# Approximate XVA for European claims 

F. Antonelli, A. Ramponi ${ }^{\dagger}$ S. Scarlatti ${ }^{\ddagger}$

July 16, 2020


#### Abstract

We consider the problem of computing the Value Adjustment of European contingent claims when default of either party is considered, possibly including also funding and collateralization requirements.

As shown in Brigo et al. (12], [13]), this leads to a more articulate variety of Value Adjustments (XVA) that introduce some nonlinear features. When exploiting a reducedform approach for the default times, the adjusted price can be characterized as the solution to a possibly nonlinear Backward Stochastic Differential Equation (BSDE). The expectation representing the solution of the BSDE is usually quite hard to compute even in a Markovian setting, and one might resort either to the discretization of the Partial Differential Equation characterizing it or to Monte Carlo Simulations. Both choices are computationally very expensive and in this paper we suggest an approximation method based on an appropriate change of numeraire and on a Taylor's polynomial expansion when intensities are represented by means of affine processes correlated with the asset's price. The numerical discussion at the end of this work shows that, at least in the case of the CIR intensity model, even the simple first-order approximation has a remarkable computational efficiency.


Keywords: Credit Value Adjustment; Defaultable Claims; Counterparty Credit Risk; Wrong Way Risk; XVA; Affine Processes.

## 1 Introduction

Many financial institutions trade contracts in over-the-counter (OTC) markets, their counterparties being other financial institutions or corporate clients. However, many of those contracts are subject, to some extent, to counterparty risk, or in other words, they are subject to some default event concerning the solvency of either one of the parties, that might take place during the lifetime of the contract. These are called defaultable. Initially, the evaluation regarded European options, named vulnerable, when the seller's default was the only risk and two approaches emerged over the years: the structural approach and the reduced form approach.

Historically, the structural approach came first introduced by Johnson and Stulz in [29] when they considered the option as the sole liability of the counterparty. In the same framework, in 31

[^0]Klein discussed more general liability structures, in [32] he included interest rate risk, and in [33] he considered a (stochastic) default barrier depending on the value of the option. More recently, [36] extended this approach to jump-diffusion models, [27] considered multiple correlations, [18] treated it by using copulas.

Then researchers developed the alternative reduced-form approach. For a comprehensive presentation of the topic, we refer the reader to [34]. In [19], and the references therein, one can find a general overview of the approach for defaultable bonds. Later, the approach's mathematical framework was carefully formalized in [5] and [6], and recently [17] and [21] extended it to defaultable claims in Levy market models.

In the last decade, after the financial crisis of 2008-09, the interest in Counterparty Credit Risk increased remarkably, and attention focused on building a general framework to define and evaluate the premium to compensate the risk connected to defaultable products (in particular of Interest Rate Swaps). This premium took the name of Credit Value Adjustment (CVA) in the seminal paper by Zhu and Pykhtin [37, and it defines the appropriate reduction of the default-free value of a portfolio, to compensate for the default risk. This discount became the crucial quantity to take into account when trading derivatives in OTC markets, spurring much research in the field: see, for instance, 4], [10, [25].

Over the years, other value adjustments were introduced in the contract's evaluation, leading to the acronym (X)VA. Here, X stands for $\mathrm{D}=$ debt, $\mathrm{L}=$ liquidity, $\mathrm{F}=$ funding, to include also the risks due to the default of both parties, funding investment strategies, lack of liquidity. We refer the reader to [26] for a comprehensive exposition on the matter. In [24], one might find an updated overview of the recent research directions under investigation. We point out that there the characterization of the adjusted value as the solution of a BSDE is very well explained. In a Markovian setting, the connection between bilateral CVA and Partial Differential Equations (PDEs) is also thoroughly investigated in [15] and further developed in [16].is

In this work, we treat a European claim, whose price is influenced by the default probabilities of either party as well by liquidity, financing, and collateralization risks when exploiting the intensity approach for the default times of both parties.

In a remarkable series of papers, ([12], [13], [14]), Brigo et al. describe in detail how introducing all the value adjustments implies the loss of an explicit expression for the adjusted value. Indeed the BSDE characterizing the contract's value is generally nonlinear and hence hardly solvable. It depends on the asset's price and many other, possibly correlated, factors such as default intensities, interest rate, stochastic volatility, so that even in a Markovian setting, the expectation representing the solution of the associated PDE becomes extremely difficult to evaluate. Hence to provide a numerical approximation, one may resort only to the discretization of the PDE characterizing the solution of the BSDE (see [30]) or to Monte Carlo simulations (as in [13]). Either approach, on average computational resources, results to be computationally very expensive.

We are interested in devising an approximation procedure simple and computationally efficient even in the presence of many stochastic factors, provided we make some modeling choices. Indeed, we suggest to view the evaluation expectation as a smooth function of the correlation parameters and to approximate it by its Taylor polynomial expansion around the zero vector (the independent case), in the hope that the first or second-order are enough to provide an accurate approximation. We apply our method to estimate the price contribution that comes from
considering stochastic default intensities correlated with the underlying's price. We remark, though, that we can straightforward extend the same technique to include further stochastic factors.

To evaluate Taylor polynomial's coefficients, we follow a two-step procedure to exploit, whenever possible, explicit formulae from option and bond's pricing theory. First, we condition the underlying's price with respect to the stochastic factors, retrieving a conditional Black \& Scholes formula. Then, assuming the intensities to be described by affine models, we represent the single terms of the expansion using a change of Numeraire technique (similar to the one in 9]) to disentangle the correlation among the asset's price and the default intensities. The affinity of the processes makes it possible to use a "bond-like" expression for the default component.

To carry out the calculations in detail and to perform the numerical analysis of the method, we represent the intensities by two Cox Ingersoll Ross (CIR) processes. The final section shows the method's efficiency using Monte Carlo simulations as a benchmark.

A strong point of this approach is that it provides a relatively simple method that one can use with many correlated processes. Correlation often destroys any affine property the dynamical system might have, making the Riccati equations/Fourier transform framework inapplicable, and one can resort only to Monte Carlo or PDE's approximations. The latter are both computationally expensive in several dimensions, hence the construction of an alternative with a remarkable gain in computational time, without loss in accuracy, becomes very important.

Our method becomes particularly convenient when the correlation structure (as Monte Carlo simulations point out for the CIR model) seems to follow a linear pattern. In this case, a firstorder Taylor's polynomial is enough to produce an accurate approximation, providing a rather handy evaluation formula. We finally remark that the conditioning and change of numeraire techniques allow us to keep the coefficients' approximations to a minimum. The expansion's zeroth term corresponds to the independent case, and we need to have a semi-explicit formula to evaluate it. This fact forced us to restrict our model choices.

The paper is structured as follows. In the next section, we describe the general problem leading to the BSDE characterization under the reduced-form approach. We specify the model and the two-step evaluation procedure to compute Taylor's approximation in Section 3, while in Section 4, we specialize the calculations when the default intensities are CIR processes. Section 5 concerns the numerical analysis of our results.

## 2 XVA Evaluation of European claims under the intensity approach

We consider a finite time interval $[0, T]$ and a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, endowed with a filtration $\left\{\mathcal{F}_{t}\right\}_{t \in[0, T]}$, augmented with the $\mathbb{P}$-null sets and made right continuous. We assume that all processes have a cádlág version.

The market is described by the interest rate process $r_{t}$ determining the money market account and by an adapted process $X_{t}$ representing an asset log-price (we will specify its dynamics later), which may also depend on additional stochastic factors. We assume

- that the filtration $\left\{\mathcal{F}_{t}\right\}_{t \in[0, T]}$ is rich enough (and possibly more) to support all the stochastic processes that describe the market;
- to be in absence of arbitrage;
- that the given probability $\mathbb{P}$ is a risk-neutral measure, already selected by some criterion.

In this market model (as in [13]) we consider two parties ( $I=$ investor, $C=$ counterparty) exchanging some European claim with default-free payoff $f\left(X_{T}\right)$, where $f$ is a function (not necessarily nonnegative) as regular as needed. We take for granted that the market processes fulfill the necessary integrability hypotheses to guarantee a good definition of all the expectations we are going to write.

Both parties might default, due to some critical credit state, with respective random times $\tau^{1}$ (Counterparty) and $\tau^{2}$ (Investor), which are not stopping times with respect to the filtration $\mathcal{F}_{t}$. In this context we define the filtration $\mathcal{G}_{t}=\mathcal{F}_{t} \vee \mathcal{H}_{t}^{1} \vee \mathcal{H}_{t}^{2}$, where $\mathcal{H}_{t}^{i}=\sigma\left(\mathbf{1}_{\left\{\tau^{i} \leq s\right\}}, s \leq t\right)$, $i=1,2$, which is the smallest filtration extension that makes both random variables stopping times. Moreover, we assume there exists a unique extension of the risk-neutral probability to $\mathcal{G}_{t}$, that we keep denoting by $\mathbb{P}$.

In general, the following fundamental assumption, known as the H-hypothesis (see e.g. [23] and [22] and the references therein), ensures price coherence:

$$
\begin{equation*}
\text { Every } \mathcal{F}_{t}-\text { martingale remains a } \mathcal{G}_{t}-\text { martingale. } \tag{H}
\end{equation*}
$$

By Lemma 7.3.5.1 in [28], (H) is automatically satisfied, under square integrability of the payoff, by the default-free price of any European contingent claim, whence we may affirm that

$$
\begin{aligned}
\mathrm{e}^{\int_{0}^{t} r_{u} d u} \mathrm{e}^{X_{t}} & =\mathbb{E}\left(\mathrm{e}^{\int_{0}^{T} r_{u} d u} \mathrm{e}^{X_{T}} \mid \mathcal{F}_{t}\right)=\mathbb{E}\left(\mathrm{e}^{\int_{0}^{T} r_{u} d u} \mathrm{e}^{X_{T}} \mid \mathcal{G}_{t}\right) \\
\mathrm{e}^{\int_{0}^{t} r_{u} d u} c(t, T) & :=\mathbb{E}\left(\mathrm{e}^{\int_{0}^{T} r_{u} d u} f\left(X_{T}\right) \mid \mathcal{F}_{t}\right)=\mathbb{E}\left(\mathrm{e}^{\int_{0}^{T} r_{u} d u} f\left(X_{T}\right) \mid \mathcal{G}_{t}\right)
\end{aligned}
$$

remain $\mathcal{G}_{t}-$ martingales under $\mathbb{P}$, for all $t \in[0, T]$.
In what follows, to stress the significance of the term "adjustment", we will point the corrections out step by step, with their signs determined by the fact that we are taking the investor?s viewpoint.

We start assuming full knowledge that is we are in the $\mathcal{G}_{t}-$ filtration. The contract makes sense only if the default of either party has not occurred yet at the evaluation time $t$. Denoting by $\tau=\min \left(\tau^{1}, \tau^{2}\right)$, this fact is represented by the indicator function $\mathbf{1}_{\{\tau>t\}}$ to be placed in front of the price.

Either party may default, so a bilateral adjustment is needed. For the moment we assume nothing is recovered at default. Denoting by $\operatorname{CVA}^{0}(t, T)$ the Credit Value Adjustment due to the counterparty's default, this quantity has to act as a discount to the default-free price to balance the investor's risk assumption. On the other hand, the Debt Value Adjustment due to the investor's default, $\mathrm{DVA}^{0}(t, T)$, has to act as an accrual of the default-free price as it compensates the counterparty's risk assumption. So, for the $\mathcal{G}_{t}$-adapted adjusted value of the European claim $c^{\mathcal{G}}(t, T)$, we may write

$$
\begin{equation*}
\mathbf{1}_{\{\tau>t\}} c^{\mathcal{G}}(t, T)=\mathbf{1}_{\{\tau>t\}}\left[c(t, T)-\mathrm{CVA}^{0}(t, T)+\mathrm{DVA}^{0}(t, T)\right] \tag{1}
\end{equation*}
$$

where $\operatorname{CVA}^{0}(t, T)$ and $\operatorname{DVA}^{0}(t, T) \geq 0$.

