# Submodularity and Local Search Approaches for Maximum Capture Problems under Generalized Extreme Value Models

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We study the maximum capture problem in facility location under random utility models, i.e., the problem of seeking to locate new facilities in a competitive market such that the captured user demand is maximized, assuming that each customer chooses among all available facilities according to a random utility maximization model. We employ the generalized extreme value (GEV) family of discrete choice models and show that the objective function in this context is monotonic and submodular. This finding implies that a simple greed heuristic can always guarantee an (1 - 1/e) approximation solution. We further develop a new algorithm combining a greedy heuristic, a gradient-based local search and an exchanging procedure to efficiently solve the problem. We conduct experiments using instances of difference sizes and under different discrete choice models, and we show that our approach significantly outperforms prior approaches in terms of both returned objective value and CPU time. Our algorithm and theoretical findings can be applied to the maximum capture problems under various random utility models in the literature, including the popular multinomial logit, nested logit, cross nested logit, and the mixed logit models.

Key words: Maximum capture, random utility, generalized extreme value, greedy heuristic, gradient-based local search

## 1. Introduction

In the last decade, the facility location problem in a competitive market has received a growing attention. In practice, modelling critical managerial decisions related to infrastructure planning, such as finding locations to locate new retail, service or product facilities in a market, often lead to facility location problems. The competitive facility location problem deals with a decision of selecting locations to open new facilities in a market to maximize the captured demand of users, where a set of incumbent competitors are already operating in order. There are two aspects that need to be considered in this problem, namely, the demand of customers and the competitors in the market. Customers are independent decision makers and their choices among different facilities might be based on a given utility that they assign to each location. Such utilities might be a function of facility attributes/features, e.g., distances, prices and transportation costs.

There are several ways to define and estimate customer demand (Berman et al. 2009). In this work, we focus on a probabilistic approach, i.e., customer demand is captured by a probability model that assigns choice probabilities to the facilities. The random utility maximization (RUM) framework (McFadden 1973, Horowitz 1986) is convenient and popular in the context. This framework is based on the assumption that each facility is associated with a random utility, which can be determined by the features/attributes of the facility. The RUM principle assumes that each customer selects a facility by maximizing his/her utilities. This way of modeling allows for predicting the probability that a customer selects a facility. The facility location problem then becomes the problem of locating new facilities in a competitive market to maximize an expected captured demand function, where customers selects a facility (a new facility or one from the competitors) according to a RUM model. Thus, the problem is also called as the *maximum capture problem* (MCP).

To the best of our knowledge, existing related studies in the literature only employ the multinomial logit (MNL) or its mixed version (mixed logit model - MMNL) (Benati and Hansen 2002, Haase 2020, Haase and Müller 2013). However, it is well-known that the MNL retains the independence from irrelevant alternatives (IIA) property, which does not hold in many contexts (McFadden 1981, McFadden and Train 2000). On the other hand, the generalized extreme value (GEV) family provides flexible ways to relax the IIA property and capture the correlation between alternative utilities (McFadden 1981). However, under the GEV family, most of the important properties that have been used to develop solution methods for the MCP under the MNL and MMNL models do not hold, or have not been proved to be true. More precisely, the objective function under the GEV family does not have a linear fractional structure, thus it is difficult to formulate the MCP into a mixed-integer linear program (MILP) as in prior work (Benati and Hansen 2002, Zhang et al. 2012). Moreover, under the GEV family, the objective function of the continuous relaxation is not either concave or convex, making the outer-approximation methods (Ljubić and Moreno 2018, Mai and Lodi 2019) not applicable. Furthermore, since the structure of the objective function is driven by a GEV choice probability generating function, which may not have a closed form and could be complicated, it is not clear whether the objective function is submodular or not. All the above remarks make the MCP under the GEV family challenging. We tackle this challenge this in paper.

Before presenting our contributions in detail, we note that, from now on, when saying a "GEV model", we refer to any choice model in the GEV family. Each GEV model can be determined

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by a choice probability generating function (CPGF)  $G(\cdot)$  (Fosgerau et al. 2013) (see our detailed definition in the next section).

Our contributions: In this paper, we formulate and solve the MCP under any GEV models. We leverage the properties of the CPGFs of GEV models (McFadden 1981, Daly and Bierlaire 2006) and show that the objective function in the context is monotonic increasing and submodular. These properties are already known for the MCP under MNL (Benati and Hansen 2002) and now we show that they also hold for any GEV models. The monotonicity and submodularity also imply that the MCP subjecting to a cardinality constraint, even though being *NP-hard*, always admits an (1 - 1/e) approximation algorithm. In other words, a simple greedy heuristic always returns a solution whose value is at least (1 - 1/e) ( $\approx 0.632$ ) times the optimal values (Nemhauser et al. 1978).

To further enhance the greedy heuristic (GH), we develop a new algorithm that adds a gradientbased local search and exchanging procedures to the GH. While the latter is simply based on steps of exchanging a location in a set of chosen locations with one outside of the set to get a better objective value, the former is motivated by the fact that if we formulate the MCP as a binary program, then the objective function is differentiable and we can make use of gradient information to direct the search. The gradient-base location search is an iterative procedure in which at each iteration we solve a subproblem to (hopefully) find a better candidate solution, and we show that such subproblems are solvable in polynomial-time. Our algorithm can be used to solve problems under any GEV models and under the MMNL model.

We conduct experiments using some datasets from the recent literature, including real-life largescale instances from a *park-and-ride* location problem in New York City (Holguin-Veras et al. 2012). We compare our algorithm, named as GGX (stands for **G**reedy Heuristic, **G**radient-based Local Search, and **Ex**changing) with some state-of-the art approaches from recent literature, i.e., the Branch & Cut method proposed by Ljubić and Moreno (2018) and outer-approximation algorithms (Bonami et al. 2011, Mai and Lodi 2019). Experiments based on MNL, MMNL and nested logit instances show that our algorithm remarkably outperforms the other approaches, in terms of both returned objective value and CPU time.

Literature review: The GEV family (McFadden 1981) covers most of the discrete choice models in the demand modeling and operations research literature. Among existing GEV models, the MNL is the simplest and most popular one. It is also well-known that the MNL retains the IIA property, which implies that the ratio between the choice probabilities of two facilities will not change no matter what other facilities are available or what attributes that other facilities have. This property has been considered as a limitation of the MNL model and should be relaxed in many applications (McFadden and Train 2000). There are several GEV models that relax this property and provide flexible ways to model the correlation between alternatives. For example, the nested logit (Ben-Akiva 1973), the cross-nested logit Vovsha and Bekhor (1998), the generalized nested logit (Wen and Koppelman 2001), the paired combinatorial logit (Koppelman and Wen 2000), the ordered generalized extreme value Small (1987), the specialized compound generalized extreme value models (Whelan et al. 2002) and network-based GEV (Daly and Bierlaire 2006, Mai et al. 2017) models. GEV models, in particular the cross-nested and network GEV models, are fully flexible, in the sense that these models can approximate any random utility maximization models (Fosgerau et al. 2013). Beside the GEV family, the MMNL is also an alternative to relax the IIA property. This model extends the MNL by assuming that choice parameters are random. Similar to GEV models, the MMNL is also able to approximate any random utilities choice model (McFadden and Train 2000). However, the choice probabilities given by the MMNL model have no closed form and often require simulation to approximate. Thus, the estimation and the application of this model is expensive in many contexts.

In the context of the MCP, most existing studies focus on the MNL model due to its simplicity. Benati and Hansen (2002) seem the first to introduce the MCP under the MNL model. They propose three methods to compute upper bounds along with a branch-and-bound method to solve small instances. The first method is based on the concavity of the continuous relaxation of the objective function. They show the submodularity of the objective function and use this property to develop the second method. The third method is an equivalent mixed-integer linear program (MILP), which is based on the fact that the objective function has a linear fractional structure and can be linearized using additional additional variables. Benati and Hansen (2002) also introduced a simple variable neighborhood search (VNS) method to solve instances with more than 50 potential locations. Some alternative MILP models, afterwards, have been proposed by Haase (2020) and Zhang et al. (2012). Haase and Müller (2013) give an evaluation and comparison between the proposed MILP models and conclude that the MILP model from Haase (2020) is the most efficient one. Freire et al. (2015) strengthen the MILP reformulation of Haase (2020) by using some tighter coefficients in some inequalities and also propose a new branch-and-bound algorithm to deal with the problem. More recently, Ljubic and Moreno (2017) propose a branch-and-cut method that combines two types of cutting planes, namely, outer-approximation (OA) cuts and submodular cuts. The first type of cuts is relied on the fact that the objective function of the continuous relaxation of the problem is concave and differentiable and the second type is based on the submodularity and separability properties of objective function. Their branch-and-cut method is an iterative procedure where cuts are generated for every demand points and a linear programming (LP) relaxation is solved at each iteration. Mai and Lodi (2019) propose a multicut outer-approximation algorithm that works in a cutting plane fashion by solving an MILP at every iteration. This algorithm generates cuts for groups of demand points instead of one cut of every demand point as in Ljubic and Moreno (2017) or one cut for all demand points as in the classical outer-approximation scheme (Duran and Grossmann 1986, Bonami et al. 2008). The branch-and-cut proposed by (Ljubić and Moreno 2018) and multicut outer approximation are considered as state-of-the art approaches for the MCP under the MNL model. Note that in the context of the MCP, MNL and MMNL instances have similar structures. Thus, all the methods developed for the MNL model can be generally applied to MMNL problem instances. There are also a couple of studies investigating the MCP under the MMNL model (Haase 2020, Haase and Müller 2013). These studies make use of MILP formulations, which is generally outperformed by the branch-and-cut approached (Ljubić and Moreno 2018).

