

Enhancement of e-Commerce via Mobile Accesses to the Internet

Ushio Sumita , Jun Yoshii

*Graduate School of Systems and Information Engineering, University of Tsukuba,
1-1-1, Tenoudai, Tsukuba, Ibaraki, 305-8573, Japan
(E-mail address: sumita@sk.tsukuba.ac.jp;yoshii40@sk.tsukuba.ac.jp)*

Abstract

The potential of the Internet has been expanded substantially by a new generation of mobile devices, opening the door for rapid growth of m-commerce. While the traditional PC access to the Internet continues to be vital for exploiting the advantages of the Internet, the mobile access appears to attract more people because of flexible accesses to the Internet in a ubiquitous manner. Accordingly, e-commerce is now in the process of being converted into m-commerce. The purpose of this paper is to develop and analyze a mathematical model for comparing e-commerce via the traditional PC access only with m-commerce which accommodates both the traditional PC access and the mobile access. The distribution of the number of products purchased by time t and the distribution of the time required for selling K products are derived explicitly, enabling one to assess the impact of mobile devices on e-businesses. Numerical examples are given for illustrating behavioral differences between m-commerce consumers and traditional e-commerce consumers.

Key words: M-commerce, E-commerce, Consumer behavior, Semi-Markov process, Sales completion time

1. Introduction

The potential of the Internet has been expanded substantially by a new generation of mobile devices, opening the door for rapid growth of m-commerce. While the traditional PC access to the Internet continues to be vital for exploiting the advantages of the Internet, the mobile access appears to attract more people because of flexible accesses to the Internet in a ubiquitous manner. Accordingly, e-commerce is now in the process of being converted into m-commerce.

Because of the fact that the mobile technology is still young, the study of the impacts of mobile devices on e-businesses is also rather new in the literature. Roto[1] and Kim[2] provide the current state of mobile devices and m-businesses. Chae and Kim[3] discuss the business implications of m-commerce, and Barwise[4] and Hammond[5] predict the evolutionary trend of m-commerce in the foreseeable future. Wu and Hisa[6] propose the hypercube innovation model for analyzing the characteristics of m-commerce with focus on three axes: changes in business models, changes in core components and stake holders. Siau, Sheng and Nah[7], and Park and Fader[8] investigate the benefits of m-commerce to consumers and how e-commerce has changed the consumer behavior. Büyüközkan[9] develops an analytical approach for determining the mobile commerce user requirements. All of these papers are either empirical, qualitative or static in their analytical nature and, to the best knowledge of the authors, no study exists in the literature for capturing behavioral differences between e-commerce and m-commerce based on a mathematical stochastic model.

The purpose of this paper is to develop and analyze a mathematical model for comparing e-commerce via the traditional PC access only with m-commerce which accommodates both the traditional PC access and the mobile access. More specifically, we consider consumers who intend to decide whether or not they should buy a product by exploring the Internet for information. In order to capture their behavioral patterns, each day is decomposed into three periods. The first period of a day represents working hours, while the second period and the third period of a day correspond to evening hours and sleeping hours at home respectively. As reported in [10], corporate employees often utilize company PCs for privately accessing the Internet. Accordingly, during the first period of a day, the PC access is assumed to be available from time to time for the private use of the Internet. The mobile access is also possible if consumers choose to do so. It is natural

to assume that the PC access supersedes the mobile access during evening hours at home. Accordingly, only the PC access is considered during the second period of a day. Since the third period of a day represents sleeping hours, the consumers are inactive in using the Internet.

Three classes of consumers are considered concerning the ways they access the Internet: those who access the Internet only through PCs throughout the period under consideration; those who access the Internet originally only through PCs but start using the mobile access at some time later; and those who access the Internet through both PCs and mobile devices from the very beginning. These classes of consumers are denoted by \mathcal{C}_{PC} , $\mathcal{C}_{PC \rightarrow BOTH}$ and \mathcal{C}_{BOTH} , with the entire consumer class defined by $\mathcal{C} = \mathcal{C}_{PC} \cup \mathcal{C}_{PC \rightarrow BOTH} \cup \mathcal{C}_{BOTH}$. (Referring to [11], an anonymous referee pointed out the importance of incorporating $\mathcal{C}_{PC \rightarrow BOTH}$ in our model, which was missing in the original version of this paper.) Each time the Internet is accessed for information, a consumer makes one of the three decisions: to purchase the product, not to purchase the product, or to remain undecided. We assume that the product is purchased at most once by any consumer in the period under consideration for our analysis.

In order to capture the stochastic behavior of a consumer in \mathcal{C} in a unified manner, we consider a semi-Markov process having six transient states and two absorbing states. Transient states i and $3 + i$ correspond to the i -th period of a day for $i = 1, 2, 3$. Absorbing states 0 and 7 describe the decision of purchasing and that of not purchasing respectively. Starting at state 1, those consumers in \mathcal{C}_{PC} continue to move states 1, 2 and 3 in a cyclic manner until they reach either state 0 or state 7. The behavior of those consumers in \mathcal{C}_{BOTH} is similar except that they start at state 4 and continue to move states 4, 5 and 6 until they reach absorption. Those consumers in $\mathcal{C}_{PC \rightarrow BOTH}$ start at state 1 as for those in \mathcal{C}_{PC} . At the end of the dwell time in state 3, however, they move to state 4 with probability $1 - r$ where $0 < r < 1$ and start using the mobile access. With remaining probability r , they remain as an exclusive PC access user of the Internet and move to state 1. From the point of view of the unified semi-Markov model, those consumers in \mathcal{C}_{PC} can be interpreted as having $r = 1$.

Through dynamic analysis of the semi-Markov process, the two stochastic performance measures of interest can be evaluated: the distribution of the number of products sold by time t and the distribution of the time required for selling K products. This analysis, in turn, enables one to assess the impact of mobile devices on e-businesses by comparing such stochastic performance

measures for m-commerce against those for traditional e-commerce.

The structure of this paper is as follows. In Section 2, a mathematical model is developed for capturing the consumer behavior in m-commerce based on a semi-Markov process approach. Section 3 is devoted to dynamic analysis of the semi-Markov model. The two stochastic performance measures are introduced in Section 4 and the associated distributions are derived explicitly, which can be computed based on the results in Section 3. Numerical examples are given in Section 5 for illustrating behavioral differences between m-commerce consumers and traditional e-commerce consumers. Finally, some concluding remarks are given in section 6.

Throughout the paper, vectors and matrices are indicated by underbar and doubleunderbar respectively, e.g. $\underline{\xi}$, $\underline{\underline{P}}(t)$, etc. The vector with all components equal to 0 is denoted by $\underline{0}$. The identity matrix is denoted by \underline{I} .

2. Development of Mathematical Model for m-Commerce Consumer Behavior: Semi-Markov Process Approach

For capturing the consumer behavior in m-commerce described in the previous section more formally, we consider a semi-Markov process $\{J(t) : t \geq 0\}$ defined on $\mathcal{N} = \{0, 1, \dots, 7\}$. Here, the i -th period of a day for those consumers in \mathcal{C}_{PC} is represented by state i , and that for those in \mathcal{C}_{BOTH} corresponds to state $i + 3$, $i = 1, 2, 3$. The two states 0 and 7 are absorbing, where the former corresponds to the decision of purchasing the product while the latter represents the decision of not purchasing the product. Given that neither the decision of purchasing nor that of not purchasing is made, we assume, for the time being, that the dwell time of the semi-Markov process in state i is absolutely continuous with probability density function (p.d.f.) $a_i(x)$, $i = 1, \dots, 6$. The corresponding distribution function, the survival function and the hazard rate function are denoted by

$$A_i(x) = \int_0^x a_i(x)dx; \quad \bar{A}_i(x) = 1 - A_i(x); \quad \eta_i(x) = \frac{a_i(x)}{\bar{A}_i(x)}. \quad (2.1)$$

It is clear that $a_i(x) = a_{i+3}(x)$ for $i = 1, 2, 3$. Because of this, we write $a_1(x) = a_4(x) = a_W(x)$, $a_2(x) = a_5(x) = a_E(x)$ and $a_3(x) = a_6(x) = a_S(x)$ interchangeably. $A_W(x)$, $\bar{A}_W(x)$, $\eta_W(x)$, etc. are defined accordingly. Since states 0 and 7 are absorbing, the dwell time in those states are infinite. The corresponding survival functions can then be written as

$$\bar{A}_0(x) = \bar{A}_7(x) = 1 \quad \text{for all } x \geq 0. \quad (2.2)$$

For those consumers in \mathcal{C}_{PC} , the Internet accesses occur in state $i = 1$ and $i = 2$ according to a Poisson processes with intensity $\lambda_1 = \lambda_{W:PC}$ and $\lambda_2 = \lambda_{E:PC}$ respectively. The corresponding probabilities of purchasing (not purchasing) for each access are denoted by $\alpha_1 = \alpha_{W:PC}$ and $\alpha_2 = \alpha_{E:PC}$ ($\beta_1 = \beta_{W:PC}$ and $\beta_2 = \beta_{E:PC}$) with $0 < \alpha_i + \beta_i < 1$, and the consumer remains undecided with probability $1 - (\alpha_i + \beta_i) > 0$, for $i = 1, 2$. Consequently, one has

$$\begin{aligned} \xi_{W:PC} &= \lambda_{W:PC} \alpha_{W:PC} & ; & & \theta_{W:PC} &= \lambda_{W:PC} \beta_{W:PC} \\ \xi_{E:PC} &= \lambda_{E:PC} \alpha_{E:PC} & ; & & \theta_{E:PC} &= \lambda_{E:PC} \beta_{E:PC} , \end{aligned} \quad (2.3)$$

where $\xi_{W:PC}$ ($\theta_{W:PC}$) is the transition intensity from state 1 to state 0 (state 7). $\xi_{E:PC}$ and $\theta_{E:PC}$ are defined similarly. At the end of the third period of a day, a consumer in \mathcal{C}_{PC} decides to start using a mobile phone with probability $1 - r$. This means that, upon completion of the dwell time in state 3, the consumer moves to state 1 with probability r and to state 4 with probability $1 - r$.

A consumer in \mathcal{C}_{BOTH} may employ both a PC and a mobile device for accessing the Internet. The Poisson intensity for PC accesses is denoted by $\lambda_{4:PC} = \lambda_{W:BOTH(PC)}$, and that for mobile accesses is written as $\lambda_{4:Mobile} = \lambda_{W:BOTH(Mobile)}$. The probabilities of purchasing (not purchasing) for each access are defined as before and are denoted by $\alpha_4 = \alpha_{W:BOTH}$ and $\alpha_5 = \alpha_{E:PC}$ ($\beta_4 = \beta_{W:BOTH}$ and $\beta_5 = \beta_{E:PC}$). In parallel with (2.3), one then has

$$\begin{aligned} \xi_{W:BOTH} &= \lambda_{W:BOTH} \cdot \alpha_{W:BOTH} , \\ \theta_{W:BOTH} &= \lambda_{W:BOTH} \cdot \beta_{W:BOTH} , \end{aligned} \quad (2.4)$$

where

$$\lambda_{W:BOTH} = \lambda_{W:BOTH(PC)} + \lambda_{W:BOTH(Mobile)} . \quad (2.5)$$

For evening hours, those consumers in \mathcal{C}_{PC} and those in \mathcal{C}_{BOTH} are indifferent and their stochastic behavioral structures are identical. We note that $\xi_{W:PC} < \xi_{W:BOTH}$ and $\theta_{W:PC} < \theta_{W:BOTH}$. These differences together with probability r representing the population growth of the mobile access users characterize the impact of mobile accesses on e-commerce in our model. The transition structure of the semi-Markov process is depicted in Figure 2.1.

