the goodwill depreciates faster, or the goodwill's single contribution to revenue is trivial. The effects of ρ_m and ρ_r on $\hat{\pi}$ are less clear and we do not observe a straightforward relationship. They mainly act as the sufficient condition for the existence of group inefficiency.

Regarding the group inefficiency level, as we can see, the curves' intersection can be found in $(0, \hat{\pi})$ (as in Figure 6) and in $(\hat{\pi}, 1)$ (Figure 5). Besides, $\Delta(\pi)$ is not always monotonic in π and the local minimum can be in the interval $(0, 1)$ (see, for example, Figure 2). Moreover, the parameters' effects are subject to the initial goodwill level. All these properties substantially increase the complexity of

[4]If $\rho_m > 2\rho_r + \delta$, $A_M^{\mathrm{J}} > A_M^{\mathrm{N}}$ for all $\pi \in (0, \pi^{**})$, and $A_M^{\mathrm{J}} < A_M^{\mathrm{N}}$ for all $\pi \in (\pi^{**}, 1)$. For the definition of π^{**} and more details, please check Proposition 4.

[5]We have run more simulations to confirm this interrelation.

the sensitivity analysis, and there is no clear conclusion about under what circumstances a higher group inefficiency level would be generated.

4.3. Discussion on the Effects of the Cost and Revenue Sharing Rates

In our model, we have assumed that the revenue sharing rate is exogenously given. This is in agreement with previous literature on the topic and, also, with most of market conditions: each agent obtains a given and previously known percentage of benefits from sales. As for the cost sharing rate, in the vertical integration setting, we have computed the optimal solution for every possible value of Φ^{J}, which is assumed to be constant and exogenous. The aim of this section is to analyze what are the effects of the cost and revenue sharing rates if we relax these assumptions.

4.3.1. Cost Sharing Rate

In Scenario N (which coincides with Scenario S if cost sharing is not allowed), it is natural so assume that $\Phi^{\mathrm{N}} = 0$. In Scenario S (cost sharing), Φ^{S} is a decision variable of the manufacturer. As illustrated in Proposition 2, if π is not too small, it is profitable for the manufacturer to support part of the retailer's advertising cost. When we move to the vertical integration scenario, if discount rates are equal, the joint effect of Φ^{J} is null, as expected. However, this is not the case if discount rates are different, as illustrated in Equation (14). Different cost sharing rates will give rise to different values of $V_M^{\mathrm{J}} + V_R^{\mathrm{J}}$ (see Proposition 3). Hence, a natural question emerges in this context: if, in Scenario J, Φ^{J} is treated as a decision variable of the coalition, when computing the time-consistent cooperative solution, is there an optimal value of Φ^{J}? An inspection of Equation (34) shows that the answer is negative: Φ^{J} does not appear in the right hand term of that expression. Therefore, the solution to the problem (14) by including Φ^{J} as a decision variable is again given just by Proposition 3. There is an infinite number (a continuum of them) of cooperative equilibria in Scenario J, each of them corresponds to a different $\Phi^{\mathrm{J}} \in [0, 1]$, giving rise to a different steady state. Such multiplicity of equilibria is a property well-documented in problems with time inconsistent preferences in an infinite horizon setting. In any case, it is interesting to study if one of these equilibria provide higher payments to the joint coalition or not. In the following, we analyze the results in Proposition 3 with respect to the changes in the cost sharing rate Φ^{J}.

First, notice that α_M^{J} is decreasing in Φ^{J}, whereas α_R^{J} increases in Φ^{J}. If we express the advertising effort of the manufacturer as a function of the cost sharing rate, $A_M^{\mathrm{J}}(\Phi^{\mathrm{J}})$, it is straightforward to check that, if $\alpha_M^{\mathrm{J}} + \alpha_R^{\mathrm{J}} > 0$ (which is the more interesting case), then $A_M^{\mathrm{J}}(\Phi^{\mathrm{J}})$ is increasing if $\rho_m > \rho_r$ and decreasing for $\rho_m < \rho_r$. On the contrary, A_R^{J} (and A_M^{J} for $\alpha_M^{\mathrm{J}} + \alpha_R^{\mathrm{J}} < 0$) are independent of Φ^{J}.

Next, we can analyze if there is a value of Φ^{J} providing a solution that is Pareto superior to the others. It is easy to check that the answer is, in general, negative.

Finally, let us study the effects of Φ^{J} on the joint payoffs. We can distinguish three different situations:

1. In the less interesting case when $\alpha_M^{\mathrm{J}} + \alpha_R^{\mathrm{J}} < 0$, since
$$\frac{\partial}{\partial \Phi^{\mathrm{J}}} \left(\alpha_M^{\mathrm{J}} + \alpha_R^{\mathrm{J}} \right) = \frac{\gamma^2 (\rho_m - \rho_r)}{2 c_r (\rho_m + \delta)(\rho_r + \delta)} \ ,$$
then, from Equations (17) and (18), if $\rho_m > \rho_r$, joint payments are higher if the manufacturer finances the whole cost of retailer advertising. On the contrary, if $\rho_m < \rho_r$, it will be profitable for the coalition that the retailer covers the totality of her advertising cost.
2. If $\alpha_M^{\mathrm{J}} > 0$ and $\alpha_R^{\mathrm{J}} > 0$, let $V_M^{\mathrm{J}} + V_R^{\mathrm{J}} = \alpha^{\mathrm{J}} G + \beta^{\mathrm{J}}$, where $\alpha^{\mathrm{J}} = \alpha_M^{\mathrm{J}} + \alpha_R^{\mathrm{J}}$ and β^{J} contains all the terms in Equations (17) and (18) not depending on G, after several calculations, it can be proved that
   - If $\rho_m > \rho_r$, then $\dfrac{\partial \alpha^{\mathrm{J}}}{\partial \Phi^{\mathrm{J}}} > 0$ and $\dfrac{\partial \beta^{\mathrm{J}}}{\partial \Phi^{\mathrm{J}}} > 0$.
   - If $\rho_m < \rho_r$, then $\dfrac{\partial \alpha^{\mathrm{J}}}{\partial \Phi^{\mathrm{J}}} < 0$ and $\dfrac{\partial \beta^{\mathrm{J}}}{\partial \Phi^{\mathrm{J}}} < 0$.