**Paper outline:** The rest of paper is structured as follow. Section 2 briefly presents the GEV family focusing on the some essential properties of the CPGF, and the MCP under the GEV family. In Section 3, we investigate the monotonicity and submodularity of the MCP under the GEV family, and present our local search algorithm. Section 4 reports computational results. Finally, Section 5 concludes.

**Notation:** Boldface characters represent matrices (or vectors), and  $a_i$  denotes the *i*-th element of vector **a**. We use [m], for any  $m \in \mathbb{N}$ , to denote the set  $\{1, \ldots, m\}$ .

## 2. Generalized Extreme Value Models and the Maximum Capture Problem

In this section we introduce some basic concepts and properties of the GEV family and formulate the MCP under GEV models.

### 2.1. Generalized Extreme Value Models

The Random Utility Maximization (RUM) framework (McFadden 1978a) is the most popular approach to model discrete choice behavior. Under the RUM principle, the decision maker is assumed to associate an utility  $u_j$  with each alternative/option j in a given choice set S that contains all possible alternatives. The additive RUM (McFadden 1978a, Fosgerau and Bierlaire 2009) assumes that each random utility is a sum of two parts  $u_j = v_j + \epsilon_j$ , where the term  $v_j$ is deterministic and can include values representing characteristics of the alternative and/or the decision maker, and the random term  $\epsilon_j$  is unknown to the analyst. There are several assumptions that have been made on the randoms terms, which leads to different types of discrete choice models in the literature, e.g., the MNL or nested logit models (McFadden 1978a, Train 2003). The deterministic terms  $v_j$  often have a linear structure, i.e.,  $v_j = \beta^T \alpha_j$ , where  $^T$  is the transpose operator and  $\beta$  is a vector of parameters to be estimated from historical data of how people make decisions, and  $\alpha_i$  is a vector of attributes of alternative j. The RUM principle then assumes that a decision is made by maximizing the random utilities, and the probability that an alternative j is selected can be computed as  $P(u_j \ge u_k, \forall k \in S)$ .

The GEV family covers most of the existing discrete choice models in the literature. This family of model is fully flexible, in the sense that it allows to construct various discrete choice models that are consistent with the RUM principle (McFadden 1981). Assume that the choice set contains malternative indexed as  $\{1, \ldots, m\}$  and let  $\mathbf{U} = \{v_1, \ldots, v_m\}$  be the vector of utilities. A GEV model can be determined by a choice probability generating function (CPGF)  $G(\mathbf{Y})$  (McFadden 1981, Fosgerau et al. 2013), where  $\mathbf{Y}$  is a vector of size m with entries  $Y_j = e^{v_j}$ . Given  $j_1, \ldots, j_k \in [m]$ , let  $\partial G_{j_1 \ldots j_k}$ , be the mixed partial derivatives of G with respect to  $Y_{j_1}, \ldots, Y_{j_k}$ . It is well-known that the CPGF  $G(\cdot)$  and the mixed partial derivatives have the the following properties (McFadden 1978b).

REMARK 1. A CPGF  $G(\mathbf{Y})$  of a GEV model, has the following properties.

- (i)  $G(\mathbf{Y}) \ge 0, \ \forall \mathbf{Y} \in \mathbb{R}^m_+,$
- (ii)  $G(\mathbf{Y})$  is homogeneous of degree one, i.e.,  $G(\lambda \mathbf{Y}) = \lambda G(\mathbf{Y})$
- (*iii*)  $G(\mathbf{Y}) \to \infty$  if  $Y_j \to \infty$

(iv) Given  $j_1, \ldots, j_k \in [m]$  distinct from each other,  $\partial G_{j_1,\ldots,j_k}(\mathbf{Y}) > 0$  if k is odd, and  $\leq$  if k is even

(v) 
$$G(\mathbf{Y}) = \sum_{j \in [m]} Y_j \partial G_j(\mathbf{Y})$$

(vi) 
$$\sum_{k \in [m]} Y_k \partial G_{jk}(\mathbf{Y}) = 0, \ \forall j \in [m].$$

Here we note that (i) - (iv) are basic properties of a GEV generating function (McFadden 1981), and Properties (v) and (vi) are direct results from the homogeneity property. We will make use of these properties throughout the rest of the paper to explore the properties of the objective function of the MCP, and derive solution algorithms for the MCP. Under a GEV model specified by a CPGF  $G(\mathbf{Y})$ , the choice probability of an alternative  $j \in [m]$ , conditional on  $\mathbf{Y}$  and G, is given by

$$P(j|\mathbf{Y}, G) = \frac{Y_i \partial G_i(\mathbf{Y})}{G(\mathbf{Y})}.$$

The GEV framework allows for correlated utilities and one can build different CPGF to model different correlation patterns among random utilities. One can build a GEV model from a network of correlation structure, which provides a very flexible way to construct choice models that are able to capture complex relationships between alternatives (Mai et al. 2017, Daly and Bierlaire 2006). In the following, we show some specific instances of the GEV family that are already popular in the demand modeling and operations research literature.

The MNL model. The MNL is one of the most widely-used models in the literature. This model results from the assumption that the random terms  $\epsilon_j$ ,  $j \in [m]$ , are independent and identically distributed (i.i.d.) and follow the standard Gumbel distribution. The CPGF function has a simple form as  $G(\mathbf{Y}) = \sum_{j \in [m]} Y_j$  and the choice probabilities have the fractional form below

$$P(j|\mathbf{Y},G) = \frac{Y_j}{\sum_{j \in [m]} Y_j} = \frac{e^{v_j}}{\sum_{j \in [m]} e^{v_j}}.$$
(1)

It is well-known that the MNL model exhibits from the IIA property, which means that the choice probability of an alternative will not be affected by the attributes or the state of the other alternatives. However, in some situations, alternatives share unobserved attributes (i.e, random terms are correlated) and the IIA property does not hold.

The nested logit model. The nested logit model (Ben-Akiva 1973) is one of the first attempts to relax the IIA property from the MNL model. In this GEV model, the choice set is partitioned into L nests, which are disjoint subsets of alternatives. Let denote by  $n_1, \ldots, n_L$  the L nests. The corresponding CPGF can be written as

$$G(\mathbf{Y}) = \sum_{l \in L} \left( \sum_{j \in n_l} Y_j^{\mu_l} \right)^{1/\mu_l}$$

where  $\mu_l \ge 1, l \in [L]$ , are the parameters of the nested model. This model is based on the observation that, in many situations, some similar or closely related alternatives can be grouped into smaller subsets. It is easy to see that the function G above satisfies the six properties above and the choice probabilities can be computed as

$$P(j|\mathbf{Y},G) = \frac{\left(\sum_{j' \in n_l} Y_{j'}^{\mu_l}\right)^{1/\mu_l}}{\sum_{l \in [L]} \left(\sum_{j' \in n_l} Y_{j'}^{\mu_l}\right)^{1/\mu_l}} \frac{Y_j^{\mu_l}}{\sum_{j' \in n_l} Y_{j'}^{\mu_l}}, \,\forall l \in [L], j \in n_l$$

The cross-nested logit model (Ben-Akiva and Bierlaire 1999) is an extension of the nested logit that allows the nests to share common alternatives. This model is known to be fully flexible, as it can approximate arbitrarily close any RUM models (Fosgerau et al. 2013). The network GEV model proposed in Daly and Bierlaire (2006) further generalizes the cross-nested model by proving a way to construct a GEV CPGF based on any rooted network of correlation structure.

Beside the GEV family, the **MMNL model** (McFadden and Train 2000) is also popular due to its flexibility in capturing utility correlation. In the MMNL model, the model parameters (and the utilities  $v_j$ ) are assumed to be random, and the choice probabilities can be obtained by taking the expectations over random coefficients. Let  $\mathbf{Y}^1, \ldots, \mathbf{Y}^K$  be K realizations sampled from the distribution of the random parameters, the choice probabilities can be approximated as

$$P(j|\mathbf{Y}^{1},...,Y^{K},G) = \frac{1}{K} \sum_{k=1}^{K} \frac{Y_{j}^{k}}{\sum_{t \in [m]} Y_{t}^{k}}.$$

The MMNL model is highly preferred in practice due to its flexibility in modeling people demand. However, the estimation and application of this model in decision-making are well-known to be expensive and complicated, due to the fact that it requires simulation to approximate the choice probabilities.