In order to deal with the case in which the three periods of a day are constant, we subsequently choose, for each $i \in \{1, \dots, 6\}$, a sequence of distribution functions $(A_i(k, x))_{k=1}^{\infty}$ such that $A_i(k, x) \rightarrow U(x - \tau_i)$ as $k \rightarrow \infty$,

where τ_i is the constant dwell time in state i and $U(x)$ is a step function defined by $U(x) = 1$ for $x \geq 0$ and $U(x) = 0$ for $x < 0$.

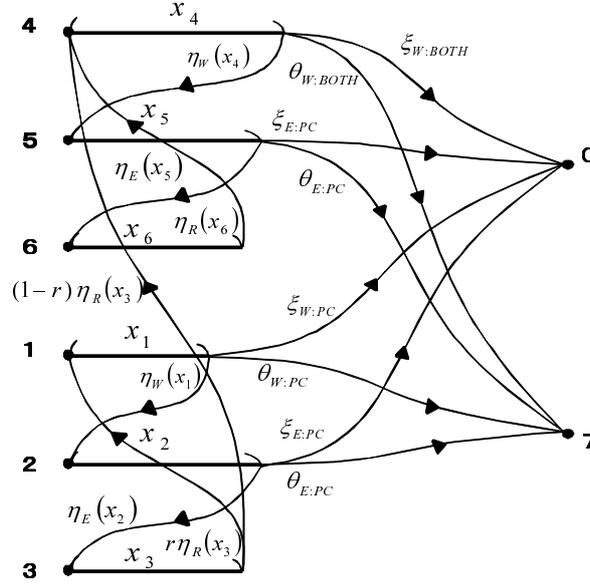


Figure 2.1 : Transition Structure of the Semi-Markov Process

3. Dynamic Analysis of the Semi-Markov Process

In this section, we derive explicitly the transition probability matrix $\underline{\underline{P}}(t)$ of the semi-Markov process $J(t)$, where $\underline{\underline{P}}(t)$ is defined by

$$\underline{\underline{P}}(t) = [P_{ij}(t)]; \quad P_{ij}(t) \stackrel{\text{def}}{=} \text{P}[J(t) = j | J(0) = i], \quad i, j \in \mathcal{N}. \quad (3.1)$$

For this purpose, the age process $X(t)$ associated with the semi-Markov process $J(t)$ is introduced as the elapsed time since the last transition of $J(t)$ into the current state at time t . Clearly the bivariate process $[J(t), X(t)]$ becomes Markov and the first step of our analysis is to evaluate the joint distribution function defined by

$$F_{ij}(x, t) = \text{P}[X(t) \leq x, J(t) = j | J(0) = i], \quad (3.2)$$

and the corresponding joint p.d.f.

$$\frac{d}{dx} F_{ij}(x, t) = f_{ij}(x, t), \quad (3.3)$$

where the delta function $\delta(t)$ is employed for describing the boundary conditions with respect to x . More specifically, one sees that

$$f_{i1}(0+, t) = \delta_{\{i=1\}}\delta(t) + r \int_0^\infty f_{i3}(x, t)\eta_S(x)dx ; \quad (3.4)$$

$$f_{i2}(0+, t) = \delta_{\{i=2\}}\delta(t) + \int_0^\infty f_{i1}(x, t)\eta_W(x)dx ; \quad (3.5)$$

$$f_{i3}(0+, t) = \delta_{\{i=3\}}\delta(t) + \int_0^\infty f_{i2}(x, t)\eta_E(x)dx ; \quad (3.6)$$

$$f_{i4}(0+, t) = \delta_{\{i=4\}}\delta(t) + \int_0^\infty \{(1-r)f_{i3}(x, t) + f_{i6}(x, t)\}\eta_S(x)dx ; \quad (3.7)$$

$$f_{i5}(0+, t) = \delta_{\{i=5\}}\delta(t) + \int_0^\infty f_{i4}(x, t)\eta_W(x)dx ; \quad (3.8)$$

$$f_{i6}(0+, t) = \delta_{\{i=6\}}\delta(t) + \int_0^\infty f_{i5}(x, t)\eta_E(x)dx . \quad (3.9)$$

Here, $\delta_{\{ST\}} = 1$ if statement ST is true, and $\delta_{\{ST\}} = 0$ otherwise. The delta function $\delta(t)$ is the unit operator associated with convolution, i.e. $g(t) = \int_0^\infty g(x)\delta(t-x)dx$ for any integrable function $g(t)$ on $[0, \infty)$.

In order to evaluate the joint p.d.f. given in (3.3), we introduce the following Laplace transforms.

$$\alpha_i(s) = \int_0^\infty e^{-sx}a_i(x)dx \quad \text{for } i = 1, \dots, 6 \quad (3.10)$$

$$\beta_i(s) = \int_0^\infty e^{-sx}\bar{A}_i(x)dx = \frac{1 - \alpha_i(s)}{s} \quad \text{for } i = 1, \dots, 6 \quad (3.11)$$

$$\underline{\underline{\hat{\zeta}}}(0+, s) = [\hat{\zeta}_{ij}(0+, s)] ; \quad \hat{\zeta}_{ij}(0+, s) \stackrel{\text{def}}{=} \int_0^\infty e^{-st}f_{ij}(0+, t)dt \quad \text{for } i, j \in \mathcal{N} \quad (3.12)$$

$$\underline{\underline{\hat{\varphi}}}(x, s) = [\hat{\varphi}_{ij}(x, s)] ; \quad \hat{\varphi}_{ij}(x, s) \stackrel{\text{def}}{=} \int_0^\infty e^{-st}f_{ij}(x, t)dt \quad \text{for } i, j \in \mathcal{N} \quad (3.13)$$

$$\underline{\underline{\hat{\psi}}}(v, s) = [\hat{\psi}_{ij}(v, s)] ; \quad \hat{\psi}_{ij}(v, s) \stackrel{\text{def}}{=} \int_0^\infty e^{-vx}\hat{\varphi}_{ij}(x, s)dx \quad \text{for } i, j \in \mathcal{N} \quad (3.14)$$

For notational convenience, we also define

$$\begin{cases} C_4 = C_{W:BOTH} = \xi_{W:BOTH} + \theta_{W:BOTH} \\ C_1 = C_{W:PC} = \xi_{W:PC} + \theta_{W:PC} \\ C_{2,5} = C_{E:PC} = \xi_{E:PC} + \theta_{E:PC} \end{cases}, \quad (3.15)$$

as well as the following functions, vectors, and matrices.

$$d_1(s) = 1 - r\alpha_W(s + C_1)\alpha_E(s + C_{2,5})\alpha_S(s) \quad (3.16)$$

$$d_2(s) = 1 - \alpha_W(s + C_4)\alpha_E(s + C_{2,5})\alpha_S(s) \quad (3.17)$$

$$\underline{\hat{\xi}}(s) = \begin{bmatrix} \hat{\xi}_1(s) \\ \hat{\xi}_2(s) \\ \hat{\xi}_3(s) \\ \hat{\xi}_4(s) \\ \hat{\xi}_5(s) \\ \hat{\xi}_6(s) \end{bmatrix} = \begin{bmatrix} \xi_{W:PC}\beta_W(s + C_1) \\ \xi_{E:PC}\beta_E(s + C_{2,5}) \\ 0 \\ \xi_{W:BOTH}\beta_W(s + C_4) \\ \xi_{E:PC}\beta_E(s + C_{2,5}) \\ 0 \end{bmatrix} \quad (3.18)$$

$$\underline{\hat{\theta}}(s) = \begin{bmatrix} \hat{\theta}_1(s) \\ \hat{\theta}_2(s) \\ \hat{\theta}_3(s) \\ \hat{\theta}_4(s) \\ \hat{\theta}_5(s) \\ \hat{\theta}_6(s) \end{bmatrix} = \begin{bmatrix} \theta_{W:PC}\beta_W(s + C_1) \\ \theta_{E:PC}\beta_E(s + C_{2,5}) \\ 0 \\ \theta_{W:BOTH}\beta_W(s + C_4) \\ \theta_{E:PC}\beta_E(s + C_{2,5}) \\ 0 \end{bmatrix} \quad (3.19)$$

$$\underline{\psi}_0(s) = [\psi_{0:1}(s), \dots, \psi_{0:6}(s)]^T, \quad (3.20)$$

where

$$\begin{aligned} \psi_{0:1}(s) &= d_2(s)\{\hat{\xi}_1(s) + \alpha_W(s + C_1)\hat{\xi}_2(s)\} \\ &\quad + (1 - r)\alpha_W(s + C_1)\alpha_E(s + C_{2,5})\alpha_S(s) \\ &\quad \times \{\hat{\xi}_4(s) + \alpha_W(s + C_4)\hat{\xi}_5(s)\}; \\ \psi_{0:2}(s) &= d_2(s)\{r\alpha_E(s + C_{2,5})\alpha_S(s)\hat{\xi}_1(s) + \hat{\xi}_2(s)\} \\ &\quad + (1 - r)\alpha_E(C_{2,5})\alpha_S(s)\{\hat{\xi}_4(s) + \alpha_W(s + C_4)\hat{\xi}_5(s)\}; \\ \psi_{0:3}(s) &= rd_2(s)\alpha_S(s)\{\hat{\xi}_1(s) + \alpha_W(s + C_1)\hat{\xi}_2(s)\} \\ &\quad + (1 - r)\alpha_S(s)\{\hat{\xi}_4(s) + \alpha_W(s + C_4)\hat{\xi}_5(s)\}; \\ \psi_{0:4}(s) &= d_1(s)\{\hat{\xi}_4(s) + \alpha_W(s + C_4)\hat{\xi}_5(s)\}; \\ \psi_{0:5}(s) &= d_1(s)\{\alpha_E(s + C_{2,5})\alpha_S(s)\hat{\xi}_4(s) + \hat{\xi}_5(s)\}; \\ \psi_{0:6}(s) &= d_1(s)\alpha_S(s)\{\hat{\xi}_4(s) + \alpha_W(s + C_4)\hat{\xi}_5(s)\}. \end{aligned}$$

