Hom complexes and homotopy in the category of graphs

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Abstract

We investigate a notion of $\times$-homotopy of graph maps that is based on the internal hom associated to the categorical product in the category of graphs. It is shown that graph $\times$-homotopy is characterized by the topological properties of the Hom complex, a functorial way to assign a poset (and hence topological space) to a pair of graphs; Hom complexes were introduced by Lovász and further studied by Babson and Kozlov to give topological bounds on chromatic number. Along the way, we also establish some structural properties of Hom complexes involving products and exponentials of graphs, as well as a symmetry result which can be used to reprove a theorem of Kozlov involving foldings of graphs. Graph $\times$-homotopy naturally leads to a notion of homotopy equivalence which we show has several equivalent characterizations. We apply the notions of $\times$-homotopy equivalence to the class of dismantlable graphs to get a list of conditions that again characterize these. We end with a discussion of graph homotopies arising from other internal homs, including the construction of 'A-theory' associated to the cartesian product in the category of reflexive graphs.

1 Introduction

In many categories, the notion of a pair of homotopic maps can be phrased in terms of a map from some specified object into an *exponential object* associated to an internal hom structure on that category (we will review these constructions below). The typical example is the category of (compactly generated) topological spaces, where a homotopy between maps $f : X \to Y$ and $g : X \to Y$ is nothing more than a map from the interval I into the topological space $\mathrm{Map}(X, Y)$. Other examples include simplicial objects, as well as the category of chain complexes of R-modules.

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For the latter, a chain homotopy between chain maps $f : C \to D$ and $g : C \to D$ can be recovered as a map from the chain complex I (defined to be the complex consisting of 0 in all dimensions except R in dimensions 0 and 1, with the identity map between them) into the complex $\mathrm{Hom}(C, D)$.

In this paper we consider these constructions in the context of the category of graphs. In particular, we investigate a notion of what we call $\times$-homotopy that arises from consideration of the well known internal hom associated to the categorical product. Here the relevant construction is the exponential H^G, a graph whose looped vertices parametrize the graph homomorphisms (maps) from G to H. We use the notion of (graph theoretic) connectivity to provide a notion of a 'path' in the exponential graph. It turns out that $\times$-homotopy classes of maps are related to the topology of the so-called Hom-complex, a functorial way to assign a poset $\mathrm{Hom}(G, H)$ (and hence topological space) to a pair of graphs G and H. Hom complexes were first introduced by Lovàsz in his celebrated proof of the Kneser conjecture (see [17]), and were later developed by Babson and Kozlov in their proof of the Lovàsz conjecture (see [2] and [3]). Elements of the poset $\mathrm{Hom}(G, H)$ are graph *multi*-homomorphisms from G to H, with the set of graph homomorphisms the atoms. Fixing one of the two coordinates of the Hom complex in each case provides a functor from graphs to topological spaces, and in Theorem 5.1 we show that $\times$-homotopy of graph maps can be characterized by the topological homotopy type of the maps induced by these functors.

Graph $\times$-homotopy of maps naturally leads us to a notion of homotopy equivalence of graphs, which in Theorem 5.2 we show can again be characterized in terms of the topological properties of relevant Hom complexes. This result also exhibits a certain symmetry in the two entries of the Hom complex and can be used to reprove a result of Kozlov from [16], here stated as Proposition 6.2. The graph operations known as 'folding' and 'unfolding' preserve homotopy type, and in fact we show that in some sense these operations generate the homotopy equivalence class of a given graph. In particular, a pair of *stiff* graphs are homotopy equivalent if and only if they are isomorphic. One particular case of interest arises when the graph can be folded down to a single looped vertex, a class of graphs called *dismantlable* in the literature (see for example [10]). We apply Theorem 5.2 to obtain several characterizations of dismantlable graphs which adds to the list established by Brightwell and Winkler in [6].

The paper is organized as follows. In Section 2 we describe the category of graphs, and gather together some facts regarding its structure. Here we focus on the internal hom structure associated with the categorical product, and review the construction of the exponential graph H^G that serves as the right adjoint. In Section 3 we recall the construction of the Hom complex and discuss some properties. We establish some structural results regarding preservation of homotopy type of the Hom complex under graph exponentiation (Proposition 3.5) as well as arbitrary limits (e.g. products) of graphs (Proposition 3.8). The latter has applications to special cases of Hedetniemi's conjecture, while the former allows us to interpret the complex $\mathrm{Hom}(G, H)$ in terms of the clique complex of the exponential graph H^G. It is this characterization that will allow us to relate the topology of the Hom complex with $\times$-homotopy classes of graph maps in later sections.

In Section 4 we introduce the notion of $\times$-homotopy of graph maps in terms of paths in the exponential graph and work out some examples. We discuss the characterization of $\times$-homotopy in terms of the topology of the relevant Hom complex. The construction of $\times$-homotopy naturally leads us to a graph theoretic notion of *homotopy equivalence* of graphs, and in Section 5 we prove some equivalent characterizations in terms of the topology of the Hom complexes. In particular, it is the symmetry involved in this characterization that allows us to reprove the result of Kozlov discussed above. We also discuss some of the categorical properties that are satisfied. In Section 6, we investigate some of the structure of these homotopy equivalence classes, and discuss the relationship with the graph operations known as foldings and unfoldings and the related notion of a stiff graph. Here we apply our previous results to obtain several characterizations of the class of dismantlable graphs. Finally, in Section 6, we briefly discuss one other notion of homotopy that arises from the internal hom associated to the cartesian product. It turns out that this construction recovers the existing notion of the so-called A-theory of graphs discussed in [4].

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2 The category of graphs

We will work in the category of graphs. A *graph* $G = (V(G), E(G))$ consists of a vertex set $V(G)$ and an edge set $E(G) \subseteq V(G) \times V(G)$ such that if $(v, w) \in E(G)$ then $(w, v) \in E(G)$. Hence our graphs are undirected and do not have multiple edges, but may have loops (if $(v, v) \in E(G)$). If $(v, w) \in E(G)$ we will say that v and w are *adjacent* and denote this as $v \sim w$. Given a pair of graphs G and H, a *graph homomorphism* (or *graph map*) is a mapping of the vertex set $f : V(G) \to V(H)$ that preserves adjacency: if $v \sim w$ in G, then $f(v) \sim f(w)$ in H (equivalently $(v, w) \in E(G)$ implies $(f(v), f(w)) \in E(H)$). With these as our objects and morphisms we obtain a category of graphs which we will denote $\mathcal{G}$. If G and H are graphs, we will use $\mathcal{G}(G, H)$ to denote the set of graph maps between them.

In this section, we review some of the structure of $\mathcal{G}$. Of particular importance for us will be the existence of an *internal hom* associated to the categorical product. We start by recalling some related constructions, all of which can be found in [9] and [11]. For undefined categorical terms, we refer to [18].

Definition 2.1. *For graphs G and H, the categorical coproduct $G \amalg H$ is the graph with vertex set $V(G) \amalg V(H)$ and with adjacency given by $(x, x') \in E(G \amalg H)$ if $(x, x') \in E(G)$ or $(x, x') \in E(H)$.*

Definition 2.2. *For graphs G and H, the categorical product $G \times H$ is a graph with vertex set*

$V(G) \times V(H)$ and adjacency given by $(g,h) \sim (g',h')$ in $G \times H$ if both $g \sim g'$ in G and $h \sim h'$ in H (see Figure 1).

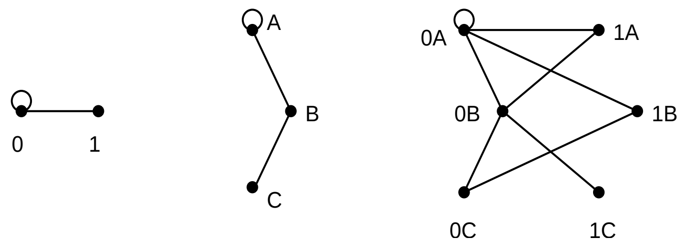

Figure 1: The graphs G, H, and $G \times H$

Definition 2.3. *For graphs G and H, the categorical exponential graph H^G is a graph with vertex set $\{f : V(G) \to V(H)\}$, the collection of all vertex set maps, with adjacency given by $f \sim f'$ if whenever $v \sim v'$ in G we have $f(v) \sim f'(v')$ in H (see Figure 2).*

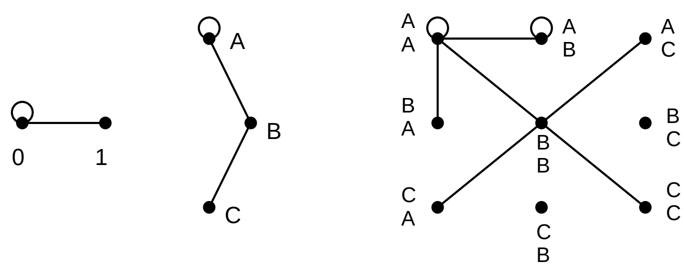

Figure 2: The graphs G, H, and H^G

The next lemma shows that the exponential graph construction provides a right adjoint to the categorical product. By definition, this gives the category of graphs the structure of an *internal hom* associated with the (monoidal) categorical product. This result is well known, and is more or less contained in [9], but we state it here in a way that is consistent with our notation.