#### 2.2. The Maximum Capture Problem

We are interested in the situation that a "newcomer" firm wants to locate new facilities in a competitive market, i.e., there are already existing facilities from competitors that can serve customers. The firm may want to maximize the expected market share achieved by attracting the customers to new facilities. To capture the customers' demand, we suppose that a customer selects a facility according to a RUM model. In this context, each customer associate each facility with a random utility and we assume that the customer will choose a facility by maximizing his/her utilities. Accordingly, the firm aims at selecting a set of locations to locate new facilities to maximize the expected number of customers. In the following, we describe in detail the MCP under GEV models.

We denote by  $\mathcal{V} = [m]$  the set of possible locations. Let I be the set of geographical zones where customers are located and  $q_i$  is the number of customers in zone  $i \in I$  and for customers at zone i, let  $v_{ij}$  be the corresponding deterministic utility of location  $j \in [m]$ . These utility values can be inferred by estimating the RUM model using historical data. The set I can be viewed as a set of customer types, e.g., customers that belong to different categories specified by, for instance, age or income. A GEV model for customers located at zone  $i \in I$  can be represented by a choice probability generating function (CPGF)  $G^i(\mathbf{Y}^i)$ , where  $\mathbf{Y}^i$  is a vector of size m with entries  $Y_i^i = e^{v_{ij}}$ .

Under a GEV model specified by a set of CPGF  $G^{i}(\mathbf{Y}^{i}), i \in I$ , taking into consideration the competitors, the choice probability of a location  $j \in [m]$  is given as

$$P(j|\mathbf{Y}^{i}, G^{i}) = \frac{Y_{j}\partial G_{j}^{i}(\mathbf{Y}^{i})}{1 + G^{i}(\mathbf{Y}^{i})}.$$

Here, without loss of generality, we assume that the total utility of the competitor is 1 for the sake of simplicity. As if it is not the case, then we can always scale the utilities  $\mathbf{Y}^{i}$  to get utilities of 1 for the competitors. More specifically, it is possible due to fact that, for any  $\alpha > 0$ ,

$$\frac{Y_j \partial G_j^i(\mathbf{Y}^i)}{\alpha + G^i(\mathbf{Y}^i)} \stackrel{(a)}{=} \frac{\frac{Y_j}{\alpha} \partial G_j^i(\mathbf{Y}^i)}{1 + G^i(\mathbf{Y}^i/\alpha)} \stackrel{(b)}{=} \frac{\frac{Y_j}{\alpha} \partial G_j^i(\mathbf{Y}^i/\alpha)}{1 + G^i(\mathbf{Y}^i/\alpha)}$$

where (a) is due the homogeneity of  $G^i(\cdot)$  ((ii) of Remark 1) and (b) is obtained by taking derivatives of the both sides of the equation  $G^i(\alpha \mathbf{Y}^i) = \alpha G^i(\mathbf{Y}^i)$  w.r.t.  $Y_i^i$ 

$$\alpha \partial G^i_j(\alpha \mathbf{y}^i) = \alpha \partial G^i_j(\mathbf{Y}^i), \text{ or } \partial G^i_j(\alpha \mathbf{y}^i) = \partial G^i_j(\mathbf{Y}^i), \text{ for any } \alpha > 0.$$

We are interested in the fact that the facilitates are located at a subset of locations  $S \subset [m]$ . Hence, the conditional choice probability can be written as

$$P(j|\mathbf{Y}^{i}, G^{i}, S) = \frac{Y_{j}^{i} \partial G_{j}^{i}(\mathbf{Y}^{i}|S)}{1 + G^{i}(\mathbf{Y}^{i}|S)}, \ \forall j \in S,$$

where the conditional CPGF  $G^i(\mathbf{Y}^i|S)$  can be computed as  $G^i(\mathbf{Y}^i|S) = G^i(\widetilde{\mathbf{Y}}^i)$ , where  $\widetilde{\mathbf{Y}}^i$  is a vector of size m with entries  $\widetilde{Y}_j^i = Y_j^i$  if  $j \in S$  and  $\widetilde{Y}_j^i = 0$  otherwise. This can be interpreted as if a location j is not in S, then its corresponding utility becomes very small, i.e.,  $v_{ij} = -\infty$ , then  $Y_j^i = e^{v_{ij}} = 0$ . The maximum capture problem under a GEV model specified by CPGFs  $G^i(\mathbf{Y}^i)$ ,  $i \in I$ , can be stated as

$$\max_{S \in \mathcal{S}} \left\{ f^{\text{GEV}}(S) = \sum_{i \in I} q_i \sum_{j \in S} P(j | \mathbf{Y}^i, G^i, S) \right\},\tag{2}$$

where S is the set of feasible solutions. Under a cardinality constraint  $|S| \leq C$ , S can be defined as  $S = \{S \subset [m] | |S| \leq C\}$ , for a given constant C such that  $1 \leq C \leq m$ . Note that the objective function can be further simplified as

$$f^{\text{GEV}}(S) = \sum_{i \in I} q_i \frac{\sum_{j \in S} Y_j^i \partial G_j^i(\mathbf{Y}^i | S)}{1 + G^i(\mathbf{Y}^i | S)}$$
$$\stackrel{(a)}{=} \sum_{i \in I} q_i - \sum_{i \in I} \frac{q_i}{1 + G^i(\mathbf{Y}^i | S)},$$

where (a) is due to Property (v) in Remark 1.

If the choice model is MNL, the objective function becomes

$$f^{\text{GEV}}(S) = \sum_{i \in I} q_i - \sum_{i \in I} \frac{q_i}{1 + \sum_{j \in m} Y_j^i},$$

and from previous studies, we know that  $f^{\text{GEV}}(S)$  is submodular and it binary representation is concave (Benati and Hansen 2002). Thus, an approach based on sub-gradient and submodular cuts can be used (Mai and Lodi 2019, Ljubić and Moreno 2018) to efficiently solve the problem. On the other hand, formulations based on GEV models would be much more complicated. For example, under the nested logit model, we can write the objective function as

$$f^{\text{GEV}}(S) = \sum_{i \in I} q_i - \sum_{i \in I} \frac{q_i}{1 + \sum_{l \in [L]} \left( \sum_{j \in n_l \cap S} (Y_j^i)^{\mu_l} \right)^{1/\mu_l}}.$$

Under a general case, e.g., the network GEV model Daly and Bierlaire (2006), Mai et al. (2017), it is even not possible to write the objective function in a closed form.

It is important to note that, if we look at the objective function under a MMNL model

$$f^{\text{MMNL}}(S) = \frac{1}{K} \sum_{k \in [K]} \left( \sum_{i \in I} q_i - \sum_{i \in I} \frac{q_i}{1 + \sum_{j \in m} Y_j^{i,k}} \right)$$
$$= \sum_{k \in [K], i \in I} \frac{q_i}{K} - \sum_{i \in I, k \in [K]} \frac{q_i/K}{1 + \sum_{j \in m} Y_j^{i,k}},$$

where  $\mathbf{Y}^{i,k}$ , k = 1, ..., K, are K realization of the random utility vector  $\mathbf{Y}^i$ . So, this objective function can be viewed as one from the MNL-based MCP problem with  $K \times |I|$  customer zones, in which there are  $q_i/K$  customers in zone (i, k)-th. So, all the results established for the MNL (and GEV in general) problem can also be used to solve the MMNL problem.

# 3. Maximum Capture Problem under GEV Models

In this section we explore the MCP under GEV models. In particular, by leveraging the properties of the GEV CPGFs shown above, we show that the objective function in the context is monotonic and submodular. This generalizes some well-known results established for MNL-based problems in previous studies (Benati and Hansen 2002). We also design a location search procedure to efficiently solve the problem.

## 3.1. Monotonicity and Submodularity

We show two key results of the paper, which demonstrates that the objective function under any GEV model is monotonic and submodular, thus providing a performance guarantee for a greedy heuristic procedure.

To prove the results, we first formulate the MCP as a binary program. That is, given a subset  $S \subset [m]$ , let  $\mathbf{x}^S$  be a binary vector of size m with entries  $\mathbf{x}_j^S = 1$  if  $j \in S$  and  $\mathbf{x}_j^S = 0$  otherwise. We see that the conditional CPGF can be written as

$$G^i(\mathbf{Y}^i|S) = G^i(\mathbf{x}^S \circ \mathbf{Y}^i),$$

where  $\circ$  is the element-by-element operator and  $\mathbf{x} \circ \mathbf{Y}^i$  is vector of size m with entries  $x_j^S Y_j^i$ ,  $j = 1, \ldots, m$ . We now can formulate (2) as

$$\max_{\mathbf{x}\in X} \left\{ f^{\text{GEV}}(\mathbf{x}) = \sum_{i\in I} q_i - \sum_{i\in I} \frac{q_i}{1 + G^i(\mathbf{x}\circ\mathbf{Y}^i)} \right\},\tag{3}$$

where  $X = \{\mathbf{x}^{S} \in \{0,1\}^{m} | \forall S \in S\}$ , i.e., the feasible set of binary solutions that corresponds to all the subsets in S. It is worth noting that if the choice model is MNL (or MMNL), then the objective  $f^{\text{GEV}}(\mathbf{x})$  is concave in  $\mathbf{x}$  and the problem can be handled efficiently by an outer-approximation method (Bonami et al. 2011, Mai and Lodi 2019, Ljubić and Moreno 2018). It is however not the case under an arbitrary GEV model.