Similarly, we define

$$\underline{\psi}_7(s) = [\psi_{7:1}(s), \dots, \psi_{7:6}(s)]^T, \quad (3.21)$$

with

$$\begin{aligned} \psi_{7:1}(s) &= d_2(s) \{ \hat{\theta}_1(s) + \alpha_W(s + C_1) \hat{\theta}_2(s) \} \\ &\quad + (1-r) \alpha_W(s + C_1) \alpha_E(s + C_{2,5}) \alpha_S(s) \\ &\quad \times \{ \hat{\theta}_4(s) + \alpha_W(s + C_4) \hat{\theta}_5(s) \}; \\ \psi_{7:2}(s) &= d_2(s) \{ r \alpha_E(s + C_{2,5}) \alpha_S(s) \hat{\theta}_1(s) + \hat{\theta}_2(s) \} \\ &\quad + (1-r) \alpha_E(C_{2,5}) \alpha_S(s) \{ \hat{\theta}_4(s) + \alpha_W(s + C_4) \hat{\theta}_5(s) \}; \\ \psi_{7:3}(s) &= r d_2(s) \alpha_S(s) \{ \hat{\theta}_1(s) + \alpha_W(s + C_1) \hat{\theta}_2(s) \} \\ &\quad + (1-r) \alpha_S(s) \{ \hat{\theta}_4(s) + \alpha_W(s + C_4) \hat{\theta}_5(s) \}; \\ \psi_{7:4}(s) &= d_1(s) \{ \hat{\theta}_4(s) + \alpha_W(s + C_4) \hat{\theta}_5(s) \}; \\ \psi_{7:5}(s) &= d_1(s) \{ \alpha_E(s + C_{2,5}) \alpha_S(s) \hat{\theta}_4(s) + \hat{\theta}_5(s) \}; \\ \psi_{7:6}(s) &= d_1(s) \alpha_S(s) \{ \hat{\theta}_4(s) + \alpha_W(s + C_4) \hat{\theta}_5(s) \}. \end{aligned}$$

$$\underline{\underline{G}}_1(s) = \begin{bmatrix} 1 & \alpha_W(s + C_1) & \alpha_W(s + C_1) \alpha_E(s + C_{2,5}) \\ r \alpha_E(s + C_{2,5}) \alpha_S(s) & 1 & \alpha_E(s + C_{2,5}) \\ r \alpha_S(s) & r \alpha_W(s + C_1) \alpha_S(s) & 1 \end{bmatrix}$$

$$\underline{\underline{g}}^T(s) = [\alpha_W(s + C_1) \alpha_E(s + C_{2,5}), \alpha_E(s + C_{2,5}), 1]$$

$$\underline{\underline{G}}_2(s) = (1-r) \alpha_S(s) [\underline{\underline{g}}(s), \alpha_W(s + C_4) \underline{\underline{g}}(s), \alpha_W(s + C_4) \alpha_E(s + C_{2,5}) \underline{\underline{g}}(s)]$$

$$\underline{\underline{G}}_3(s) = \begin{bmatrix} 1 & \alpha_W(s + C_4) & \alpha_W(s + C_4) \alpha_E(s + C_{2,5}) \\ \alpha_E(s + C_{2,5}) \alpha_S(s) & 1 & \alpha_E(s + C_{2,5}) \\ \alpha_S(s) & \alpha_W(s + C_4) \alpha_S(s) & 1 \end{bmatrix}$$

$$\underline{\underline{G}}(s) = \begin{bmatrix} d_2(s) \underline{\underline{G}}_1(s) & \underline{\underline{G}}_2(s) \\ \underline{\underline{0}} & d_1(s) \underline{\underline{G}}_3(s) \end{bmatrix} \quad (3.22)$$

Then the following theorem holds.

Theorem 3.1. *Let $\hat{\zeta}(0+, s)$ and $\hat{\hat{\varphi}}(v, s)$ be as in (3.12) and (3.14) respectively. One then has:*

$$a) \quad \hat{\zeta}(0+, s) = \frac{1}{d_1(s) d_2(s)} \begin{bmatrix} 0 & \underline{\underline{0}}^T & 0 \\ \underline{\psi}_0(s) & \underline{\underline{G}}(s) & \underline{\psi}_7(s) \\ 0 & \underline{\underline{0}}^T & 0 \end{bmatrix},$$

where $d_1(s)$, $d_2(s)$, $\underline{\psi}_0(s)$, $\underline{\psi}_7(s)$ and $\underline{G}(s)$ are as given in (3.16), (3.17), (3.20), (3.21) and (3.22) respectively.

$$\begin{aligned} b) \quad \underline{\hat{\varphi}}(v, s) &= \underline{\hat{\zeta}}(0+, s) \\ &\times \text{diag} \left\{ \frac{1}{s+v}, \beta_W(s+v+C_1), \beta_E(s+v+C_{2,5}), \beta_S(s+v), \right. \\ &\quad \left. \beta_W(s+v+C_4), \beta_E(s+v+C_{2,5}), \beta_S(s+v), \frac{1}{s+v} \right\}, \end{aligned}$$

where $\text{diag}\{a_1, \dots, a_n\}$ denotes an $n \times n$ diagonal matrix with diagonal elements a_1, \dots, a_n .

Proof. In addition to the boundary conditions in (3.4) through (3.9) for states 1 through 6 respectively, one sees, for states 0 and 7, that

$$\begin{aligned} f_{i0}(0+, t) &= \xi_{W:PC} \int_0^\infty f_{i1}(x, t) dx \\ &+ \xi_{E:PC} \int_0^\infty \{f_{i2}(x, t) + f_{i5}(x, t)\} dx \\ &+ \xi_{W:BOTH} \int_0^\infty f_{i4}(x, t) dx \end{aligned} \quad (3.23)$$

and

$$\begin{aligned} f_{i7}(0+, t) &= \theta_{W:PC} \int_0^\infty f_{i1}(x, t) dx \\ &+ \theta_{E:PC} \int_0^\infty \{f_{i2}(x, t) + f_{i5}(x, t)\} dx \\ &+ \theta_{W:BOTH} \int_0^\infty f_{i4}(x, t) dx. \end{aligned} \quad (3.24)$$

By taking the Laplace transform of (3.4) through (3.9) and the above two equations with respect to t , one finds that

$$\begin{aligned} \hat{\zeta}_{i0}(0+, s) &= \xi_{W:PC} \hat{\zeta}_{i1}(0+, s) \beta_W(s+C_1) \\ &+ \xi_{E:PC} \left\{ \hat{\zeta}_{i2}(0+, s) + \hat{\zeta}_{i5}(0+, s) \right\} \beta_E(s+C_{2,5}) \\ &+ \xi_{W:BOTH} \hat{\zeta}_{i4}(0+, s) \beta_W(s+C_4); \end{aligned} \quad (3.25)$$

$$\hat{\zeta}_{i1}(0+, s) = \delta_{\{i=1\}} + r\hat{\zeta}_{i3}(0+, s)\alpha_S(s); \quad (3.26)$$

$$\hat{\zeta}_{i2}(0+, s) = \delta_{\{i=2\}} + \hat{\zeta}_{i1}(0+, s)\alpha_W(s + C_1); \quad (3.27)$$

$$\hat{\zeta}_{i3}(0+, s) = \delta_{\{i=3\}} + \hat{\zeta}_{i2}(0+, s)\alpha_E(s + C_{2,5}); \quad (3.28)$$

$$\hat{\zeta}_{i4}(0+, s) = \delta_{\{i=4\}} + \left\{ (1-r)\hat{\zeta}_{i3}(0+, s) + \hat{\zeta}_{i6}(0+, s) \right\} \alpha_S(s); \quad (3.29)$$

$$\hat{\zeta}_{i5}(0+, s) = \delta_{\{i=5\}} + \hat{\zeta}_{i4}(0+, s)\alpha_W(s + C_4); \quad (3.30)$$

$$\hat{\zeta}_{i6}(0+, s) = \delta_{\{i=6\}} + \hat{\zeta}_{i5}(0+, s)\alpha_E(s + C_{2,5}); \quad (3.31)$$

$$\begin{aligned} \hat{\zeta}_{i7}(0+, s) &= \theta_{W:PC}\hat{\zeta}_{i1}(0+, s)\beta_W(s + C_1) \\ &+ \theta_{E:PC}\left\{ \hat{\zeta}_{i2}(0+, s) + \hat{\zeta}_{i5}(0+, s) \right\} \beta_E(s + C_{2,5}) \\ &+ \theta_{W:BOTH}\hat{\zeta}_{i4}(0+, s)\beta_W(s + C_4). \end{aligned} \quad (3.32)$$

By describing (3.25) through (3.32) in matrix form, it follows that

$$\underline{\underline{\hat{\zeta}}}(0+, s) = \begin{bmatrix} \underline{0}^T \\ \underline{u}_1^T \\ \underline{u}_2^T \\ \underline{u}_3^T \\ \underline{u}_4^T \\ \underline{u}_5^T \\ \underline{u}_6^T \\ \underline{0}^T \end{bmatrix} + \underline{\underline{\hat{\zeta}}}(0+, s) \underline{\underline{\gamma}}(s), \quad (3.33)$$

where \underline{u}_i is the i -th unit vector in R^8 and

$$\underline{\underline{\gamma}}(s) = \begin{bmatrix} 0 & \underline{0}^T & 0 \\ \hat{\underline{\xi}}(s) & \underline{\underline{B}}(s) & \hat{\underline{\theta}}(s) \\ 0 & \underline{0}^T & 0 \end{bmatrix}, \quad (3.34)$$

with

$$\underline{\underline{B}}(s) = \begin{bmatrix} 0 & \alpha_W(s + C_1) & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_E(s + C_{2,5}) & 0 & 0 & 0 \\ r\alpha_R(s) & 0 & 0 & (1-r)\alpha_R(s) & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha_W(s + C_4) & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha_E(s + C_{2,5}) \\ 0 & 0 & 0 & \alpha_R(s) & 0 & 0 \end{bmatrix}. \quad (3.35)$$

Equation (3.33) can be solved for $\hat{\underline{\zeta}}(0+, s)$ as

$$\hat{\underline{\zeta}}(0+, s) = \begin{bmatrix} \underline{0}^T \\ \underline{u}_1^T \\ \underline{u}_2^T \\ \underline{u}_3^T \\ \underline{u}_4^T \\ \underline{u}_5^T \\ \underline{u}_6^T \\ \underline{0}^T \end{bmatrix} \left[\underline{I} - \underline{\gamma}(s) \right]^{-1}. \quad (3.36)$$

It can be shown from (3.34), after a little algebra, that

$$\left[\underline{I} - \underline{\gamma}(s) \right]^{-1} = \frac{1}{d_1(s)d_2(s)} \begin{bmatrix} d_1(s)d_2(s) & \underline{0}^T & 0 \\ \underline{\psi}_0(s) & \underline{G}(s) & \underline{\psi}_7(s) \\ 0 & \underline{0}^T & d_1(s)d_2(s) \end{bmatrix},$$

and part a) follows from (3.36).

For part b), we note that

$$f_{i0}(x, t) = f_{i0}(0+, t - x)\bar{A}_0(x) ; \bar{A}_0(x) = 1 ; \quad (3.37)$$

$$f_{i1}(x, t) = f_{i1}(0+, t - x)\bar{A}_W(x)e^{-C_1x} ; \quad (3.38)$$

$$f_{i2}(x, t) = f_{i2}(0+, t - x)\bar{A}_E(x)e^{-C_{2,5}x} ; \quad (3.39)$$

$$f_{i3}(x, t) = f_{i3}(0+, t - x)\bar{A}_S(x) ; \quad (3.40)$$

$$f_{i4}(x, t) = f_{i4}(0+, t - x)\bar{A}_W(x)e^{-C_4x} ; \quad (3.41)$$

$$f_{i5}(x, t) = f_{i5}(0+, t - x)\bar{A}_E(x)e^{-C_{2,5}x} ; \quad (3.42)$$

$$f_{i6}(x, t) = f_{i6}(0+, t - x)\bar{A}_S(x) ; \quad (3.43)$$

$$f_{i7}(x, t) = f_{i7}(0+, t - x)\bar{A}_7(x) ; \bar{A}_7(x) = 1. \quad (3.44)$$

These equations can be interpreted in the following manner. Since states 0 and 7 are absorbing, for the process to be in one of the two states at time t with age x , it should have entered the state at time $t - x$, explaining (3.37) and (3.44). For the process to be in state j at time t for $j = 1, 2, 4, 5$, as shown in (3.38), (3.39), (3.41) and (3.42), it should have entered the state at time $t - x$, and there has been no transition to any other state until time t . The case for state 3 and state 6 in (3.40) and (3.43) is similar except that transitions from state 3 or state 6 to state 0 or state 7 are not possible.