Lemma 2.4. *For graphs A, B and C, we have a natural isomorphism of sets*

$$\varphi : \mathcal{G}(A \times B, C) \to \mathcal{G}(A, C^B)$$

given by $(\varphi(f)(v))(w) = f(v,w)$ for all $f \in \mathcal{G}(A \times B, C)$, $v \in V(A)$, $w \in V(B)$.

Proof. Let $f : A \times B \to C$ be an element of $\mathcal{G}(A \times B, C)$. To see that $\varphi(f) \in \mathcal{G}(A, C^B)$, suppose that $a \sim a'$ are adjacent vertices in A. We need $\varphi(f)(a)$ and $\varphi(f)(a')$ to be adjacent vertices in C^B. To check this, suppose $b \sim b'$ in B. Then we have $\varphi(f)(a)(b) = f(a,b)$ and $\varphi(f)(a')(b') = f(a',b')$, which are adjacent vertices of C since f is a graph map.

To check naturality, suppose $f : A \to A'$ and $g : C \to C'$ are graph maps. We need to verify that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{G}(A\times B, C) & \xleftarrow{(f\times B, g)} & \mathcal{G}(A'\times B, C') \\ \varphi\downarrow & & \downarrow\varphi \\ \mathcal{G}(A, C^B) & \xleftarrow[(f, g^B)]{} & \mathcal{G}(A', (C')^B) \end{array}$$

For this, let $\alpha \in \mathcal{G}(A' \times B, C)$. Then on the one hand we have $(\varphi(f \times B, g))(\alpha)(a)(b) = (f \times B, g)(\alpha)(a, b) = g(\alpha(f(a), b))$. In the other direction, we have $((f, g^B)(\varphi))(\alpha)(a)(b) = g(\varphi(\alpha)(f(a))(b)) = g(\alpha(f(a), b))$. Hence the diagram commutes, and so the isomorphism φ is natural. $\square$

We close this section with a few additional definitions and remarks. We let $\mathbf{1}$ denote the graph consisting of a single looped vertex. We point out that $\mathbf{1}$ is the *terminal object* in $\mathcal{G}$ in the sense that there exists a unique map $G \to \mathbf{1}$ for all G. Similarly, the graph $\emptyset$ is the graph whose vertex set is the empty set. It is the *initial object* in the sense that there exists a unique map $\emptyset \to G$ for all G.

A *reflexive* graph G is a graph with loops on all its vertices ($v \sim v$ for all $v \in V(G)$). A map of reflexive graphs will be a graph map on the underlying graph. We will use $\mathcal{G}^\circ$ to denote the category of reflexive graphs.

We see that $\mathcal{G}^\circ$ is a subcategory of $\mathcal{G}$, and we let $i : \mathcal{G}^\circ \to \mathcal{G}$ denote the inclusion functor. Let $S : \mathcal{G} \to \mathcal{G}^\circ$ denote the functor given by taking the subgraph induced by looped vertices, and $L : \mathcal{G} \to \mathcal{G}^\circ$ denote the functor given by adding loops to all vertices (see Figure 3). One can check that i is a left adjoint to S, whereas i is a right adjoint to L. As functors $\mathcal{G} \to \mathcal{G}$, one can check that L (strictly speaking iL) is a left adjoint to S (strictly speaking iS). We will make some use of these facts in a later section.

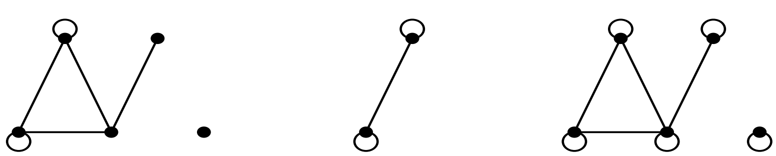

Figure 3: The graph G, and the reflexive graphs $S(G)$ and $L(G)$.

If v and w are vertices of a graph G, the *distance* $d(v, w)$ is the length of the shortest path in G from v to w. The *diameter* of a finite connected graph G, denoted $\mathrm{diam}(G)$ is the maximum distance between two vertices of G. The *neighborhood* of a vertex v, denoted $N_G(v)$ (or $N(v)$ if the context is clear), is the set of vertices that are adjacent to v (so that $v \in N(v)$ if and only if v has a loop).

There are several simplicial complexes one can associate with a given graph G. One such construction is the *clique complex* $\Delta(G)$, a simplicial complex with vertices given by all *looped* vertices of G, and with faces given by all cliques (complete subgraphs) on the looped vertices of G.

3 The Hom complex and some properties

Next we recall the construction of the Hom complex associated to a pair of graphs. As discussed in the introduction, (a version of) the Hom complex was first introduced by Lovász in [17], and later studied by Babson and Kozlov in [2].

Definition 3.1. *For graphs G, H, we define* $\mathrm{Hom}(G,H)$ *to be the poset whose elements are given by all functions $\eta : V(G) \to 2^{V(H)} \backslash \{\emptyset\}$, such that if $(x,y) \in E(G)$ then $(\tilde{x},\tilde{y}) \in E(H)$ for every $\tilde{x} \in \eta(x)$ and $\tilde{y} \in \eta(y)$. The relation is given by containment, so that $\eta \leq \eta'$ if $\eta(x) \subseteq \eta'(x)$ for all $x \in V$ (see Figure 4).*

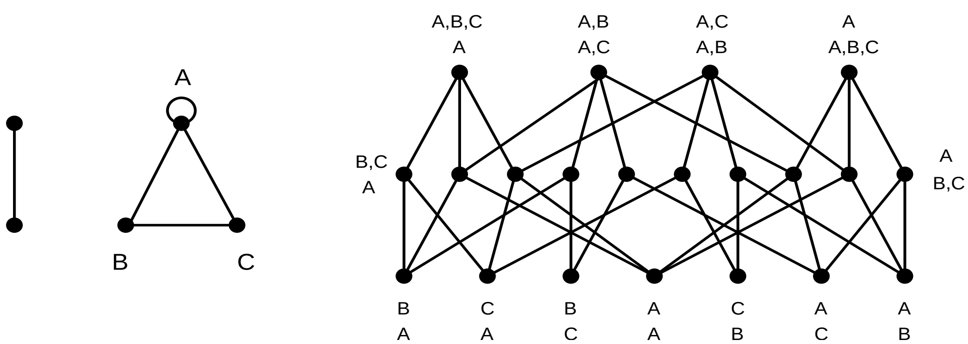

Figure 4: The graphs G and H, and the poset $\mathrm{Hom}(G,H)$

We will often refer to $\mathrm{Hom}(G,H)$ as a topological space; by this we mean the geometric realization of the order complex of the poset. The *order complex* of a poset P is the simplicial complex whose faces are the chains of P (see Figure 5).

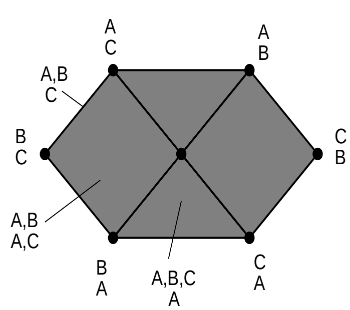

Figure 5: The realization of the poset $\mathrm{Hom}(G,H)$ (up to barycentric subdivision)

Note that if G and H are both finite, then (the order complex of) this $\mathrm{Hom}(G,H)$ yields a simplicial complex which is isomorphic to the barycentric subdivision of the polyhedral Hom complex as defined in [2].

The Hom complexes were originally used to obtain 'topological' lower bounds on the chromatic number of graphs. The main results of [17] and [3] in this context are the following theorems. Here $\chi(G)$ is the chromatic number of a graph G, $\mathrm{conn}(X)$ denotes the (topological) connectivity of the space X, and C_{2r+1} is the odd cycle of length $2r+1$.