The following proposition tells us that the objective function is monotonic, which implies that adding more facilities always yields better objective values.

**Proposition 1 (Monotonicity).** Adding more facilities always yields better objective values, i.e.,  $f^{\text{GEV}}(S \cup \{i\}) > f^{\text{GEV}}(S)$  for any  $i \notin S$ .

*Proof:* To prove the claim, let  $\mathbf{x} \in \{0, 1\}^m$  be the binary vector representing set S. For any  $j \in [m]$  such that  $x_j = 0$ , we need to prove  $f^{\text{GEV}}(\mathbf{x} + \mathbf{e}^j) > f^{\text{GEV}}(\mathbf{x})$ , where  $\mathbf{e}^j$  is a vector of size m with zero entries except the j-th element that is equal to 1. To prove this, let us consider  $G^i(\mathbf{x} \circ \mathbf{Y}^i)$ . Taking the derivative of  $G^i(\mathbf{x} \circ \mathbf{Y}^i)$  w.r.t. an  $x_j, j \in [m]$ , we have

$$\frac{\partial G^{i}(\mathbf{x} \circ \mathbf{Y}^{i})}{\partial x_{j}} = Y_{j}^{i} \partial G_{j}^{i}(\mathbf{x} \circ \mathbf{Y}^{i}) \stackrel{(b)}{>} 0, \tag{4}$$

where (b) is due to Property (iv) of Remark 1. This implies that  $G^i(\mathbf{x} \circ \mathbf{Y}^i)$  is (strictly) monotonic increasing in any  $x_j, j \in [m]$ . Thus

$$G^{i}((\mathbf{x} + \mathbf{e}^{j}) \circ \mathbf{Y}^{i}) > G^{i}(\mathbf{x} \circ \mathbf{Y}^{i}).$$

$$\tag{5}$$

Together with the definition of  $f^{\text{GEV}}(\mathbf{x})$  in (3), we have  $f^{\text{GEV}}(\mathbf{x} + \mathbf{e}^j) > f^{\text{GEV}}(\mathbf{x})$  as desired.

Since  $f^{\text{GEV}}(S)$  is monotonic, if we consider a cardinality constraint  $|S| \leq C$  for a scalar  $C \in \{1, \ldots, m\}$ , then an optimal solution  $S^*$  always achieves the maximum cardinality, i.e.,  $|S^*| = C$ . Thus, we can replace the cardinality constraint by an equality one, i.e., |S| = C. Note that a similar claim has been validated for the MNL-based problems in prior work (Mai and Lodi 2019).

The submodularity is well-known for the objective function under the MNL model (Benati and Hansen 2002). The theorem below shows that it is also the case under any models in the GEV family.

## **Theorem 1 (Submodularity).** $f^{\text{GEV}}(S)$ is submodular.

*Proof:* To prove the submodularity, we will show that for any set  $A \subset B \subset [m]$  and for any  $j \in [m] \setminus B$  we have

$$f^{\text{GEV}}(A \cup \{j\}) - f^{\text{GEV}}(A) \ge f^{\text{GEV}}(B \cup \{j\}) - f^{\text{GEV}}(B)$$

$$\tag{6}$$

Let use denote each component of  $f^{\text{GEV}}(S)$  as

$$g^{i}(S) = \frac{q_{i}}{1 + G^{i}(\mathbf{Y}^{i}|S)}, \forall i \in I$$

then (6) can be validated if we can prove

$$g^i(A \cup \{j\}) - g^i(A) \le g^i(B \cup \{j\}) - g^i(B),$$

or equivalently,

$$\frac{1}{1+G^{i}((\mathbf{x}^{A}+\mathbf{e}^{j})\circ\mathbf{Y}^{i})} - \frac{1}{1+G^{i}(\mathbf{x}^{A}\circ\mathbf{Y}^{i})} \leq \frac{1}{1+G^{i}((\mathbf{x}^{A}+\mathbf{e}^{j})\circ\mathbf{Y}^{i})} - \frac{1}{1+G^{i}(\mathbf{x}^{A}\circ\mathbf{Y}^{i})}$$

$$\Leftrightarrow \frac{G^{i}((\mathbf{x}^{A}+\mathbf{e}^{j})\circ\mathbf{Y}^{i}) - G^{i}((\mathbf{x}^{A})\circ\mathbf{Y}^{i})}{[1+G^{i}((\mathbf{x}^{A}+\mathbf{e}^{j})\circ\mathbf{Y}^{i})][1+G^{i}(\mathbf{x}^{A}\circ\mathbf{Y}^{i})]} \geq \frac{G^{i}((\mathbf{x}^{B}+\mathbf{e}^{j})\circ\mathbf{Y}^{i}) - G^{i}(\mathbf{x}^{B}\circ\mathbf{Y}^{i})}{[1+G^{i}((\mathbf{x}^{A}+\mathbf{e}^{j})\circ\mathbf{Y}^{i})][1+G^{i}(\mathbf{x}^{A}\circ\mathbf{Y}^{i})]} \quad (7)$$

Now, for ease of notation, for any  $\mathbf{x} \in \{0,1\}^m$  and  $j \in [m]$  such that  $x_j = 0$ , let

$$\phi(\mathbf{x}) = G^i((\mathbf{x} + \mathbf{e}^j) \circ \mathbf{Y}^i) - G^i(\mathbf{x} \circ \mathbf{Y}^i)$$

For any  $k \in [m]$  such that  $x_k = 0$  and  $k \neq j$  we take the partial derivative of  $\phi(\mathbf{x})$  w.r.t.  $x_k$  and get

$$\frac{\partial \phi(\mathbf{x})}{\partial x_k} = Y_k^i \left( \partial G_k^i((\mathbf{x} + \mathbf{e}^j) \circ \mathbf{Y}^i) - \partial G_k^i(\mathbf{x} \circ \mathbf{Y}^i) \right)$$
(8)

Now, let define another function  $\rho(\mathbf{x}) = \partial G_k^i(\mathbf{x} \circ \mathbf{Y}^i)$ . Taking the partial derivative of  $\rho(\mathbf{x})$  w.r.t.  $x_j$  we get

$$\frac{\partial \rho(\mathbf{x})}{\partial x_j} = Y_j^i \partial G_{kj}^i(\mathbf{x} \circ \mathbf{Y}^i),$$

and since  $\partial G_{kj}^i(\mathbf{x} \circ \mathbf{Y}^i) \leq 0$  (Property (*iv*) of Remark 1), we have  $\partial \rho(\mathbf{x}) / \partial x_j \leq 0$ . Thus,  $\rho(\mathbf{x})$  is monotonic decreasing in  $x_j$ , which implies

$$\partial G_k^i((\mathbf{x} + \mathbf{e}^j) \circ \mathbf{Y}^i) - \partial G_k^i(\mathbf{x} \circ \mathbf{Y}^i) \le 0.$$
(9)

Combine (8) and (9) we have  $\partial \phi(\mathbf{x}) / \partial x_k \leq 0$ . Thus,  $\phi(\mathbf{x})$  is monotonic decreasing in  $x_k$ , leading to the inequality

$$G^{i}((\mathbf{x} + \mathbf{e}^{j}) \circ \mathbf{Y}^{i}) - G^{i}(\mathbf{x} \circ \mathbf{Y}^{i}) \ge G^{i}((\mathbf{x} + \mathbf{e}^{j} + \mathbf{e}^{k}) \circ \mathbf{Y}^{i}) - G^{i}((\mathbf{x} + \mathbf{e}^{k}) \circ \mathbf{Y}^{i}).$$

Consequently, we have

$$G^{i}((\mathbf{x}^{A} + \mathbf{e}^{j}) \circ \mathbf{Y}^{i}) - G^{i}((\mathbf{x}^{A}) \circ \mathbf{Y}^{i}) \ge G^{i}((\mathbf{x}^{B} + \mathbf{e}^{j}) \circ \mathbf{Y}^{i}) - G^{i}((\mathbf{x}^{B}) \circ \mathbf{Y}^{i}),$$
(10)

for any  $A \subset B \subset [m]$  and  $j \notin B$ . Moreover, using (5) from the proof of Proposition 1, since  $A \subset B$ , we have

$$\begin{aligned} G^{i}(\mathbf{x}^{A} \circ \mathbf{Y}^{i}) &\leq G^{i}(\mathbf{x}^{B} \circ \mathbf{Y}^{i}) \\ G^{i}((\mathbf{x}^{A} + \mathbf{e}^{j}) \circ \mathbf{Y}^{i}) &\leq G^{i}((\mathbf{x}^{B} + \mathbf{e}^{j}) \circ \mathbf{Y}^{i}). \end{aligned}$$

Thus,

$$[1+G^{i}((\mathbf{x}^{A}+\mathbf{e}^{j})\circ\mathbf{Y}^{i})][1+G^{i}(\mathbf{x}^{A}\circ\mathbf{Y}^{i})] \leq [1+G^{i}((\mathbf{x}^{B}+\mathbf{e}^{j})\circ\mathbf{Y}^{i})][1+G^{i}(\mathbf{x}^{B}\circ\mathbf{Y}^{i})]$$
(11)

Combine (10) and (11) we obtain (7) and then (6) as desired.