By taking the Laplace transform of (3.37) through (3.44) with respect to t , it can be seen that

$$\hat{\varphi}_{i0}(x, s) = \hat{\zeta}_{i0}(0+, s)e^{sx}; \quad (3.45)$$

$$\hat{\varphi}_{i1}(x, s) = \hat{\zeta}_{i1}(0+, s)e^{(s+C_1)x}\bar{A}_W(x); \quad (3.46)$$

$$\hat{\varphi}_{i2}(x, s) = \hat{\zeta}_{i2}(0+, s)e^{(s+C_{2,5})x}\bar{A}_E(x); \quad (3.47)$$

$$\hat{\varphi}_{i3}(x, s) = \hat{\zeta}_{i3}(0+, s)e^{sx}\bar{A}_S(x); \quad (3.48)$$

$$\hat{\varphi}_{i4}(x, s) = \hat{\zeta}_{i4}(0+, s)e^{(s+C_4)x}\bar{A}_W(x); \quad (3.49)$$

$$\hat{\varphi}_{i5}(x, s) = \hat{\zeta}_{i5}(0+, s)e^{(s+C_{2,5})x}\bar{A}_E(x); \quad (3.50)$$

$$\hat{\varphi}_{i6}(x, s) = \hat{\zeta}_{i6}(0+, s)e^{sx}\bar{A}_S(x); \quad (3.51)$$

$$\hat{\varphi}_{i7}(x, s) = \hat{\zeta}_{i7}(0+, s)e^{sx}. \quad (3.52)$$

Again by taking the Laplace transform of (3.45) through (3.52) with respect to x and putting the results into matrix form, the theorem follows. \square

Let the Laplace transform of $\underline{\underline{P}}(t)$ with respect to t be denoted by $\underline{\underline{\pi}}(s)$, i.e.

$$\underline{\underline{\pi}}(s) = \int_0^\infty e^{-st}\underline{\underline{P}}(t)dt. \quad (3.53)$$

From the definition of $\underline{\underline{P}}(t)$ in (3.1), one easily sees that $\underline{\underline{\pi}}(s) = \underline{\underline{\hat{\varphi}}}(0, s)$. The next theorem is then immediate from Theorem 3.1.

Theorem 3.2.

$$\begin{aligned} \underline{\underline{\pi}}(s) = \underline{\underline{\hat{\varphi}}}(0, s) = \underline{\underline{\hat{\zeta}}}(0+, s) \\ \times \text{diag} \left\{ \frac{1}{s}, \beta_W(s + C_1), \beta_E(s + C_{2,5}), \beta_S(s), \right. \\ \left. \beta_W(s + C_4), \beta_E(s + C_{2,5}), \beta_S(s), \frac{1}{s} \right\} \end{aligned}$$

So far, we have assumed that the dwell time of the semi-Markov process in state i is absolutely continuous with p.d.f. $a_i(x)$, $i = 1, \dots, 6$, given that neither the decision of purchasing nor that of not purchasing is made. In reality, however, the three periods of a day should be treated as constants $\tau_1 = \tau_4 = \tau_W, \tau_2 = \tau_5 = \tau_E$ and $\tau_3 = \tau_6 = \tau_S$. This case can be dealt with by considering a sequence of Laplace transforms of p.d.f.'s $(\alpha_i(k, s))_{k=1}^\infty$ where

$\alpha_i(k, s) \rightarrow e^{-s\tau_i}$ as $k \rightarrow \infty$, $i = 1, \dots, 6$. We emphasize the limit by using the symbol \sim , i.e. $\tilde{\alpha}_i(s) = e^{-s\tau_i}$. At the limit, the corresponding Laplace transform $\tilde{\pi}(s)$ of the transition probability matrix $\tilde{P}(t)$ can be obtained by substituting $\tilde{\alpha}_i(s) = e^{-s\tau_i}$ into Theorems 3.1 and 3.2. Assuming that a day starts with the first period, of particular interest to our analysis are $\tilde{\pi}_{10}(s)$ and $\tilde{\pi}_{17}(s)$, which are the Laplace transform of the probability of a consumer having decided to purchase the product by time t and that of a consumer having decided not to purchase the product by time t . These results can be obtained directly from Theorems 3.1 and 3.2 with substitution of $\tilde{\alpha}_i(s) = e^{-s\tau_i}$ and by employing the initial probability vector \underline{u}_1^T , as summarized in the next theorem.

Theorem 3.3. *Suppose that the three periods of a day are represented by constants τ_W, τ_E and τ_R . Let $\tau = \tau_W + \tau_E + \tau_R$, $\tau(PC) = C_1\tau_W + C_{2,5}\tau_E$ and $\tau(BOTH) = C_4\tau_W + C_{2,5}\tau_E$, where $C_1, C_{2,5}$ and C_4 are as in (3.15). One then has:*

$$\begin{aligned}
a) \tilde{\pi}_{10}(s) &= \frac{1}{s} \cdot \frac{1}{1 - re^{-\tau(PC)}e^{-s\tau}} \\
&\quad \times \left\{ \xi_{W:PC} \frac{1 - e^{-(s+C_1)\tau_W}}{s + C_1} + \xi_{E:PC} e^{-(s+C_1)\tau_W} \frac{1 - e^{-(s+C_{2,5})\tau_E}}{s + C_{2,5}} \right\} \\
&\quad + \frac{1}{s} \cdot \frac{(1-r)e^{-\tau(PC)}e^{-s\tau}}{(1 - re^{-\tau(PC)}e^{-s\tau})(1 - e^{-\tau(BOTH)}e^{-s\tau})} \\
&\quad \times \left\{ \xi_{W:BOTH} \frac{1 - e^{-(s+C_4)\tau_W}}{s + C_4} + \xi_{E:PC} e^{-(s+C_4)\tau_W} \frac{1 - e^{-(s+C_{2,5})\tau_E}}{s + C_{2,5}} \right\} \\
b) \tilde{\pi}_{17}(s) &= \frac{1}{s} \cdot \frac{1}{1 - re^{-\tau(PC)}e^{-s\tau}} \\
&\quad \times \left\{ \theta_{W:PC} \frac{1 - e^{-(s+C_1)\tau_W}}{s + C_1} + \theta_{E:PC} e^{-(s+C_1)\tau_W} \frac{1 - e^{-(s+C_{2,5})\tau_E}}{s + C_{2,5}} \right\} \\
&\quad + \frac{1}{s} \cdot \frac{(1-r)e^{-\tau(PC)}e^{-s\tau}}{(1 - re^{-\tau(PC)}e^{-s\tau})(1 - e^{-\tau(BOTH)}e^{-s\tau})} \\
&\quad \times \left\{ \theta_{W:BOTH} \frac{1 - e^{-(s+C_4)\tau_W}}{s + C_4} + \theta_{E:PC} e^{-(s+C_4)\tau_W} \frac{1 - e^{-(s+C_{2,5})\tau_E}}{s + C_{2,5}} \right\}.
\end{aligned}$$

We are now in a position to prove the following main theorem by inverting the results in Theorem 3.3 a) and b) into the real domain. For notational

convenience, the following intervals are introduced for $k = 0, 1, 2, \dots$.

$$Int[k, W] = \{t : k\tau \leq t < k\tau + \tau_W\} \quad (3.54)$$

$$Int[k, E] = \{t : k\tau + \tau_W \leq t < k\tau + \tau_W + \tau_E\} \quad (3.55)$$

$$Int[k, S] = \{t : k\tau + \tau_W + \tau_E \leq t < (k+1)\tau\} \quad (3.56)$$

Here, $Int[k, W]$, $Int[k, E]$ and $Int[k, S]$ represent the working hours, the evening hours and the sleeping hours, respectively, of the k -th day. We also write $\lfloor x \rfloor$ to mean the integer part of a real number x . Proof of the theorem is rather lengthy and cumbersome, and only the outline is provided in a succinct manner in Appendix.

Theorem 3.4. *Let $Int[k, W]$, $Int[k, E]$ and $Int[k, S]$ be as in (3.54), (3.55) and (3.56) respectively. Let τ and C_i be as in Theorem 3.3 and define $M(t) = \lfloor \frac{t}{\tau} \rfloor$. For notational convenience, we also define the following functions.*

$$H_{\xi:a}(m, t) = \frac{\xi_{W:PC}}{C_1} (1 - e^{-C_1\tau_W}) \frac{1 - \{re^{-\tau(PC)}\}^m}{1 - re^{-\tau(PC)}} \quad (3.57)$$

$$H_{\xi:b}(m, t) = \frac{\xi_{E:PC}}{C_{2,5}} e^{-C_1\tau_W} (1 - e^{-C_{2,5}\tau_E}) \frac{1 - \{re^{-\tau(PC)}\}^m}{1 - re^{-\tau(PC)}} \quad (3.58)$$

$$H_{\xi:c}(m, t) = \frac{\xi_{W:BOTH}}{C_4} (1 - e^{-C_4\tau_W}) \frac{1 - r}{r - e^{-(C_4 - C_1)\tau_W}} \times \left\{ \frac{1 - \{re^{-\tau(PC)}\}^m}{1 - re^{-\tau(PC)}} - \frac{1 - e^{-\tau(BOTH)m}}{1 - e^{-\tau(BOTH)}} \right\} \quad (3.59)$$

$$H_{\xi:d}(m, t) = \frac{\xi_{E:PC}}{C_{2,5}} e^{-C_4\tau_W} (1 - e^{-C_{2,5}\tau_E}) \frac{1 - r}{r - e^{-(C_4 - C_1)\tau_W}} \times \left\{ \frac{1 - \{re^{-\tau(PC)}\}^m}{1 - re^{-\tau(PC)}} - \frac{1 - e^{-\tau(BOTH)m}}{1 - e^{-\tau(BOTH)}} \right\} \quad (3.60)$$

Then the probability $\tilde{P}_{10}(t)$ can be obtained as follows.

i) If $t \in \text{Int}[M(t), W]$, then

$$\begin{aligned}\tilde{P}_{10}(t) &= H_{\xi:a}(M(t), t) + H_{\xi:b}(M(t), t) + H_{\xi:c}(M(t), t) + H_{\xi:d}(M(t), t) \\ &+ \frac{\xi_{W:PC}}{C_1} \{r e^{-\tau(PC)} e^{C_1\tau}\}^{M(t)} (e^{-C_1 M(t)\tau} - e^{-C_1 t}) \\ &+ \frac{\xi_{W:BOTH}}{C_4} \{r e^{-\tau(PC)} e^{C_4\tau}\}^{M(t)} (e^{-C_4 M(t)\tau} - e^{-C_4 t}) \\ &\times \frac{1-r}{r - e^{-(C_4-C_1)\tau W}} \left\{ 1 - \{r^{-1} e^{-(C_4-C_1)\tau W}\}^{M(t)} \right\}.\end{aligned}$$

ii) If $t \in \text{Int}[M(t), E]$, then

$$\begin{aligned}\tilde{P}_{10}(t) &= H_{\xi:a}(M(t) + 1, t) + H_{\xi:b}(M(t), t) + H_{\xi:c}(M(t) + 1, t) + H_{\xi:d}(M(t), t) \\ &+ \frac{\xi_{E:PC}}{C_{2,5}} e^{-(C_1-C_{2,5})\tau W} \{r e^{-\tau(PC)} e^{C_{2,5}\tau}\}^{M(t)} (e^{-C_{2,5}(M(t)\tau+\tau W)} - e^{-C_{2,5}t}) \\ &+ \frac{\xi_{W:PC}}{C_{2,5}} e^{-(C_4-C_{2,5})\tau W} \{r e^{-\tau(PC)} e^{C_{2,5}\tau}\}^{M(t)} (e^{-C_{2,5}(M(t)\tau+\tau W)} - e^{-C_{2,5}t}) \\ &\times \frac{1-r}{r - e^{-(C_4-C_1)\tau W}} \left\{ 1 - \{r^{-1} e^{-(C_4-C_1)\tau W}\}^{M(t)} \right\}.\end{aligned}$$

iii) If $t \in \text{Int}[M(t), S]$, then

$$\tilde{P}_{10}(t) = H_{\xi:a}(M(t)+1, t) + H_{\xi:b}(M(t)+1, t) + H_{\xi:c}(M(t)+1, t) + H_{\xi:d}(M(t)+1, t).$$

The counterpart of Theorem 3.4 for $\tilde{P}_{17}(t)$ can be obtained in a similar manner, where $\xi_{W:BOTH}$, $\xi_{W:PC}$ and $\xi_{E:PC}$ should be replaced by $\theta_{W:BOTH}$, $\theta_{W:PC}$ and $\theta_{E:PC}$ respectively. In parallel with (3.57) through (3.60), we define $H_{\theta:a}(m, t)$ through $H_{\theta:d}(m, t)$ by replacing $\xi_{W:BOTH}$, $\xi_{W:PC}$ or $\xi_{E:PC}$ in the first factor by $\theta_{W:BOTH}$, $\theta_{W:PC}$ or $\theta_{E:PC}$ respectively. The result is summarized in Theorem 3.5 below.