   As a result, independently of the initial goodwill level G, joint payments are higher for $\Phi^{\mathrm{J}} = 1$ if $\rho_m > \rho_r$ and, on the contrary, if $\rho_m < \rho_r$, it is better for the coalition to take $\Phi^{\mathrm{J}} = 0$.
3. The situation becomes much more complicated if $\alpha_M^{\mathrm{J}} + \alpha_R^{\mathrm{J}} > 0$ but α_M^{J} or α_R^{J} is strictly negative. In such a case, the value of Φ^{J} maximizing $V_M^{\mathrm{J}} + V_R^{\mathrm{J}}$ can be at any point in the interval $[0, 1]$. If the optimal solution is interior, then the "optimal" value of Φ^{J} will be, in general, a linear function of G. But the goodwill level evolves along time. Therefore, there is not a (constant) value of Φ^{J} maximizing the joint payments. In addition, by taking Φ^{J} as a linear function of G, say $\Phi^{\mathrm{J}} = aG + b \in [0, 1]$, with the idea of looking for later on the values of the parameters a and b maximizing the joint payments, we lose linearity of the decision rule for the manufacturer and the quadratic structure of the value functions. It is unclear how this problem could be solved.

Summarizing, in the search of time-consistent equilibria in the vertical integration setting, if the cost sharing rate is treated as a decision variable, there is a continuum of solutions obtained for different values of Φ^{J}. This property, that seems to be surprising, is one of the effects of introducing time inconsistent preferences. It is not possible, in general, to select a particular value of Φ^{J} giving rise to an equilibrium Pareto dominating the others. As for the joint payments, in some cases (the second situation we discussed above), it is possible to identify a value $\Phi^{\mathrm{J}} \in \{0, 1\}$ maximizing the joint payments. This appropriate selection of an "optimal" (when there exists) value of the cost sharing rate can mitigate (but not completely eliminate, depending on the values of the parameters) the group inefficiency effect. This result can be checked from an inspection of the difference $(V_M^{\mathrm{J}}(G) + V_R^{\mathrm{J}}(G)) - (V_M^{\mathrm{N}}(G) + V_R^{\mathrm{N}}(G))$ for all values of Φ^{J}. In particular, the linear term in G in Equation (20) becomes
$$\frac{\gamma^2}{2 c_r (\rho_m + \delta)(\rho_r + \delta)} \left[(\rho_m - \rho_r) \Phi^{\mathrm{J}} - (\rho_m - 2\rho_r - \delta) \pi^2 \right] \ .$$
For every Φ^{J} there are values of the parameters ρ_m, ρ_r, π and δ guaranteeing that the expression above is negative, so if the goodwill level G is high enough, there will be group inefficiency for every $\Phi^{\mathrm{J}} \in [0, 1]$.

4.3.2. Revenue Sharing Rate

In Scenarios N and S, if the retailer can decide the revenue sharing rate, she will take it at its lowest possible value (Equations (26) and (30)). On the contrary, if this variable is decided by the manufacturer, in Scenario N, she will take it at its highest value (Equation (25)). The situation is less clear in the cost sharing scenario, if π can be decided by the manufacturer. An inspection to Equation (31) shows that, in that case, the maximum of the right hand term in π is achieved when

$$\pi^* = \frac{1}{2 - \Phi^{\mathrm{S}}} \left[1 + \frac{\mu c_r \left(1 - \Phi^{\mathrm{S}}\right)^2}{\gamma^2} + \frac{c_r \left(1 - \Phi^{\mathrm{S}}\right)^2 \theta}{\gamma^2 G} \right] .$$

If $\pi^* < 1$, then it will be profitable for the manufacturer to choose this revenue sharing rate. However, it can be shown that this can not happen for the values of Φ^{S^*}. Hence, as in Scenario N, the manufacturer will choose the maximum possible revenue sharing rate.

With respect to the vertical integration scenario, we obtain similar results to the previous ones on the cost sharing rate: if the revenue sharing rate is treated as a decision variable, there is a continuum of time-consistent cooperative equilibria obtained for all the different values of π. It is also not possible, in general, to select a particular value of π providing an equilibrium Pareto dominating the others. As for the joint payments, the discussion follows the similar patterns as in that of the cost sharing rate.

5. Concluding Remarks

In this paper, we have considered a differential game of advertising in a two-echelon supply chain. We have contributed to the literature by introducing two extensions. First of all, we allow for the possibility that each chain member could exhibit distinct time discount rates, which could result in a trade-off between efficiency and time-consistency. Besides, we have extended the sales model by combining the separate effect of goodwill, the synergistic effect of goodwill and retailer's advertising, and the decreasing marginal returns to goodwill. We have characterized the feedback Nash equilibrium, the stage-wise Stackelberg equilibrium and the time-consistent cooperative solution for scenarios of non-cooperation, cost-sharing program and vertical integration, respectively. We have made a detailed comparison of the advertising strategies and outcomes among different cooperative and non-cooperative settings.

Our results reveal that when the manufacturer is the leader and the retailer has limited influence on the sales, the cost-sharing program will be implemented if the revenue sharing rate of the retailer is not much larger than that of the manufacturer. Both members of the supply chain increase their advertising budget when the cost-sharing program is applied, which generates a bigger market size, thus implying a Pareto superior outcome to the one under non-cooperation. Similar results can be found in the literature (for instance, Jørgensen et al., 2000, 2001b, 2003; Buratto et al., 2007). Recall that, by modifying the sales function, we have enlarged the difference of the marketing influence of each member in the supply chain. As in the other leader-follower games where all agents' marketing activities affect goodwill and/or sales, we also find that the necessary condition for giving positive advertising support is related to the margins of each member, and the Stackelberg payments are Pareto superior to Nash payments.

Under vertical integration, when the two agents differ in their time preference rates but act in a time-consistent way, depending on the parameters of the model, the manufacturer's advertising rate may be zero. If we focus on the parameter set such that the manufacturer's investment is positive, it depends on the revenue and cost sharing rates. This result differs from that of the standard case, where agents have the same discount rate. Moreover, even the payoffs of the grand coalition is affected by these two side payments.

The most novel result of our study is related to the efficiency analysis. There is a consensus that a centralized channel is more efficient, according to all the studies up to date. Nonetheless, under the hypothesis of asymmetric time-discounting, the vertical integration scenario does not necessarily yield a better outcome for the coalition. Particularly, we find that if the retailer is much more impatient than the manufacturer, group inefficiency emerges when both initial goodwill level and revenue sharing rate are small. On the contrary, if the manufacturer's discount rate is much greater than that of the retailer, there exists group inefficiency for all levels of initial goodwill when the revenue sharing rate does not prioritize the retailer. We also observe that, for the latter case, the group inefficiency likelihood is increased by the retailer's advertising's higher cost-effectiveness and larger contribution to revenue, as well as the goodwill's higher depreciation rate and smaller influence on revenue; whereas the group inefficiency level is elevated by the manufacturer's better advertising performance (more effective and less costly).