Theorem 1 generalizes the submodularity of the MNL-based MCP problem shown in previous studies (Benati and Hansen 2002). Together with the fact that  $f^{\text{GEV}}(S)$  is monotonic (Proposition 1), we know that a simple greedy local search algorithm will guarantee an (1-1/e) approximation solution, i.e., a greedy will return a solution  $\overline{S}$  such that  $f^{\text{GEV}}(\overline{S}) \ge (1-1/e) \max_{S \in S} f^{\text{GEV}}(S)$  (Nemhauser et al. 1978).

Corollary 1 (Performance guarantee for a greedy heuristic). Under a cardinality constraint, a greedy heuristic algorithm can guarantee an (1-1/e) approximation solution.

#### 3.2. Gradient-based Local Search

Due to the submodularity property, a greedy heuristic can guarantee an (1 - 1/e) approximation solution. In this section, we design a new local search procedure to further improve this greedy solution. Our approach is motivated by the fact that the objective function  $f^{\text{GEV}}(\mathbf{x})$  is differentiable, suggesting that we could use gradient information to direct the search. The general idea to design an iterative procedure, where at each step we build a model function (linear or quadratic) to approximate the objective function using gradient and/or Hessian information. We then maximize the model function to find a new iterate. A key component of our approach is that the model function can be only an adequate representation of the objective function in a local neighbourhood of the current solution. Thus, we only maximize the model function within a restricted region. This approach is inspired by the trust-region method widely used in continuous optimization (Conn et al. 2000).

To start our exposition, let us define a model function based on Taylor series built around a solution candidate  $\overline{\mathbf{x}}$ 

$$f^{\text{GEV}}(\mathbf{x}) \approx f^{\text{GEV}}(\overline{\mathbf{x}}) + \nabla f^{\text{GEV}}(\overline{\mathbf{x}})^{\mathrm{T}}(\mathbf{x} - \overline{\mathbf{x}}) + \frac{1}{2}(\mathbf{x} - \overline{\mathbf{x}})^{\mathrm{T}}\mathbf{B}(\mathbf{x} - \overline{\mathbf{x}}),$$

where **B** is the Hessian matrix or an approximation of it at  $\overline{\mathbf{x}}$ . In our context, the Hessian can be computed easily, but maximizing the model function will involve solving a binary quadratic maximization problem, which is expensive. Thus, we set  $\mathbf{B}_k = 0$ . In other words, we use a linear model function to approximate  $f^{\text{GEV}}(\mathbf{x})$ .

At each iteration, we need to solve the following sub-problem

$$\max_{\mathbf{x}} \quad \nabla f^{\text{GEV}}(\overline{\mathbf{x}})^{\text{T}} \mathbf{x}$$
(P1)

subject to 
$$\sum_{j \in [m]} x_j = C$$
 (12)

$$\sum_{\substack{j \in [m] \\ \mathbf{x} \in \{0, 1\}^m}} |x_j - \overline{x}_j| \le \Delta$$
(13)

where (12) is the cardinality constraint, and (13) is to ensure that the new solution candidate is within a region of size  $\Delta$  around  $\overline{\mathbf{x}}$ . Note that (13) can be linearized as

$$\sum_{j \in [m], \overline{x}_j = 1} (1 - x_j) + \sum_{j \in [m], \overline{x}_j = 0} x_j \le \Delta,$$

so as (P1) becomes a integer linear program, which can be handled by an MILP solver. In the following, we will look closely to (P1) and show that it can be solved to optimality in polynomial time.

Solving Subproblems: We further look into the subproblem of the gradient-based local search (P1) to design an efficient algorithm to solve it. To facilitate our exposition, we first note that the constraint  $\sum_{j\in[m]} |\overline{x}_j - x_j| \leq \Delta$  implies that there are at most  $\Delta/2$  locations that either appears in S or in  $\overline{S}$ , but not in both, where  $S, \overline{S}$  are the subsets representing  $\mathbf{x}$  and  $\overline{\mathbf{x}}$ , respectively. For this reason,  $\Delta$  should be integer and even, and the constraint  $\sum_{j\in[m]} |\overline{x}_j - x_j| \leq \Delta$  is equivalent to  $|S \Delta \overline{S}| \leq \Delta$ , where  $\Delta$  is the symmetric difference operator, i.e.,  $S \Delta \overline{S} = (S \setminus \overline{S}) \cup (\overline{S} \cup S)$ . We therefore can rewrite (P1) as

$$\max_{S \subset [m]} \qquad \sum_{j \in S} d_j \tag{P2}$$

subject to 
$$|S| = C$$
 (14)

$$|S \triangle \overline{S}| \le \Delta,\tag{15}$$

where  $\overline{S} \subset [m]$  is the subset that corresponds to the binary vector  $\overline{\mathbf{x}}$  and  $d_j = \nabla f^{\text{GEV}}(\overline{x})_j, j \in [m]$ . Under the cardinality constraint |S| = C, we see that  $|S \setminus \overline{S}| = |\overline{S} \setminus S| = |S \bigtriangleup \overline{S}|/2$ . The following proposition shows that  $d_j$  are non-negative, for all  $j \in [m]$ .

**Proposition 2.** All the coefficients of the objective function of (P2) are non-negative.

*Proof:* We prove the claim by showing that, for any  $\mathbf{x} \in [0,1]^m$ ,  $\nabla_{\mathbf{x}} f^{\text{GEV}}(\mathbf{x}) \ge 0$ . Given  $j \in [m]$ , taking the derivative of  $f^{\text{GEV}}(\mathbf{x})$  w.r.t.  $x_j$  we have

$$\frac{\partial f^{\text{GEV}}(\mathbf{x})}{\partial x_j} = \sum_{i \in I} \frac{\partial G^i(\mathbf{x} \circ \mathbf{Y}^i)}{\partial x_j} \frac{q_i}{(1 + G^i(\mathbf{x} \circ \mathbf{Y}^i))^2}$$
$$= \sum_{i \in I} \frac{q_i Y_j^i \partial G_j^i(\mathbf{x} \circ \mathbf{Y}^i)}{(1 + G^i(\mathbf{x} \circ \mathbf{Y}^i))^2} \ge 0$$
(16)

where (16) is due to the fact that  $\partial G_j^i(\mathbf{x} \circ \mathbf{Y}^i) > 0$  (Property (*iv*) of Remark 1). We obtain the desired inequality.

In Algorithm 1 we describe our main steps to solve (P2). In Step 1, we find  $\Delta/2$  smallest coefficients  $d_j$  in  $\overline{S}$  and  $\Delta/2$  largest coefficients  $d_j$  in  $[m]\backslash\overline{S}$ . This is motivated by the fact we only seek subsets generated by exchanging at most  $\Delta/2$  elements in  $\overline{S}$  with some outside  $\overline{S}$ . Thus, to achieve best objective values, we should exchange elements of lowest coefficients in  $\overline{S}$  with those of highest coefficients in  $[m]\backslash\overline{S}$ . In the second step,  $\gamma(t)$  represents the best gain obtained by exchanging t elements, and in the third step we just select the best  $\gamma(t)$  to get the best solution. Proposition 3 below shows that Algorithm 1 will efficiently return an optimal solution to (P2).