Theorem 3.5. *Let $\text{Int}[k, W]$, $\text{Int}[k, E]$ and $\text{Int}[k, S]$ be as in (3.54), (3.55) and (3.56) respectively. Let τ , C_i and $M(t)$ be as in Theorem 3.3. Then the probability $\tilde{P}_{17}(t)$ can be obtained as follows.*

i) If $t \in \text{Int}[M(t), W]$, then

$$\begin{aligned}\tilde{P}_{17}(t) &= H_{\theta:a}(M(t), t) + H_{\theta:b}(M(t), t) + H_{\theta:c}(M(t), t) + H_{\theta:d}(M(t), t) \\ &+ \frac{\theta_{W:PC}}{C_1} \{r e^{-\tau(PC)} e^{C_1\tau}\}^{M(t)} (e^{-C_1 M(t)\tau} - e^{-C_1 t}) \\ &+ \frac{\theta_{W:BOTH}}{C_4} \{r e^{-\tau(PC)} e^{C_4\tau}\}^{M(t)} (e^{-C_4 M(t)\tau} - e^{-C_4 t}) \\ &\times \frac{1-r}{r - e^{-(C_4-C_1)\tau W}} \left\{ 1 - \{r^{-1} e^{-(C_4-C_1)\tau W}\}^{M(t)} \right\}.\end{aligned}$$

ii) If $t \in \text{Int}[M(t), E]$, then

$$\begin{aligned}\tilde{P}_{17}(t) &= H_{\theta:a}(M(t) + 1, t) + H_{\theta:b}(M(t), t) + H_{\theta:c}(M(t) + 1, t) + H_{\theta:d}(M(t), t) \\ &+ \frac{\theta_{E:PC}}{C_{2,5}} e^{-(C_1-C_{2,5})\tau W} \{r e^{-\tau(PC)} e^{C_{2,5}\tau}\}^{M(t)} (e^{-C_{2,5}(M(t)\tau + \tau W)} - e^{-C_{2,5}t}) \\ &+ \frac{\theta_{W:PC}}{C_{2,5}} e^{-(C_4-C_{2,5})\tau W} \{r e^{-\tau(PC)} e^{C_{2,5}\tau}\}^{M(t)} (e^{-C_{2,5}(M(t)\tau + \tau W)} - e^{-C_{2,5}t}) \\ &\times \frac{1-r}{r - e^{-(C_4-C_1)\tau W}} \left\{ 1 - \{r^{-1} e^{-(C_4-C_1)\tau W}\}^{M(t)} \right\}.\end{aligned}$$

iii) If $t \in \text{Int}[M(t), S]$, then

$$\tilde{P}_{17}(t) = H_{\theta:a}(M(t)+1, t) + H_{\theta:b}(M(t)+1, t) + H_{\theta:c}(M(t)+1, t) + H_{\theta:d}(M(t)+1, t).$$

From Theorems 3.4 and 3.5, the two absorption probabilities e_{10} and e_{17} can be obtained by letting $t \rightarrow \infty$.

Theorem 3.6. *Starting at state 1 at time 0, let e_{10} and e_{17} be the absorption probabilities in state 0 and state 7 respectively. One then has*

$$\begin{aligned}e_{10} &= \tilde{P}_{10}(\infty) \\ &= \left\{ \frac{\xi_{W:PC}}{C_1} (1 - e^{-C_1\tau W}) + \frac{\xi_{E:PC}}{C_{2,5}} e^{-C_1\tau W} (1 - e^{-C_{2,5}\tau E}) \right\} \frac{1}{1 - r e^{\tau(PC)}} \\ &+ \left\{ \frac{\xi_{W:BOTH}}{C_4} (1 - e^{-C_4\tau W}) + \frac{\xi_{E:PC}}{C_{2,5}} e^{-C_4\tau W} (1 - e^{-C_{2,5}\tau E}) \right\} \\ &\times \frac{1-r}{r - e^{-(C_4-C_1)\tau W}} \left\{ \frac{1}{1 - r e^{-\tau(PC)}} - \frac{1}{1 - e^{-\tau(BOTH)}} \right\};\end{aligned}$$

$$\begin{aligned}
e_{17} &= \tilde{P}_{17}(\infty) \\
&= \left\{ \frac{\theta_{W:PC}}{C_1} (1 - e^{-C_1\tau W}) + \frac{\theta_{E:PC}}{C_{2,5}} e^{-C_1\tau W} (1 - e^{-C_{2,5}\tau E}) \right\} \frac{1}{1 - re^{\tau(PC)}} \\
&+ \left\{ \frac{\theta_{W:BOTH}}{C_4} (1 - e^{-C_4\tau W}) + \frac{\theta_{E:PC}}{C_{2,5}} e^{-C_4\tau W} (1 - e^{-C_{2,5}\tau E}) \right\} \\
&\times \frac{1 - r}{r - e^{-(C_4 - C_1)\tau W}} \left\{ \frac{1}{1 - re^{-\tau(PC)}} - \frac{1}{1 - e^{-\tau(BOTH)}} \right\}.
\end{aligned}$$

For those users who have both the PC access and the mobile access to the Internet from the beginning, the initial state would be state 4. Accordingly, also of interest to our analysis would be the probabilities $\tilde{P}_{40}(t)$ and $\tilde{P}_{47}(t)$. These probabilities can be obtained merely by adopting the initial state vector \underline{u}_4^T in place of \underline{u}_1^T . Theorems 3.7 and 3.8 below provide the counterparts of Theorems 3.4 and 3.5.

Theorem 3.7. *Let $Int[k, W]$, $Int[k, E]$ and $Int[k, S]$ be as in (3.54), (3.55) and (3.56) respectively. Let τ , C_i and $M(t)$ be as in Theorem 3.3. Then the probability $\tilde{P}_{40}(t)$ can be obtained as follows.*

i) If $t \in Int[M(t), W]$, then

$$\begin{aligned}
\tilde{P}_{40}(t) &= \frac{\xi_{W:BOTH}}{C_4} (1 - e^{-C_4\tau W}) \frac{1 - e^{-\tau(BOTH)M(t)}}{1 - e^{-\tau(BOTH)}} \\
&+ \frac{\xi_{W:BOTH}}{C_4} (e^{-\tau(BOTH)} e^{C_4\tau})^{M(t)} (e^{-C_4M(t)\tau} - e^{-C_4t}) \\
&+ \frac{\xi_{E:PC}}{C_{2,5}} e^{-C_4\tau W} (1 - e^{-C_{2,5}\tau E}) \frac{1 - e^{-\tau(BOTH)M(t)}}{1 - e^{-\tau(BOTH)}}.
\end{aligned}$$

ii) If $t \in Int[M(t), E]$, then

$$\begin{aligned}
\tilde{P}_{40}(t) &= \frac{\xi_{W:BOTH}}{C_4} (1 - e^{-C_4\tau W}) \frac{1 - e^{-\tau(BOTH)(M(t)+1)}}{1 - e^{-\tau(BOTH)}} \\
&+ \frac{\xi_{E:PC}}{C_{2,5}} e^{-C_4\tau W} (1 - e^{-C_{2,5}\tau E}) \frac{1 - e^{-\tau(BOTH)M(t)}}{1 - e^{-\tau(BOTH)}} \\
&+ \frac{\xi_{E:PC}}{C_{2,5}} e^{-(C_4 - C_{2,5})\tau W} (e^{-\tau(BOTH)} e^{C_{2,5}\tau})^{M(t)} \\
&\times (e^{-C_{2,5}(M(t)\tau + \tau W)} - e^{-C_{2,5}t}).
\end{aligned}$$

iii) If $t \in \text{Int}[M(t), S]$, then

$$\begin{aligned}\tilde{P}_{40}(t) &= \frac{\xi_{W:BOTH}}{C_4} (1 - e^{-C_4\tau W}) \frac{1 - e^{-\tau(BOTH)(M(t)+1)}}{1 - e^{-\tau(BOTH)}} \\ &+ \frac{\xi_{E:PC}}{C_{2,5}} e^{-C_4\tau W} (1 - e^{-C_{2,5}\tau E}) \frac{1 - e^{-\tau(BOTH)(M(t)+1)}}{1 - e^{-\tau(BOTH)}}.\end{aligned}$$

Theorem 3.8. Let $\text{Int}[k, W]$, $\text{Int}[k, E]$ and $\text{Int}[k, S]$ be as in (3.54), (3.55) and (3.56) respectively. Let τ , C_i and $M(t)$ be as in Theorem 3.3. Then the probability $\tilde{P}_{47}(t)$ can be obtained as follows.

i) If $t \in \text{Int}[M(t), W]$, then

$$\begin{aligned}\tilde{P}_{47}(t) &= \frac{\theta_{W:BOTH}}{C_4} (1 - e^{-C_4\tau W}) \frac{1 - e^{-\tau(BOTH)M(t)}}{1 - e^{-\tau(BOTH)}} \\ &+ \frac{\theta_{W:BOTH}}{C_4} (e^{-\tau(BOTH)} e^{C_4\tau})^{M(t)} (e^{-C_4M(t)\tau} - e^{-C_4t}) \\ &+ \frac{\xi_{E:PC}}{C_{2,5}} e^{-C_4\tau W} (1 - e^{-C_{2,5}\tau E}) \frac{1 - e^{-\tau(BOTH)M(t)}}{1 - e^{-\tau(BOTH)}}.\end{aligned}$$

ii) If $t \in \text{Int}[M(t), E]$, then

$$\begin{aligned}\tilde{P}_{47}(t) &= \frac{\theta_{W:BOTH}}{C_4} (1 - e^{-C_4\tau W}) \frac{1 - e^{-\tau(BOTH)(M(t)+1)}}{1 - e^{-\tau(BOTH)}} \\ &+ \frac{\theta_{E:PC}}{C_{2,5}} e^{-C_4\tau W} (1 - e^{-C_{2,5}\tau E}) \frac{1 - e^{-\tau(BOTH)M(t)}}{1 - e^{-\tau(BOTH)}} \\ &+ \frac{\xi_{E:PC}}{C_{2,5}} e^{-(C_4 - C_{2,5})\tau W} (e^{-\tau(BOTH)} e^{C_{2,5}\tau})^{M(t)} \\ &\quad \times (e^{-C_{2,5}(M(t)\tau + \tau W)} - e^{-C_{2,5}t}).\end{aligned}$$

iii) If $t \in \text{Int}[M(t), S]$, then

$$\begin{aligned}\tilde{P}_{47}(t) &= \frac{\theta_{W:BOTH}}{C_4} (1 - e^{-C_4\tau W}) \frac{1 - e^{-\tau(BOTH)(M(t)+1)}}{1 - e^{-\tau(BOTH)}} \\ &+ \frac{\theta_{E:PC}}{C_{2,5}} e^{-C_4\tau W} (1 - e^{-C_{2,5}\tau E}) \frac{1 - e^{-\tau(BOTH)(M(t)+1)}}{1 - e^{-\tau(BOTH)}}.\end{aligned}$$

Corresponding to Theorem 3.6, one has the following theorem by letting $t \rightarrow \infty$ in Theorems 3.7 and 3.8.