Theorem 3.2 (Lovász)**.** *For any graph G,*

$$\chi(G) \geq \mathrm{conn}\big(\mathrm{Hom}(K_2, G)\big) + 3.$$

Theorem 3.3 (Babson and Kozlov)**.** *For any graph G,*

$$\chi(G) \geq \mathrm{conn}\big(\mathrm{Hom}(C_{2r+1}, G)\big) + 4.$$

In [2] Babson and Kozlov establish some basic functorial properties of the Hom complex which we briefly discuss. Fixing one of the coordinates of the Hom complexes provides a covariant functor $\mathrm{Hom}(T, ?)$, and a contravariant functor $\mathrm{Hom}(?, T)$, from $\mathcal{G}$ to the category of posets. If $f : G \to H$ is a graph map, we have in the first case an induced poset map $f_T : \mathrm{Hom}(T,G) \to \mathrm{Hom}(T,H)$ given by $f_T(\alpha)(t) = \{f(g) : g \in \alpha(t)\}$ for $\alpha \in \mathrm{Hom}(T,G)$ and $t \in V(T)$. In the other case, we have $f^T : \mathrm{Hom}(H,T) \to \mathrm{Hom}(G,T)$ given by $f^T(\beta)(g) = \beta(f(g))$ for $\beta \in \mathrm{Hom}(H,T)$ and $g \in V(G)$.

A graph map $f : G \to H$ induces a natural transformation $\bar{f} : \mathrm{Hom}(?, G) \to \mathrm{Hom}(?, H)$ in the following way. For each $T \in \mathrm{Ob}(\mathcal{G})$ we have a map $\bar{f}_T : \mathrm{Hom}(T,G) \to \mathrm{Hom}(T,H)$ given by $(\bar{f}_T(\alpha))(t) = \{f(g) : g \in \alpha(t)\}$ for $\alpha \in \mathrm{Hom}(T,G)$ and $t \in V(T)$. If $g : S \to T$ is a graph map, the diagram

$$\begin{tikzcd}
\mathrm{Hom}(S,G) \arrow[d, "\bar{f}_S"] & \mathrm{Hom}(T,G) \arrow[l, "g^G"'] \arrow[d, "\bar{f}_T"] \\
\mathrm{Hom}(S,H) & \mathrm{Hom}(T,H) \arrow[l, "g^H"']
\end{tikzcd}$$

commutes since if $\alpha \in \mathrm{Hom}(T,G)$ and $s \in V(S)$ then on the one hand we have

$$((\bar{f}_S g^G)(\alpha))(s) = \{f(x) : x \in ((g^G(\alpha))(s)\} = \{f(x) : x \in \alpha(g(s))\},$$

and on the other

$$((g^H \bar{f}_T)(\alpha))(s) = ((\bar{f}_T)(\alpha))(g(s)) = \{f(x) : x \in \alpha(g(s))\}.$$

The function induced by composition $\mathrm{Hom}(G, H) \times \mathrm{Hom}(H, K) \to \mathrm{Hom}(G, K)$ is a poset map; see [13] for a proof of this fact.

Many operations in the category of graphs interact nicely with the topology of the Hom complexes. We now gather together some of these results. The first observation comes from [2].

Lemma 3.4. *Let A, B, and C be graphs. Then there is an isomorphism of posets*

$$\mathrm{Hom}(A \amalg B, C) \cong \mathrm{Hom}(A, C) \times \mathrm{Hom}(B, C).$$

Also, if A is connected and not a single vertex, then

$$\mathrm{Hom}(A, B \amalg C) \cong \mathrm{Hom}(A, B) \amalg \mathrm{Hom}(A, C).$$

As we will see, other graph operations are preserved by the Hom complexes *up to homotopy type.*

Recall that for graphs A, B, and C the exponential graph construction provides the adjunction $\mathcal{G}(A \times B, C) = \mathcal{G}(A, C^B)$, an isomorphism of sets. The next proposition shows that this map induces a homotopy equivalence of the associated Hom complexes. In the proof of Propositions 3.5 and 3.8 we will use the following notion from poset topology (see [5] for a good reference). If P is a poset, and $c : P \to P$ is a poset map such that $c \circ c = c$ and $c(p) \geq p$ for all $p \in P$ then c is called a *closure map.* It can be shown (see [5]) that in this case $c : P \to c(P)$ induces a strong deformation retract of the associated spaces.

Proposition 3.5. *Let A, B, and C be graphs. Then $\mathrm{Hom}(A \times B, C)$ can be included in $\mathrm{Hom}(A, C^B)$ so that $\mathrm{Hom}(A \times B, C)$ is the image of a closure map on $\mathrm{Hom}(A, C^B)$. In particular, there is an inclusion of a strong deformation retract*

$$|\mathrm{Hom}(A \times B, C)| \xhookrightarrow[\simeq]{} |\mathrm{Hom}(A, C^B)|.$$

Proof. Let $P = \mathrm{Hom}(A \times B, C)$ and $Q = \mathrm{Hom}(A, C^B)$ be the respective posets. Our plan is to define an inclusion map $j : P \to Q$ and a closure map $c : Q \to Q$ such that $\mathrm{im}(j) = \mathrm{im}(c)$, from which the result would follow.

We define a map of posets $j : P \to Q$ according to

$$j(\alpha)(a) = \big\{f : V(B) \to V(C) | f(b) \in \alpha(a, b)\ \forall b \in B\big\},$$

for every $a \in V(A)$ and $\alpha \in P$. To show that $j(\alpha)$ is in fact an element of Q, we need to verify that if $a \sim a'$ in A then we have $f \sim f'$ in C^B for all $f \in j(\alpha)(a), f' \in j(\alpha)(a')$. If $b \sim b'$ in B then $(a, b) \sim (a', b')$ in $A \times B$. Hence $c \sim c'$ in C for $c \in \alpha(a, b)$ and all $c' \in \alpha(a', b')$. In particular, $f(b) \sim f(b')$ in C and we conclude that $f \sim f'$, as desired. Hence $j(\alpha)$ is indeed an element of $\mathrm{Hom}(A, C^B)$.

We claim that j is injective. To see this, let $\alpha \neq \alpha'$ be distinct elements of the poset P, with $\alpha(a,b) \neq \alpha'(a,b)$ for some $(a,b) \in V(A \times B)$. Without loss of generality, suppose $c \in \alpha(a,b) \backslash \alpha'(a,b)$. Then we have some $f \in j(\alpha)(a)$ such that $f(b) = c$, and yet $f \notin j(\alpha')(a)$. We conclude that $j(\alpha) \neq j(\alpha')$, and hence j is injective.

Next we define a closure map of posets $c : Q \to Q$. If $\gamma : V(A) \to 2^{V(C^B)} \backslash \{\emptyset\}$ is an element of $Q = \mathrm{Hom}(A, C^B)$, define $c(\gamma) \in Q$ as follows: fix some $a \in V(A)$, and for every $b \in V(B)$ let $C^{\gamma}_{ab} = \{f(b) \in V(C) : f \in \gamma(a)\}$. Then define $c(\gamma)$ according to

$$c(\gamma)(a) = \big\{g : V(B) \to V(C) | g(b) \in C^{\gamma}_{ab} \ \forall b \in B\big\}.$$

We first verify that c maps into Q, so that $c(\gamma) \in \mathrm{Hom}(A, C^B)$. For this suppose $a \sim a'$ in A and let $f \in c(\gamma)(a)$ and $g \in c(\gamma)(a')$. To show that $f \sim g$ in C^B we consider some $b \in b'$ in B. Then by construction there is some $f' \in \gamma(a)$, $g' \in \gamma(a')$ such that $f(b) = f'(b)$ and $g(b') = g'(b')$. Hence $f(b) \sim g(b')$ in C as desired.

It is clear that $c(\gamma) \geq \gamma$ and $(c \circ c)(\gamma) = c(\gamma)$ for all $\gamma \in Q$. Thus c is a closure map.