**Proposition 3.** Algorithm 1 returns an optimal solution to (P1) with complexity  $\mathcal{O}(m\Delta/2)$ .

#### Algorithm 1: Solving sub-problems

# Step 1: Take smallest coefficients in  $\overline{S}$  and largest coefficients in  $[m]\setminus\overline{S}$ Choose  $\sigma_1^1, \ldots, \sigma_{\Delta/2}^1 \in \overline{S}$  and  $\sigma_1^2, \ldots, \sigma_{\Delta/2}^2 \in [m]\setminus\overline{S}$  such that

$$\begin{split} d_{\sigma_1^1} &\leq d_{\sigma_2^1} \leq \ldots \leq d_{\sigma_{\Delta/2}^1} \leq \min_{j \in \overline{S} \setminus \{\sigma_1^1, \ldots, \sigma_{\Delta/2}^1\}} d_j \\ d_{\sigma_1^2} &\geq d_{\sigma_2^2} \geq \ldots \geq d_{\sigma_{\Delta/2}^2} \geq \max_{j \in [m] \setminus \overline{S} \setminus \{\sigma_1^2, \ldots, \sigma_{\Delta/2}^2\}} d_j \end{split}$$

# Step 2: Select the best set for each local region size  $|S \triangle \overline{S}| = 2t$ , for  $t = 1, ..., \Delta/2$ for  $t = 1, ..., \Delta/2$  do

$$\gamma(t) = \sum_{h=1}^{t} \left( d_{\sigma_h^2} - d_{\sigma_h^1} \right)$$

Select  $t^* = \operatorname{argmax}_{t=1,\dots,\Delta/2} \gamma(t)$ 

# Step 3: Return the best solution within the local region  $|S \bigtriangleup \overline{S}| \le \Delta$ Return

$$S^* \leftarrow \overline{S} \cup \{\sigma_1^2, \dots, \sigma_{t^*}^2\} \setminus \{\sigma_1^1, \dots, \sigma_{t^*}^1\}$$

Proof: To prove the convergence, we let S be a feasible solution of (P2), i.e., |S| = C and  $|S \triangle \overline{S}| \le \Delta$ . We will prove that  $\sum_{j \in S} d_j \le \sum_{j \in S^*} d_j$ , where  $S^*$  is the solution returned. from Algorithm 1. Let  $t = |S \triangle \overline{S}|/2$ , then we know that S can be obtained by exchanging t elements between S and  $\overline{S}$ . Let  $\pi_1^1, \ldots, \pi_t^1$  be the indexes of t elements in  $\overline{S}$  that are exchanged with t elements in t, indexed as  $\pi_1^2, \ldots, \pi_t^2$ . We have

$$\sum_{j \in S} d_j = \sum_{j \in \overline{S}} d_j - \sum_{h=1}^t d_{\pi_h^1} + \sum_{h=1}^t d_{\pi_h^2}$$
$$\stackrel{(c)}{\leq} \sum_{j \in \overline{S}} d_j - \sum_{h=1}^t d_{\sigma_h^1} + \sum_{h=1}^t d_{\sigma_h^2}$$
$$= \sum_{j \in \overline{S}} d_j + \gamma(t)$$
$$\stackrel{(d)}{\leq} \sum_{j \in \overline{S}} d_j + \gamma(t^*) = \sum_{j \in S^*} d_j,$$

where (c) is due to the way we select  $\sigma_h^1$  and  $\sigma_h^2$ ,  $h = 1, ..., \Delta/2$ , and (d) is due to the way  $t^*$  is selected. This implies that  $S^*$  is an optimal solution to (P2) as desired.

For the complexity, we see that Step 1 would take  $\mathcal{O}(\Delta/2|\overline{S}| + \Delta/2(m - |\overline{S}|)) = \mathcal{O}(m\Delta/2)$ . Step 2 would require  $\mathcal{O}(\Delta^2/4)$ , which would be much smaller than  $\mathcal{O}(m\Delta/2)$ . Adding all together, the complexity of Algorithm 1 is  $\mathcal{O}(m\Delta/2)$ .

## 3.3. GGX Algorithm

Our main algorithm consists of three main phases. In the first phase (warm up), we perform a greedy heuristic, which can be done by starting from the null set and adding locations one at a time, taking at each step the location that increases  $f^{\text{GEV}}(\cdot)$  the most. This phase finishes when we reach the the maximum capacity, i.e., |S| = C. After this phase, due to the submodularity, it is guaranteed that the obtained solution yields at least a factor (1 - 1/e) times the optimal value. In the second phase, we iteratively solve the sub-problem (P1) to seek better solutions. This phase ends when we cannot find any better solutions. In the last phase, we further enhance the solution obtained by performing a simple greedy local search based on exchanging some locations in the current set S with some others from  $[m] \backslash S$ . We describe the three phases in detail in Algorithm 2.

```
Algorithm 2: GGX algorithm
```

# 1: Greedy heuristics (warm up step)  $S = \emptyset$  $\begin{array}{l} \mathbf{for} \ j = 1, \dots, C \ \mathbf{do} \\ & \int j^* = \mathrm{argmax}_{j \in [m] \setminus S} f^{\mathrm{GEV}}(S \cup \{j\}) \\ & S \leftarrow S \cup \{j^*\} \\ \# \ 2: \ \mathbf{G} radient \text{-based local search} \end{array}$  $k = 0; S^0 = S$ do Solve (P1) based on a local region around  $\mathbf{x}^{S^k}$  to get a new solution candidate  $\overline{S}$  $\begin{array}{l} \text{if } f^{\text{GEV}}(\overline{S}) > f^{\text{GEV}}(S^k) \text{ then} \\ \ \ \, \bigsqcupsin S^{k+1} \leftarrow \overline{S} \\ \text{else} \\ \ \ \, \bigsqcupsin S^{k+1} = S^k \end{array}$ # 3: **Ex**changing phase do  $\begin{array}{l} (j^*,t^*) = \operatorname*{argmax}_{\substack{j \in S \\ t \in [m] \setminus S}} \left\{ f^{\text{GEV}}(S^k \cup \{t\} \setminus \{j\}) \right\} \\ \overline{S} = S^k \cup \{k^*\} \setminus \{j^*\} \\ \text{if } f^{\text{GEV}}(\overline{S}) > f^{\text{GEV}}(S^k) \text{ then} \\ \buildrel S^{k+1} \leftarrow \overline{S} \\ \text{else} \\ \buildrel S^{k+1} = S^k \\ b \mid a \mid b \mid a \mid 1 \end{array}$ Return  $S^k$ .

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In the context of assortment optimization under GEV models, Mai and Lodi (2019) also propose a gradient-based local search (named as Binary Trust Region - BiTR) procedure to solve the binary nonlinear formulation of the assortment optimization problem. There are major differences between Algorithm 2 and the one proposed in Mai and Lodi (2019). First, our algorithm starts with a greedy heuristic that guarantees an (1 - 1/e) approximation solution, while there is no performance guarantee for the BiTR. Second, we explore the the structure of the MCP under the GEV family, e.g., the coefficients of the sub-problem's objective function are non-negative and we only care about fixed-size subsets, to build more efficient method to solve the sub-problems.

# 4. Numerical Experiments

In this section, we provide experimental results to compare our GGX algorithm with existing approaches. We use three datasets from recent literature (Ljubić and Moreno 2018, Mai and Lodi 2019) and generate instances under three popular discrete choice models, i.e., the MNL, MMNL and nested logit models.

#### 4.1. Experimental Settings

We will compare our algorithm with the standard greedy heuristic (GH - Step 1 of our GGX algorithm), the multicut and singlecut outer-approximation algorithms (MOA and OA) (Mai and Lodi 2019) and the Branch-and-Cut (BC) (Ljubić and Moreno 2018). In particular, the for nested logit instances, we only compare the GGX with GH, OA, MOA approaches, as BC is not designed to handle such instances. Note that for the MNL and MMNL instances, it is possible to formulate the MCP as an MILP and solve by an MILP solver (e.g. IBM's CPLEX). However, as shown in Ljubić and Moreno (2018) and Mai and Lodi (2019), this approach is outperformed by the MOA and BC methods. Thus, we do not include the MILP solver in our experiments.

We use the following three datasets as benchmark instances and we refer the reader to Freire et al. (2015) for more detailed descriptions. These datasets have been also used in some recent MCP studies (Ljubić and Moreno 2018, Mai and Lodi 2019).

• HM14: The dataset includes 15 problems generated randomly in a plane, with  $|I| \in \{50, 100, 200, 400, 800\}$  and  $m \in \{25, 50, 100\}$ .

• ORlib: The dataset includes 11 problems where there are four instances with (|I|, m) = (50, 25), four instances with (|I|, m) = (50, 50) and three instances with (|I|, m) = (1000, 100).

• **PR-NYC** (or **NYC**): the dataset comes from a large-scale park-and-ride location problem in New York city with |I| = 82341 and m = 59. As reported in previous studies, these are the largest and most challenging instances. We employ the same settings of parameters as in previous studies (Ljubic and Moreno 2017, Mai and Lodi 2020). The number of facilities that need to be opened C is varied from 2 to 10. The deterministic part of the utility is defined as  $v_{ij} = -\beta c_{ij}$  for a location  $j \in M$  and  $v_{ij'} = -\beta \alpha c_{ij'}$ for each competitor j', where  $c_{ij}$  is the distance between zone/client  $i \in I$  and location  $j \in [m]$ , the parameter  $\beta$  is the sensitivity of customers about the perceived utilities and  $\alpha$  represents the competitiveness of the competitors. These parameters are chosen as  $\alpha = \{0.01, 0.1, 1\}$  and  $\beta =$  $\{1, 5, 10\}$  for datasets HM14 and ORlib, and  $\alpha = \{0.5, 1, 2\}$  and  $\beta = \{0.5, 1, 2\}$  for the NYC dataset. Therefore, for each discrete choice model chosen, each problem above has 81 different instances and the total numbers of instances for HM14, ORlib, NYC are 972, 891, 81, respectively.