Theorem 3.9. *Starting at state 4 at time 0, Let e_{40} and e_{47} be the absorption probabilities in state 0 and state 7 respectively. One then has*

$$\begin{aligned} e_{40} &= \tilde{P}_{40}(\infty) \\ &= \left\{ \frac{\xi_{W:BOTH}}{C_4} (1 - e^{-C_4\tau_W}) + \frac{\xi_{E:PC}}{C_{2,5}} e^{-C_4\tau_W} (1 - e^{-C_{2,5}\tau_E}) \right\} \frac{1}{1 - e^{\tau(BOTH)}} ; \end{aligned}$$

$$\begin{aligned} e_{47} &= \tilde{P}_{47}(\infty) \\ &= \left\{ \frac{\theta_{W:BOTH}}{C_4} (1 - e^{-C_4\tau_W}) + \frac{\theta_{E:PC}}{C_{2,5}} e^{-C_4\tau_W} (1 - e^{-C_{2,5}\tau_E}) \right\} \frac{1}{1 - e^{\tau(BOTH)}} . \end{aligned}$$

4. Analysis of Dynamic Sales Volume and Sales Completion Time

Using the results of the semi-Markov model discussed in Section 3, we are now in a position to assess the impact of the mobile access to the Internet on enhancement of e-commerce. Let the population of \mathcal{C}_{PC} , $\mathcal{C}_{PC \rightarrow BOTH}$ and \mathcal{C}_{BOTH} be defined by

$$N_{PC} = |\mathcal{C}_{PC}| ; N_{PC \rightarrow BOTH} = |\mathcal{C}_{PC \rightarrow BOTH}| ; N_{BOTH} = |\mathcal{C}_{BOTH}| , \quad (4.1)$$

where $|A|$ denotes the cardinality of a set A . Given N_{PC} , $N_{PC \rightarrow BOTH}$ and N_{BOTH} , of interest then is the distribution of the sales volume at time t . Also, of equal importance would be the distribution of the sales completion time for K products. In this section, we derive these two distributions explicitly.

In order to capture individual consumer behaviors better from an application point of view, we redefine the state space of the semi-Markov model $\mathcal{N} = \{0, 1, \dots, 7\}$ as $\mathcal{S} = \{Buy, UD, \neg Buy\}$, where Buy corresponds to state 0, UD (*UnDecided*) aggregates the six states $\{1, \dots, 6\}$, and $\neg Buy$ means state 7. Furthermore, for distinguishing consumers who belong to different classes, we write $\underline{\tilde{P}}_r(t) = [\tilde{P}_{r:ij}(t)]$ where $\underline{\tilde{P}}_1(t)$ is the transition probability matrix of the semi-Markov process with $r = 1$ and $\underline{\tilde{P}}_r(t)$ denotes that with $0 < r < 1$. Accordingly, we define

$$\begin{aligned} P_{PC:Buy}(t) &= \tilde{P}_{1:10}(t) ; P_{PC:UD}(t) = \sum_{j=1}^6 \tilde{P}_{1:1j}(t) ; \\ P_{PC:\neg Buy}(t) &= \tilde{P}_{1:17}(t) , \end{aligned} \quad (4.2)$$

$$\begin{aligned}
P_{PC \rightarrow BOTH:Buy}(t) &= \tilde{P}_{r:10}(t); P_{PC \rightarrow BOTH:UD}(t) = \sum_{j=1}^6 \tilde{P}_{r:1j}(t); \\
P_{PC \rightarrow BOTH:\neg Buy}(t) &= \tilde{P}_{r:17}(t),
\end{aligned} \tag{4.3}$$

and

$$\begin{aligned}
P_{BOTH:Buy}(t) &= \tilde{P}_{r:40}(t); P_{BOTH:UD}(t) = \sum_{j=1}^6 \tilde{P}_{r:4j}(t); \\
P_{BOTH:\neg Buy}(t) &= \tilde{P}_{r:47}(t),
\end{aligned} \tag{4.4}$$

which can be readily computed from Theorems 3.4, 3.5, 3.7 and 3.8.

For $VAR \in \{PC, PC \rightarrow BOTH, BOTH\}$, we now introduce the following trivariate generating functions capturing the state of individual consumers at time t .

$$\chi_{VAR:IND}(u, v, w, t) = P_{VAR:Buy}(t)u + P_{VAR:UD}(t)v + P_{VAR:\neg Buy}(t)w \tag{4.5}$$

Let $N_{VAR:Buy}(t)$, $N_{VAR:UD}(t)$ and $N_{VAR:\neg Buy}(t)$ be the number of consumers in class \mathcal{C}_{VAR} who have bought the product by time t , the number of consumers in class \mathcal{C}_{VAR} who have not decided in either way by time t and the number of consumers in class \mathcal{C}_{VAR} who have decided not to buy the product by time t , respectively. We note that $N_{VAR} = N_{VAR:Buy}(t) + N_{VAR:UD}(t) + N_{VAR:\neg Buy}(t)$ for any $t \geq 0$. Assuming that individual consumers behave independently of each other, the collective consumer behavior can then be described by

$$\begin{aligned}
\chi_{VAR:ALL}(u, v, w, t) &= E[u^{N_{VAR:Buy}(t)}v^{N_{VAR:UD}(t)}w^{N_{VAR:\neg Buy}(t)}] \\
&= \{ \chi_{VAR:IND}(u, v, w, t) \}^{N_{VAR}}.
\end{aligned} \tag{4.6}$$

Accordingly, the joint probability of $N_{VAR:Buy}(t)$, $N_{VAR:UD}(t)$ and $N_{VAR:\neg Buy}(t)$ is given by

$$\begin{aligned}
P[N_{VAR:Buy}(t) = n_1, N_{VAR:UD}(t) = n_2, N_{VAR:\neg Buy}(t) = n_3] \\
= \binom{N_{VAR}}{n_1, n_2, n_3} P_{VAR:Buy}(t)^{n_1} P_{VAR:UD}(t)^{n_2} P_{VAR:\neg Buy}(t)^{n_3}.
\end{aligned}$$

Based on these observations, the next theorem can be shown.

Theorem 4.1. For $VAR \in \{PC, PC \rightarrow BOTH, BOTH\}$, let N_{VAR} be as in (4.1) and define $K_{VAR}(t)$ to be the number of products sold to those consumers in \mathcal{C}_{VAR} by time t . Then $K_{VAR}(t)$ has the binomial distribution with mean $N_{VAR} \cdot P_{VAR:Buy}(t)$, i.e. $Q_{VAR}(k, t) \stackrel{def}{=} P[K_{VAR}(t) = k]$ for $k \in \{0, 1, \dots, N_{VAR}\}$ is given by

$$Q_{VAR}(k, t) = \binom{N_{VAR}}{k} P_{VAR:Buy}(t)^k \{1 - P_{VAR:Buy}(t)\}^{N_{VAR}-k}.$$

Proof. Since $E[u^{N_{VAR:Buy}(t)}] = \chi_{N_{VAR:IND}}(u, 1, 1, t)$, one sees from (4.5) and (4.6) that

$$E[u^{N_{VAR:Buy}(t)}] = \{P_{VAR:Buy}(t)u + (1 - P_{VAR:Buy}(t))\}^{N_{VAR}},$$

proving the theorem. □

For $VAR \in \{PC, PC \rightarrow BOTH, BOTH\}$, we next turn our attention to the sales completion time for K products among those consumers in \mathcal{C}_{VAR} where $0 < K \leq N_{VAR}$. More formally, let $T_{VAR}(K)$ be the time until K products have been sold among \mathcal{C}_{VAR} , i.e.

$$T_{VAR}(K) = \inf\{t : K_{VAR}(t) = K\}. \quad (4.7)$$

Let $\bar{H}_{VAR(K)}(t)$ be the survival function of $T_{VAR}(K)$, i.e.

$$\bar{H}_{VAR(K)}(t) = P[T_{VAR}(K) > t]. \quad (4.8)$$

The next theorem then holds true.

Theorem 4.2. Let $Q_{VAR}(k, t)$ and $\bar{H}_{VAR(K)}(t)$ be as in Theorem 4.1 and (4.8) respectively, where $0 < K \leq N_{VAR}$. One then has

$$\bar{H}_{VAR(K)}(t) = \sum_{k=0}^{K-1} Q_{VAR}(k, t).$$

Proof. From (4.7), one easily sees that $T_{VAR}(K) > t$ if and only if $K_{VAR}(t) < K$. This dual relationship between $T_{VAR}(K)$ and $K_{VAR}(t)$ then implies that

$$\bar{H}_{VAR(K)}(t) = P[T_{VAR}(K) > t] = P[K_{VAR}(t) < K],$$

and theorem follows from Theorem 4.1. □

The above analysis for the individual classes of consumers should be integrated so as to capture the stochastic nature of the consumer behaviors in the entire market. More specifically, let

$$N = N_{PC} + N_{PC \rightarrow BOTH} + N_{BOTH}, \quad (4.9)$$

and define $K(t)$ to be the number of products sold by time t in the entire market. As before, we also define $T(K)$ to be the time required for selling K products in the entire market where $0 < K \leq N$. One then has the following theorem.

Theorem 4.3. *Let N , $K(t)$, K and $T(K)$ be as described above. Let $Q(k, t) = P[K(t) = k]$ and define the survival function of $T(K)$ by $\bar{H}_{ALL(K)}(t) = P[T(K) > t]$. Then the following statements hold true.*

a) *For $k \in \{0, 1, \dots, N\}$, one has*

$$Q(k, t) = \sum_{i=0}^k Q_{\setminus BOTH}(k-i, t) Q_{BOTH}(i, t)$$

where

$$Q_{\setminus BOTH}(k, t) = \sum_{i=0}^k Q_{PC}(k-i, t) Q_{PC \rightarrow BOTH}(i, t),$$

with mathematical convention that $Q_{VAR}(k, t) \stackrel{def}{=} 0$ for $k > N_{VAR}$ for $VAR \in \{PC, PC \rightarrow BOTH, BOTH\}$.

$$b) \bar{H}_{ALL(K)}(t) = \sum_{i=0}^{K-1} Q(k, t)$$

Proof. Since $K(t) = K_{PC}(t) + K_{PC \rightarrow BOTH}(t) + K_{BOTH}(t)$, part a) follows immediately from Theorem 4.1 and the discrete convolution theorem. Part b) can be shown as for the proof of Theorem 4.2. □

5. Numerical Examples

The purpose of this section is to explore numerically how the mobile access to the Internet would enhance e-commerce. For this purpose, the basic values of the underlying parameters are set as in Table 5.1. It is assumed that the decision making probabilities are indifferent, regardless of different access times in a day and regardless of the PC access or the mobile access, where the decision for purchasing is made with probability 0.03 and that for not purchasing with probability 0.01 for each Internet access. The total number of consumers is given as $N = 10000$ and the following five cases are considered.