Next we claim that $c(Q) \subseteq j(P)$. To see this, suppose $\gamma \in Q$. We define $\alpha : V(A \times B) \to 2^{V(C)} \backslash \{\emptyset\}$ according to $\alpha(a,b) = C^{\gamma}_{ab}$, where $C^{\gamma}_{ab} \subseteq V(C)$ is as above. To see that $\alpha \in \mathrm{Hom}(A \times B, C)$ suppose $(a,b) \sim (a',b')$, and let $c \in \alpha(a,b) = C^{\gamma}_{ab}$, $c' \in \alpha(a',b') = C^{\gamma}_{a'b'}$. Since $a \sim a'$ we have $f \sim f'$ in C^B for all $f \in \gamma(a), f' \in \gamma(a')$. Hence since $b \sim b'$ we get $f(b) \sim f'(b')$, and in particular obtain $c \sim c'$ in C as desired.

Finally, we get $j(P) \subseteq c(Q)$ since $j(P) \subseteq Q$ and $c(j(P)) = j(P)$. Thus $j(P) = c(Q)$, implying that $\mathrm{Hom}(A \times B, C) \simeq \mathrm{Hom}(A, C^B)$ via this inclusion. $\square$

Remark 3.6. *As a result of Proposition 3.5, for all graphs G and H there is a homotopy equivalence*

$$\mathrm{Hom}(G,H) = \mathrm{Hom}(\mathbf{1} \times G, H) \simeq \mathrm{Hom}(\mathbf{1}, H^G),$$

where $\mathbf{1}$ denotes a single looped vertex. The last of these posets is the face poset of the clique complex on the looped vertices of H^G, and hence its realization is the barycentric subdivision of the clique complex of H^G. Since the looped vertices in H^G are precisely the graph homomorphisms $G \to H$, we see that $\mathrm{Hom}(G,H)$ can be realized up to homotopy type as the clique complex of the subgraph of H^G induced by the graph homomorphisms.

By a *diagram of graphs* $D = \{D_i\}$, we mean a collection of graphs $\{D_i\}$ with a specified collection of maps between them (the image of a category D under some functor to $\mathcal{G}$). For a graph T, any such diagram of graphs gives rise to a diagram of posets $\mathrm{Hom}(T, D)$ obtained by applying the functor $\mathrm{Hom}(T, ?)$ to each object and each morphism (see Figure 6).

$$\begin{tikzcd} & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \end{tikzcd} \qquad \begin{tikzcd} & D_1 \arrow[d, "f"] \\ D_3 \arrow[r, "g"'] & D_2 \end{tikzcd} \qquad \begin{tikzcd} & \mathrm{Hom}(T, D_1) \arrow[d, "f_T"] \\ \mathrm{Hom}(T, D_3) \arrow[r, "g_T"'] & \mathrm{Hom}(T, D_2) \end{tikzcd}$$

Figure 6: A category D, a diagram of graphs, and the induced diagram of posets

We can combine the facts from Lemma 2.4 and Proposition 3.5 to see that Hom complexes preserve (up to homotopy type) limits of such diagrams.

Proposition 3.7. *Let D be a diagram of graphs with limit* $\lim(D)$*. Then for every graph T we have a homotopy equivalence:*

$$\big|\mathrm{Hom}\big(T, \lim(D)\big)\big| \simeq \big|\lim\big(\mathrm{Hom}(T, D)\big)\big|.$$

Proof. Let T be a graph. We will express the functor $\mathrm{Hom}(T, ?)$ as a composition of functors that each preserve limits. First we note that the functor $(?)^T : \mathcal{G} \to \mathcal{G}$ given by $G \mapsto G^T$ preserves limits since it has the left adjoint given by the functor $? \times T$; this was the content of Proposition 3.5. Hence for any diagram of graphs D, we get $\big(\lim(D)\big)^T = \lim(D^T)$.

Next we note that the functor $L : \mathcal{G} \to \mathcal{G}^\circ$ that takes the induced subgraph on the looped vertices (described above) also preserves limits since it has the left adjoint given by the inclusion functor $\mathcal{G}^\circ \to \mathcal{G}$. So we have $L\big(\lim(D)\big) = \lim\big(L(D)\big)$.

Now we claim that the functor $\mathrm{Hom}(\mathbf{1}, ?)$ preserves limits up to homotopy type. To see this, we recall that $\mathrm{Hom}(\mathbf{1}, ?)$, as a functor from the category of reflexive graphs, associates to a given (reflexive) graph G the face poset of its *clique complex*, $\Delta(G)$. Hence, taking geometric realization, we get $|\mathrm{Hom}(\mathbf{1}, G)| \simeq |\Delta(G)|$ for all reflexive graphs G. Now, as a functor to flag simplicial complexes, the clique complex Δ has an inverse functor given by taking the 1-skeleton and adding loops to each vertex. In particular, this shows that $\Delta(?)$ preserves limits, and we get $\Delta(\lim \tilde{D}) = \lim\big(\Delta(\tilde{D})\big)$, for any diagram of reflexive graphs $\tilde{D}$.

Finally, we can put these observations together to get the following string of isomorphisms $(=)$ and homotopy equivalences $(\simeq)$:

$$\begin{aligned} \big|\mathrm{Hom}\big(T, \lim(D)\big)\big| &\simeq \Big|\mathrm{Hom}\Big(\mathbf{1}, \big(\lim(D)\big)^T\Big)\Big| = \big|\mathrm{Hom}\big(\mathbf{1}, \lim(D^T)\big)\big| \\ &= \Big|\mathrm{Hom}\Big(\mathbf{1}, L\big(\lim(D^T)\big)\Big)\Big| = \Big|\mathrm{Hom}\Big(\mathbf{1}, \lim\big(L(D^T)\big)\Big)\Big| \\ &\simeq \Big|\Delta\Big(\lim\big(L(D^T)\big)\Big)\Big| = \Big|\lim\Big(\Delta\big(L(D^T)\big)\Big)\Big| \\ &\simeq \Big|\lim\Big(\mathrm{Hom}\big(\mathbf{1}, L(D^T)\big)\Big)\Big| = \Big|\lim\Big(\big(\mathrm{Hom}(\mathbf{1}, (D^T)\big)\Big)\Big| \\ &\simeq \big|\lim\big(\mathrm{Hom}(T, D)\big)\big|. \end{aligned}$$

The first and last homotopy equivalences are as in Proposition 3.5. ∎

Recall that the product $G \times H$ is a limit (pullback) of the diagram $G \to \mathbf{1} \leftarrow H$. Since $\mathrm{Hom}(T, \mathbf{1})$ is a point for every graph T, this implies that $|\mathrm{Hom}(T,G)| \times |\mathrm{Hom}(T,H)|$ is homotopy equivalent to $|\mathrm{Hom}(T, G \times H)|$ for all graphs T, G, and H. In fact in the case of the product we can exhibit this homotopy equivalence as a closure map on the level of posets.

Proposition 3.8. *Let T, G, and H be graphs. Then the poset* $\mathrm{Hom}(T,G) \times \mathrm{Hom}(T,H)$ *can be included into* $\mathrm{Hom}(T, G \times H)$ *so that* $\mathrm{Hom}(T,G) \times \mathrm{Hom}(T,H)$ *is the image of a closure map on* $\mathrm{Hom}(T, G \times H)$. *In particular, there is an inclusion of a strong deformation retract*

$$|\mathrm{Hom}(T,G)| \times |\mathrm{Hom}(T,H)| \xhookrightarrow[\simeq]{} |\mathrm{Hom}(T, G \times H)|.$$

Proof. We let $Q = \mathrm{Hom}(T,G) \times \mathrm{Hom}(T,H)$ and $P = \mathrm{Hom}(T, G \times H)$ be the respective posets. Once again our plan is to define an inclusion $i : Q \to P$ and a closure map $c : P \to P$ such that $\mathrm{im}(i) = \mathrm{im}(c)$.

We define a map $i : Q \to P$ according to $i(\alpha, \beta)(v) = \alpha(v) \times \beta(v)$, for every vertex $v \in V(T)$. Note that if v and w are adjacent vertices of T then $\tilde{v} \sim \tilde{w}$ in G and v', w' in H for all $\tilde{v} \in \alpha(v)$, $\tilde{w} \in \alpha(w)$, $v' \in \beta(v)$, and $w' \in \beta(w)$. Hence $(\tilde{v}, \tilde{w}) \sim (v', w')$ are adjacent in $G \times H$, so that $i(\alpha, \beta)$ is indeed an element of $\mathrm{Hom}(T, G \times H)$. It is clear that i is injective.