We believe that our research could be a useful aid for managers to decide whether to cooperate and which coordination mechanism to choose. From our model, a co-op advertising program is promising and offers mutual prosperity. Consequently, we would advise the manufacturer to subsidize the retailer's advertising campaigns, as long as she has advantage in revenue sharing. Furthermore, our findings suggest that the decision makers should take into account the possible divergence in the discount rates (for instance, due to differences in firm size, financial health, legislative restrictions, crisis intensity rate, and so on) between the two entities when considering a vertical integration in the form of a coalition, merger, acquisition, etc. A channel centralization is not recommended if the retailer is shortsighted and takes large part of the revenue, unless the brand is well known. Similarly, this coordination mechanism is not advantageous for a supply chain consisting of one relatively myopic manufacturer, i.e., with a high discount rate, and one farsighted retailer when the revenue sharing rule does not favor the retailer.

Appendix

Proof of Lemma 1. First, note that, if $A_M(s) = A_m G(s) + B_m$, for all $s \in [t, \infty)$, $t \geq 0$, then the solution to $\dot{G}(s) = k_m A_M(s) - \delta G(s)$, with initial condition $G(t) = G$, is

$$G(s) = \left(G - \frac{k_m A_m}{\delta - A_m k_m} \right) e^{-(\delta - A_m k_m)(s-t)} + \frac{k_m B_m}{\delta - A_m k_m} \ . \tag{22}$$

Next, if $A_M(s) = A_m G(s) + B_m$ and $A_R(s) = A_r \sqrt{G(s)}$, then

$$V_M(G) = \int_t^\infty e^{-\rho_m (s-t)} \left\{ \pi \left[\theta + (\mu + \gamma A_r) G(s) \right] - \frac{c_m}{2} (A_m G(s) + B_m)^2 - \frac{c_r}{2} \Phi(s) (A_r)^2 G(s) \right\} ds \tag{23}$$

and

$$V_R(G) = \int_t^\infty e^{-\rho_r (s-t)} \left\{ (1-\pi) \left[\theta + (\mu + \gamma A_r) G(s) \right] - \frac{c_r}{2} (1 - \Phi(s)) (A_r)^2 G(s) \right\} ds \ . \tag{24}$$

The result follows by substituting equation (22) in (23) and (24). □

Proof of Proposition 1. Denoting $V_M^{\mathrm{N}}(G)$, $V_R^{\mathrm{N}}(G)$ the value functions of the manufacturer and the retailer respectively, the Hamilton-Jacobi-Bellman (HJB) equations are

$$\rho_m V_M^{\mathrm{N}} = \max_{\{A_M^{\mathrm{N}} \geq 0\}} \left\{ \pi \left(\theta + \mu G + \gamma A_R^{\mathrm{N}} \sqrt{G} \right) - \frac{c_m}{2} (A_M^{\mathrm{N}})^2 + (V_M^{\mathrm{N}})' (k_m A_M^{\mathrm{N}} - \delta G) \right\} \ , \tag{25}$$

$$\rho_r V_R^{\mathrm{N}} = \max_{\{A_R^{\mathrm{N}} \geq 0\}} \left\{ (1-\pi) \left(\theta + \mu G + \gamma A_R^{\mathrm{N}} \sqrt{G} \right) - \frac{c_r}{2} (A_R^{\mathrm{N}})^2 + (V_R^{\mathrm{N}})' (k_m A_M^{\mathrm{N}} - \delta G) \right\} \ . \tag{26}$$

Assuming an interior solution, maximizing the right-hand sides of these two equations yields $A_M^{\mathrm{N}*} = k_m (V_M^{\mathrm{N}})'/c_m$ and $A_R^{\mathrm{N}*} = (1-\pi)\gamma\sqrt{G}/c_r$. Note that the strategy of the retailer is already fixed. As a result, when substituting it in the objective functional of the manufacturer (3), we obtain a standard linear-quadratic optimal control problem, whose unique solution is known to be linear. For these strategies, from Lemma 1 we know that, in such a case, value functions must be of the form $V_M^{\mathrm{N}}(G) = (\lambda_M^{\mathrm{N}}/2) G^2 + \alpha_M^{\mathrm{N}} G + \beta_M^{\mathrm{N}}$, $V_R^{\mathrm{N}}(G) = \alpha_R^{\mathrm{N}} G + \beta_R^{\mathrm{N}}$. Substituting $A_M^{\mathrm{N}*}$, $A_R^{\mathrm{N}*}$, together with $V_M^{\mathrm{N}}(G)$, into (25), we obtain

$$\rho_m \left(\frac{\lambda_M^{\mathrm{N}}}{2} G^2 + \alpha_M^{\mathrm{N}} G + \beta_M^{\mathrm{N}} \right) = \pi \left(\theta + \mu G + \frac{\gamma^2 (1-\pi)}{c_r} G \right) \tag{27}$$

$$- \frac{k_m^2}{2 c_m} \left((\lambda_M^{\mathrm{N}})^2 G^2 + 2 \lambda_M^{\mathrm{N}} \alpha_M^{\mathrm{N}} G + (\alpha_M^{\mathrm{N}})^2 \right) + (\lambda_M^{\mathrm{N}} G + \alpha_M^{\mathrm{N}}) \left(\frac{k_m^2 (\lambda_M^{\mathrm{N}} G + \alpha_M^{\mathrm{N}})}{c_m} - \delta G \right) \ .$$