Now, let us admit the defaulting party might partially compensate for the loss due to his/her default. In this case, we have to include other two nonnegative terms, CVA ${ }^{\text {rec }}(t, T)$ and $\mathrm{DVA}^{r e c}(t, T)$ (respectively for the counterparty and the investor), and we can rewrite the above as

$$
\mathbf{1}_{\{\tau>t\}} c^{\mathcal{G}}(t, T)=\mathbf{1}_{\{\tau>t\}}\left[c(t, T)-\mathrm{CVA}^{0}(t, T)+\mathrm{DVA}^{0}(t, T)+\mathrm{CVA}^{r e c}(t, T)-\mathrm{DVA}^{r e c}(t, T)\right]
$$

Moreover, as explained in [14], the two parties might be asked to collateralize their participation to the contract, they might need to borrow money to finance this participation and/or the risky asset(s) from a repo market to rea, lize their hedging strategies. All this leads to funding and liquidity risks that, again, have to be included for the correct contract's evaluation. Thus, we should write

$$
\begin{align*}
\mathbf{1}_{\{\tau>t\}} c^{\mathcal{G}}(t, T)= & \mathbf{1}_{\{\tau>t\}}\left[c(t, T)-\mathrm{CVA}^{0}(t, T)+\mathrm{DVA}^{0}(t, T)\right. \\
& \left.+\mathrm{CVA}^{r e c}(t, T)-\mathrm{DVA}^{r e c}(t, T)+\operatorname{FVA}(t, T)+\operatorname{LVA}(t, T)\right] \tag{2}
\end{align*}
$$

with $\operatorname{FVA}(t, T), \operatorname{LVA}(t, T) \in \mathbb{R}$.The first represents the Funding Value Adjustment, the second the Liquidity Value Adjustment, and they are both determined by strategy financing and collateralization.

It is then necessary to model these terms to get to a manageable formula. The range of possible choices of mechanisms to include in the formation of prices is quite broad, and we refer the reader again to [12], 13 and 14 for a detailed discussion. Of course, there is an interplay among the different cash flows. For instance, collateralization changes the parties? exposures, the amount of cash borrowed at rate r increases its value at a rate $r_{s}$.

Here we use the following set of assumptions.

1. The claim pays no dividends.
2. The adjustment processes all depend on a close-out value, $\epsilon_{t}$, determined by a contractual agreement. It is natural to consider it $\mathcal{F}_{t}$-adapted since it is established on the basis of the information before default. Usually, it is taken as the default-free price or as the price of the defaultable claim itself.
3. We denote the collateralization process by $C_{s}$ and it is a, possibly time-varying, percentage of the close-out value

$$
C_{s}=\left\{\begin{array}{lll}
\alpha_{s} \epsilon_{s}^{+}, & \text {when due by the counterparty } & 0<\alpha_{s}<1,  \tag{3}\\
\alpha_{s} \epsilon_{s}^{-}, & \text {when due by the investor } & \forall s \in[0, T]
\end{array}\right.
$$

Thus the net exposure is $\left(\epsilon_{s}-C_{s}\right)^{+}=\left(1-\alpha_{s}\right) \epsilon_{s}^{+}$for the investor and $\left(\epsilon_{s}-C_{s}\right)^{-}=\left(1-\alpha_{s}\right) \epsilon_{s}^{-}$ for the counterparty.
Moreover, we assume that collateralizing happens at rate $r_{s}^{c}$.
4. We denote by $R_{1}(s)$ the recovery percentage of the close-out value in case of counterparty 's default and by $R_{2}(s)$, when investor's default occurs. Mirror-like we define the Loss Given Default as $L_{i}(s)=\left(1-R_{i}(s)\right), i=1,2$.

| Symbol | Definition | Symbol | Definition |
| :---: | :---: | :---: | :---: |
| $r_{t}$ | Risk-free rate | $\tau_{1}$ | Default time Counterparty |
| $r_{t}^{\phi}$ | Funding rate | $\tau_{2}$ | Default time Investor |
| $r_{t}^{c}$ | Collateral rate | $\epsilon_{t}$ | Close-out value |
| $h_{t}$ | Hedging rate | $\lambda_{t}^{i}$ | Default intensities |
| $\alpha_{t}$ | collateralization level | $f(\cdot)$ | Option payoff |
| $R_{i}(t)$ | Recovery rates $i=1,2$ | $\bar{v}_{t}$ | $\int_{t}^{T} v_{s} d s$ |
| $\tilde{r}_{t}$ | $r_{t}^{\phi}-h_{t}$ | $\hat{r}_{t}$ | $r_{t}^{\phi}-r_{t}^{c}$ |

Table 1: Summary of notations.
5. To build investing strategies, the parties may invest in the riskless asset at a rate $r^{\phi}$ and the risky asset(s) at a rate $h_{t}$, the latter happening in a parallel repo market. We denote by $\phi_{u}$ the quantity of riskless asset the contract globally requires (either positive or negative) and by $H_{t}$ the value of the portion of the risky asset(s) (either positive or negative) traded on the repo market.
Since at the same time the investor's purchase generates wealth at a rate $r_{s}$, and as well the borrow/sale of the risky asset generates wealth at a rate $r^{\phi}$, also this aspect will have to be taken into account.

As we said, the recovery and the collateral agreements are usually a fraction of the close-out value, and therefore they should be $\mathcal{F}_{t}$-adapted. On the contrary, the funding and hedging processes $(\phi, H)$ might incorporate the contribution of the default events, and therefore they could be a priori $\mathcal{G}_{t}$-adapted.

Finally, the price should be given by the three components

$$
\begin{equation*}
c^{\mathcal{G}}(t, T)=\phi_{t}+H_{t}+C_{t} . \tag{4}
\end{equation*}
$$

Following the crystal clear exposition in [13] (but also in [12] and [14] ), keeping in mind hypothesis (H) and (4), one can obtain the following BSDE in the $\mathcal{G}$-filtration

$$
\begin{align*}
& \mathbf{1}_{\{\tau>t\}} c^{\mathcal{G}}(t, T)=\mathbf{1}_{\{\tau>t\}}\left\{\mathbb{E}\left[\mathrm{e}^{-\int_{t}^{T} r_{u} d u} f\left(X_{T}\right) \mathbf{1}_{\{\tau>T\}} \mid \mathcal{G}_{t}\right]\right. \\
& +\mathbb{E}\left[\mathrm{e}^{-\int_{t}^{\tau} r_{u} d u} \mathbf{1}_{\{\tau \leq T\}}\left(\epsilon_{\tau}-\left(1-\alpha_{\tau}\right)\left[L_{1}(\tau) \epsilon_{\tau}^{+} \mathbf{1}_{\left\{\tau^{1}=\tau\right\}}-L_{2}(\tau) \epsilon_{\tau}^{-} \mathbf{1}_{\left\{\tau^{2}=\tau\right\}}\right]\right) \mid \mathcal{G}_{t}\right]  \tag{5}\\
& \left.+\left[\int_{t}^{\tau \wedge T} \mathrm{e}^{-\int_{t}^{s} r_{u} d u}\left\{\left[r_{s}-r_{s}^{\phi}\right] c^{\mathcal{G}}(s, T) d s+\left[r_{s}^{\phi}-r_{s}^{c}\right] C_{s}+\left[h_{s}-r_{s}\right] H_{s}\right\} d s \mid \mathcal{G}_{t}\right]\right\} .
\end{align*}
$$

The random variables $\tau^{i}, i=1,2$, are not $\mathcal{F}_{t}$-stopping times, hence the traders can observe only whether the default events happened or not, conditioned to the available information. Thus, any risk-neutral evaluation that would naturally take place in the $\mathcal{G}$-filtration, needs translating in terms of $\left\{\mathcal{F}_{t}\right\}$. For that, we have the following well known Key Lemma, to be found in [6] or [4], just to quote some references.

Lemma 2.1 Given a $\mathcal{G}_{t}$-stopping time $\tau$, for any integrable $\mathcal{G}_{T}$-measurable r.v. $Y$, the following equality holds

$$
\begin{equation*}
\mathbb{E}\left[\mathbf{1}_{\{\tau>t\}} Y \mid \mathcal{G}_{t}\right]=\mathbf{1}_{\{\tau>t\}} \frac{\mathbb{E}\left[\mathbf{1}_{\{\tau>t\}} Y \mid \mathcal{F}_{t}\right]}{\mathbb{P}\left(\tau>t \mid \mathcal{F}_{t}\right)} \tag{6}
\end{equation*}
$$

This Lemma calls for the conditional distributions of the default times that we are going to treat within the (Cox) reduced-form framework. We denote the conditional distribution of the random times as

$$
\begin{equation*}
F_{t}^{i}=\mathbb{P}\left(\tau^{i} \leq t \mid \mathcal{F}_{t}\right), \quad i=1,2 \quad \forall t \geq 0, \tag{7}
\end{equation*}
$$

and we assume that they both verify $F_{t}^{i}<1$. Hence we can define the corresponding $\mathcal{F}$ - hazard processes of the $\tau^{i}$,s as

$$
\begin{equation*}
\Gamma_{t}^{i}:=-\ln \left(1-F_{t}^{i}\right) \quad \Rightarrow \quad F_{t}^{i}=1-\mathrm{e}^{-\Gamma_{t}^{i}} \quad \forall t>0, \quad \Gamma_{0}=0 \tag{8}
\end{equation*}
$$

which we assume to be differentiable, defining the so-called $\mathcal{F}_{t}$-adapted intensity processes $\lambda^{i}$ by

$$
\Gamma_{t}^{i}=\int_{0}^{t} \lambda_{u}^{i} d u \quad \Rightarrow \quad F_{t}^{i}=1-\mathrm{e}^{-\int_{0}^{t} \lambda_{u}^{i} d u}
$$

As in the classical framework of [20], we assume conditional independence for the default times, i.e. for any $t>0$ and $t_{1}, t_{2} \in[0, t]$

$$
\mathbb{P}\left(\tau^{1}>t_{1}, \tau^{2}>t_{2} \mid \mathcal{F}_{t}\right)=\mathbb{P}\left(\tau^{1}>t_{1} \mid \mathcal{F}_{t}\right) \mathbb{P}\left(\tau^{2}>t_{2} \mid \mathcal{F}_{t}\right),
$$

so that we may conclude that $\lambda_{t}:=\lambda_{t}^{1}+\lambda_{t}^{2}$ is the intensity process of $\tau=\inf \left\{\tau^{1}, \tau^{2}\right\}$.
Remark 2.2 It is worth noting that the independence assumption certainly simplifies computations, but it does not take into consideration default contagion effects. Within the intensity framework, more realistic models allowing default dependence were recently proposed (see [77, [8] and the references therein), and we remark that we could extend our method to the correlated case, provided we introduce an additional parameter.

Exploiting the key Lemma and the intensity processes as in [3], the above equation gets projected on the smaller filtration, obtaining

$$
\begin{align*}
& \mathbf{1}_{\{\tau>t\}} \mathcal{C}^{\mathcal{G}}(t, T)=\mathbf{1}_{\{\tau>t\}} \mathbb{E}\left[\mathrm{e}^{-\int_{t}^{T}\left(r_{u}+\lambda_{u}\right) d u} f\left(X_{T}\right)\right. \\
& +\int_{t}^{T} \mathrm{e}^{-\int_{t}^{s}\left(r_{u}+\lambda_{u}\right) d u}\left[\lambda_{s} \epsilon_{s}-\left(1-\alpha_{s}\right)\left(\lambda_{s}^{1} L_{1}(s) \epsilon_{s}^{+}-\lambda_{s}^{2} L_{2}(s) \epsilon_{s}^{-}\right)\right] d s  \tag{9}\\
& \left.+\int_{t}^{T} \mathrm{e}^{-\int_{t}^{s}\left(r_{u}+\lambda_{u}\right) d u}\left[\left(r_{s}-r_{s}^{\phi}\right) c^{\mathcal{G}}(s, T)+\left(r_{s}^{\phi}-r_{s}^{c}\right) \alpha_{s} \epsilon_{s}+\left(h_{s}-r_{s}\right) H_{s}\right] d s \mid \mathcal{F}_{t}\right]
\end{align*}
$$

Applying the Key Lemma and Lemma 2 in [13] (extension of the key lemma) to (9), we may conclude that there exists an $\mathcal{F}_{t}$-adapted adjusted price of the European claim, $c^{a}(t, T)$ and an adapted hedging strategy (the part hedging the default-free risks) $\tilde{H}$ such that

$$
c^{a}(t, T) \mathbf{1}_{\{\tau>t\}}=c^{\mathcal{G}}(t, T) \mathbf{1}_{\{\tau>t\}}, \quad \tilde{H}_{t} \mathbf{1}_{\{\tau>t\}}=H_{t} \mathbf{1}_{\{\tau>t\}},
$$

and we may conclude that on $\{\tau>t\}$

$$
\begin{align*}
& \mathbf{1}_{\{\tau>t\}} c^{a}(t, T)=\mathbf{1}_{\{\tau>t\}} \mathbb{E}\left[\mathrm{e}^{-\int_{t}^{T}\left(r_{u}+\lambda_{u}\right) d u} f\left(X_{T}\right)\right. \\
& +\int_{t}^{T} \mathrm{e}^{-\int_{t}^{s}\left(r_{u}+\lambda_{u}\right) d u}\left[\lambda_{s} \epsilon_{s}-\left(1-\alpha_{s}\right)\left(\lambda_{s}^{1} L_{1}(s) \epsilon_{s}^{+}-\lambda_{s}^{2} L_{2}(s) \epsilon_{s}^{-}\right)\right] d s  \tag{10}\\
& \left.+\int_{t}^{T} \mathrm{e}^{-\int_{t}^{s}\left(r_{u}+\lambda_{u}\right) d u}\left[\left(r_{s}-r_{s}^{\phi}\right) c^{a}(s, T)+\left(r_{s}^{\phi}-r_{s}^{c}\right) \alpha_{s} \epsilon_{s}+\left(h_{s}-r_{s}\right) \tilde{H}_{s}\right] d s \mid \mathcal{F}_{t}\right]
\end{align*}
$$

Remark 2.3 Following [14], a few issues about the above BSDE need to be addressed.