Next, we define a closure operator $c : P \to P$, whose image will coincide with that of the map i. For $\gamma \in P := \mathrm{Hom}(T, G \times H)$, we define $c(\gamma) \in P$ as follows: for every $v \in V(T)$ we have minimal vertex subsets $A_v \subseteq V(G)$, $B_v \subseteq V(H)$ such that $\gamma(v) \subseteq \{(a,b) : a \in A_v, b \in B_v\}$. Define $c(\gamma)(v) := \{(a,b)\} = A_v \times B_v$ to be this minimal set of vertices of $G \times H$.

We first verify that c maps into P, so that $c(\gamma) \in \mathrm{Hom}(T, G \times H)$. Suppose $v \sim w$ are adjacent vertices of T. If $(\tilde{a}, \tilde{b}) \in c(\gamma)(v)$ and $(a', b') \in c(\gamma)(w)$ then we have $(\tilde{a}, \tilde{y}), (\tilde{x}, \tilde{b}) \in \gamma(v)$ and $(a', y'), (x', b') \in \gamma(w)$ for some $\tilde{x}, x' \in G$ and $\tilde{y}, y' \in H$. Hence $\tilde{a} \sim a'$ in G and also $\tilde{b}, b'$ in H, so that $(\tilde{a}, \tilde{b}) \sim (a', b')$ in $G \times H$ as desired.

Since $c(\gamma) \geq \gamma$ and $(c \circ c)(\gamma) = c(\gamma)$ for all $\gamma \in P$, we see that $c : P \to P$ is a closure operator.

Next we claim that $c(P) \subseteq i(Q)$. Suppose $c(\gamma) \in c(P)$, so that for all $v \in T$ we have $c(\gamma)(v) = A_v \times B_v$ for some $A_v \subseteq V(G)$ and $B_v \subseteq V(H)$. Define $\alpha : V(T) \to 2^{V(G)} \backslash \{\emptyset\}$ by $\alpha(v) = A_v$, and $\beta : V(T) \to 2^{V(H)} \backslash \{\emptyset\}$ by $\beta(v) = B_v$. We claim that $\alpha \in \mathrm{Hom}(T,G)$ and $\beta \in \mathrm{Hom}(T,H)$. Indeed, if $w \in T$ is a vertex adjacent to v and $\alpha(w) = A_w$, then if $a_i \in A_v$ and $a_{i'} \in A_w$, we have $(a_i, y) \in \gamma(v)$ and $(a_{i'}, y') \in \gamma(w)$ for some $y, y' \in H$. Hence (a_i, y) and $(a_{i'}, y')$ are adjacent vertices in $G \times H$ (since $\gamma \in \mathrm{Hom}(T, G \times H)$). But this implies that a_i and $a_{i'}$ are adjacent in G, as desired.

Finally, $i(Q) \subseteq c(P)$ since $i(Q) \subseteq P$ and $c(i(Q)) = i(Q)$. Thus $i(Q) = c(P)$ and hence $\mathrm{Hom}(T,G) \times \mathrm{Hom}(T,H) \simeq \mathrm{Hom}(T, G \times H)$ via this inclusion. $\square$

Remark 3.9. *Proposition 3.8 can be used to prove special cases of Hedetmieni's conjecture, which is the simple statement that* $\chi(G \times H) = \min\{\chi(G), \chi(H)\}$ *for all graphs* G *and* H*. Since it is clear that* $\chi(G \times H) \leq \min\{\chi(G), \chi(H)\}$*, the content of the conjecture is the other inequality. Now, combining Proposition 3.8 together with (say) Theorem 3.2 we obtain*

$$\begin{aligned}\chi(G \times H) &\geq \operatorname{conn}\big(\operatorname{Hom}(K_2, G \times H)\big) + 3 \\ &= \operatorname{conn}\big(\operatorname{Hom}(K_2, G) \times \operatorname{Hom}(K_2, H)\big) + 3 \\ &= \min\big\{\operatorname{conn}\big(\operatorname{Hom}(K_2, G)\big), \operatorname{conn}\big(\operatorname{Hom}(K_2, H)\big)\big\} + 3.\end{aligned}$$

Here we apply the simple observation that $\operatorname{conn}(X \times Y) = \min\{\operatorname{conn}(X), \operatorname{conn}(Y)\}$ *for topological spaces* X *and* Y*. This then proves the conjecture for the case when the topological bounds on the chromatic numbers of* G *and* H *are tight (e.g., when* G *and* H *are both taken to be either Kneser graphs or generalized Mycielski graphs).*

4 Graph $\times$-homotopy and Hom complexes

In this section, we define a notion of homotopy for graph maps and describe its interaction with the Hom complexes. The motivation comes from the internal hom structure in the category $\mathcal{G}$ as described above.

Recall that a vertex set map $f : V(G) \to V(H)$ is a looped vertex in H^G if and only if f is a *graph* map $G \to H$. Hence the set of graph maps $\mathcal{G}(G, H)$ are precisely the looped vertices in the internal hom graph H^G. The (path) connected components of the graph H^G then provide a natural notion of 'homotopy' for graph maps: two maps $f, g : G \to H$ will be considered $\times$-*homotopic* if we can find a path along the looped vertices H^G that starts at f and ends at g. The use of the $\times$ is to emphasize the fact that we are using the exponential graph construction which is adjunct to the *categorical product*; in the last section we will consider other exponentials.

To make the notion of a path truly graph theoretic we want to think of it as a map from a path-like graph object into the graph H^G.

Definition 4.1. *We let* I_n *denote the graph with vertices* $\{0, 1, \ldots, n\}$ *and with adjacency given by* $i \sim i$ *for all* i *and* $(i-1) \sim i$ *for all* $1 \leq i \leq n$ *(see Figure 7).*

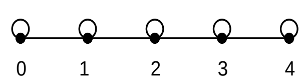

Figure 7: The graph I_4.

Note that $N(n) = \{n, n-1\} \subseteq \{n, n-1, n-2\} = N(n-1)$, and hence we can fold the endpoint of I_n. This gives us the following property.

Lemma 4.2. $\mathrm{Hom}(T, I_n)$ *is contractible for all* $n \geq 0$ *and every graph* T.

Proof. We proceed by induction on n. For $n = 0$, we have that $\mathrm{Hom}(T, I_0) = \mathrm{Hom}(T, \mathbf{1})$ is a point. For $n > 0$, we use the fact that $N(n) \subseteq N(n-1)$ to get $\mathrm{Hom}(T, I_n) \simeq \mathrm{Hom}(T, I_{n-1})$ by Proposition 6.2. The latter complex is contractible by induction. $\square$

Definition 4.3. *Let* $f, g : G \to H$ *be graph maps. We say that* f *and* g *are* $\times$*-homotopic if there exists an integer* $n \geq 1$ *and a map of graphs* $F : I_n \to H^G$ *such that* $F(0) = f$ *and* $F(n) = g$. *In this case we will also say the maps are* n*-homotopic.*

We will denote $\times$-homotopic maps as $f \simeq_\times g$, or simply $f \simeq g$ if the context is clear. Graph $\times$-homotopy determines an equivalence relation on the set of graph maps between G and H, and we let $[G, H]_\times$ (or simply $[G, H]$) denote the set of $\times$-homotopy classes of maps between graphs G and H.

Example 4.4. *As an example we can take* $G = K_2$ *and* $H = K_3$ *to be the complete graphs on 2 and 3 vertices. The graph* H^G *is displayed in Figure 8.*

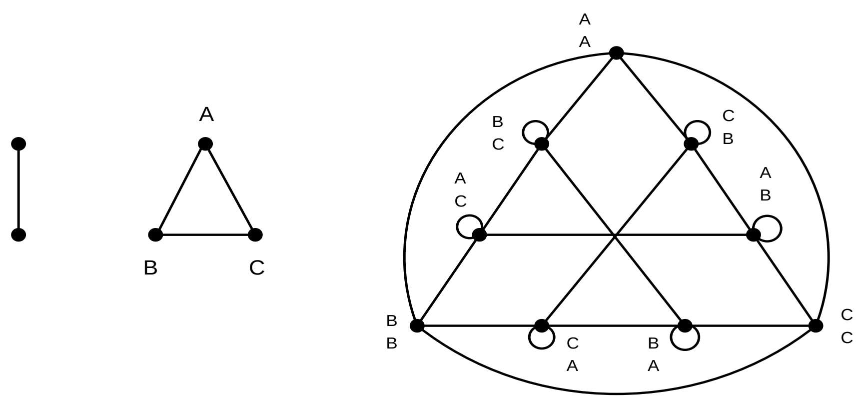

Figure 8: The graphs $G = K_2$, $H = K_3$, *and* H^G.