By identifying the terms in G^2 in the equation above we obtain

$$\frac{\rho_m}{2} \lambda_M^{\mathrm{N}} = \frac{(\lambda_M^{\mathrm{N}})^2 k_m^2}{2 c_m} - \lambda_M^{\mathrm{N}} \delta \ ,$$

which has two solutions: $\lambda_M^{\mathrm{N}} = 0$ and $\lambda_M^{\mathrm{N}} = (\rho_m + 2\delta) c_m / (k_m)^2$. We analyze first the existence of a feedback Nash equilibrium in constant strategies for the manufacturer. For $\lambda_M^{\mathrm{N}} = 0$, after rearranging terms, equations (25) and (26) become

$$\left\{ \rho_m \alpha_M^{\mathrm{N}} - \frac{\pi \left[c_r \mu + (1-\pi) \gamma^2 \right]}{c_r} + \delta \alpha_M^{\mathrm{N}} \right\} G = -\rho_m \beta_M^{\mathrm{N}} + \pi \theta + \frac{k_m^2}{2 c_m} (\alpha_M^{\mathrm{N}})^2 \ , \tag{28}$$

$$\left\{\rho_r\alpha_R^{\mathrm{N}} - \frac{(1-\pi)\left[2c_r\mu + (1-\pi)\gamma^2\right]}{2c_r} + \delta\alpha_R^{\mathrm{N}}\right\} G = -\rho_r\beta_R^{\mathrm{N}} + (1-\pi)\theta + \frac{k_m^2}{c_m}\alpha_M^{\mathrm{N}}\alpha_R^{\mathrm{N}}\,. \tag{29}$$

It is straightforward to check that

$$\alpha_M^{\mathrm{N}} = \frac{c_r\pi\mu + \pi(1-\pi)\gamma^2}{c_r(\rho_m+\delta)}, \quad \alpha_R^{\mathrm{N}} = \frac{2c_r(1-\pi)\mu + (1-\pi)^2\gamma^2}{2c_r(\rho_r+\delta)},$$
$$\beta_M^{\mathrm{N}} = \frac{k_m^2}{2c_m\rho_m}(\alpha_M^{\mathrm{N}})^2 + \frac{\pi\theta}{\rho_m}, \quad \beta_R^{\mathrm{N}} = \frac{k_m^2}{c_m\rho_r}\alpha_M^{\mathrm{N}}\alpha_R^{\mathrm{N}} + \frac{(1-\pi)\theta}{\rho_r}$$

satisfy (28) and (29). It is straightforward to check that the sufficient transversality conditions $\lim_{t\to\infty} e^{-\rho_m t}V_i\left(G(t)\right) = 0$, $i = M, R$ is met (the solution converges to a steady state). Note also that, since $\alpha_M^{\mathrm{N}} > 0$, then $A_M^{\mathrm{N}*} > 0$, in agreement with our hypothesis concerning the existence of an interior solution. Finally, it can be checked that, for the other candidate $\lambda_M^{\mathrm{N}} = (\rho_m + 2\delta)c_m/(k_m)^2$, $\lim_{t\to\infty} e^{-\rho_m t}V_M\left(G(t)\right) = \infty$. □

Proof of Proposition 2. We solve the problem by backward induction. Denoting $V_M^{\mathrm{S}}(G)$, $V_R^{\mathrm{S}}(G)$ the value functions of the manufacturer and the retailer respectively, we start from determining the retailer's advertising strategies. The retailer's HJB equation is

$$\rho_r V_R^{\mathrm{S}} = \max_{\{A_R^{\mathrm{S}}\geq 0\}} \left\{(1-\pi)\left(\theta + \mu G + \gamma A_R^{\mathrm{S}}\sqrt{G}\right) - \frac{c_r}{2}(1-\Phi^{\mathrm{S}})(A_R^{\mathrm{S}})^2 + (V_R^{\mathrm{S}})'(k_m A_M^{\mathrm{S}} - \delta G)\right\}\ . \tag{30}$$

If $\Phi^{\mathrm{S}} \neq 1$, maximizing the right-hand side yields $A_R^{\mathrm{S}*} = \dfrac{(1-\pi)\gamma\sqrt{G}}{c_r(1-\Phi^{\mathrm{S}})}$. Substituting $A_R^{\mathrm{S}*}$ into the manufacturer's HJB equation we obtain

$$\begin{aligned}\rho_m V_M^{\mathrm{S}} = \max_{\{A_M^{\mathrm{S}}\geq 0,\, 0\leq\Phi^{\mathrm{S}}\leq 1\}} &\left\{\pi\left[\theta + \mu G + \frac{(1-\pi)\gamma^2}{c_r(1-\Phi^{\mathrm{S}})}G\right] - \frac{c_m}{2}(A_M^{\mathrm{S}})^2 - \frac{(1-\pi)^2\gamma^2\Phi^{\mathrm{S}}}{2c_r(1-\Phi^{\mathrm{S}})^2}G\right.\\ &\left. + (V_M^{\mathrm{S}})'(k_m A_M^{\mathrm{S}} - \delta G)\right\}\ .\end{aligned} \tag{31}$$

The manufacturer's strategies are derived by maximizing the right-hand side of (31), whose result is, in the case of interior solutions,

$$A_M^{\mathrm{S}*} = \frac{k_m}{c_m}(V_M^{\mathrm{S}})', \quad \Phi^{\mathrm{S}*} = \begin{cases} \frac{3\pi-1}{\pi+1} & \text{if} \quad \frac{1}{3} < \pi < 1, \\ 0 & \text{if} \quad 0 < \pi \leq \frac{1}{3}. \end{cases}$$

In accordance with our hypothesis, $\Phi^{\mathrm{S}*} \neq 1$, since $\Phi^{\mathrm{S}*} = 1$ only happens when $\pi = 1$. Note also that when $0 < \pi \leq \frac{1}{3}$, the outcome is consistent with that of scenario N. In the case of $\frac{1}{3} < \pi < 1$, since the strategy of the retailer is already fixed, from (3) and (1) we have to look for the linear solution of the corresponding linear-quadratic optimal control problem for the manufacturer. From Lemma 1, value functions are of the form $V_M^{\mathrm{S}}(G) = (\lambda_M^{\mathrm{S}})G^2 + \alpha_M^{\mathrm{S}}G + \beta_M^{\mathrm{S}}$, $V_R^{\mathrm{S}}(G) = \alpha_R^{\mathrm{S}}G + \beta_R^{\mathrm{S}}$.