1. We remark that this equation has a unique strong solution as long as we take square integrable close-out value and intensities and, for instance, we assume the processes $r, r^{c}, r^{\phi}, h$ to be bounded. This is going to be our standing assumption.
2. The process $\tilde{H}_{t}$ is linked to the solution of the BSDE. If we restrict to a diffusion setting with deterministic coefficients, the theory of BSDE's gives an explicit representation for the process $\tilde{H}$. To deal with this, we extend the observation made in [14] when they assume deterministic intensities.
More precisely, we assume that the stock price, $S_{u}=\mathrm{e}^{X_{u}}$, and the intensities processes, under the given risk-neutral probability, verify

$$
\begin{aligned}
d S_{u} & =r_{u} S_{u} d u+\sigma\left(t, S_{u}\right) d Y_{u}, \quad \text { and } \\
d \lambda_{u}^{i} & =a_{i}\left(u, \lambda_{u}^{i}\right) d u+b_{i}\left(u, \lambda_{u}^{i}\right) d B_{u}^{i}, \quad i=1,2
\end{aligned}
$$

for correlated Brownian motions $Y, B^{1}, B^{2}$ and deterministic coefficients $\sigma(u, x), a_{i}(u, \lambda), b_{i}(u, \lambda)$ chosen to ensure the existence and uniqueness of strong solutions. Then (10) can be equivalently written on $\{\tau>t\}$ as

$$
\begin{align*}
& \mathrm{e}^{-\int_{0}^{t}\left(r_{u}+\lambda_{u}\right) d u} c^{a}(t, T)=c^{a}(0, T)+\int_{0}^{t} Z_{s} d Y_{s}+M_{t} \\
& -\int_{0}^{t} \mathrm{e}^{-\int_{0}^{s}\left(r_{u}+\lambda_{u}\right) d u}\left[\lambda_{s} \epsilon_{s}-\left(1-\alpha_{s}\right)\left(\lambda_{s}^{1} L_{1}(s) \epsilon_{s}^{+}-\lambda_{s}^{2} L_{2}(s) \epsilon_{s}^{-}\right)\right] d s  \tag{11}\\
& -\int_{0}^{t} \mathrm{e}^{-\int_{0}^{s}\left(r_{u}+\lambda_{u}\right) d u}\left[\left(r_{s}-r_{s}^{\phi}\right) c^{a}(s, T)+\left(r_{s}^{\phi}-r_{s}^{c}\right) \alpha_{s} \epsilon_{s}+\left(h_{s}-r_{s}\right) \tilde{H}_{s}\right] d s
\end{align*}
$$

where $Z$ is the component of the solution of the BSDE coming from the martingale representation theorem, while $M$ is a martingale depending on the intensities and possibly on some other stochastic factors (again represented by diffusions). In this context, $c^{a}(t, T)$ is a deterministic function of the state variables, and assuming enough regularity of this function, $\tilde{H}$ should represent the $\delta$-hedging of the contract

$$
\tilde{H}_{u}=\frac{\partial c^{a}(u, T)}{\partial S} S_{u}
$$

On the other hand, the Markovian setting gives also that $Z$ is given by

$$
Z_{u}=\sigma\left(u, S_{u}\right) \frac{\partial c^{a}(u, T)}{\partial S} \Rightarrow \quad \tilde{H}_{u}=\frac{S_{u}}{\sigma\left(u, S_{u}\right)} Z_{u}
$$

provided that $\sigma(u, x)>0$ for all $u, x$.
From now on, in addition to the hypotheses stated in the first of the previous remarks, we assume that

$$
0<\sigma_{0} x \leq \sigma(u, x) \leq \sigma_{1} x, \quad \forall u, x
$$

for some constants $\sigma_{0}$ and $\sigma_{1}$.
This implies, as in [13] or [14, that we may apply Girsanov's theorem to change the Brownian motion driving the above BSDE to include the term $\tilde{H}$. Indeed,

$$
B_{t}=Y_{t}+\int_{0}^{t}\left(r_{u}-h_{u}\right) \frac{S_{u}}{\sigma\left(u, S_{u}\right)} d u
$$

is a new Brownian motion with respect to the probability defined by the Radon-Nykodim derivative

$$
\frac{d \mathbb{Q}}{d \mathbb{P}}=\mathrm{e}^{-\int_{0}^{T}\left(r_{u}-h_{u}\right) \frac{S_{u}}{\sigma\left(u, S_{u}\right)} d Y_{u}+\frac{1}{2} \int_{0}^{T}\left(r_{u}-h_{u}\right)^{2} \frac{S_{u}^{2}}{\sigma^{2}\left(u, S_{u}\right)} d u}
$$

which verifies the Novikov condition. Consequently, under $\mathbb{Q}$ the asset price equation and (11) become

$$
\begin{align*}
& d S_{t}=S_{t} h_{t} d t+\sigma\left(t, S_{t}\right) d B_{t} \\
& \mathrm{e}^{-\int_{0}^{t}\left(r_{u}+\lambda_{u}\right) d u} c^{a}(t, T)=c^{a}(0, T)+\int_{0}^{t} Z_{s} d B_{s}+M_{t} \\
& -\int_{0}^{t} \mathrm{e}^{-\int_{0}^{s}\left(r_{u}+\lambda_{u}\right) d u}\left[\lambda_{s} \epsilon_{s}-\left(1-\alpha_{s}\right)\left(\lambda_{s}^{1} L_{1}(s) \epsilon_{s}^{+}-\lambda_{s}^{2} L_{2}(s) \epsilon_{s}^{-}\right)\right] d s  \tag{12}\\
& -\int_{0}^{t} \mathrm{e}^{-\int_{0}^{s}\left(r_{u}+\lambda_{u}\right) d u}\left[\left(r_{s}-r_{s}^{\phi}\right) c^{a}(s, T)+\left(r_{s}^{\phi}-r_{s}^{c}\right) \alpha_{s} \epsilon_{s}\right] d s .
\end{align*}
$$

Passing again to the conditional expectation and multiplying both sides by $\mathrm{e}_{0}^{t}\left(r_{u}+\lambda_{u}\right) d u$, we obtain

$$
\begin{align*}
& \mathbf{1}_{\{\tau>t\}} c^{a}(t, T)=\mathbf{1}_{\{\tau>t\}} \mathbb{E}_{\mathbb{Q}}\left[\mathrm{e}^{-\int_{t}^{T}\left(r_{u}+\lambda_{u}\right) d u} f\left(X_{T}\right)\right. \\
& +\int_{t}^{T} \mathrm{e}^{-\int_{t}^{s}\left(r_{u}+\lambda_{u}\right) d u}\left[\lambda_{s} \epsilon_{s}-\left(1-\alpha_{s}\right)\left(\lambda_{s}^{1} L_{1}(s) \epsilon_{s}^{+}-\lambda_{s}^{2} L_{2}(s) \epsilon_{s}^{-}\right)\right] d s  \tag{13}\\
& \left.+\int_{t}^{T} \mathrm{e}^{-\int_{t}^{s}\left(r_{u}+\lambda_{u}\right) d u}\left[\left(r_{s}-r_{s}^{\phi}\right) c^{a}(s, T)+\left(r_{s}^{\phi}-r_{s}^{c}\right) \alpha_{s} \epsilon_{s}\right] d s \mid \mathcal{F}_{t}\right] .
\end{align*}
$$

The latter equation is linear or nonlinear depending on the choice of $\epsilon_{s}$. In the literature there are fundamentally two possible choices: either $\epsilon_{s}=c(s, T)$ (the default-free value of the claim) or $\epsilon_{s}=c^{a}(s, T)$.

The first choice will always give a solvable linear BSDE. With the second choice, we might obtain a solvable linear BSDE if the adjusted value stays always nonnegative (or nonpositive), otherwise the negative and positive parts generate a nonlinear, not explicitly solvable, BSDE.

To exploit explicit formulas, when possible, we decide to choose always $\epsilon_{s}=c(s, T)$ (that corresponds to asking collateralization proportional to the default-free price rather than to the current price), to guarantee the solvability of the BSDE for all European claims.

With this choice (13) becomes on $\{\tau>t\}$

$$
c^{a}(t, T)=\mathbb{E}_{\mathbb{Q}}\left[\mathrm{e}^{-\int_{t}^{T}\left(r_{u}+\lambda_{u}\right) d u} f\left(X_{T}\right)+\int_{t}^{T} \mathrm{e}^{-\int_{t}^{s}\left(r_{u}+\lambda_{u}\right) d u}\left[\Psi_{s}+\left(r_{s}-r_{s}^{\phi}\right) c^{a}(s, T)\right] d s \mid \mathcal{F}_{t}\right]
$$

where

$$
\Psi_{s}=\left[\lambda_{s}+\left(r_{s}^{\phi}-r_{s}^{c}\right) \alpha_{s}\right] c(s, T)-(1-\alpha)\left[\lambda_{s}^{1} L_{1}(s) c(s, T)^{+}-\lambda_{s}^{2} L_{2}(s) c(s, T)^{-}\right],
$$

which can be solved obtaining

$$
\begin{equation*}
\mathbf{1}_{\{\tau>t\}} c^{a}(t, T)=\mathbf{1}_{\{\tau>t\}} \mathbb{E}_{\mathbb{Q}}\left[\mathrm{e}^{-\int_{t}^{T}\left(r_{u}^{d}+\lambda_{u}\right) d u} f\left(X_{T}\right)+\int_{t}^{T} \mathrm{e}^{-\int_{t}^{s}\left(r_{u}^{\phi}+\lambda_{u}\right) d u} \Psi_{s} d s \mid \mathcal{F}_{t}\right] \tag{14}
\end{equation*}
$$

We remark we could have proposed a more general situation, considering different collateral rates and recovery processes and close-out values for the two parties. All these generalizations would have led to a more articulate, but not mathematically more difficult, equation. Indeed, the main nonlinearity is due to the recovery terms, once one decides to consider as close-out value the adjusted price of the contract.

In the next section, we introduce the market model and in the following two, we describe our evaluation procedure by steps, leading to approximations handier than Monte Carlo simulations.

Remark 2.4 We remark that if we are in absence of default of either part, $\lambda^{1}=\lambda^{2}=0$, funding, collateralization, rehypothecation are considered and the close-out value is taken equal to the contract's current value, then the solution of (13) becomes

$$
c^{a}(t, T)=\mathbb{E}_{\mathbb{Q}}\left[\mathrm{e}^{-\int_{t}^{T}\left[\left(1-\alpha_{u}\right) r_{u}^{\phi}+\alpha_{u} r_{u}^{c}\right] d u} f\left(X_{T}\right) \mid \mathcal{F}_{t}\right],
$$

which reduces to the usual Black $\mathcal{B}$ Scholes setting, only if the collateralization, funding, repo rates all coincide with the risk-free rate.

From now on we omit the probability $\mathbb{Q}$ in the notation of the expectation and we will always be referring to (14).

## 3 The evaluation procedure

In what follows we specify the market model, where the asset price is represented as a stochastic exponential, and the default intensities are assumed to be affine processes. Then we illustrate a conditioning procedure that helps to exploit explicit expressions for the default-free price, as it happens in the Black \& Scholes model when considering European Vanilla Options or Futures. Finally, we apply a change of Numeraire that allows using the well-known expression for ZeroCoupon Bonds when interest rates are affine processes. This last step helps to disentangle the contribution due to the intensities and the one coming from the derivative.

In section 4 we specialize this procedure to the case when the intensities are CIR processes. We will be able to derive semi-explicit formulas, that we approximate by means of a Taylor's expansion with respect to the correlation parameters, up to the first or second order. We do not consider the other very popular affine Vasicek model since it is well known explicit formulas can be derived in this case.

### 3.1 The model

We keep denoting by $t \in[0, T]$ the initial time and we make the following simplifying hypotheses for (14):

1. all the rates, $r, r^{c}, r^{\phi}, h$ are deterministic;
2. for $i=1,2,(1-\alpha) L_{i}$ are constant and we will keep denoting them simply by $L_{i}$.