We see that each of the six graph maps $f : G \to H$ *is represented by a looped vertex in the exponential graph* H^G. *In this case, any two maps* f *and* g *are connected by a path along other looped vertices, and hence in our setup all maps from* $G = K_2$ *to* $H = K_3$ *will be considered* $\times$*-homotopic (so that there is a single homotopy class of maps).*

Example 4.5. *On the other hand, if we take* $G = K_2$, *and this time* $H = K_2$, *we get two distinct* $\times$*-homotopy classes of maps. The graph* H^G *is displayed in Figure 9.*

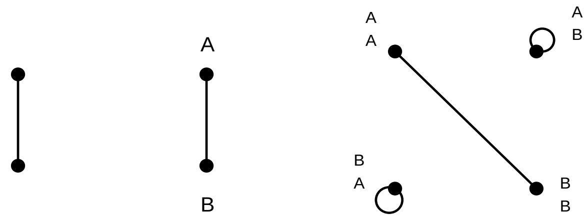

Figure 9: The graphs $G = K_2$, $H = K_2$, and H^G

We see that the two graph maps $G \to H$ are represented by looped vertices in H^G, but this time are disconnected from one another. Hence in this example, each of the two graph maps is in its own $\times$-homotopy class.

We can understand $\times$-homotopy in other ways by considering the adjoint properties available to us. Note that for all $m \leq n$, we have a map $\iota_m : G \to G \times I_n$ given by $v \mapsto (v, m)$, an isomorphism onto its image. A map $F : I_n \to H^G$ corresponds to a map $\tilde{F} : G \times I_n \to H$ with the property that $\tilde{F} \times 0 = f$ and $\tilde{F} \times n = g$. It is this formulation that we will most often use to check for $\times$-homotopy. We record this observation as a lemma.

Lemma 4.6. *Let $f, g : G \to H$ be graph maps. Then f and g are $\times$-homotopic if and only if there exists an integer n and a graph map $F : G \times I_n \to H$ such that $F_0 := F \circ \iota_0 = f : G \to H$ and $F_n := F \circ \iota_n = g : G \to H$.*

$$\begin{tikzcd}
G \arrow[d, "\iota_0"'] \arrow[dr, "f"] & \\
G \times I_n \arrow[r, "F"] & H \\
G \arrow[u, "\iota_n"] \arrow[ur, "g"'] &
\end{tikzcd}$$

Next we investigate how $\times$-homotopy of graph maps interacts with the Hom complex. It turns out that $\times$-homotopy equivalence classes of maps are characterized by the topology of the Hom complex in the following way.

Proposition 4.7. *Let G and H be graphs, and suppose $f, g : G \to H$ are graph maps. Then f and g are $\times$-homotopic if and only if they are in the same path-connected component of* $\mathrm{Hom}(G, H)$. *In particular, the number of $\times$-homotopy classes of maps from G to H is equal to the number of path components in* $\mathrm{Hom}(G, H)$.

Proof. Suppose $f, g : G \to H$ are graph maps such that f and g are in the same component of $\mathrm{Hom}(G, H)$. Then we can find a path from f to g in $|\mathrm{Hom}(G, H)|$, which can be approximated as a finite walk $(f, x_1, x_2, \ldots, g)$ on the 1-skeleton. We claim that we can extend this to a walk

$(f = f_0, x_1, f_1, x_2, f_2, \dots, f_n = g)$, where each $f_i : G \to H$ is a graph map (i.e., $f_i(v)$ consists of a single element for each $v \in V(G)$).

To see this, note that $f \le x_1$ in $\mathrm{Hom}(G, H)$. First suppose that $x_1 \le x_2$. Then for each $v \in V(G)$, we choose (by the choice axiom, say) a single element of $x_1(v)$ to get our map $f_1 : G \to H$ such that $f_1 \le x_1 \le x_2$. Next suppose $x_2 \le x_1$. If x_2 is already a graph map, take $f_1 = x_2$, and otherwise for each $v \in V(G)$ choose a single element of $x_2(v)$ to get a map $f_1 : G \to H$.

Now, to get our homotopy, we define a map $F : G \times I_n \to H$ by $F(v, i) = f_i(v)$. Then F is indeed a graph map since we have an $x_i \in \mathrm{Hom}(G, H)$ such that $f_{i-1}, f_i \le x_i$ for each $0 < i \le n$. Hence the maps $f = f_0$ and $g = f_n$ are $\times$-homotopic.

For the other direction, suppose that $f, g : G \to H$ are distinct maps that are $\times$-homotopic for $n = 1$. We define a function $\xi : V(G) \to 2^{V(H)} \backslash \{\emptyset\}$ by $v \mapsto \{f(v), g(v)\}$. We claim that ξ is a cell in $\mathrm{Hom}(G, H)$. To see this, suppose $v \sim w$ are adjacent vertices of G. Then both $(f(v), f(w))$ and $(g(v), g(w))$ are edges in H since f and g are graph maps. Also, $(0v, 1w)$ and $(0w, 1v)$ are edges in $G \times I_1$ and since f and g are 1-homotopic this implies that $(f(v), g(w))$ and $(f(w), g(v))$ are both edges in H. Hence vertices of $\xi(v)$ are adjacent to vertices of $\xi(w)$ as desired. It is clear that both f and g are vertices of ξ and hence we have a path from f to g. Now, suppose f and g are $\times$-homotopic for some choice of n and let $F : G \times I_n \to H$ be the homotopy. Let $f_i : G \mapsto H$ be the graph map given by $v \mapsto F(\iota_i(v))$. Then by induction we have a path in $\mathrm{Hom}(G, H)$ from f to f_{n-1} and the above construction gives a path from f_{n-1} to $f_n = g$. ∎

We end this section with the following observation.

Lemma 4.8. *Let G be a graph, $k \le n$ integers, and let $\iota_k : G \to G \times I_n$ denote the graph map given by $\iota(g) = (g, k)$. Then for a graph T, the induced map $\iota_{k_T} : \mathrm{Hom}(T, G) \to \mathrm{Hom}(T, G \times I_n)$ is a homotopy equivalence.*

Proof. Let $i : \mathrm{Hom}(T, G) \times \mathrm{Hom}(T, I_n) \hookrightarrow \mathrm{Hom}(T, G \times I_n)$ denote the inclusion, a homotopy equivalence by Proposition 3.8. Let $\phi_k : \mathrm{Hom}(T, G) \to \mathrm{Hom}(T, G) \times \mathrm{Hom}(T, I_n)$ denote the inclusion given by $x \mapsto (x, c_k)$, where $c_k \in \mathrm{Hom}(T, I_n)$ is the constant map sending all elements of $V(T)$ to k. We note that ϕ_k is a homotopy equivalence by Lemma 4.2. We then have the following commutative diagram showing that $\iota_{k_T} = i \circ \phi_k$ is a homotopy equivalence.

$$\begin{tikzcd}
\mathrm{Hom}(T, G) \arrow[r, "\iota_{k_T}"] \arrow[d, "\phi_k", "\simeq"'] & \mathrm{Hom}(T, G \times I_n) \\
\mathrm{Hom}(T, G) \times \mathrm{Hom}(T, I_n) \arrow[ur, "i", "\simeq"'] &
\end{tikzcd}$$

∎

5 Homotopy equivalence of graphs

If $f, g : G \to H$ are graph maps, the functors obtained by fixing a graph T in one coordinate of the Hom complex in each case provides a pair of topological maps. For a fixed test graph T, the functor $\mathrm{Hom}(T, ?)$ provides the pair of maps $f_T, g_T : \mathrm{Hom}(T, G) \to \mathrm{Hom}(T, H)$, while $\mathrm{Hom}(?, T)$ provides the maps $f^T, g^T : \mathrm{Hom}(H, T) \to \mathrm{Hom}(G, T)$ (discussed above). If f and g are $\times$-homotopic, we can ask how these induced maps are related up to (topological) homotopy. It turns out that the induced maps are homotopic, and in fact provide a characterization of graph $\times$-homotopy in each case. More precisely, we have the following result.

Theorem 5.1. *Let $f, g : G \to H$ be graph maps. Then the following are equivalent:*

(1) f and g are $\times$-homotopic.

(2) For every graph T, the induced maps $f_T, g_T : \mathrm{Hom}(T, G) \to \mathrm{Hom}(T, H)$ are homotopic.

(3) The induced maps $f_G, g_G : \mathrm{Hom}(G, G) \to \mathrm{Hom}(G, H)$ are homotopic.