Substituting $A_M^{\mathrm{S}*}$, $A_R^{\mathrm{N}*}$, together with $V_M^{\mathrm{N}}(G)$, into (31), we obtain

$$\rho_m\left(\frac{\lambda_M^{\mathrm{S}}}{2}G^2 + \alpha_M^{\mathrm{S}}G + \beta_M^{\mathrm{S}}\right) = \pi\left(\theta + \mu G + \frac{\gamma^2(1-\pi)}{c_r(1-\phi^{\mathrm{S}*})}G\right)$$

$$-\frac{k_m^2}{2c_m}\left((\lambda_M^{\mathrm{S}})^2G^2+2\lambda_M^{\mathrm{S}}\alpha_M^{\mathrm{S}}G+(\alpha_M^{\mathrm{S}})^2\right)-\frac{(1-\pi)^2\gamma^2\Phi^{\mathrm{S}*}}{2c_r(1-\Phi^{\mathrm{S}*})^2}G+(\lambda_M^{\mathrm{S}}G+\alpha_M^{\mathrm{S}})\left(\frac{k_m^2(\lambda_M^{\mathrm{S}}G+\alpha_M^{\mathrm{S}})}{c_m}-\delta G\right)\ .$$

By identifying the terms in G^2 in the equation above we obtain

$$\frac{\rho_m}{2}\lambda_M^{\mathrm{S}}=\frac{(\lambda_M^{\mathrm{S}})^2k_m^2}{2c_m}-\lambda_M^{\mathrm{S}}\delta\ ,$$

which has two solutions: $\lambda_M^{\mathrm{S}}=0$ and $\lambda_M^{\mathrm{S}}=(\rho_m+2\delta)c_m/k_m^2$. So, in particular, there exists a feedback Stackelberg equilibrium in constant strategies for the manufacturer. In order to compute this equilibrium, we substitute, for $\lambda_M^{\mathrm{S}}=0$, $A_R^{\mathrm{S}*}$, $A_M^{\mathrm{S}*}$ and $\Phi^{\mathrm{S}*}$ into (30) and (31) to yield

$$\left[\rho_r\alpha_R^{\mathrm{S}}-(1-\pi)\mu-\frac{(1-\pi^2)\gamma^2}{4c_r}+\delta\alpha_R^{\mathrm{S}}\right]G=-\rho_r\beta_R^{\mathrm{S}}+(1-\pi)\theta+\frac{k_m^2}{c_m}\alpha_M^{\mathrm{S}}\alpha_R^{\mathrm{S}}\ ,\tag{32}$$

$$\left[\rho_m\alpha_M^{\mathrm{S}}-\pi\mu-\frac{(1+\pi)^2\gamma^2}{8c_r}+\delta\alpha_M^{\mathrm{S}}\right]G=-\rho_m\beta_M^{\mathrm{S}}+\pi\theta+\frac{k_m^2}{2c_m}(\alpha_M^{\mathrm{S}})^2\ .\tag{33}$$

It is easy to check that

$$\alpha_M^{\mathrm{S}}=\frac{8c_r\pi\mu+(1+\pi)^2\gamma^2}{8c_r(\rho_m+\delta)},\quad \alpha_R^{\mathrm{S}}=\frac{4c_r(1-\pi)\mu+(1-\pi^2)\gamma^2}{4c_r(\rho_r+\delta)},$$
$$\beta_M^{\mathrm{S}}=\frac{k_m^2}{2c_m\rho_m}(\alpha_M^{\mathrm{S}})^2+\frac{\pi\theta}{\rho_m},\quad \beta_R^{\mathrm{S}}=\frac{k_m^2}{c_m\rho_r}\alpha_M^{\mathrm{S}}\alpha_R^{\mathrm{S}}+\frac{(1-\pi)\theta}{\rho_r}$$

satisfy (32) and (33). For this solution, it is straightforward to check that $\lim_{t\to\infty}e^{-\rho_m t}V_i\left(G(t)\right)=0$, $i=M,R$ is met (the solution converges to a steady state). These conditions are not satisfied by the other (nonlinear) decision rule for the manufacturer. Finally, notice that $\alpha_M^{\mathrm{S}}>0$ and then $A_M^{\mathrm{S}*}>0$, i.e., the solution is interior, as we had assumed. □

Proof of Proposition 3. We follow the approach of de Paz et al. (2013) to obtain the time-consistent equilibria. Denoting $V_M^{\mathrm{J}}(G)$, $V_R^{\mathrm{J}}(G)$ the value functions of the manufacturer and the retailer respectively, the Dynamic Programming Equation (DPE) of the coalition is

$$\begin{aligned}\rho_mV_M^{\mathrm{J}}+\rho_rV_R^{\mathrm{J}}=\max_{\{A_M^{\mathrm{J}}\ge0,\,A_R^{\mathrm{J}}\ge0\}}&\Big\{\theta+\mu G+\gamma A_R^{\mathrm{J}}\sqrt{G}-\frac{c_m}{2}(A_M^{\mathrm{J}})^2-\frac{c_r}{2}(A_R^{\mathrm{J}})^2\\&+((V_M^{\mathrm{J}})'+(V_R^{\mathrm{J}})')\left(k_mA_M^{\mathrm{J}}-\delta G\right)\Big\}\ .\end{aligned}\tag{34}$$

Maximization gives, in case of interior solution, $A_M^{\mathrm{J}*}=k_m((V_M^{\mathrm{J}})'+(V_R^{\mathrm{J}})')/c_m$, $A_R^{\mathrm{J}*}=\gamma\sqrt{G}/c_r$. As in the previous scenarios, we focus our attention in the existence of linear strategies for the manufacturer. Since $V_M^{\mathrm{J}}(G)=(\lambda_M^{\mathrm{J}})G^2+\alpha_M^{\mathrm{J}}G+\beta_M^{\mathrm{J}}$, $V_R^{\mathrm{J}}(G)=\alpha_R^{\mathrm{J}}G+\beta_R^{\mathrm{J}}$ (Lemma 1), the dynamic programming equation for the manufacturer becomes

$$\rho_m\left(\frac{\lambda_M^{\mathrm{J}}}{2}G^2+\alpha_M^{\mathrm{J}}G+\beta_M^{\mathrm{J}}\right)=\pi(\theta+\mu G+\gamma A_R^{\mathrm{J}*}\sqrt{G})-\frac{c_m}{2}(A_M^{\mathrm{J}*})^2-\frac{c_r}{2}\Phi(A_R^{\mathrm{J}*})^2+\alpha_M^{\mathrm{J}}\left[k_m^2A_M^{\mathrm{J}*}-\delta G)\right]$$

$$=\pi\left(\theta+\mu G+\frac{\gamma^2}{c_r}G\right)-\frac{k_m^2}{2c_m}(\lambda_M^{\mathrm{J}}G+\alpha_M^{\mathrm{J}}+\alpha_R^{\mathrm{J}})^2-\frac{\gamma^2\Phi}{2c_r}G+(\lambda_M^{\mathrm{J}}G+\alpha_M^{\mathrm{J}})\left[\frac{k_m^2}{c_m}(\lambda_M^{\mathrm{J}}G+\alpha_M^{\mathrm{J}}+\alpha_R^{\mathrm{J}})-\delta G)\right]\ .$$