So we have $\Psi_{s}=\left[\lambda_{s}+\left(r_{s}^{\phi}-r_{s}^{c}\right) \alpha\right] c(s, T)-\left[\lambda_{s}^{1} L_{1} c(s, T)^{+}-\lambda_{s}^{2} L_{2} c(s, T)^{-}\right]$. We also choose the following model for our state variables for fixed initial conditions $\left(t, x, \lambda_{1}, \lambda_{2}\right) \in \mathbb{R}^{+} \times \mathbb{R} \times \mathbb{R}^{+} \times \mathbb{R}^{+}$, $\forall s \in[t, T]$

$$
\begin{align*}
X_{s} & =x+\int_{t}^{s}\left(h_{u}-\frac{\sigma^{2}}{2}\right) d u+\sigma\left(B_{s}-B_{t}\right) \quad x \in \mathbb{R}  \tag{15}\\
\lambda_{s}^{i} & =\lambda_{i}+\int_{t}^{s}\left[\gamma_{u}^{i} \lambda_{u}^{i}+\beta_{u}^{i}\right] d u+\int_{t}^{s}\left[\eta_{u}^{i} \lambda_{u}^{i}+\delta_{u}^{i}\right]^{\frac{1}{2}} d B_{u}^{i}, \quad \lambda_{i}>0, \quad i=1,2 \tag{16}
\end{align*}
$$

where $\sigma>0$ and $r, \gamma^{i}, \beta^{i}, \eta^{i}, \delta^{i}, i=1,2$ are all deterministic bounded functions of time, while $\left(B^{1}, B^{2}, B^{3}\right)$ is a 3 -dimensional Brownian motion, with

$$
B_{s}=\rho_{1} B_{s}^{1}+\rho_{2} B_{s}^{2}+\sqrt{1-\rho_{1}^{2}-\rho_{2}^{2}} B_{s}^{3}, \quad \rho_{1}^{2}+\rho_{2}^{2} \leq 1
$$

The processes $X_{s}, \lambda_{s}^{1}, \lambda_{s}^{2}$ are Markovian, therefore $c(s, T)$ and $c^{a}(s, T)$ are deterministic functions respectively of the state variables $X$ and $\left(X, \lambda^{1}, \lambda^{2}\right)$, and depending also on the correlation parameters $\boldsymbol{\rho}=\left(\rho_{1}, \rho_{2}\right)$.

For any $t \leq s \leq T$, we define the processes

$$
\begin{equation*}
N_{i}(u, s):=\mathbb{E}\left(\mathrm{e}^{-\int_{t}^{s} \lambda_{v}^{i} d v} \mid \mathcal{F}_{u}\right), \quad i=1,2 \tag{17}
\end{equation*}
$$

which are martingales for $t \leq u \leq s$ and that, having chosen the intensities as affine processes, by Fourier transform have an explicit expression for their initial values

$$
\begin{equation*}
N_{i}(t, s)=\mathrm{e}^{A_{i}(t, s) \lambda_{i}+B_{i}(t, s)} \Rightarrow N_{i}(u, s)=\mathrm{e}^{A_{i}(u, s) \lambda_{i}+B_{i}(u, s)-\int_{t}^{u} \lambda_{v}^{i} d v} \tag{18}
\end{equation*}
$$

where $\lambda_{i}$ is the initial condition of the intensity and $A_{i}$ and $B_{i}$ are deterministic functions verifying a set of Riccati equations. We remark that by independence of the intensities we also have

$$
N(u, s):=\mathbb{E}\left(\mathrm{e}^{-\int_{t}^{s} \lambda_{v} d v} \mid \mathcal{F}_{u}\right)=\mathbb{E}\left(\mathrm{e}^{-\int_{t}^{s}\left(\lambda_{v}^{1}+\lambda_{v}^{2}\right) d v} \mid \mathcal{F}_{u}\right)=N_{1}(u, s) N_{2}(u, s)
$$

which is still a martingale as product of independent martingales. By applying Itô's formula, the dynamics of these martingales are given by

$$
\begin{align*}
d N_{i}(u, s) & =N_{i}(u, s) A_{i}(u, s)\left(\eta_{u}^{i} \lambda_{u}^{i}+\delta_{u}^{i}\right)^{\frac{1}{2}} d B_{u}^{i} \\
d N(u, s) & =N(u, s)\left[A_{1}(u, s)\left(\eta_{u}^{1} \lambda_{u}^{1}+\delta_{u}^{1}\right)^{\frac{1}{2}} d B_{u}^{1}+A_{2}(u, T)\left(\eta_{u}^{2} \lambda_{u}^{2}+\delta_{u}^{2}\right)^{\frac{1}{2}} d B_{u}^{2}\right] \tag{19}
\end{align*}
$$

In some classical specifications of the affine modeling framework:

- $\gamma_{u}^{i}=-\gamma_{i}, \beta^{i}(\lambda)=\gamma_{i} \theta_{i}, \quad \delta_{u}^{i}=\delta_{i}^{2}, \eta_{u}^{i}=0$ (Vasicek)
- $\gamma_{u}^{i}=-\gamma_{i}, \beta_{i}(\lambda)=\gamma_{i} \theta_{i}, \delta_{u}^{i}=0, \eta_{u}^{i}=\eta_{i}^{2} \quad(\mathrm{CIR})$,
for $\gamma_{i}, \theta_{i}, i=1,2$ positive constants, it is possible to compute $A_{i}(t, s)$ and $B_{i}(t, s)$ in closed form.


### 3.2 Conditioning

In this subsection, we express an alternative formulation for the expectations in (14), which may be useful to write (conditionally) whenever possible, the explicit formula for the defaultfree price. To simplify notation, from now on we denote by $\mathbb{E}_{t}$ the conditional expectation with respect to $\mathcal{F}_{t}$.

Since the interest rate $r^{\phi}$ is deterministic, we rewrite (14) as

$$
\begin{align*}
\mathbf{1}_{\{\tau>t\}} c^{a}(t, T) & =\mathbf{1}_{\{\tau>t\}}\left\{\mathrm{e}^{-\int_{t}^{T} r_{u}^{\phi} d u} \mathbb{E}_{t}\left(\mathrm{e}^{-\int_{t}^{T} \lambda_{u} d u} f\left(X_{T}\right)\right)\right. \\
& +\mathbf{1}_{\{\tau>t\}} \int_{t}^{T} \mathrm{e}^{-\int_{t}^{s} r_{u}^{\phi} d u} \mathbb{E}_{t}\left(\mathrm{e}^{-\int_{t}^{s} \lambda_{u} d u} \Psi_{s}\right) d s \tag{20}
\end{align*}
$$

and we focus on the inner expectations.
Proposition 3.1 Let

$$
\mathcal{A}_{s}^{t}=\mathcal{F}_{s}^{B^{1}, B^{2}} \vee \mathcal{F}_{t}=\sigma\left(\left\{B_{u}^{1}, B_{u}^{2}, u \leq s\right\}\right) \vee \mathcal{F}_{t}, \quad t \leq s \leq T
$$

Then

$$
\begin{gathered}
\mathbb{E}_{t}\left[\mathrm{e}^{-\int_{t}^{T} \lambda_{u} d u} f\left(X_{T}\right)\right]=\mathrm{e}^{\int_{t}^{T} h_{u} d u} \mathbb{E}_{t}\left[\mathrm{e}^{-\int_{t}^{T} \lambda_{u} d u} \mathbb{E}\left(\mathrm{e}^{-\int_{t}^{T} h_{u} d u} f\left(X_{T}\right) \mid \mathcal{A}_{T}^{t}\right)\right] \\
\text { where } X_{T} \left\lvert\, \mathcal{A}_{T}^{t} \sim \mathcal{N}\left(\zeta_{T}(\boldsymbol{\rho})+\int_{t}^{T}\left(h_{u} d u-\frac{\Sigma^{2}(\boldsymbol{\rho})}{2}\right) d u ; \Sigma^{2}(\boldsymbol{\rho})(T-t)\right)\right. \text { and } \\
\zeta_{T}(\boldsymbol{\rho})=x+\sigma\left(B_{T}^{1}-B_{t}^{1}\right) \rho_{1}+\sigma\left(B_{T}^{2}-B_{t}^{2}\right) \rho_{2}-\frac{\sigma^{2}|\boldsymbol{\rho}|^{2}}{2}(T-t), \quad \Sigma(\boldsymbol{\rho})=\sigma \sqrt{1-|\boldsymbol{\rho}|^{2}} .
\end{gathered}
$$

Proof: From (15) the log-price at time $T$ is

$$
X_{T}=\zeta_{T}(\boldsymbol{\rho})+\int_{t}^{T} h_{u} d u+\Sigma(\boldsymbol{\rho})\left(B_{T}^{3}-B_{t}^{3}\right)-\frac{\Sigma^{2}(\boldsymbol{\rho})}{2}(T-t)
$$

and a simple application of the conditional expectation's tower-property gives

$$
\begin{aligned}
\mathbb{E}_{t}\left[\mathrm{e}^{-\int_{t}^{T} \lambda_{u} d u} f\left(X_{T}\right)\right] & =\mathbb{E}_{t}\left[\mathbb{E}\left(\mathrm{e}^{-\int_{t}^{T} \lambda_{u} d u} f\left(X_{T}\right) \mid \mathcal{A}_{T}^{t}\right)\right]=\mathbb{E}_{t}\left[\mathrm{e}^{-\int_{t}^{T} \lambda_{u} d u} \mathbb{E}\left(f\left(X_{T}\right) \mid \mathcal{A}_{T}^{t}\right)\right] \\
& =\mathrm{e}^{\int_{t}^{T} h_{u} d u} \mathbb{E}_{t}\left[\mathrm{e}^{-\int_{t}^{T} \lambda_{u} d u} \mathbb{E}\left(\mathrm{e}^{-\int_{t}^{T} h_{u} d u} f\left(X_{T}\right) \mid \mathcal{A}_{T}^{t}\right)\right] .
\end{aligned}
$$

### 3.3 Changing Numeraires

As a final step to evaluate the expectations $E_{t}$ in the previous expression, we apply the following family of changes of probability

$$
\begin{equation*}
\left.\frac{d \mathbb{Q}^{s}}{d \mathbb{Q}}\right|_{\mathcal{F}_{s}}=\frac{N(s, s)}{N(t, s)}, \tag{21}
\end{equation*}
$$

defining the $s$-forward measures, for any $t \leq s \leq T$. Recalling (19), by Girsanov's theorem, under $\mathbb{Q}^{s}$

$$
W_{v}^{i}=B_{v}^{i}-\int_{t}^{v} A_{i}(u, s)\left(\eta_{u}^{i} \lambda_{u}^{i}+\delta_{u}^{i}\right)^{\frac{1}{2}} d u, \quad i=1,2, t \leq v \leq s
$$

define independent Brownian motions and the market dynamics, for $t \leq v \leq s \leq T$, become

$$
\begin{align*}
& X_{v}=x+\int_{t}^{v}\left(h_{u}-\frac{\sigma^{2}}{2}+\sigma \sum_{i=1,2} \rho_{i} A_{i}(u, s)\left(\eta_{u}^{i} \lambda_{u}^{i}+\delta_{u}^{i}\right)^{\frac{1}{2}}\right) d u+\sigma\left(W_{s}-W_{t}\right)  \tag{22}\\
& \lambda_{v}^{i}=\lambda_{i}+\int_{t}^{v}\left[\left(\gamma_{u}^{i}+A_{i}(u, s) \eta_{u}^{i}\right) \lambda_{u}^{i}+\left(\beta_{u}^{i}+A_{i}(u, s) \delta_{u}^{i}\right)\right] d u+\int_{t}^{v}\left[\eta_{u}^{i} \lambda_{u}^{i}+\delta_{u}^{i}\right]^{\frac{1}{2}} d W_{u}^{i}, \tag{23}
\end{align*}
$$

where $\left(W^{1}, W^{2}, B^{3}\right)$ is a 3 -dimensional Brownian motion, on $[t, s]$ with

$$
W_{v}=\rho_{1} W_{v}^{1}+\rho_{2} W_{v}^{2}+\sqrt{1-\rho_{1}^{2}-\rho_{2}^{2}} B_{v}^{3}, \quad \rho_{1}^{2}+\rho_{2}^{2} \leq 1
$$

and we may conclude that the affine structure of the model is preserved. We remark that for each fixed $s$, different Brownian motions are generated. We keep denoting them in the same manner, as they all have the same distributional properties.

Hence, for any $t \leq s \leq T$ and any $\mathcal{F}_{s}-$ measurable random variable $Y$, we have

$$
\begin{equation*}
\mathbb{E}_{t}\left(\mathrm{e}^{\int_{t}^{s} \lambda_{u} d u} Y\right)=N(t, s) \mathbb{E}_{t}^{s}(Y), \tag{24}
\end{equation*}
$$

where $\mathbb{E}_{t}^{s}$, denotes expectations under $\mathbb{Q}^{s}$.