(4) For every graph T, the induced maps $f^T, g^T : \mathrm{Hom}(H, T) \to \mathrm{Hom}(G, T)$ are homotopic.

(5) The induced maps $f^H, g^H : \mathrm{Hom}(H, H) \to \mathrm{Hom}(G, H)$ are homotopic.

Proof. We first prove (1) $\Rightarrow$ (2). Suppose $f, g : G \to H$ are $\times$-homotopic via a graph map $F : G \times I_n \to H$. Then (with notation as above) we have a commutative diagram in $\mathcal{G}$ and, via the functor $\mathrm{Hom}(T, ?)$, the induced diagram in $\mathcal{TOP}$, the category of topological spaces and continuous maps, of the form:

$$\begin{tikzcd}
G \arrow[d, "\iota_0"'] \arrow[dr, "f"] & \\
G \times I_n \arrow[r, "F"] & H \\
G \arrow[u, "\iota_n"] \arrow[ur, "g"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Hom}(T, G) \arrow[d, "\iota_{0_T}"'] \arrow[dr, "f_T"] & \\
\mathrm{Hom}(T, G \times I_n) \arrow[r, "F_T"] & \mathrm{Hom}(T, H) \\
\mathrm{Hom}(T, G) \arrow[u, "\iota_{n_T}"] \arrow[ur, "g_T"'] &
\end{tikzcd}$$

Now, $\mathrm{Hom}(T, I_n)$ is path connected (contractible) by Lemma 4.2. Let $\gamma : I = [0, 1] \to \mathrm{Hom}(T, I_n)$ be a path such that $\gamma(0) = c_0$ and $\gamma(1) = c_n$ (where again $c_i \in \mathrm{Hom}(T, I_n)$ is the constant map sending all vertices of T to i). Let $j_i : \mathrm{Hom}(T, G) \to \mathrm{Hom}(T, G) \times I$ be the (topological) map given by (id, i). We then obtain the following diagram in $\mathcal{TOP}$ (where $(T, G) = \mathrm{Hom}(T, G)$, etc.):

$$\begin{tikzcd}
 & (T,G) \arrow[dl, "j_0"'] \arrow[dr, "\iota_{0_T}"] & & \\
(T,G)\times I \arrow[r, "id\times\gamma"] & (T,G)\times(T,I_n) \arrow[r, hook] & (T,G\times I_n) \arrow[r, "F_T"] & (T,H) \\
 & (T,G) \arrow[ul, "j_1"] \arrow[ur, "\iota_{n_T}"'] & &
\end{tikzcd}$$

We claim that this diagram commutes. To see this, suppose $\alpha \in \mathrm{Hom}(T,G)$. Then for all $t \in V(T)$ we have $\iota_{0_T}(\alpha)(t) = \{\iota_0(x) : x \in \alpha(t)\} = \{(x,0) : x \in \alpha(t)\} \in \mathrm{Hom}(T,G) \times \mathrm{Hom}(T,I^n)$, so that $\iota_{0_T}(\alpha) = (\alpha, c_0)$. On the other hand, $(id \times \gamma)(j_0)(\alpha) = (id \times \gamma)(\alpha, 0) = (\alpha, c_0)$. The bottom square is similar.

Now, let $\Phi : \mathrm{Hom}(T,G) \times I \to \mathrm{Hom}(T,H)$ be the composition from above. We have that $\Phi \circ j_0 = F_T \circ \iota_{0_T} = f_T : \mathrm{Hom}(T,G) \to \mathrm{Hom}(T,H)$ and similarly $\Phi \circ j_1 = g_T$, so that f_T and g_T are homotopic.

The implication (2) $\Rightarrow$ (3) is clear, so we next turn to (3) $\Rightarrow$ (1). For this, suppose $f, g : G \to H$ are not $\times$-homotopic. Then f and g are in different path components of $\mathrm{Hom}(G,H)$ by Proposition 4.7. We claim that the induced maps $f_G, g_G : \mathrm{Hom}(G,G) \to \mathrm{Hom}(G,H)$ are also not homotopic. To obtain a contradiction, suppose they are and let $\Phi : \mathrm{Hom}(G,G) \times I \to \mathrm{Hom}(G,H)$ be a (topological) homotopy between them. Note that if $\mathrm{id} \in \mathrm{Hom}(G,G)$ is the identity map, then $f_G(\mathrm{id}) = f$ and $g_G(\mathrm{id}) = g$ since, for instance, we have $f_G(\mathrm{id})(x) = \{f(y) : y \in \mathrm{id}(x)\} = \{f(y) : y \in \{x\}\} = \{f(x)\}$ for all $x \in V(G)$. So then the restriction $\Phi|_{\{\mathrm{id}\}\times I} : \mathrm{Hom}(G,G) \times I \to \mathrm{Hom}(G,H)$ gives a path in $\mathrm{Hom}(G,H)$ from f to g, a contradiction.

We next prove (1) $\Rightarrow$ (4). Again, suppose $f, g : G \to H$ are $\times$-homotopic via $F : G \times I_n \to H$. Then this time we have the commutative diagram in $\mathcal{G}$ and the induced diagram in $\mathcal{TOP}$ of the form:

$$\begin{tikzcd}
G \arrow[d, "\iota_0"'] \arrow[dr, "f"] & \\
G \times I_n \arrow[r, "F"] & H \\
G \arrow[u, "\iota_n"] \arrow[ur, "g"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathrm{Hom}(G,T) \\
\mathrm{Hom}(H,T) \arrow[ur, "f^T"] \arrow[r, "F^T"] \arrow[dr, "g^T"'] & \mathrm{Hom}(G \times I_n, T) \arrow[u, "\iota_0^T"'] \arrow[d, "\iota_n^T"] \\
 & \mathrm{Hom}(G,T)
\end{tikzcd}$$

To show that f^T and g^T are homotopic, we will find a map $\Psi : \mathrm{Hom}(H,T) \to \mathrm{Hom}(G,T)^I$ such that $p_0\Psi = f^T$ and $p_1\Psi = g^T$. First, we define a map $\varphi : \mathrm{Hom}(I_n, T^G) \times \{0, \frac{1}{n}, \frac{2}{n}, \ldots, 1\} \to \mathrm{Hom}(\mathbf{1}, T^G)$ via $\varphi(\alpha, \frac{i}{n})(v) = \alpha(i)$ for $v \in \mathbf{1}$, $\alpha \in \mathrm{Hom}(I_n, T^G)$, and $0 \le i \le n$. This extends to a map $\varphi : \mathrm{Hom}(I_n, T^G) \times I \to \mathrm{Hom}(\mathbf{1}, T^G)$ since the maps $\varphi_j : \mathrm{Hom}(I_n, T^G) \to \mathrm{Hom}(\mathbf{1}, T^G)$ are all homotopic for $0 \leq j \leq n$ (recall $\iota_j : \mathbf{1} \to I_n$ induces a homotopy equivalence). Let $\tilde{\varphi} : \mathrm{Hom}(I_n, T^G) \to \mathrm{Hom}(\mathbf{1}, T^G)^I$ be the adjoint map. Next, from the above proposition, we have a map $\psi : \mathrm{Hom}(\mathbf{1}, T^G) \to \mathrm{Hom}(G, T)$ that is a homotopy inverse to the inclusion $\mathrm{Hom}(\mathbf{1} \times G, T) \to \mathrm{Hom}(\mathbf{1}, T^G)$. Let $\tilde{\psi} : \mathrm{Hom}(\mathbf{1}, T^G)^I \to \mathrm{Hom}(G, T)^I$ be the induced map on the path spaces. Define $\Phi : \mathrm{Hom}(I_n, T^G) \to \mathrm{Hom}(G, T)^I$ by the composition $\Phi = \tilde{\varphi}\tilde{\psi}$. Finally, we get the desired map Ψ as the horizontal composition in the commutative diagram below.

$$\begin{tikzcd}
& & (G,T) & \\
(H,T) \arrow[r, "F^T"] & (G \times I_n, T) \arrow[ur, "\iota_{0_T}"] \arrow[dr, "\iota_{n_T}"'] \arrow[r, hook] & (I_n, T^G) \arrow[r, "\Phi"] & (G,T)^I \arrow[ul, "p_0"'] \arrow[dl, "p_1"] \\
& & (G,T) &
\end{tikzcd}$$