By identifying the terms in G^2 in the equation above we obtain

$$\frac{\rho_m}{2}\lambda_M^{\mathrm{J}} = \frac{(\lambda_M^{\mathrm{J}})^2 k_m^2}{2c_m} - \lambda_M^{\mathrm{J}}\delta \, ,$$

which has two solutions: $\lambda_M^{\mathrm{J}} = 0$ and $\lambda_M^{\mathrm{J}} = (\rho_m + 2\delta)c_m/k_m^2$. Let us compute the time-consistent cooperative equilibrium in constant strategies for the manufacturer ($\lambda_M^{\mathrm{J}} = 0$). By substituting $A_M^{\mathrm{J}*}$ and $A_R^{\mathrm{J}*}$ into the individual DPEs, we obtain

$$\rho_m(\alpha_M^{\mathrm{J}}G+\beta_M^{\mathrm{J}}) = \pi\left(\theta + \mu G + \frac{\gamma^2}{c_r}G\right) - \frac{k_m^2}{2c_m}(\alpha_M^{\mathrm{J}}+\alpha_R^{\mathrm{J}})^2 - \frac{\gamma^2\Phi}{2c_r}G + \alpha_M^{\mathrm{J}}\left[\frac{k_m^2}{c_m}(\alpha_M^{\mathrm{J}}+\alpha_R^{\mathrm{J}}) - \delta G)\right] \, , \quad (35)$$

$$\begin{aligned}\rho_r(\alpha_R^{\mathrm{J}}G + \beta_R^{\mathrm{J}}) &= (1-\pi)(\theta + \mu G + \gamma A_R^{\mathrm{J}*}\sqrt{G}) - \frac{c_r}{2}(1-\Phi)(A_R^{\mathrm{J}*})^2 + \alpha_R^{\mathrm{J}}\left[k_m^2 A_M^{\mathrm{J}*} - \delta G)\right] \quad (36)\\ &= (1-\pi)\left(\theta + \mu G + \frac{\gamma^2}{c_r}G\right) - \frac{\gamma^2(1-\Phi)}{2c_r}G + \alpha_R^{\mathrm{J}}\left[\frac{k_m^2}{c_m}(\alpha_M^{\mathrm{J}}+\alpha_R^{\mathrm{J}}) - \delta G)\right] \, .\end{aligned}$$

By identifying terms, the coefficients are given by

$$\begin{aligned}\alpha_M^{\mathrm{J}} &= \frac{2c_r\pi\mu + (2\pi - \Phi)\gamma^2}{2c_r(\rho_m+\delta)}, \quad \alpha_R^{\mathrm{J}} = \frac{2c_r(1-\pi)\mu + (1+\Phi-2\pi)\gamma^2}{2c_r(\rho_r+\delta)},\\ \beta_M^{\mathrm{J}} &= \frac{k_m^2}{2c_m\rho_m}((\alpha_M^{\mathrm{J}})^2 - (\alpha_R^{\mathrm{J}})^2) + \frac{\pi\theta}{\rho_m}, \quad \beta_R^{\mathrm{J}} = \frac{k_m^2}{c_m\rho_r}(\alpha_M^{\mathrm{J}}\alpha_R^{\mathrm{J}} + (\alpha_R^{\mathrm{J}})^2) + \frac{(1-\pi)\theta}{\rho_r}.\end{aligned}$$

This is the solution when $\alpha_M^{\mathrm{J}} + \alpha_R^{\mathrm{J}} \geq 0$. Note that in this case (with $\lambda_M^{\mathrm{J}} = 0$) the solution converges to a steady state, as needed. On the contrary, it can be checked that the solution obtained for $\lambda_M^{\mathrm{J}} = (\rho_m + 2\delta)c_m/k_m^2$ does not converge to a steady state.

It remains to compute the corner solution. By reproducing the same calculations for $A_M^{\mathrm{J}*} = 0$, $A_R^{\mathrm{J}*} = \frac{\gamma\sqrt{G}}{c_r}$, from the individual DPEs we derive the same values of α_M^{J}, α_R^{J}, and $\beta_M^{\mathrm{J}} = \frac{\pi\theta}{\rho_m}$, $\beta_R^{\mathrm{J}} = \frac{(1-\pi)\theta}{\rho_r}$. □

Proof of Proposition 4.

1. We first compare A_M^{N} and A_M^{S}. From (5) and (9), it is straightforward to get $A_M^{\mathrm{N}} \leq A_M^{\mathrm{S}}$ (when $\pi \in (\frac{1}{3}, 1)$ a strict inequality holds).
2. Then we compare A_M^{S} and A_M^{J}. From (9) and (15), we obtain

$$A_M^{\mathrm{J}} - A_M^{\mathrm{S}} = \frac{k_m}{8c_mc_r(\rho_m+\delta)(\rho_r+\delta)}f_1(\pi),$$

where

$$\begin{aligned}f_1(\pi) =& -\gamma^2(\rho_r+\delta)\pi^2 + \left[6\gamma^2(\rho_r+\delta) - 8(\gamma^2 + c_r\mu)(\rho_m+\delta)\right]\pi\\ &+ 4(\gamma^2 + 2c_r\mu)(\rho_m+\delta) - \gamma^2(\rho_r+\delta) \, .\end{aligned}$$

Note that $f_1\left(\frac{1}{3}\right) = \frac{8}{9}\gamma^2(\rho_r+\delta) + \left(\frac{4}{3}\gamma^2 + \frac{16}{3}c_r\mu\right)(\rho_m+\delta) > 0$ and $f_1(1) = 4\gamma^2(\rho_r - \rho_m)$. Then, if $\rho_r > \rho_m$, $f_1(\pi) > 0$ for all $\pi \in (\frac{1}{3}, 1)$. If $\rho_r < \rho_m$, there exists a (unique) root $\pi^* \in (\frac{1}{3}, 1)$ of $f_1(\pi)$ such that $f_1(\pi) > 0$ for $\pi \in (\frac{1}{3}, \pi^*)$ and $f_1(\pi) < 0$ for $\pi \in (\pi^*, 1)$. Since $f_1(0) > 0$ for $\rho_r < \rho_m$, then $f_1(\pi) > 0$ for $\pi \in (0, \pi^*)$ in this case.