## 4 Semiexplicit formulae

In this section, we restrict to considering a European call with strike price $\mathrm{e}^{\kappa}$ and maturity $T$, for which we may exploit the Black \& Scholes formula, at least in a conditional fashion. We remark that in this case, by exploiting the put-call parity, it is possible to extend the evaluation method also to forward contracts.

We treat the case when the intensities are both described by a CIR process. We do not consider here the Vasicek model, since not appropriate for intensities, as it does not guarantee the positivity of the process, even though it has been previously considered in credit risk modeling (see for instance [21]) as it allows to write very computable explicit formulas.

### 4.1 The CIR specification

In this case, the dynamics of the market, for any $t \leq s \leq T$, are given by

$$
\begin{align*}
X_{s} & =x+\int_{t}^{s}\left(h_{u}-\frac{\sigma^{2}}{2}\right) d u+\sigma\left(B_{s}-B_{t}\right)  \tag{25}\\
\lambda_{s}^{i} & =\lambda_{i}+\int_{t}^{s} \gamma_{i}\left(\theta_{i}-\lambda_{u}^{i}\right) d u+\eta_{i} \int_{t}^{s} \sqrt{\lambda_{u}^{i}} d B_{u}^{i}, \quad i=1,2 . \tag{26}
\end{align*}
$$

We denote by $\tilde{r}_{u}=r_{u}^{\phi}-h_{u}$ and, $\hat{r}_{u}=r_{u}^{\phi}-r_{u}^{c}$ we have to compute

$$
\begin{align*}
\mathbf{1}_{\{\tau>t\}} c^{a}(t, T) & =\mathbf{1}_{\{\tau>t\}}\left\{\mathrm{e}^{-\int_{t}^{T} \tilde{r}_{u} d u} \mathbb{E}_{t}\left[\mathrm{e}^{-\int_{t}^{T} \lambda_{u} d u} \mathrm{e}^{-\int_{t}^{T} h_{u} d u} f\left(X_{T}\right)\right]\right. \\
& \left.+\int_{t}^{T} \mathrm{e}^{-\int_{t}^{s} r_{u}^{\phi} d u} \mathbb{E}_{t}\left[\mathrm{e}^{-\int_{t}^{s} \lambda_{u} d u} \Lambda_{s} c(s, T)\right] d s\right\} \tag{27}
\end{align*}
$$

where

$$
\Lambda_{s}=\lambda_{s}+\alpha \hat{r}_{s}-L_{1} \lambda_{s}^{1} .
$$

Proposition 4.1 Let $f(x)=\left(\mathrm{e}^{x}-\mathrm{e}^{\kappa}\right)^{+}$and

$$
\begin{aligned}
c(s, T) & \equiv c(s, T)^{+}=c_{B S}\left(X_{s}, s, \bar{v}_{s}, \sigma\right) \\
c_{B S}\left(x, s, \bar{v}_{s}, \sigma\right) & =\mathrm{e}^{x} \mathcal{N}\left(d_{1}\left(x, s, \bar{v}_{s}, \sigma\right)\right)-\mathrm{e}^{\kappa-\bar{v}_{s}} \mathcal{N}\left(d_{2}\left(x, s, \bar{v}_{s}, \sigma\right)\right) \\
d_{1,2}\left(x, s, \bar{v}_{s}, \sigma\right) & =\frac{x-\kappa+\bar{v}_{s} \pm \frac{\sigma^{2}}{2}(T-s)}{\sigma \sqrt{(T-s)}}
\end{aligned}
$$

where we denoted by $\bar{v}_{s}=\int_{s}^{T} v_{u} d u$, for any $v:[0, T] \longrightarrow \mathbb{R}$. Then we have

$$
\begin{align*}
\mathbf{1}_{\{\tau>t\}} c^{a}(t, T ; \boldsymbol{\rho}) & =\mathbf{1}_{\{\tau>t\}}\left\{\mathrm{e}^{-\int_{t}^{T} \tilde{r}_{u} d u} \mathbb{E}_{t}\left[\mathrm{e}^{-\int_{t}^{T} \lambda_{u} d u} c_{B S}\left(\zeta_{T}(\boldsymbol{\rho}), t, \bar{h}_{t}, \Sigma(\boldsymbol{\rho})\right)\right]\right. \\
& \left.+\int_{t}^{T} \mathrm{e}^{-\int_{t}^{s} r_{u}^{\phi} d u} \mathbb{E}_{t}\left[\mathrm{e}^{-\int_{t}^{s} \lambda_{u} d u} \Lambda_{s} c_{B S}\left(X_{s}(\boldsymbol{\rho}), s, \bar{r}_{s}, \sigma\right)\right] d s\right\} . \tag{28}
\end{align*}
$$

Proof: Applying inside the first expectation the conditioning with respect to $\mathcal{A}_{T}^{t}$, we obtain

$$
\begin{aligned}
& \mathbb{E}_{t}\left[\mathrm{e}^{-\int_{t}^{T} \lambda_{u} d u} \mathrm{e}^{-\int_{t}^{T} h_{u} d u} f\left(X_{T}\right)\right]=\mathbb{E}_{t}\left[\mathbb{E}_{t}\left(\mathrm{e}^{-\int_{t}^{T} \lambda_{u} d u} \mathrm{e}^{-\int_{t}^{T} h_{u} d u} f\left(X_{T}\right) \mid \mathcal{A}_{T}^{t}\right)\right] \\
= & \mathbb{E}_{t}\left[\mathrm{e}^{-\int_{t}^{T} \lambda_{u} d u} \mathbb{E}_{t}\left(\mathrm{e}^{-\int_{t}^{T} h_{u} d u} f\left(X_{T}\right) \mid \mathcal{A}_{T}^{t}\right)\right]=\mathbb{E}_{t}\left[\mathrm{e}^{-\int_{t}^{T} \lambda_{u} d u} c_{B S}\left(\zeta_{T}^{t}(\boldsymbol{\rho}), t, \bar{h}_{t}, \Sigma(\boldsymbol{\rho})\right)\right]
\end{aligned}
$$

and we may view the second expectation in (27) as $\mathbb{E}_{t}\left[\mathrm{e}^{-\int_{t}^{s} \lambda_{u} d u} \Lambda_{s} c_{B S}\left(X_{s}(\boldsymbol{\rho}), s, \bar{r}_{s}, \sigma\right)\right]$ where ,for $t \leq s \leq T$, setting $M_{s}^{i}=B_{s}^{i}-B_{t}^{i}$, for $i=1,2$, we have

$$
X_{s}(\boldsymbol{\rho})=x+\int_{t}^{s}\left(h_{u}-\frac{\sigma^{2}}{2}\right) d u+\sigma\left(M_{s}^{1} \rho_{1}+M_{s}^{2} \rho_{2}+M_{s}^{3} \sqrt{1-|\boldsymbol{\rho}|^{2}}\right) .
$$

Consequently, we have

$$
d_{1,2}\left(\zeta_{T}(\boldsymbol{\rho}), t, \bar{h}_{t}, \Sigma(\boldsymbol{\rho})\right)=\left\{\begin{array}{l}
{\left[d_{1}\left(x, t, \bar{h}_{t}, \sigma\right)+\frac{M_{T}^{1}}{\sigma \sqrt{T-t}} \rho_{1}+\frac{M_{T}^{2}}{\sigma \sqrt{T-t}} \rho_{2}-\sigma \sqrt{T-t}|\boldsymbol{\rho}|^{2}\right] \frac{1}{\sqrt{1-|\boldsymbol{\rho}|^{2}}}} \\
{\left[d_{2}\left(x, s, \bar{h}_{t}, \sigma\right)+\frac{M_{T}^{1}}{\sigma \sqrt{T-t}} \rho_{1}+\frac{M_{T}^{2}}{\sigma \sqrt{T-t}} \rho_{2}\right] \frac{1}{\sqrt{1-|\boldsymbol{\rho}|^{2}}}}
\end{array}\right.
$$

Pointing out the dependence on $\boldsymbol{\rho}$ of $c^{a}(t, T)$, we get (28).
We want to approximate (28) by a Taylor expansion with respect to the correlation parameters $\boldsymbol{\rho}=\left(\rho_{1}, \rho_{2}\right)$ around $\mathbf{0}=(0,0)$ on $\{\tau>t\}$. The first-order approximation would hence be

$$
c^{a}(t, T ; \boldsymbol{\rho}) \approx c^{a}(t, T ; \mathbf{0})+\frac{\partial c^{a}(t, T ; \mathbf{0})}{\partial \rho_{1}} \rho_{1}+\frac{\partial c^{a}(t, T ; \mathbf{0})}{\partial \rho_{2}} \rho_{2} .
$$

Remark 4.2 For the sake of exposition, we decided to restrict our discussion to the first order approximation, which may turn to be extremely satisfying when the model seems to exhibit a roughly linear dependence upon the correlation parameters. This was highlighted by the Monte Carlo simulations for the CIR intensity setting (section (5) and the accuracy of our method turned out to be very good. If the dependence on the correlation parameters is more markedly nonlinear, one may develop Taylor's polynomial to a higher order to capture this behavior. We explicitly wrote also a second-order formula: it is computationally longer, but it does not present any additional theoretical complexity. We did not report it here to keep the exposition lighter.

Since the integrability conditions are satisfied, the derivatives pass under the integral and expectation signs and the problem is reduced to computing the derivatives with respect to the correlation parameters of $c_{B S}\left(\zeta_{T}(\boldsymbol{\rho}), t, T, \Sigma(\boldsymbol{\rho})\right)$ and of $c_{B S}\left(X_{s}(\boldsymbol{\rho}), s, T, \sigma\right)$ and evaluating them at $\mathbf{0}$. After some calculations, one arrives at the following expressions

$$
c_{B S}\left(\zeta_{T}(\boldsymbol{\rho}), t, \bar{h}_{t}, \Sigma(\boldsymbol{\rho})\right) \approx c_{B S}\left(x, t, \bar{h}_{t}, \sigma\right)+\sigma \mathrm{e}^{x} \mathcal{N}\left(d_{1}\left(x, t, \bar{h}_{t}, \sigma\right)\right)\left[M_{T}^{1} \rho_{1}+M_{T}^{2} \rho_{2}\right]
$$

and

$$
c_{B S}\left(X_{s}(\boldsymbol{\rho}), s, \bar{r}_{s}, \sigma\right) \approx c_{B S}\left(X_{s}(\mathbf{0}), s, \bar{r}_{s}, \sigma\right)+\sigma \mathrm{e}^{X_{s}(\mathbf{0})} \mathcal{N}\left(d_{1}\left(X_{s}(\mathbf{0}), s, \bar{r}_{s}, \sigma\right)\right)\left[M_{s}^{1} \rho_{1}+M_{s}^{2} \rho_{2}\right]
$$

to be plugged into (27), with each term to be computed following the procedure outlined in the previous sections. Thus, exploiting the independence between $X_{s}(\mathbf{0})$ and $B^{1}, B^{2}$ we have

$$
\begin{aligned}
c^{a}(t, T ; \boldsymbol{\rho}) \approx & \mathrm{e}^{-\int_{t}^{T} \widetilde{r}_{u} d u}\left\{N(t, T) c_{B S}\left(x, t, \bar{h}_{t}, \sigma\right)+\sigma \mathrm{e}^{x} \mathcal{N}\left(d_{1}\left(x, t, \bar{h}_{t}, \sigma\right)\right) \mathbb{E}_{t}\left[\mathrm{e}^{-\int_{t}^{T} \lambda_{u} d u}\left(M_{T}^{1} \rho_{1}+M_{T}^{2} \rho_{2}\right)\right]\right\} \\
& +\int_{t}^{T} \mathrm{e}^{-\int_{t}^{s} r_{u}^{d} d u}\left\{\mathbb{E}_{t}\left[\mathrm{e}^{s_{t}^{s} \lambda_{u} d u} \Lambda_{s}\right] \mathbb{E}_{t}\left[c_{B S}\left(X_{s}(\mathbf{0}), s, \bar{r}_{s}, \sigma\right)\right]\right. \\
& \left.+\sigma \mathbb{E}_{t}\left[\mathrm{e}^{X_{s}(\mathbf{0})} \mathcal{N}\left(d_{1}\left(X_{s}(\mathbf{0}), s, \bar{r}_{s}, \sigma\right)\right)\right] \sum_{i=1}^{2} \mathbb{E}_{t}\left(\mathrm{e}^{-\int_{t}^{s} \lambda_{u} d u} \Lambda_{s} M_{s}^{i}\right) \rho_{i}\right\} d s
\end{aligned}
$$

and we have to compute every single expectation. We proceed by steps, showing that we may reduce to computing some basic cases.