- 1) All consumers have only the PC access with $N = N_{PC}$.
- 2) There exist three different types of consumers with $N_{PC} = 2500$, $N_{PC \rightarrow BOTH} = 5000$, and $N_{BOTH} = 2500$, where the probability $1 - r$ representing the growth of the mobile users is varied for :
 - 2-1) $r = 0.8$;
 - 2-2) $r = 0.5$;
 - and 2-3) $r = 0.2$.
- 3) All consumers have both the PC access and the mobile access from the beginning with $N = N_{BOTH}$.

It should be noted that the degree of the mobile use is strengthened in the order of 1), 2-1), 2-2), 2-3) and 3).

$\alpha_{W:PC}$	$\alpha_{E:PC}$	$\alpha_{W:BOTH}$	$\beta_{W:PC}$	$\beta_{E:PC}$	$\beta_{W:BOTH}$	
0.03	0.03	0.03	0.01	0.01	0.01	
$\lambda_{W:PC}$	$\lambda_{E:PC}$	$\lambda_{W:BOTH}$	r	τ_W	τ_E	τ_R
1/24	1/24	1/12	0.8	8	8	8

Table 5.1. Basic Values of the Underlying Parameters

In Figure 5.1, the survival functions for $K(240)$, i.e. the number of products sold by time $t = 240$, are plotted for the five cases in the order of 1), 2-1), 2-2), 2-3) and 3) from left to right. It can be readily seen that $K(240)$ increases stochastically in this order. With probability 0.7, for example, 1736 products or more can be sold for case 1), while this number increases from 2115, 2203 and 2229 to 2450 as the case moves from 2-1), 2-2) and 2-3) to 3) respectively. The increase from 1736 for case 1) to 2115 for case 2-1), 21.8% increase, is rather large considering the fact that the sifting probability $1 - r$

is increased from 0 to merely 0.2 at $r = 0.8$. However, the subsequent increase diminishes from 2115 to 2229, only 5.3% increase, as $1 - r$ increases from 0.2 to 0.8. Similar observations can be made for the expected values depicted in Figure 5.2. The monotonicity of the variance also given in Figure 5.2 reflects the fact that the support interval of $K(240)$ increases as the case moves from 1), 2-1), 2-2) and 2-3) to 3).

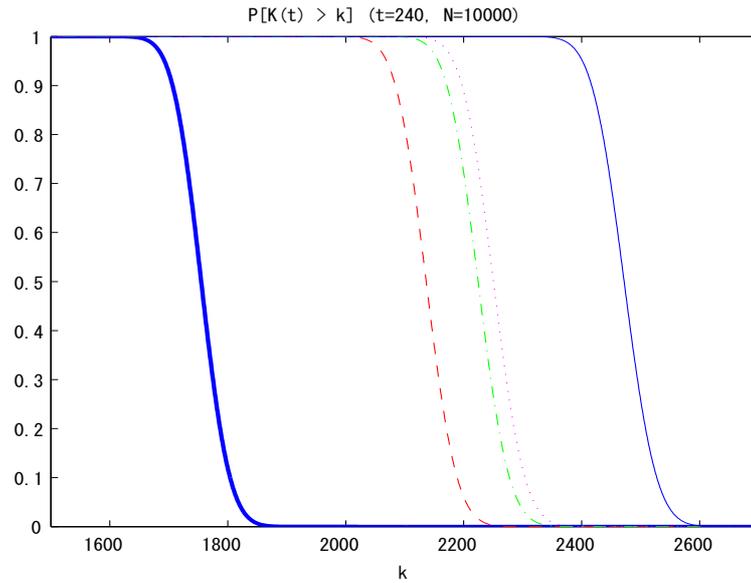


Figure 5.1. Survival Function of $K(t)$ ($t = 240$, $N = 10000$)

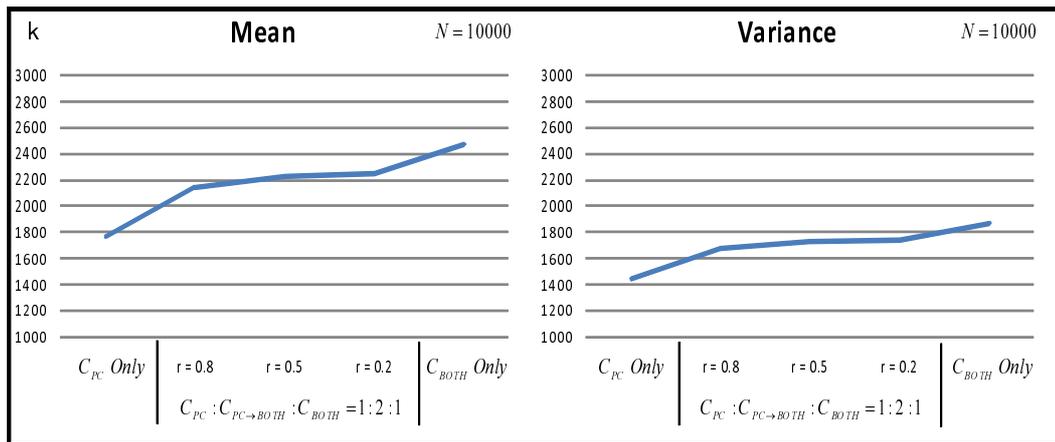


Figure 5.2 Mean and Variance of $K(t)$ ($t = 240$, $N = 10000$)

Figures 5.3 and 5.4 provide the counterparts of Figures 5.1 and 5.2 for the survival function for $T(2000)$, i.e. the sales completion time for $K = 2000$ products, except that the left-most curve now corresponds to case 3) and the right-most curve represents case 1). We observe that $T(2000)$ decreases stochastically as the case moves from 1), 2-1), 2-2) and 2-3) to 3). With probability 0.7, $T(2000)$ is greater than or equal to 273 for case 1), while this number decreases from 219, 203 and 200 to 176 as the case moves from 2-1), 2-2) and 2-3) to 3) respectively. Again, the initial decrease from 273 ($r = 1$) to 219 ($r = 0.8$), 19.8% decrease, is large in comparison with the subsequent decrease from 219 ($r = 0.8$) to 200 ($r = 0.2$), 8.7% decrease. The expected sales completion time and its variance are depicted in Figure 5.4. While the monotonicity of the expected value is observed again, the variance fluctuates visibly in a rather strange manner. This fluctuation phenomenon may be explained by the fact that the third period of a day, denoted by τ_S , affects the distribution of $T(2000)$ differently for different cases. The flat parts observed in Figure 5.3 correspond to τ_S representing sleeping hours during which consumers are inactive in the use of the Internet. One realizes that the flat parts appear differently for five different curves which may result in the fluctuation of the variance.

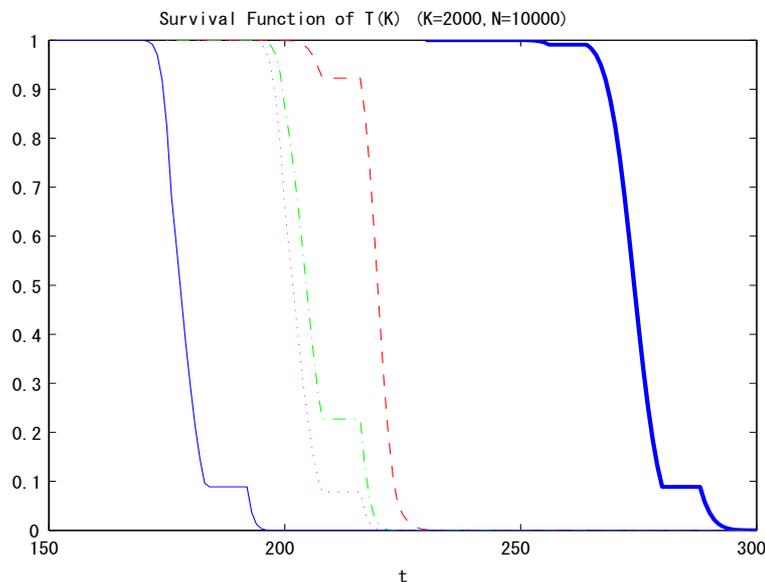


Figure 5.3. Survival Function of $T(K)$ ($K = 2000$, $N = 10000$)

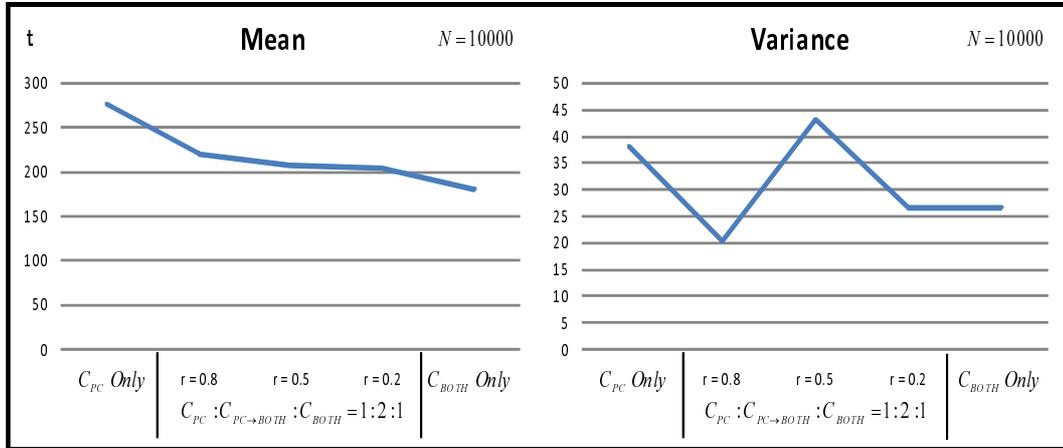


Figure 5.4. Mean and Variance of $T(K)$ ($K = 2000$, $N = 10000$)

6. Concluding Remarks

Through a new generation of mobile devices rapidly spread in society, the way the Internet is used has been going under revolution, where the traditional e-commerce is in the process of being converted into m-commerce. However, because of the fact that the mobile technology is still young, the study of the impact of the mobile access to the Internet on e-businesses is rather limited, where pioneering papers are either empirical, qualitative or static in their analytical nature and, to the best knowledge of the authors, no study exists in the literature for capturing behavioral differences between e-commerce and m-commerce based on a mathematical stochastic model. The purpose of this paper is to fill this gap by developing and analyzing a mathematical model for comparing e-commerce via the traditional PC access only with m-commerce which accommodates both the traditional PC access and the mobile access.

Three classes of consumers are considered concerning the ways they access the Internet. A class of consumers who access the Internet only through PCs throughout the period under consideration is denoted by \mathcal{C}_{PC} . The remaining two classes written as $\mathcal{C}_{PC \rightarrow BOTH}$ and \mathcal{C}_{BOTH} consist of those who access the Internet originally only through PCs but start using the mobile access at some time later and those who access the Internet through both PCs and mobile devices from the very beginning, respectively. The entire market is then represented by $\mathcal{C} = \mathcal{C}_{PC} \cup \mathcal{C}_{PC \rightarrow BOTH} \cup \mathcal{C}_{BOTH}$. Each time the Internet

is accessed for information, it is assumed that a consumer makes one of the three decisions: to purchase the product, not to purchase the product, or to remain undecided.