3. Then we compare A_M^{N} and A_M^{J}. Use (5) and (15) to compute

$$A_M^{\mathrm{N}} - A_M^{\mathrm{J}} = \frac{k_m}{2c_m c_r(\rho_m+\delta)(\rho_r+\delta)} f_2(\pi), \tag{37}$$

where

$$f_2(\pi) = -2\gamma^2(\rho_r+\delta)\pi^2 + 2(\gamma^2+c_r\mu)(\rho_m+\delta)\pi - (\gamma^2+2c_r\mu)(\rho_m+\delta). \tag{38}$$

Note that $f_2(0) = -(\gamma^2+2c_r\mu)(\rho_m+\delta) < 0$ and $f_2(1) = \gamma^2(\rho_m - 2\rho_r - \delta)$.
If $\rho_m > 2\rho_r + \delta$, then $f_2(1) > 0$ and there must exist $\pi^{**} \in (0,1)$ solving $f_2(\pi) = 0$. Since $f_2(\pi)$ is a second degree polynomial, π^{**} is unique. This implies that (37) is negative for $\pi \in (0, \pi^{**})$ and positive for $\pi \in (\pi^{**}, 1)$.

It remains to analyze the case when $f_2(1) < 0$. If $\rho_m < 2\rho_r + \delta$, i.e. $\rho_m + \delta < 2(\rho_r + \delta)$, let us compute the maximum of the second-degree polynomial $f_2(\pi)$. The solution to $f_2'(\pi) = 0$ is

$$\bar{\pi} = \frac{(\gamma^2+c_r\mu)(\rho_m+\delta)}{2\gamma^2(\rho_r+\delta)}\ . \tag{39}$$

A necessary condition for the existence of $\pi^{**} \in (0,1)$ such that $f_2(\pi^{**}) = 0$ is that $\bar{\pi} < 1$, so

$$\rho_r + \delta > \frac{(\gamma^2+c_r\mu)(\rho_m+\delta)}{2\gamma^2}\ . \tag{40}$$

Therefore, if $(\rho_m+\delta)/2 < \rho_r + \delta < (\gamma^2+c_r\mu)(\rho_m+\delta)/(2\gamma^2)$, then $\bar{\pi} > 1$ and $f_2(\pi)$ is negative for all $\pi \in (0,1)$. It remains to consider the case when condition (40) is verified. In that case, $\bar{\pi} \in (0,1)$ and a necessary and sufficient condition for the existence of a root of $f_2(\pi)$ in the interval $(0,1)$ is that $f_2(\bar{\pi}) > 0$. By substituting (39) in (38) we obtain that $f_2(\bar{\pi}) > 0$ if, and only if,

$$\rho_r + \delta < \frac{(\gamma^2+c_r\mu)^2(\rho_m+\delta)}{2\gamma^2(\gamma^2+2c_r\mu)}\ ,$$

but this is in contradiction with condition (40). Therefore, for $\rho_m < 2\rho_r + \delta$, $f_2(\pi)$ is negative for all $\pi \in (0,1)$.

From the previous proof in points 2 and 3, it is straightforward that when $\rho_r < \rho_m < 2\rho_r + \delta$, $\frac{1}{3} < \pi^* < 1 < \pi^{**}$.

When $\rho_m > 2\rho_r + \delta$, $\pi^* \in (\frac{1}{3}, 1)$ and $\pi^{**} \in (0,1)$. Assume $\pi^{**} < \pi^*$, then from the pairwise comparison between A_M^{J} and A_M^{S} (in point 2), we have $A_M^{\mathrm{J}} > A_M^{\mathrm{S}}$ if $\pi \in (0, \pi^*)$. From the pairwise comparison between A_M^{J} and A_M^{N} (in point 3), we have $A_M^{\mathrm{N}} > A_M^{\mathrm{J}}$ if $\pi \in (\pi^{**}, 1)$. Summarizing, if $\pi \in (\pi^{**}, \pi^*)$, $A_M^{\mathrm{N}} > A_M^{\mathrm{J}} > A_M^{\mathrm{S}}$. However, it is contradictory to the result in point 1 where we obtain $A_M^{\mathrm{N}} \leq A_M^{\mathrm{S}} \forall \pi \in (0,1)$. Therefore, $\pi^* < \pi^{**}$.

Summarizing all the pairwise comparison we made previously, the results follow. □

Proof of Proposition 6.

1. For $\pi \in (0, \frac{1}{3}]$, $V_M^{\mathrm{N}} = V_M^{\mathrm{S}}$ and $V_R^{\mathrm{N}} = V_R^{\mathrm{S}}$. For $\pi \in (\frac{1}{3}, 1)$ use (7) and (12) to compute

$$V_M^{\mathrm{N}} - V_M^{\mathrm{S}} = -(3\pi - 1)^2\gamma^2 \left\{ \frac{1}{8c_r(\rho_m + \delta)} G + \frac{k_m^2 \left[(-7\pi^2 + 10\pi + 1)\gamma^2 + 16c_r\pi\mu \right]}{128 c_m c_r^2 \rho_m (\rho_m + \delta)^2} \right\}$$

where $-7\pi^2 + 10\pi + 1 > 0 \quad \forall \pi \in (\frac{1}{3}, 1)$, implying $V_M^{\mathrm{N}} - V_M^{\mathrm{S}} < 0$. Next use (8) and (13) to compute

$$V_R^{\mathrm{N}} - V_R^{\mathrm{S}} = -(1 - \pi)(3\pi - 1)\gamma^2 \left\{ \frac{1}{4c_r(\rho_r + \delta)} G + \frac{k_m^2 \left[(-5\pi^2 + 10\pi - 1)\gamma^2 + 4c_r\mu(5\pi - 1) \right]}{32 c_m c_r^2 \rho_r (\rho_m + \delta)(\rho_r + \delta)} \right\}$$

where $-5\pi^2 + 10\pi - 1 > 0 \quad \forall \pi \in (\frac{1}{3}, 1)$, implying $V_R^{\mathrm{N}} - V_R^{\mathrm{S}} < 0$.

2. From the previous proof, we have $\alpha_M^{\mathrm{S}} \geq \alpha_M^{\mathrm{N}}$ and $\alpha_R^{\mathrm{S}} \geq \alpha_R^{\mathrm{N}}$ (when $\pi \in (\frac{1}{3}, 1)$ the strict inequality holds).