1. Noticing that

$$
\begin{aligned}
& M_{s}^{3} \sim N\left(0 ; \sigma^{2}(s-t)\right) \\
& \begin{aligned}
& X_{s}(\mathbf{0})=x+\int_{t}^{s}\left(h_{u}-\frac{\sigma^{2}}{2}\right) d u+M_{s}^{3} \sim N\left(x+\int_{t}^{s}\left(h_{u}-\frac{\sigma^{2}}{2}\right) d u ; \sigma^{2}(s-t)\right) \\
& d_{i}\left(X_{s}(\mathbf{0}), s, \bar{r}_{s}, \sigma\right)=\frac{X_{s}(\mathbf{0})-k+\bar{r}_{s} \pm \frac{\sigma^{2}}{2}(T-s)}{\sigma \sqrt{T-s}} \\
& \quad=\frac{M_{s}^{3}}{\sqrt{T-s}}+d_{i}\left(x, s, \bar{r}_{s}, \sigma\right)+\frac{1}{\sigma \sqrt{T-s}} \int_{t}^{s}\left(h_{u}-\frac{\sigma^{2}}{2}\right) d u \\
& \quad \sim N\left(d_{i}\left(x, s, \bar{r}_{s}, \sigma\right)+\frac{1}{\sigma \sqrt{T-s}} \int_{t}^{s}\left(h_{u}-\frac{\sigma^{2}}{2}\right) d u, \frac{s-t}{T-s}\right), \quad i=1,2 \\
& \mathbb{E}_{t}\left[c_{B S}\left(X_{s}(\mathbf{0}), s, \bar{r}_{s}, \sigma\right)\right]=\mathbb{E}_{t}\left[\mathrm{e}^{X_{s}(\mathbf{0})} \mathcal{N}\left(d_{1}\left(X_{s}(\mathbf{0}), s, \bar{r}_{s}, \sigma\right)\right)\right]-\mathrm{e}^{\kappa-\bar{r}_{s}} \mathbb{E}_{t}\left[\mathcal{N}\left(d_{2}\left(X_{s}(\mathbf{0}), s, \bar{r}_{s}, \sigma\right)\right)\right]
\end{aligned}
\end{aligned}
$$

the Gaussian integrals can be computed explicitly

$$
\begin{aligned}
\mathbb{E}_{t}\left[\mathrm{e}^{X_{s}(\mathbf{0})} \mathcal{N}\left(d_{1}\left(X_{s}(\mathbf{0}), s, \bar{r}_{s}, \sigma\right)\right)\right] & =\mathrm{e}^{x+\int_{t}^{s} h_{u} d u} \mathcal{N}\left(d_{1}\left(x+\left(\bar{r}_{s}-\bar{h}_{s}\right), t, \bar{h}_{t}, \sigma\right)\right) \\
\mathbb{E}_{t}\left[\mathcal{N}\left(d_{2}\left(X_{s}(\mathbf{0}), s, \bar{r}_{s}, \sigma\right)\right)\right] & =\mathcal{N}\left(d_{2}\left(x+\left(\bar{r}_{s}-\bar{h}_{s}\right), t, \bar{h}_{t}, \sigma\right)\right),
\end{aligned}
$$

by applying the following
Lemma 4.3 Let $p \in \mathbb{R}$ and $X \sim N\left(\mu, \nu^{2}\right)$, then

$$
\mathbb{E}\left(\mathrm{e}^{p X} \mathcal{N}(X)\right)=\mathrm{e}^{p \mu+\frac{(p \nu)^{2}}{2}} \mathcal{N}\left(\frac{\mu+p \nu^{2}}{\sqrt{1+\nu^{2}}}\right)
$$

where by $\mathcal{N}$ we denote the standard Normal distribution function.
Proof: see Zacks (1981) for $p=0$, the general case follows by a "completing the squares" argument.
Therefore we may conclude that

$$
\begin{equation*}
\mathbb{E}_{t}\left[c_{B S}\left(X_{s}(\mathbf{0}), s, \bar{r}_{s}, \sigma\right)\right]=\mathrm{e}^{-\left(\bar{r}_{s}-\bar{h}_{s}\right)+\int_{t}^{s} h_{u} d u} c_{B S}\left(x+\left(\bar{r}_{s}-\bar{h}_{s}\right), t, \bar{h}_{t}, \sigma\right) \tag{29}
\end{equation*}
$$

2. It remains to evaluate the expectations

$$
\mathbb{E}_{t}\left(\mathrm{e}^{-\int_{t}^{s} \lambda_{u} d u} \Lambda_{s}\right), \quad \mathbb{E}_{t}\left(\mathrm{e}^{-\int_{t}^{s} \lambda_{u} d u}\left(B_{s}^{i}-B_{t}^{i}\right)\right), \quad \mathbb{E}_{t}\left(\mathrm{e}^{-\int_{t}^{s} \lambda_{u} d u} \Lambda_{s}\left(B_{s}^{i}-B_{t}^{i}\right)\right) \quad i=1,2
$$

Recalling that $\Lambda_{s}=\lambda_{s}+\alpha \hat{r}_{s}-L_{1} \lambda_{s}^{1}$ the above expressions reduce to computing

$$
\mathbb{E}_{t}\left[\mathrm{e}^{-\int_{t}^{s} \lambda_{u} d u}\left(\lambda_{s}^{i}\right)^{\alpha}\left(B_{s}^{j}-B_{t}^{j}\right)^{k}\right]
$$

for $i, j=1,2$, and $\alpha, k=0,1$.

$$
\mathbb{E}_{t}\left[\mathrm{e}^{-\int_{t}^{s} \lambda_{u} d u}\left(\lambda_{s}^{i}\right)^{\alpha}\left(B_{s}^{j}-B_{t}^{j}\right)^{k}\right] .
$$

To do so, we apply the change of Numeraire described in subsection 3.3, obtaining

$$
\mathbb{E}_{t}\left[\mathrm{e}^{-\int_{t}^{s} \lambda_{u} d u}\left(\lambda_{s}^{i}\right)^{\alpha}\left(B_{s}^{j}-B_{t}^{j}\right)^{k}\right]=N(t, s) \mathbb{E}_{t}^{s}\left[\left(\lambda_{s}^{i}\right)^{\alpha}\left[\left(W_{s}^{j}-W_{t}^{j}\right)+\eta_{j} \int_{t}^{s} A_{j}(u, s) \sqrt{\lambda_{u}^{j}} d u\right]^{k}\right] .
$$

We can exploit the independence of $W^{1}$ and $W^{2}$, so that the last expectation, for $i \neq j$ becomes

$$
\eta_{j} \mathbb{E}_{t}^{s}\left[\left(\lambda_{s}^{i}\right)^{\alpha}\right]\left[\int_{t}^{s} A_{j}(u, s) \mathbb{E}_{t}^{s}\left(\sqrt{\lambda_{u}^{j}}\right) d u\right]^{k}
$$

where for $t \leq u \leq s$

$$
\lambda_{u}^{i}=\lambda_{i}+\int_{t}^{u}\left[\gamma_{i} \theta_{i}-\left(\gamma_{i}-\eta_{i}^{2} A_{i}(v, s)\right) \lambda_{u}^{i}\right] d v+\eta_{i} \int_{t}^{u} \sqrt{\lambda_{v}^{i}} d W_{v}^{i} .
$$

When $i=j$, if $k=0$, clearly we have only the first expectation, if $\alpha=0$ only the second, and for $\alpha=k=1$, we end up with

$$
\mathbb{E}_{t}^{s}\left[\lambda_{s}^{i}\left(W_{s}^{i}-W_{t}^{i}\right)\right]+\eta_{j} \int_{t}^{s} A_{i}(u, s) \mathbb{E}_{t}^{s}\left[\lambda_{s}^{i} \sqrt{\lambda_{u}^{i}}\right] d u
$$

3. Thus we have reduced the problem to considering the expectations for $u \leq s$

$$
\begin{align*}
& \mathbb{E}_{t}^{s}\left(\lambda_{s}^{i}\right), \quad \mathbb{E}_{t}^{s}\left(\sqrt{\lambda_{u}^{i}}\right), \quad \mathbb{E}_{t}^{s}\left(\lambda_{s}^{i} \sqrt{\lambda_{u}^{i}}\right),  \tag{30}\\
& \mathbb{E}_{t}^{s}\left(\lambda_{s}^{i}\left(W_{s}^{i}-W_{t}^{i}\right)\right), \tag{31}
\end{align*}
$$

The third of (30), again by the independence of the increments, can be written as

$$
\mathbb{E}_{t}^{s}\left(\lambda_{s}^{i} \sqrt{\lambda_{u}^{i}}\right)=\mathbb{E}_{t}^{s}\left(\left(\lambda_{s}^{i}-\lambda_{u}^{i}\right) \sqrt{\lambda_{u}^{i}}\right)+\mathbb{E}_{t}^{s}\left(\left(\lambda_{u}^{i}\right)^{\frac{3}{2}}\right)=\mathbb{E}_{t}^{s}\left(\lambda_{s}^{i}-\lambda_{u}^{i}\right) \mathbb{E}_{t}^{s}\left(\sqrt{\lambda_{u}^{i}}\right)+\mathbb{E}_{t}^{s}\left(\left(\lambda_{u}^{i}\right)^{\frac{3}{2}}\right) .
$$

By applying Itô's formula and taking expectations, for $t \leq u \leq s \leq T$ we have

$$
\begin{aligned}
\mathbb{E}_{t}^{s}\left(\lambda_{u}^{i}\right) & =\mathrm{e}^{-\int_{t}^{u}\left[\gamma_{i}-\eta_{i}^{2} A_{i}(\xi, s)\right] d \xi}\left\{\lambda_{i}+\gamma_{i} \theta_{i} \int_{t}^{u} \mathrm{e}^{\int_{t}^{v}\left[\gamma_{i}-\eta_{i}^{2} A_{i}(\xi, s)\right] d \xi} d v\right\}, \\
\mathbb{E}_{t}^{s}\left[\sqrt{\lambda_{u}^{i}}\right] & =\mathrm{e}^{-\frac{1}{2} \int_{t}^{u}\left[\gamma_{i}-\eta_{i}^{2} A_{i}(\xi, s)\right] d \xi}\left[\sqrt{\lambda_{i}}+\frac{1}{2}\left[\gamma_{i} \theta_{i}-\frac{\eta_{i}^{2}}{4}\right] \int_{t}^{u} \mathrm{e}^{\frac{1}{2} \int_{t}^{v}\left[\gamma_{i}-\eta_{i}^{2} A_{i}(\xi, s)\right] d \xi} \mathbb{E}_{t}^{s}\left[\frac{1}{\sqrt{\lambda_{v}^{i}}}\right] d v\right], \\
\mathbb{E}_{t}^{s}\left[\left(\lambda_{u}^{i}\right)^{\frac{3}{2}}\right] & =\mathrm{e}^{-\frac{3}{2} \int_{t}^{u}\left[\gamma_{i}-\eta_{i}^{2} A_{i}(\xi, s)\right] d \xi}\left[\left(\lambda_{i}\right)^{\frac{3}{2}}+\frac{3}{2}\left[\gamma_{i} \theta_{i}+\frac{\eta_{i}^{2}}{4}\right] \int_{t}^{u} \mathrm{e}^{\frac{3}{2} \int_{t}^{v}\left[\gamma_{i}-\eta_{i}^{2} A_{i}(\xi, s)\right] d \xi} \mathbb{E}_{t}^{s}\left[\sqrt{\lambda_{v}^{i}}\right] d v\right],
\end{aligned}
$$

and we approximate $\frac{1}{\sqrt{\lambda_{v}^{i}}}$ by $\frac{1}{\sqrt{\lambda_{i}}}$ or $\frac{1}{\sqrt{\theta_{i}}}$, freezing the process either at the initial condition or at the mean reversion parameter. This choice usually provides simple and numerically quite accurate approximations of the powers of a CIR process. Finally, we may use integration by parts for the expectation (31) and we may conclude

In conclusion, all the pieces appearing in (4.1) can be computed explicitly, provided we perform the mentioned freezing for $\left(\lambda_{u}^{i}\right)^{-\frac{1}{2}}$.