The experiments are done on a PC with processor AMD Ryzen 7-3700X CPU @ 3.80 GHz and 16 gigabytes of RAM. We use MATLAB 2020 to implement and run the algorithms, and we link to IBM ILOG-CPLEX 12.10 to solve MILPs under default settings. We also take the code used in Ljubic and Moreno (2017) to generate results for the MNL and MMNL instances with the BC approach.

#### 4.2. Multinomial Logit - MNL

We take MNL instances from previous work (Ljubić and Moreno 2018, Mai and Lodi 2019) and report numerical results in Table 4.2 below. Each row of the table corresponds to 81 instances and we indicate the largest number of instances solved with the best objective values in **bold**. We use the same settings as in Mai and Lodi (2020). We do not show the CPU times for GH as it runs very fast. The GH finishes 26/27 problems in less than 0.01 seconds and it just needs around 0.5 seconds to finish all the instances of the largest dataset (i.e. the NYC one). On the other hand, solutions obtained by GH are relatively good, in the sense that the percentage gaps between the objective values yielded by those solutions and the best objective values vary only from 0 to 2.94%. In terms of number instances with the best objective values, GGX performs the best as it gives the largest number instances with the best objective values in 26/27 problems. Moreover, GGX solves 81/81instances with the best objective values in 25/27 problems. On the other hand, GGX only requires short CPU times to finish (the average CPU times are always less than 1.5 seconds except for the NYC instances). Furthermore, when comparing GGX with the OA, MOA, and BC approaches, the average CPU times required by GGX are about 78 times lower than OA, 28 times lower than MOA, and 12 times lower than BC. For small instances with  $|I| \leq 100$ , BC achieves good performance. It provides the best objective values for all 81 instances of each problem with the lowest CPU times. However, for larger problem instances (|I| > 100), BC becomes more expensive, especially for the three large problems in the ORlib dataset with (|I|, |M|) = (800, 100) and for the NYC instances (the average CPU times are always more than 90 seconds). The MOA has the best performance

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Instance		m		stances	with b	est obje	ective	Average CPU time (s)				
Instance	I	m	GGX	GH	OA	MOA	BC	GGX	OA	MOA	BC	
OUR	50	25	81	81	81	81	81	0.14	19.15	0.12	0.01	
OUR	50	50	81	81	73	81	81	$0,\!15$	109.45	0.15	0.01	
OUR	50	100	79	79	58	81	81	0.24	188.91	0.32	0.05	
OUR	100	25	81	80	73	81	81	0.14	138.30	0.18	0.01	
OUR	100	50	81	77	69	81	81	0.15	170.80	0.28	0.03	
OUR	100	100	81	77	59	81	81	0.26	187.43	0.60	0.13	
OUR	200	25	81	81	72	81	81	0.14	146.79	0.34	0.02	
OUR	200	50	81	80	64	81	80	0.16	189.46	0.80	0.06	
OUR	200	100	81	77	59	81	81	0.33	235.27	15.99	0.29	
OUR	400	25	81	73	71	81	80	0.14	116.44	0,62	0.04	
OUR	400	50	81	78	60	80	81	0.18	200.32	12.13	0.13	
OUR	400	100	81	73	58	76	81	0.49	291.38	99.45	0.65	
OUR	800	25	81	76	59	81	81	0.14	160.71	2.27	0.11	
OUR	800	50	81	62	59	64	75	0.23	251.8	160.47	0.48	
OUR	800	100	80	72	55	58	75	0.94	363.54	234.67	14.29	
cap101	50	25	81	81	81	81	81	0.14	0.21	0.20	0.01	
cap102	50	25	81	81	81	81	81	0.14	0.24	0.25	0.01	
cap103	50	25	81	81	81	81	81	0.14	0.19	0.23	0.01	
cap104	50	25	81	80	81	81	81	0.14	0.31	0.23	0.01	
cap131	50	50	81	74	81	81	81	0.15	0.37	0.31	0.02	
cap132	50	50	81	81	81	81	81	0.15	0.32	0.35	0.02	
cap133	50	50	81	80	81	81	81	0.15	0.38	0.34	0.02	
cap134	50	50	81	81	81	81	81	0.15	0.36	0.34	0.02	
capa	1000	100	81	58	81	81	81	1.31	141.10	226.68	114.74	
capb	1000	100	81	61	68	66	81	1.23	113.27	220.45	94.72	
capc	1000	100	81	53	81	81	81	1.42	145.22	232.68	149.88	
NYC	82341	59	81	72	77	81	80	33.87	164.01	2.32	161.71	
Average			80.89	75.19	71.30	78.74	80.44					

 Table 1
 Numerical results for MNL instances, grouped by the problem name (81 instances per row).

for the NYC dataset, as it only requires 2.32 seconds to give the best objective values for all the 81 instances. In general, GGX achieves the best performance for the MNL instances, as compared to the other approaches.

#### 4.3. Mixed Logit - MMNL

In this section, we report numerical results for MMNL instances. To generate such instances, we assume that each utility  $v_{ij}$ ,  $i \in I, j \in [m]$ }, contains a random error component that follows a normal distribution of zero mean. We also assume that the variance of the random number is proportional to the distance  $c_{ij}$ . More precisely, each  $v_{ij}$  associated with customer zone (or client)  $i \in I$  and location  $j \in [m]$  is defined as  $v_{ij} = -\theta c_{ij} + c_{ij}\tau_{ij}/3$ , where  $\tau_{ij}$  is a standard normal random number. We also keep the utilities associated with the competitors deterministic. For each problem, we approximate the objective function by the Monte Carlo method. To do so, we choose a sample size N = 100 for the HM14 and ORlib datasets and N = 10 for the NYC one. For the latter, we only choose small N because the NYC problem is already large even under the MNL model. As mentioned, we consider and solve these MMNL instances as extended MNL ones, in which the

number of customer zones is 5000, 5000 and 823,410 for instances from the HM14, ORlib and NYC datasets, respectively. We give a CPU time budget of 600 seconds for all instances.

In Table 2, for each problem we report the number of instances with the best objective values and the average CPU times over 81 instances for five approaches, i.e., GGX, GH, OA, MOA, and BC. We indicate in bold the largest numbers of instances solved with the best objective values. The results clearly show that GGX generally outperforms other approaches. More precisely, GGX manages to return best objective values for all instances considered (i.e. 2187/2187 instances). Moreover, GH also performs very well, in the sense that the percentage gaps between the objective values given by GH and the best objective values only vary from 0% to 2.92%. In terms of CPU time, GH is still the fastest approach when it just requires less than 2 seconds to solve every instance except the NYC ones, which take only about 6.5 seconds in average. The GGX approach, even though being slower than the GH, but is still much faster than the others. We also observe that the OA, MOA, and BC approaches need much more time to solve MMNL instances, as compared to solving the MNL instances. The average CPU times required by these three approaches are more than 250 seconds. In overall, GGX dominates GH in terms of returned objective value, and outperforms OA, MOA and BC in terms of both returned objective value and CPU time.

#### 4.4. Nested Logit Model

This section reports numerical results for nested logit instances. We perform a comparison between 4 approaches, namely, the GGX, GH, OA and MOA algorithms. We do not include the BC approach in this experiment, as it is not designed to handle nested logit instances. For the OA and MOA approaches, since it is quite straightforward to generate outer-approximation cuts using gradient information, we apply these algorithms to solve the nested logit instances to see how they perform. Note that, in the context, the objective function is no-longer concave, thus OA and MOA become heuristic with no performance guarantee, to the best of our knowledge. To generate nested logit instances, we build a customer nested logit model by partitioning the set of locations into L = 5 different and disjoint groups of equal size. In particular, the NYC dataset has 59 locations (m = 59), so for this problem we partition the locations into four groups with 10 locations and one with 9 locations. We also choose the nested logit parameters as  $\mu = (1.1, 1.2, 1.3, 1.4, 1.5)$ , noting that more nests and/or other nested logit parameters can be chosen. Our selections here are just to illustrate the performance of different algorithms in handling GEV instances. We also give a time budget of 600 seconds for all the algorithms.