   Next, for $\pi \in (\frac{1}{3}, 1)$, use α_M^{S} and α_M^{J} as defined in Propositions 2 and 3 and take $\Phi^{\mathrm{J}} = 0$ to compute

$$\alpha_M^{\mathrm{S}} - \alpha_M^{\mathrm{J}} = \frac{\gamma^2}{8c_r(\rho_m + \delta)} \left[\pi^2 - 6\pi + 1 \right] < 0 \, .$$

   In a similar way, by using α_R^{N} and α_R^{J} defined in Propositions 1 and 3,

$$\alpha_R^{\mathrm{N}} - \alpha_R^{\mathrm{J}} = \frac{\pi^2\gamma^2}{2c_r(\rho_r + \delta)} > 0 \, .$$

□

Proof of Remark 1.

1. If $\pi \in (\frac{1}{3}, 1)$, use (9) and (15) and take $\Phi^{\mathrm{J}} = \Phi^{\mathrm{S}}$ in (10) to compute

$$A_M^{\mathrm{J}} - A_M^{\mathrm{S}} = \frac{k_m(1 - \pi)}{8c_m c_r (1 + \pi)(\rho_m + \delta)(\rho_r + \delta)} f_3(\pi), \tag{41}$$

   where

$$\begin{aligned} f_3(\pi) =& \gamma^2(\rho_r + \delta)\pi^2 + \left[8c_r\mu(\rho_m + \delta) + 4\gamma^2(2\rho_m - \rho_r + \delta) \right] \pi \\ &+ 8c_r\mu(\rho_m + \delta) + 3\gamma^2(\rho_r + \delta). \end{aligned}$$

   It is straightforward to check that $f_3(1)$ and $f_3(\frac{1}{3})$ are positive. It suffices to check that there is no $\widetilde{\pi}$ verifying $f_3'(\widetilde{\pi}) = 0$ with $f_3(\widetilde{\pi}) < 0$ in the interval $\widetilde{\pi} \in (\frac{1}{3}, 1)$. First, the stationary point of function $f_3(\pi)$ is

$$\widetilde{\pi} = 2 - 4 \left(\frac{\rho_m + \delta}{\rho_r + \delta} \right) \left(\frac{c_r\mu}{\gamma^2} + 1 \right) \, .$$

   Condition $\frac{1}{3} < \widetilde{\pi} < 1$ becomes

$$\frac{12}{5} \left(\frac{c_r\mu}{\gamma^2} + 1 \right) < \frac{\rho_r + \delta}{\rho_m + \delta} < 4 \left(\frac{c_r\mu}{\gamma^2} + 1 \right) \, . \tag{42}$$

Define $g(\pi) = f_3(\pi) - \gamma^2(\rho_r + \delta)\pi^2$. It is clear that a necessary condition for the existence of negative values of $f_3(\pi)$ (and also for the existence of a positive $\widetilde{\pi}$) is that the coefficient in the linear term must be negative, hence $g(\pi)$ is decreasing. As a result,

$$f_3(\widetilde{\pi}) = \gamma^2(\rho_r + \delta)\widetilde{\pi}^2 + g(\widetilde{\pi}) > g(\widetilde{\pi}) > g(1) \; .$$

Condition $f_3(\widetilde{\pi}) < 0$ implies $g(1) < 0$, i.e.

$$\frac{\rho_r + \delta}{\rho_m + \delta} > 8 + 16\left(\frac{c_r\mu}{\gamma^2}\right) \; ,$$

in contradiction with (42), so there is no solution for $f_3(\pi) = 0$ in the interval $(\frac{1}{3}, 1)$ and (41) is positive for $\pi \in (\frac{1}{3}, 1)$.

2. It follows from (11) and (16).
3. If $\pi \in (\frac{1}{3}, 1)$, use α_M^{S}, α_M^{J}, α_R^{S} and α_R^{J} as defined in Propositions 2 and 3, and take $\Phi^{\mathrm{J}} = \Phi^{\mathrm{S}}$ in (10) to compute

$$\alpha_M^{\mathrm{S}} - \alpha_M^{\mathrm{J}} = -\frac{(3-\pi)(1-\pi)^2\gamma^2}{8c_r(1+\pi)(\rho_m + \delta)} < 0 \; ,$$

and

$$\alpha_R^{\mathrm{S}} - \alpha_R^{\mathrm{J}} = \frac{\gamma^2(1-\pi)^3}{4c_r(\rho_r + \delta)(\pi + 1)} > 0 \; .$$

□

Proof of $\Delta(0) > 0$ and $\Delta(1) < 0$ in Section 4.2.

1. $\Delta(0) = \dfrac{k_m^2}{8c_m c_r^2 \rho_m \rho_r (\rho_m + \delta)^2 (\rho_r + \delta)^2} \left[(\gamma^2 + 2c_r\mu)^2(\rho_m + \delta)^2(2\rho_m - \rho_r)\right] \; ,$
   $\rho_m > 2\rho_r + \delta$ implies $\Delta(0) > 0$.
2.

$$\begin{aligned}\Delta(1) = &- \frac{\gamma^2 k_m^2}{8c_m c_r^2 \rho_m \rho_r (\rho_m + \delta)^2(\rho_r + \delta)^2} \\ &\{2c_r\mu(\rho_r + \delta)\left[2\rho_m(\rho_m + \delta) - 4\rho_r(\rho_r + \delta)\right] \\ &-\gamma^2(\rho_m - 2\rho_r - \delta)\left[(2\rho_m - \rho_r)(\rho_m + \delta) - 2\rho_r(\rho_r + \delta)\right]\} \; .\end{aligned} \tag{43}$$

$$\alpha_M^{\mathrm{J}} + \alpha_R^{\mathrm{J}} = \frac{\left[-2(\rho_m - \rho_r)(\gamma^2 + c_r\mu)\pi + (\rho_m + \delta)(\gamma^2 + 2c_r\mu)\right]}{2c_r(\rho_m + \delta)(\rho_r + \delta)}$$

is decreasing in π when $\rho_m > \rho_r$. The assumption $\alpha_M^{\mathrm{J}} + \alpha_R^{\mathrm{J}} \geq 0$ for all $\pi \in (0, 1)$ is equivalent to

$$\alpha_M^{\mathrm{J}} + \alpha_R^{\mathrm{J}}|_{\pi=1} = \frac{2(c_r\mu + \gamma^2)(\rho_r + \delta) - \gamma^2(\rho_m + \delta)}{2c_r(\rho_m + \delta)(\rho_r + \delta)} \geq 0 \; ,$$

implying

$$2c_r\mu(\rho_r + \delta) \geq \gamma^2(\rho_m - 2\rho_r - \delta) \; . \tag{44}$$

Using (44) and $\rho_m > 2\rho_r + \delta$, we have that (43) is negative.

□

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