Summarizing

$$
\begin{equation*}
c^{a}(t, T ; \boldsymbol{\rho}) \approx g_{0}(t, T ; \mathbf{0})+g_{1}(t, T ; \mathbf{0}) \rho_{1}+g_{2}(t, T ; \mathbf{0}) \rho_{2} \tag{32}
\end{equation*}
$$

where the zeroth term is (with $R_{1}=1-L_{1}$ )

$$
\begin{align*}
& g_{0}(t, T ; \mathbf{0})=\mathrm{e}^{-\int_{t}^{T} \tilde{r}_{u} d u} N(t, T) c_{B S}\left(x, t, \bar{h}_{t}, \sigma\right) \\
+ & \int_{t}^{T} \mathrm{e}^{-\int_{t}^{s} \tilde{r}_{u} d u-\left(\bar{r}_{s}-\bar{h}_{s}\right)} N(t, s)\left[R_{1} \mathbb{E}_{t}^{s}\left(\lambda_{s}^{1}\right)+\mathbb{E}_{t}^{s}\left(\lambda_{s}^{2}\right)+\alpha \hat{r}_{s}\right] c_{B S}\left(x+\left(\bar{r}_{s}-\bar{h}_{s}\right), t, \bar{h}_{t}, \sigma\right) d s \tag{33}
\end{align*}
$$

and the first-order coefficients are

$$
\begin{align*}
& g_{1}(t, T ; \mathbf{0})=\sigma\left\{\eta_{1} \mathrm{e}^{x-\int_{t}^{T} \tilde{r}_{u} d u} N(t, T) \mathcal{N}\left(d_{1}\left(x, t, \bar{h}_{t}, \sigma\right)\right) \int_{t}^{T} A_{1}(s, T) \mathbb{E}_{t}^{T}\left(\sqrt{\lambda_{s}^{1}}\right) d s\right. \\
& +\int_{t}^{T} \mathrm{e}^{x-\int_{t}^{s} \tilde{r}_{u} d u} N(t, s) \mathcal{N}\left(d_{1}\left(x+\left(\bar{r}_{s}-\bar{h}_{s}\right), t, \bar{h}_{t}, \sigma\right)\right)\left[R_{1} \mathbb{E}_{t}^{s}\left(\lambda_{s}^{1}\left(W_{s}^{1}-W_{t}^{1}\right)\right)\right.  \tag{34}\\
& \left.\left.+\eta_{1} \int_{t}^{s} A_{1}(u, s)\left[\mathbb{E}_{t}^{s}\left(\lambda_{s}^{1}-\lambda_{u}^{1}\right) \mathbb{E}_{t}^{s}\left(\sqrt{\lambda_{u}^{1}}\right)+\mathbb{E}_{t}^{s}\left(\left(\lambda_{u}^{1}\right)^{\frac{3}{2}}\right)+\left(\mathbb{E}_{t}^{s}\left(\lambda_{s}^{2}\right)+\alpha \hat{r}_{s}\right) \mathbb{E}_{t}^{s}\left(\sqrt{\lambda_{u}^{1}}\right)\right] d u\right] d s\right\} \\
& g_{2}(t, T ; \mathbf{0})=\sigma\left\{\eta_{2} \mathrm{e}^{x-\int_{t}^{T} \tilde{r}_{u} d u} N(t, T) \mathcal{N}\left(d_{1}\left(x, t, \bar{h}_{t}, \sigma\right)\right) \int_{t}^{T} A_{2}(s, T) \mathbb{E}_{t}^{T}\left(\sqrt{\lambda_{s}^{2}}\right) d s\right. \\
& +\int_{t}^{T} \mathrm{e}^{x-\int_{t}^{s} \tilde{r}_{u} d u} N(t, s) \mathcal{N}\left(d_{1}\left(x+\left(\bar{r}_{s}-\bar{h}_{s}\right), t, \bar{h}_{t}, \sigma\right)\right)\left[\mathbb{E}_{t}^{s}\left(\lambda_{s}^{2}\left(W_{s}^{2}-W_{t}^{2}\right)\right)\right.  \tag{35}\\
& \left.\left.+\eta_{2} \int_{t}^{s} A_{2}(u, s)\left[\left(R_{1} \mathbb{E}_{t}^{s}\left(\lambda_{s}^{1}\right)+\alpha \hat{r}_{s}\right) \mathbb{E}_{t}^{s}\left(\sqrt{\lambda_{u}^{2}}\right)+\mathbb{E}_{t}^{s}\left(\lambda_{s}^{2}-\lambda_{u}^{2}\right) \mathbb{E}_{t}^{s}\left(\sqrt{\lambda_{u}^{2}}\right)+\mathbb{E}_{t}^{s}\left(\left(\lambda_{u}^{2}\right)^{\frac{3}{2}}\right)\right] d u\right] d s\right\}
\end{align*}
$$

where, for $t \leq s \leq T$ and $i=1,2$, we have

$$
N_{i}(t, s)=\mathrm{e}^{A_{i}(t, s) \lambda_{i}+B_{i}(t, s)}, \quad N(t, s)=N_{1}(t, s) N_{2}(t, s)
$$

with

$$
\begin{aligned}
h_{i}=\sqrt{\gamma_{i}^{2}+2 \eta_{i}^{2}}, \quad A_{i}(t, T) & =-\frac{2\left(\mathrm{e}^{h_{i}(T-t)}-1\right)}{h_{i}-\gamma_{i}+\left(h_{i}+\gamma_{i}\right) \mathrm{e}^{h_{i}(T-t)}} \\
B_{i}(t, T) & =\frac{2 \gamma_{i} \theta_{i}}{\eta_{i}^{2}} \ln \left(\frac{2 h_{i} \mathrm{e}^{\gamma_{i}+h_{i}(T-t)}}{h_{i}-\gamma_{i}+\left(h_{i}+\gamma_{i}\right) \mathrm{e}^{h_{i}(T-t)}}\right) .
\end{aligned}
$$

## 5 Numerical results

In this section, we present some numerical results of our approximation method for the call price. As a first step, we assess the performance of the first-order approximation (32) by using the Monte Carlo evaluations with control variates as a benchmark, employing the default-free price as control: in the considered cases, this reduces the length of the confidence interval by at least one order of magnitude. For the simulations, we generated $M=10^{6}$ sample paths with a
time step equal to $10^{-3}$ for any considered maturity. The benchmark Monte Carlo method was implemented to approximate the call price (14) by using the Euler discretization scheme with full truncation for the intensity processes $\lambda_{t}^{1}$ and $\lambda_{t}^{2}$ (see [35]) and with an exact simulation of the Brownian motion for the underlying $X_{t}$. The running integrals appearing in the expectations were evaluated by means of trapezoidal routine. All the algorithms were implemented in MatLab (R2019b).

The evaluation of the zeroth and first-order terms of our approximation ( $(\sqrt{33)}, \sqrt{34}),(\sqrt{35})$ ) requires the computation of nested one-dimensional integrals of well-behaved functions once for each set of chosen parameters and this step was implemented through the vectorized global adaptive quadrature MatLab algorithm.

The parameters of the intensity processes were set as in [11] and [3] (see Table 2) and they agree with calibrated default intensities. The strike price was fixed to $K=\mathrm{e}^{\kappa}=100$ and we considered two maturities, $T=0.5$ and $T=2$. Lastly, without loss of generality, we took $t=0$, the log-asset's initial value was set to 4.6052 , and its volatility to $\sigma=40 \%$. The remaining parameters were chosen as $r=h=0.001, r^{\phi}=0.005, r^{c}=0.002$ and $\alpha=0.5$.

The accuracy of the first-order approximation is summarized in Tables (3), (4), listing the errors with respect to the benchmark MC prices (see also figure (17) for different choices of the default parameters for the Investor and the Counterparty and to the time-to-maturity $T$ of the contract. It is apparent how the approximation is highly satisfactory for short term maturity while it tends to deteriorate a little when the horizon increases.

In Table (5) we highlight the separate contributions of the zeroth and first-order terms, $g_{0}(0, T ; \mathbf{0}), g_{1}(0, T ; \mathbf{0})$ and $g_{2}(0, T ; \mathbf{0})$ in (32), which are not significantly affected in relative magnitude by changes in the values of the parameters. In particular, the contribution due to the correlation between the underlying and the intensities is quite sizeable and it supports the choice of stochastic processes versus deterministic functions to represent the intensities. We notice that the contribution of the term $g_{1}$ is more significant compared to that of $g_{2}$ which appears to be always rather small. This is to be expected since we are considering a call option and default of the Investor is bound to have a limited impact on the overall value; on the contrary, the term $g_{1}$ is more relevant being connected to the counterparty's default and, as natural, it decreases as the collateralization tends to one.

The contribution coming from the stochastic nature of the intensities can be better appreciated by looking at the results of the further set of numerical experiments reported in Table (6). There, in order to compare with the results in [14], we considered the rates $r=0.001, h=0.005$, $r^{\phi}=0.005, r^{c}=0.002$ and we chose $\lambda_{0}^{1}=0.04, \lambda_{0}^{2}=0.02$ and the other parameters as in (2). The losses given default were set to $L_{1}=L_{2}=60 \%$ and we took $T=0.5$. The correction that we obtain with respect to the prices in [14] is of the order of $10^{-2}$, which can, of course, become very relevant as the volume of the transaction grows.

As a final remark, we write explicitly our evaluation formula when constant intensities $\lambda_{t}^{i} \equiv \lambda^{i}$ are taken. It is immediately seen by using (29) that the price (27) becomes

$$
\begin{align*}
c^{a}(t, T)= & \mathrm{e}^{\left(\lambda^{1}+\lambda^{2}-\left(r^{\phi}-h\right)\right)(T-t)} c_{B S}(x, t, \bar{h}, \sigma)+\left(\lambda^{1}+\lambda^{2}+\left(r^{\phi}-r^{c}\right) \alpha-\lambda^{1} L_{1}\right) \times \\
& \int_{t}^{T} \mathrm{e}^{-\left(\lambda^{1}+\lambda^{2}+\left(r^{\phi}-h\right)\right)(s-t)} \mathrm{e}^{-(r-h)(T-s)} c_{B S}(x+(r-h)(T-s), t, \bar{h}, \sigma) d s \tag{36}
\end{align*}
$$

which, as noticed in [13] and [14], shows that the interplay among all the rates in this framework

|  | $\lambda_{0}$ | $\gamma$ | $\theta$ | $\eta$ | 6 -months surv. prob. | 2-years surv. prob. |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\tau_{1}$ (counterparty) | 0.03 | 0.02 | 0.161 | 0.08 | 0.9848 | 0.9371 |
| $\tau_{2}$ (investor) | 0.035 | 0.35 | 0.45 | 0.15 | 0.9660 | 0.7399 |

Table 2: Parameter sets for the CIR default intensities.

|  | -0.6 | -0.4 | -0.2 | 0 | 0.2 | 0.4 | 0.6 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| -0.6 | $-7.478 \mathrm{e}-04$ | $-5.850 \mathrm{e}-04$ | $-3.852 \mathrm{e}-04$ | $-1.951 \mathrm{e}-04$ | $-4.362 \mathrm{e}-05$ | $7.122 \mathrm{e}-05$ | $1.881 \mathrm{e}-04$ |
| -0.4 | $-5.338 \mathrm{e}-04$ | $-3.423 \mathrm{e}-04$ | $-1.508 \mathrm{e}-04$ | $5.306 \mathrm{e}-05$ | $1.955 \mathrm{e}-04$ | $3.118 \mathrm{e}-04$ | $3.636 \mathrm{e}-04$ |
| -0.2 | $-3.104 \mathrm{e}-04$ | $-9.415 \mathrm{e}-05$ | $8.240 \mathrm{e}-05$ | $2.456 \mathrm{e}-04$ | $3.640 \mathrm{e}-04$ | $4.693 \mathrm{e}-04$ | $5.321 \mathrm{e}-04$ |
| 0 | $-1.194 \mathrm{e}-04$ | $8.440 \mathrm{e}-05$ | $2.489 \mathrm{e}-04$ | $4.203 \mathrm{e}-04$ | $5.234 \mathrm{e}-04$ | $6.252 \mathrm{e}-04$ | $7.105 \mathrm{e}-04$ |
| 0.2 | $5.723 \mathrm{e}-05$ | $2.527 \mathrm{e}-04$ | $4.217 \mathrm{e}-04$ | $5.816 \mathrm{e}-04$ | $7.102 \mathrm{e}-04$ | $8.091 \mathrm{e}-04$ | $9.161 \mathrm{e}-04$ |
| 0.4 | $2.584 \mathrm{e}-04$ | $4.708 \mathrm{e}-04$ | $6.296 \mathrm{e}-04$ | $7.458 \mathrm{e}-04$ | $8.760 \mathrm{e}-04$ | $9.736 \mathrm{e}-04$ | $1.079 \mathrm{e}-03$ |
| 0.6 | $4.854 \mathrm{e}-04$ | $6.768 \mathrm{e}-04$ | $8.431 \mathrm{e}-04$ | $9.614 \mathrm{e}-04$ | $1.074 \mathrm{e}-03$ | $1.167 \mathrm{e}-03$ | $1.241 \mathrm{e}-03$ |

Table 3: Approximation errors, Set 1 for $\tau_{1}$, Set 2 for $\tau_{2}, T=0.5$. The average length of the $95 \%$ confidence interval for the MC estimates is $5.3939 e-04$.
accounts for a significant contribution to the global price.
Last but not least, we would like to point out that our approximation implies a very big reduction of the computational time as it allows avoiding the costly multi-dimensional Monte Carlo Simulations or PDE discretization.