Table 3 reports comparison results of the four approaches. Each row of the table corresponds to 81 solved instances and we also indicate the largest numbers of instances solved with best objective values in bold. The results clearly show that GGX outperforms the other approaches in terms of

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Dechle	1	m	# instances with best objective					Average CPU time (s)					
Problem	I		GGX	GH	OA	MOA	BC	GGX	GH	OA	MOA	BC	
OUR	50	25	81	81	66	58	81	0.24	0.01	236.36	189.23	1.66	
OUR	50	50	81	81	46	54	81	0.83	0.02	312.73	232.16	6.45	
OUR	50	100	81	81	41	29	81	3.49	0.05	390.60	425.57	40.78	
OUR	100	25	81	75	53	55	81	0.29	0.01	279.68	212.33	11.37	
OUR	100	50	81	81	47	31	81	1.11	0.03	356.28	430.85	36.57	
OUR	100	100	81	75	41	22	78	4.76	0.10	528.64	519.79	133.67	
OUR	200	25	81	81	54	33	81	0.35	0.02	317.28	375.46	47.32	
OUR	200	50	81	77	39	27	78	1.51	0.05	402.01	464.52	126.53	
OUR	200	100	81	77	34	28	77	6.75	0.23	456.24	328.89	174.75	
OUR	400	25	81	77	52	32	77	0.44	0.03	336.85	404.77	138.9	
OUR	400	50	81	74	40	25	74	2.03	0.13	472.38	515.90	237.10	
OUR	400	100	81	63	29	15	63	9.34	0.67	569.4	570.10	325.16	
OUR	800	25	81	66	52	29	68	0.60	0.07	345.27	396.63	229.83	
OUR	800	50	81	73	39	27	74	3.06	0.34	406.18	467.89	341.22	
OUR	800	100	81	72	32	18	72	14.29	1.25	562.63	565.37	421.74	
cap101	50	25	81	70	57	54	77	0.24	0.01	388.27	485.02	276.30	
cap102	50	25	81	72	57	54	80	0.24	0.01	388.19	497.24	268.27	
cap103	50	25	81	66	50	51	78	0.24	0.01	365.41	457.82	218.40	
cap104	50	25	81	67	56	51	74	0.25	0.01	369.85	484.69	268.10	
cap131	50	50	81	81	33	38	81	0.84	0.02	459.78	619.15	425.71	
cap132	50	50	81	80	33	38	80	0.84	0.02	452.13	620.70	422.28	
cap133	50	50	81	81	33	35	81	0.83	0.02	459.68	643.83	435.96	
cap134	50	50	81	81	33	40	81	0.84	0.02	458.49	652.69	429.13	
capa	1000	100	81	81	14	14	81	16.50	1.52	677.73	677.73	600.00	
capb	1000	100	81	81	44	14	81	16.46	1.52	600.91	691.05	600.00	
capc	1000	100	81	81	38	13	81	16.47	1.52	600.91	637.31	600.00	
NYC	82341	59	81	72	75	68	72	114.42	6.50	178.18	135.24	381.48	
Average			81	75.81	44.00	35.30	77.56						

**Table 2** Numerical results for MMNL instances, grouped by the problem name (81 instances per row).

the number of instances solved with the best objective values. More precisely, GGX gives the best objective values for all problem instances while GH only performs the best for 9/27 problems. In terms of CPU time, GGX is not very fast. In particular, for the NYC instances, the average CPU time is about 355.36 seconds and is much larger than the average CPU times required by GH, OA and MOA. The reason is that the objective function in this context is quite expensive to evaluate, as compared to the cases of the MMNL and MNL models, and the *exchanging* procedure of the GGX (Phase 3) requires calculating the objective function several times to find a pair of locations to swap. The GH is still very fast and the returned objective values are pretty close to the best values given by GGX. The percentage gaps between the objective values obtained from GH and the best objective values only vary from 0 to 3.32%. The OA and MOA approaches, even though run very fast, but give bad solutions. This can be explained by the fact that the objective function under a nested logit model is highly non-concave, thus a subgradient cut (or an outer-approximation cut) could potentially remove good solutions during the cutting-plane procedure.

We look more closely to the NYC problem (the largest problem) to see how the algorithm works. In table 4, we report comparison results for the NYC instances in detail. Each row of the table

Problem	I	m	# instances with best objective				Average CPU time (s)				
r robiem	11	m	GGX	GH	OA	MOA	GGX	GH	OA	MOA	
OUR	50	25	81	81	24	20	0.35	0.01	0.15	0.09	
OUR	50	50	81	81	2	0	1.94	0.03	2.83	0.12	
OUR	50	100	81	81	27	27	13.08	0.10	104.39	0.19	
OUR	100	25	81	80	18	15	0.43	0.01	0.66	0.16	
OUR	100	50	81	75	3	0	2.44	0.04	0.39	0.16	
OUR	100	100	81	76	0	1	16.04	0.11	83.31	0.27	
OUR	200	25	81	81	2	0	0.57	0.02	0.80	0.18	
OUR	200	50	81	80	1	0	3.41	0.19	1.85	0.23	
OUR	200	100	81	81	0	0	20.76	0.15	111.70	0.46	
OUR	400	25	81	72	24	20	0.85	0.16	0.75	0.18	
OUR	400	50	81	77	5	0	5.37	0.21	1.83	0.35	
OUR	400	100	81	72	0	0	32.02	0.08	167.55	1.15	
OUR	800	25	81	77	0	0	0.93	0.04	0.89	0.31	
OUR	800	50	81	66	0	0	6.38	0.10	2.14	0.61	
OUR	800	100	81	69	22	18	41.99	0.30	299.35	5.64	
cap101	50	25	81	81	1	0	0.35	0.01	0.09	0.10	
cap102	50	25	81	81	1	0	0.35	0.01	0.10	0.10	
cap103	50	25	81	81	0	0	0.35	0.01	0.09	0.10	
cap104	50	25	81	78	0	0	0.35	0.01	0.09	0.09	
cap131	50	50	81	74	0	0	1.96	0.03	0.10	0.11	
cap132	50	50	81	81	0	0	1.96	0.03	0.11	0.11	
cap133	50	50	81	78	0	0	1.97	0.03	0.11	0.11	
cap134	50	50	81	80	0	0	1.96	0.03	0.10	0.11	
capa	1000	100	81	60	0	0	46.17	0.37	1.02	66.50	
capb	1000	100	81	54	0	0	46.23	0.37	10.91	66.24	
capc	1000	100	81	50	0	0	46.18	0.37	8.38	56.44	
NYC	82341	59	81	72	5	4	355.36	6.89	0.85	0.65	
Average			81	74.78	5.00	3.89					

Table 3 Numerical results for nested-logit instances, grouped by the problem name (81 instances per row).

corresponds to 9 instances with a value of C, varying from 2 to 10. GGX performs the best in terms of objective value, as it gives best objective values for all the instances while GH only gives 6/9best objective values for  $C \in \{2,3,4\}$ . The numbers of instances with best objective values given by OA and MOA are very low. They both have 4 instances with the best objective values when C = 2 and the OA has one more instance with the best objective value when C = 5. This clearly shows that OA and MOA are outperformed by GGX and GH. On the other hand, in terms of CPU time, GGX is much more expensive than the other approaches. The average CPU times required by GGX is about 52 times, 418 times, and 547 times higher than those required by GH, OA, and MOA approaches, respectively. In summary, for these large instances, GH performs much better as compared to OA and MOA, and GGX manages to significantly improve the objective values returned by GH.

# 5. Conclusion

In this paper we have studied the maximum capture problem in facility location where customer behavior is captured by any GEV model. By leveraging the properties of the GEV generating

	1	// :	-			Δ				
			stance		Average					
$\mathbf{C}$	with b	oest o	bjecti	ve values	(	CPU ti	me(s)	)		
	GGX	GH	OA	MOA	GGX	GH	OA	MOA		
2	9	6	4	4	81.90	2.28	1.03	0.74		
3	9	6	0	0	234.77	3.53	1.24	0.83		
4	9	6	0	0	454.85	4.75	1.26	0.94		
5	9	9	1	0	551.34	5.74	1.13	0.67		
6	9	9	0	0	556.38	6.94	1.14	0.60		
7	9	9	0	0	541.49	8.04	0.52	0.54		
8	9	9	0	0	538.21	9.19	0.49	0.52		
9	9	9	0	0	537.17	10.27	0.48	0.51		
10	9	9	0	0	541.68	11.25	0.41	0.52		
Average	9	8	0.56	0.44						

Table 4Comparison results for NYC instances, grouped by C, 9 instances per row.

function, we have showed that the objective function is monotonic and submodular, implying that a simple greedy heuristic can always give a solution whose value is at least (1-1/e) times the optimal values. We have further developed an algorithm based on a greedy heuristic, a gradient-based local search and an exchanging procedure to solve the problem under any GEV model and the MMNL model. We have tested and compared our algorithm with some state-of-the-art algorithms using MNL, MMNL and nested logit instances and our numerical experiments clearly demonstrate the advantages of our approach, in terms of both returned objective value and CPU time.

Our theoretical findings and algorithm can be applied to the maximum capture problem under any GEV model, including the popular MNL model and other complex GEV models in the literature. Future directions would be to formulate and solve a maximum capture problem in the situation that the choice parameters are not known with certainty, or to consider a combination of facility location and security planning under the MNL/MMNL or any GEV models.

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