In order to capture the stochastic behavior of a consumer in \mathcal{C} in a unified manner, a semi-Markov process is formulated with six transient states and two absorbing states. Through dynamic analysis of the semi-Markov process, the two stochastic performance measures of interest can be evaluated: the distribution of the number of products sold by time t and the distribution of the time required for selling K products. This analysis, in turn, enables one to assess the impact of mobile devices on e-businesses by comparing such stochastic performance measures for m-commerce against those for traditional e-commerce. Numerical examples are given for demonstrating the effectiveness of the computational procedures proposed in this paper. However, this research is still in its infancy. Extensive numerical experiments would be needed to extract some useful rules of thumb from the managerial point of view in conducting m-commerce. In addition, efforts should be made for estimating the values of the parameters involved in the analytical model from real data. These studies are in progress and will be reported elsewhere.

Acknowledgement

The authors wish to thank an anonymous referee and the guest editor for many helpful comments which have contributed to improve the original version of this paper substantially. In particular, the suggestion of the referee to incorporate $\mathcal{C}_{PC \rightarrow BOTH}$ in our analysis is greatly appreciated.

References

- [1] V. Roto, Browsing on Mobile Phones, Nokia Research Center, 2005.
- [2] S.H. Kim, Impact of Mobile-Commerce: Benefits, Technological and Strategic Issues and Implementation, Journal of Applied Sciences 6 (12) (2006) 2523–2531.
- [3] M. Chae and J. Kim, What 's So Different About the Mobile Internet?, Communications of the ACM, 2003.
- [4] P. Barwise, TV, PC, or Mobile? Future Media for Consumer e-Commerce, Business Strategy Review, Vol.12 (2001) 35–42.

- [5] K. Hammond, B2C e-Commerce 2000-2010: What Experts Predict, Business Strategy Review, 2001.
- [6] J.H. Wu and T.L. Hisa, Analysis of E-commerce innovation and impact: a hypercube model, Electronic Commerce Research and Applications 3 (2004) 389–404.
- [7] K. Siau and H. Sheng and F. Nah, The Value of Mobile Commerce to Customers, Proceedings of the Third Annual Workshop on HCI Research in MIS, 2004.
- [8] Y.H. Park and P.S. Fader, Modeling Browsing Behavior at Multiple Websites, Marketing Science 23 (3) (2004) 280–303.
- [9] G. Büyüközkan, Determining the mobile commerce user requirements using an analytic approach, Computer Standards & Interfaces 31 (2009) 144–152.
- [10] MakeYouGoHm, Employee non-work internet activity during work hours, <http://www.makeyougohm.com/20060329/3090/> (2006)
- [11] BBC NEWS, Mobile internet usage on the rise, <http://news.bbc.co.uk/2/hi/technology/7748372.stm> (2008)

Appendix Outline of Proof of Theorem 3.4

From Theorem 3.3, one sees that

$$\begin{aligned}
s\tilde{\pi}_{10}(s) &= \frac{1}{1 - re^{-\tau(PC)}e^{-s\tau}} \\
&\times \left\{ \xi_{W:PC} \frac{1 - e^{-(s+C_1)\tau_W}}{s + C_1} + \xi_{E:PC} e^{-(s+C_1)\tau_W} \frac{1 - e^{-(s+C_{2,5})\tau_E}}{s + C_{2,5}} \right\} \\
&+ \frac{(1-r)e^{-\tau(PC)}e^{-s\tau}}{(1 - re^{-\tau(PC)}e^{-s\tau})(1 - e^{-\tau(BOTH)}e^{-s\tau})} \\
&\times \left\{ \xi_{W:BOTH} \frac{1 - e^{-(s+C_4)\tau_W}}{s + C_4} + \xi_{E:PC} e^{-(s+C_4)\tau_W} \frac{1 - e^{-(s+C_{2,5})\tau_E}}{s + C_{2,5}} \right\}
\end{aligned}$$

The first factor in the second term of the right hand side of the above equation can be written as a sum of two terms given by

$$\begin{aligned}
&\frac{(1-r)e^{-\tau(PC)}e^{-s\tau}}{(1 - re^{-\tau(PC)}e^{-s\tau})(1 - e^{-\tau(BOTH)}e^{-s\tau})} \\
&= \frac{X}{1 - re^{-\tau(PC)}e^{-s\tau}} + \frac{Y}{1 - e^{-\tau(BOTH)}e^{-s\tau}}
\end{aligned}$$

where

$$X = \frac{1-r}{r - e^{-(C_4-C_1)\tau_W}} \quad \text{and} \quad Y = -\frac{1-r}{r - e^{-(C_4-C_1)\tau_W}}.$$

Consequently, $s\tilde{\pi}_{10}(s)$ can be expressed as a sum of geometric series' as shown below.

$$\begin{aligned}
s\tilde{\pi}_{10}(s) &= \frac{\xi_{W:PC}}{s + C_1} \sum_{k=0}^{\infty} (re^{-\tau(PC)})^k e^{-sk\tau} \\
&- \frac{\xi_{W:PC}e^{-C_1\tau_W}}{s + C_1} \sum_{k=0}^{\infty} (re^{-\tau(PC)})^k e^{-s(k\tau+\tau_W)} \\
&+ \frac{\xi_{E:PC}e^{-C_1\tau_W}}{s + C_{2,5}} \sum_{k=0}^{\infty} (re^{-\tau(PC)})^k e^{-s(k\tau+\tau_W)} \\
&- \frac{\xi_{E:PC}e^{-\tau(PC)}}{s + C_{2,5}} \sum_{k=0}^{\infty} (re^{-\tau(PC)})^k e^{-s(k\tau+\tau_W+\tau_E)}
\end{aligned}$$

$$\begin{aligned}
& + X \frac{\xi_{W:BOTH}}{s + C_4} \sum_{k=0}^{\infty} (re^{-\tau(PC)})^k e^{-sk\tau} \\
& - X \frac{\xi_{W:BOTH}}{s + C_4} e^{-C_4\tau W} \sum_{k=0}^{\infty} (re^{-\tau(PC)})^k e^{-s(k\tau + \tau W)} \\
& + X \frac{\xi_{E:PC}}{s + C_{2,5}} e^{-C_4\tau W} \sum_{k=0}^{\infty} (re^{-\tau(PC)})^k e^{-s(k\tau + \tau W)} \\
& - X \frac{\xi_{E:PC}}{s + C_{2,5}} e^{-\tau(BOTH)} \sum_{k=0}^{\infty} (re^{-\tau(PC)})^k e^{-s(k\tau + \tau W + \tau E)} \\
& + Y \frac{\xi_{W:BOTH}}{s + C_4} \sum_{k=0}^{\infty} e^{-\tau(BOTH)k} e^{-sk\tau} \\
& - Y \frac{\xi_{W:BOTH}}{s + C_4} e^{-C_4\tau W} \sum_{k=0}^{\infty} e^{-\tau(BOTH)k} e^{-s(k\tau + \tau W)} \\
& + Y \frac{\xi_{E:PC}}{s + C_{2,5}} e^{-C_4\tau W} \sum_{k=0}^{\infty} e^{-\tau(BOTH)k} e^{-s(k\tau + \tau W)} \\
& - Y \frac{\xi_{E:PC}}{s + C_{2,5}} e^{-\tau(BOTH)} \sum_{k=0}^{\infty} e^{-\tau(BOTH)k} e^{-s(k\tau + \tau W + \tau E)}
\end{aligned}$$

Since the inversion of the Laplace transform $\frac{1}{s+\alpha}e^{-s\beta}$ into the real domain is given by

$$\mathcal{L}^{-1} \left[\frac{1}{s + \alpha} e^{-s\beta} \right] = \int_0^t e^{-\alpha(t-y)} \delta(y - \beta) dy = \delta_{\{0 \leq \beta \leq t\}} e^{-\alpha(t-\beta)},$$

$s\tilde{\pi}_{10}(s)$ can be inverted, with $\tilde{P}_{10}(0) = 0$, as

$$\begin{aligned}
\frac{d}{dt} \tilde{P}_{10}(t) & = \xi_{W:PC} e^{-C_1 t} \sum_{k=0}^{\infty} \delta_{\{t \in \text{Int}[k, W]\}} (re^{-\tau(PC)} e^{C_1 \tau})^k \\
& + \xi_{E:PC} e^{-(C_1 - C_{2,5})\tau W} e^{-C_{2,5} t} \sum_{k=0}^{\infty} \delta_{\{t \in \text{Int}[k, E]\}} (re^{-\tau(PC)} e^{C_{2,5} \tau})^k
\end{aligned}$$

$$\begin{aligned}
& + X\xi_{W:BOTHE} e^{-C_4 t} \sum_{k=0}^{\infty} \delta_{\{t \in \text{Int}[k, W]\}} \left(r e^{-\tau(PC)} e^{C_4 \tau} \right)^k \\
& + X\xi_{E:PC} e^{-(C_4 - C_{2,5})\tau W} e^{-C_{2,5} t} \sum_{k=0}^{\infty} \delta_{\{t \in \text{Int}[k, E]\}} \left(r e^{-\tau(PC)} e^{C_{2,5} \tau} \right)^k \\
& + Y\xi_{W:BOTHE} e^{-C_4 t} \sum_{k=0}^{\infty} \delta_{\{t \in \text{Int}[k, W]\}} \left(e^{-\tau(BOTHE)} e^{C_4 \tau} \right)^k \\
& + Y\xi_{E:PC} e^{-(C_4 - C_{2,5})\tau W} e^{-C_{2,5} t} \sum_{k=0}^{\infty} \delta_{\{t \in \text{Int}[k, E]\}} \left(e^{-\tau(BOTHE)} e^{C_{2,5} \tau} \right)^k .
\end{aligned}$$

By integrating both sides of the the above equation from 0 to t , it then follows that

$$\begin{aligned}
\tilde{P}_{10}(t) & = \xi_{W:PC} \sum_{k=0}^{\infty} \left(r e^{-\tau(PC)} e^{C_1 \tau} \right)^k \int_0^t \delta_{\{t' \in \text{Int}[k, W]\}} e^{-C_1 t'} dt' \\
& + \xi_{E:PC} e^{-(C_1 - C_{2,5})\tau W} \sum_{k=0}^{\infty} \left(r e^{-\tau(PC)} e^{C_{2,5} \tau} \right)^k \int_0^t \delta_{\{t' \in \text{Int}[k, E]\}} e^{-C_{2,5} t'} dt' \\
& + X\xi_{W:BOTHE} \sum_{k=0}^{\infty} \left(r e^{-\tau(PC)} e^{C_1 \tau} \right)^k \int_0^t \delta_{\{t \in \text{Int}[k, W]\}} e^{-C_4 t'} dt' \\
& + X\xi_{E:PC} e^{-(C_4 - C_{2,5})\tau W} \sum_{k=0}^{\infty} \left(r e^{-\tau(PC)} e^{C_{2,5} \tau} \right)^k \int_0^t \delta_{\{t \in \text{Int}[k, E]\}} e^{-C_{2,5} t'} dt' \\
& + Y\xi_{W:BOTHE} \sum_{k=0}^{\infty} \left(e^{-\tau(BOTHE)} e^{C_1 \tau} \right)^k \int_0^t \delta_{\{t \in \text{Int}[k, W]\}} e^{-C_4 t'} dt' \\
& + Y\xi_{E:PC} e^{-(C_4 - C_{2,5})\tau W} \sum_{k=0}^{\infty} \left(e^{-\tau(BOTHE)} e^{C_{2,5} \tau} \right)^k \int_0^t \delta_{\{t \in \text{Int}[k, E]\}} e^{-C_{2,5} t'} dt' .
\end{aligned}$$

The theorem can now be proven by specifying the active terms for given t and conducting the exponential integrals.