| $\rho_{2} \backslash \rho_{1}$ | -0.6 | -0.4 | -0.2 | 0 | 0.2 | 0.4 | 0.6 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| -0.6 | $-6.619 \mathrm{e}-02$ | $-5.728 \mathrm{e}-02$ | $-4.887 \mathrm{e}-02$ | $-3.983 \mathrm{e}-02$ | $-3.114 \mathrm{e}-02$ | $-2.303 \mathrm{e}-02$ | $-1.427 \mathrm{e}-02$ |
| -0.4 | $-5.191 \mathrm{e}-02$ | $-4.320 \mathrm{e}-02$ | $-3.409 \mathrm{e}-02$ | $-2.552 \mathrm{e}-02$ | $-1.726 \mathrm{e}-02$ | $-9.017 \mathrm{e}-03$ | $-7.615 \mathrm{e}-04$ |
| -0.2 | $-3.706 \mathrm{e}-02$ | $-2.828 \mathrm{e}-02$ | $-1.938 \mathrm{e}-02$ | $-1.138 \mathrm{e}-02$ | $-3.327 \mathrm{e}-03$ | $4.780 \mathrm{e}-03$ | $1.299 \mathrm{e}-02$ |
| 0 | $-2.246 \mathrm{e}-02$ | $-1.338 \mathrm{e}-02$ | $-5.165 \mathrm{e}-03$ | $2.822 \mathrm{e}-03$ | $1.095 \mathrm{e}-02$ | $1.877 \mathrm{e}-02$ | $2.686 \mathrm{e}-02$ |
| 0.2 | $-7.224 \mathrm{e}-03$ | $1.505 \mathrm{e}-03$ | $9.585 \mathrm{e}-03$ | $1.776 \mathrm{e}-02$ | $2.559 \mathrm{e}-02$ | $3.352 \mathrm{e}-02$ | $4.164 \mathrm{e}-02$ |
| 0.4 | $8.800 \mathrm{e}-03$ | $1.771 \mathrm{e}-02$ | $2.568 \mathrm{e}-02$ | $3.327 \mathrm{e}-02$ | $4.091 \mathrm{e}-02$ | $4.864 \mathrm{e}-02$ | $5.639 \mathrm{e}-02$ |
| 0.6 | $2.543 \mathrm{e}-02$ | $3.414 \mathrm{e}-02$ | $4.206 \mathrm{e}-02$ | $4.961 \mathrm{e}-02$ | $5.704 \mathrm{e}-02$ | $6.453 \mathrm{e}-02$ | $7.191 \mathrm{e}-02$ |

Table 4: Approximation errors, Set 1 for $\tau_{1}$, Set 2 for $\tau_{2}, T=2$. The average length of the $95 \%$ confidence interval for the MC estimates is 0.0086 .

| $T$ | $g_{0}$ | $g_{1}$ | $g_{2}$ |
| :---: | :---: | :---: | :---: |
| 0.5 | 11.3300 | -0.0071 | 0.0003 |
| 2 | 22.4224 | -0.0435 | 0.0370 |

Table 5: Contribution of zero-th and first order terms in the expansion approximation with Set 1 for $\tau_{1}$ and Set 2 for $\tau_{2}$. The corresponding default-free prices according to the $\mathrm{B} \& \mathrm{~S}$ formula are $c_{B S}\left(X_{0}, 0, \bar{r}_{s}, \sigma\right)=11.2685(T=0.5)$ and $c_{B S}\left(X_{0}, 0, \bar{r}_{s}, \sigma\right)=22.3480(T=2)$.

## References

[1] F. Antonelli, S. Scarlatti, Pricing Options under stochastic volatility: a power series approach, Finance and Stochastics, 13, 269-303 (2009).
[2] F. Antonelli, A. Ramponi, S. Scarlatti, Random time forward-starting options, Int. J. of Theor. and Appl. Finance, 19, 8 (2016).
[3] F. Antonelli, A. Ramponi, S. Scarlatti, CVA and vulnerable options by correlation expansion , Annals of Operations Research, https://doi.org/10.1007/s10479-019-03367-z, 1-27 (2019).
[4] T. R. Bielecki, S. Crepey, D. Brigo, Counterparty Risk and Funding: A Tale of Two Puzzles. Chapman and Hall/CRC (2014).
[5] T. R. Bielecki, M. Rutkowski, Credit Risk: Modeling, Valuation and Hedging, Springer Finance Series (2002).
[6] T. R. Bielecki, M. Jeanblanc, M. Rutkowski, Valuation and Hedging of Credit Derivatives, Lecture notes CIMPA- UNESCO Morocco School, (2009).
[7] L. Bo, A. Capponi, P. Chen, Credit portfolio selection with decaying contagion intensities, Matematical Finance, 29, 137-173, (2019).
[8] L. Bo, C. Ceci, Locally Risk-Minimizing Hedging of Counterparty Risk for Portfolio of Credit Derivatives, Applied Mathematics \& Optimization, (2019)

|  | $K=90$ |  |  | $K=100$ |  |  | $K=110$ |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | $\alpha=0$ | $\alpha=0.5$ | $\alpha=1$ | $\alpha=0$ | $\alpha=0.5$ | $\alpha=1$ | $\alpha=0$ | $\alpha=0.5$ | $\alpha=1$ |
| $g_{0}$ | 16.3455 | 16.4559 | 16.5663 | 11.2208 | 11.2965 | 11.3723 | 7.4639 | 7.5142 | 7.5646 |
| $g_{1}$ | -0.0317 | -0.0155 | 0.0007 | -0.0254 | -0.0124 | 0.0006 | -0.0193 | -0.0094 | 0.0004 |
| $g_{2}$ | 0.0004 | 0.0004 | 0.0003 | 0.0004 | 0.0003 | 0.0002 | 0.0003 | 0.0002 | 0.0001 |

Table 6: Values of the zero-th and first order terms of the expansion approximation with different strikes $K$ and levels of collateralization $\alpha$.

| $m$ | -0.2 | -0.1 | 0 | 0.1 | 0.2 |
| :---: | :---: | :---: | :---: | :---: | :---: |
|  | $\alpha=0$ |  |  |  |  |
| $c^{\text {const }}$ | 5.5458 | 8.3127 | 11.9943 | 16.7047 | 22.5212 |
| $g_{0}$ | 5.2034 | 7.7995 | 11.2539 | 15.6736 | 21.1312 |
| $c^{\text {const }}$ | 5.5728 | 8.3532 | $\alpha=0.5$ | 12.0527 | 16.7862 |
| $g_{0}$ | 5.2307 | 7.8405 | 11.3130 | 15.7561 | 21.2410 |
|  | $\alpha=1$ |  |  |  |  |
| $c^{\text {const }}$ | 5.5998 | 8.3937 | 12.1112 | 16.8676 | 22.7408 |
| $g_{0}$ | 5.2581 | 7.8815 | 11.3722 | 15.8385 | 21.3535 |

Table 7: Values of the zero-th and first order terms of the expansion approximation with different moneyness $m$ and levels of collateralization $\alpha$. The prices $c^{\text {const }}$ are obtained from (36).
[9] D. Brigo, F. Vrins, Disentangling wrong-way risk: pricing credit valuation adjustment via change of measures, European Journal of Operational Research, 269, 1154-1164, (2018).
[10] D. Brigo, M. Morini, A. Pallavicini, Counterparty Credit Risk, Collateral and Funding: With Pricing Cases For All Asset Classes, Wiley (2013).
[11] D. Brigo, T. Hvolby, F. Vrins, Wrong-Way Risk adjusted exposure: Analytical Approximations for Options in Default Intensity Models, In Innovations in Insurance, Risk and Asset Management, WSPC Proceedings (2018).
[12] D. Brigo, Q. D. Liu, A. Pallavicini, D. Sloth, Nonlinear Valuation Under collateralization, Credit Risk, and Funding Costs, In Innovations in Derivatives Markets, Springer Proceedings In Mathematics \& Statistics, 165 (2016).
[13] D. Brigo, M. Francischello, A. Pallavicini, Analysis of Nonlinear Valuation Equations Under Credit and Funding Effects, In Innovations in Derivatives Markets, Springer Proceedings In Mathematics \& Statistics, 165 (2016).
[14] D. Brigo, M. Francischello, A. Pallavicini, Nonlinear valuation under credit, funding, and margins: Existence, uniqueness, invariance, and disentanglement, European Journal of Operational Research, 274, 2, 788-805 (2019).


Figure 1: MC prices (dot) vs approximated prices (lines). On the left $T=0.5$, on the right $T=2$.
[15] C. Burgard, M. Kjaer, Partial differential equation representations of derivatives with counterparty risk and funding costs, The Journal of Credit Risk, 7 (3), 1-19 (2011).
[16] C. Burgard, M. Kjaer, Derivatives funding, netting and accounting, Risk, 30 (March), 100104 (2017).
[17] A. Capponi, S. Pagliarani, T. Vargiolu, Pricing vulnerable claims in a Levy driven model, Finance and Stochastics, 18, $755-789$ (2014).
[18] U. Cherubini, E. Luciano, Pricing Vulnerable Options with Copulas, Journal of Risk Finance, 5, 27-39 (2003).
[19] D. Duffie, K. J. Singleton, Modeling term structures of defaultable bonds, Review Financial Studies, 12 , 687-720 (1999).
[20] D. Duffie, M. Huang, Swap rates and credit quality, J. Financ., 51, 3, 687-720 (1996).
[21] F. A. Fard, Analytical pricing of vulnerable options under a generalized jump-diffusion model, Insurance Mathematics and Economics, 60, 19-28 (2015).
[22] P. V. Gapeev, Some extensions of Norros' lemma in models with several defaults. Inspired by Finance, The Musiela Festschrift. Kabanov Yu. M., Rutkowski M., Zariphopoulou Th. eds. Springer, 273-281 (2014).
[23] P. V. Gapeev, M. Jeanblanc, L. Li, M. Rutkowski, Constructing random measures with given survival processes and applications to valuation of credit derivatives, Contemporary Quantitative Finance, Essays in Honour of Eckhard Platen. Chiarella, C., Novikov, A. eds. Springer, 255-280 (2010).
[24] K. Glau, Z. Grbac, M. Scherer, R. Zagst (eds.), Innovations in Derivatives Markets, Springer Proceedings in Mathematics \& Statistics 165 (2016).
[25] J. Gregory, Counterparty credit risk and credit value adjustment, Wiley (2012).
[26] A.Green XVA: Credit, Funding and Capital Valuation Adjustments XVA: Credit, Funding and Capital Valuation Adjustments, Wiley (2016)
[27] L. J. Kao, Credit valuation adjustment of cap and floor with counterparty risk: a structural pricing model for vulnerable European options, Review of Deriv. Research, 19, 41-64 (2016).
[28] M. Jeanblanc, M. Yor, M. Chesney, Mathematical methods for financial markets, Springer (2009).
[29] H. Johnson, R. Stulz, The Pricing of Options with Default Risk, Journal of Finance, 42, 267-280 (1987).
[30] J. Kim, T. Leung, Pricing derivatives with counterparty risk and collateralization: A fixed point approach, European Journal of Operational Research, 249 (2), 525-539 (2016).
[31] P. Klein, Pricing Black-Scholes options with correlated credit risk, Journal of Banking \& Finance, 20, 1211-1229 (1996).
[32] P. Klein, M. Inglis, Valuation of European options subject to financial distress and interest rate risk, Journal of derivatives, 6, 44-56 (1999).
[33] P. Klein, M. Inglis, Pricing vulnerable European option's when the option payoff can increase the risk of financial distress, Journal of Banking \& Finance, 25, 993-1012 (2001).
[34] D. Lando, Credit Risk Modeling, Princeton University Press (2004).
[35] R. Lord, R. Koekkoek, D. Van DijK, A comparison of biased simulation schemes for the stochastic volatility models, Quantitative Finance, 10 (2) (2010) 177-194.
[36] L. Tian, G. Wang, X. Wang, Y. Wang, Pricing vulnerable options with correlated credit risk under jump-diffusion processes, Journal of Futures Markets, 34, 957-979 ( 2014).
[37] S. Zhu, M. Pykhtin, A Guide to Modeling Counterparty Credit Risk, GARP Risk Review, July/August (2007)


[^0]:    *University of L'Aquila, fabio. antonelli@univaq.it
    ${ }^{\dagger}$ Dept. Economics and Finance, University of Roma - Tor Vergata, alessandro.ramponi@uniroma2.it
    ${ }^{\ddagger}$ Dept. Enterprise Engineering, University of Roma - Tor Vergata, sergio.scarlatti@uniroma2.it