The implication (4) $\Rightarrow$ (5) is again clear, and so we are left with only (5) $\Rightarrow$ (1). For this, suppose $f, g : G \to H$ are not $\times$-homotopic, so that f and g are in different path components of $\mathrm{Hom}(G, H)$. We claim that the induced maps $f^H, g^H : \mathrm{Hom}(H, H) \to \mathrm{Hom}(G, H)$ are not homotopic. Suppose not, so that we have $f^H, g^H : \mathrm{Hom}(H, H) \to \mathrm{Hom}(G, H)$ are homotopic via a (topological) map $\Phi : \mathrm{Hom}(H, H) \times I \to \mathrm{Hom}(G, H)$. Here note that if $\mathrm{id} \in \mathrm{Hom}(H, H)$ is the identity map, then $f^H(\mathrm{id}) = f$ and $g^H(\mathrm{id}) = g$ since, for instance, $f^H(\mathrm{id})(x) = \mathrm{id}(f(x)) = f(x)$. Hence the restriction $\Phi|_{\{\mathrm{id}\} \times I} : \mathrm{Hom}(H, H) \times I \to \mathrm{Hom}(G, H)$ gives a path in $\mathrm{Hom}(G, H)$ from f to g, a contradiction. The result follows. $\square$

The notion of $\times$-homotopy of graph maps provides a natural candidate for the notion of $\times$-homotopy equivalence of graphs. Again, this has several equivalent formulations, which we establish next.

Theorem 5.2. *Let $f : G \to H$ be maps of graphs. Then the following are equivalent.*

(1) There exists a map $g : H \to G$ such that $f \circ g \simeq_\times \mathrm{id}_H$ and $g \circ f \simeq_\times \mathrm{id}_G$ (call g a homotopy inverse to f).

(2) For every graph T, the induced map $f_T : \mathrm{Hom}(T, G) \to \mathrm{Hom}(T, H)$ is a homotopy equivalence.

(3) For every graph T, the induced map $(f_T)_0 : \pi_0\big(\mathrm{Hom}(T, G)\big) \to \pi_0\big(\mathrm{Hom}(T, H)\big)$ is an isomorphism (bijection).

(4) For every graph T, the induced map $f_T : [T, G]_\times \to [T, H]_\times$ is a bijection.

(5) The maps $f_G : \mathrm{Hom}(G, G) \to \mathrm{Hom}(G, H)$ and $f_H : \mathrm{Hom}(H, G) \to \mathrm{Hom}(H, H)$ both induce isomorphisms on the path components.

(6) For every graph T, the induced map $f^T : \mathrm{Hom}(H,T) \to \mathrm{Hom}(G,T)$ is a homotopy equivalence.

(7) The maps $f^G : \mathrm{Hom}(H,G) \to \mathrm{Hom}(G,G)$ and $f^H : \mathrm{Hom}(H,H) \to \mathrm{Hom}(G,H)$ both induce isomorphisms on path components.

$$\begin{tikzcd}
& \mathrm{Hom}(G,H) & \\
\mathrm{Hom}(G,G) \arrow[ur, "f_G"] & & \mathrm{Hom}(H,H) \arrow[ul, "f^H"'] \\
& \mathrm{Hom}(H,G) \arrow[ul, "f^G"] \arrow[ur, "f_H"'] &
\end{tikzcd}$$

Proof. For (1) $\Rightarrow$ (2), g_T is a homotopy inverse by Theorem 5.1.

(2) $\Rightarrow$ (3) is clear.

(3) $\iff$ (4) follows from Proposition 4.7.

(3) $\Rightarrow$ (5) is clear.

For (5) $\Rightarrow$ (1), we assume $(f_H)_0 : \pi_0\big(\mathrm{Hom}(H,G)\big) \to \pi_0\big(\mathrm{Hom}(H,H)\big)$ is an isomorphism. Let ϕ be its inverse and let $(\mathrm{id}_H)_0$ denote the connected component of id_H in $\mathrm{Hom}(H,H)$. Let $g \in \phi\big((\mathrm{id}_H)_0\big)$ be a vertex of $\mathrm{Hom}(H,G)$ (i.e., a graph map). We claim that g satisfies the conditions of (1). To see this note that $\big((f_H)_0\phi\big)\big((\mathrm{id}_H)_0\big) = (\mathrm{id}_H)_0$ and since $g \in \phi\big((\mathrm{id}_H)_0\big)$ we have that $fg = f_H(g)$ is in the same component as id_H in $\mathrm{Hom}(H,H)$. Hence $fg \simeq_\times \mathrm{id}_H$, as desired. A similar consideration of the isomorphism $(f_G)_0 : \pi_0\big(\mathrm{Hom}(G,G)\big) \to \pi_0\big(\mathrm{Hom}(G,H)\big)$ shows that $gf \simeq_\times \mathrm{id}_G$.

For (1) $\Rightarrow$ (6), g^T again provides the inverse by Theorem 5.1.

(6) $\Rightarrow$ (7) is clear.

Finally, we check (7) $\Rightarrow$ (1). For this we assume $(f^G)_0 : \pi_0\big(\mathrm{Hom}(H,G)\big) \to \pi_0\big(\mathrm{Hom}(G,G)\big)$ is an isomorphism. Let ψ be the inverse and let $(\mathrm{id}_G)_0$ denote the connected component of id_G. Let $g \in \psi\big((\mathrm{id}_G)_0\big)$ be a graph map $g : H \to G$. We claim that g satisfies the conditions that we need. Note that $\big((f^G)_0\psi\big)\big((\mathrm{id}_G)_0\big) = (\mathrm{id}_G)_0$ and $f^G(g) = gf$, and hence $gf \simeq_\times \mathrm{id}_G$. Similarly we get $fg \simeq_\times \mathrm{id}_H$ and the result follows. $\square$

Definition 5.3. *A graph map $f : G \to H$ is called a $\times$-homotopy equivalence (or simply homotopy equivalence) if it satisfies any of the above conditions. Homotopy equivalence of graphs is an equivalence relation, and we let $[G]$ denote the homotopy equivalence class of G.*

Aside from certain qualitative similarities, homotopy equivalences of graphs satisfy many of the formal properties enjoyed by equivalences in any abstract homotopy theory, [12] and [19]. We close this section with a couple of observations along these lines.

Definition 5.4. *Let $\mathcal{M}$ be a class of maps in a category $\mathcal{C}$. $\mathcal{M}$ is said to satisfy the 2 out of 3 property if, for all maps f and g, whenever any two of f, g, gf are in $\mathcal{M}$, then so is the third.*

Lemma 5.5. *Homotopy equivalences of graphs satisfy the 2 out of 3 property.*

Proof. Let $f : X \to Y$ and $g : Y \to Z$ be maps of graphs, and let T be a graph. We will be considering the following diagrams.

$$\mathrm{Hom}(T,X) \underset{a_T}{\overset{f_T}{\rightleftarrows}} \mathrm{Hom}(T,Y)$$

$$\mathrm{Hom}(T,Y) \underset{b_T}{\overset{g_T}{\rightleftarrows}} \mathrm{Hom}(T,Z)$$

$$\mathrm{Hom}(T,X) \underset{c_T}{\overset{gf_T}{\rightleftarrows}} \mathrm{Hom}(T,Y)\ .$$

First suppose f and g are both homotopy equivalences, with homotopy inverse maps $a : Y \to X$ and $b : Z \to Y$ respectively. We claim ab is the homotopy inverse to fg. To see this, note that $(abgf)_T = a_T b_T g_T f_T \simeq a_T f_T \simeq \mathrm{id}_X$. Similarly, $(gfab)_T \simeq \mathrm{id}_Z$, so that gf is a homotopy equivalence.

Next suppose that f and gf are homotopy equivalences, and let $c : Z \to X$ be the homotopy inverse to gf. We claim $fc : Z \to Y$ is the homotopy inverse to g. For this we compute $(gfc)_T = g_T f_T c_T \simeq \mathrm{id}_Z$ and $(fcg)_T = f_T c_T g_T \simeq f_T c_T g_T f_T a_T \simeq f_T a_T \simeq \mathrm{id}_Y$. We conclude that g is a homotopy equivalence.

Finally, we claim that if g and gf are homotopy equivalences then $cg : Y \to X$ is the homotopy inverse to f. This follows from the fact that $(fcg)_T = f_T c_T g_T \simeq b_T g_T f_T c_T g_T \simeq b_T g_T \simeq \mathrm{id}_Y$ and also $(cgf)_T = c_T g_T f_T \simeq \mathrm{id}_Z$. ∎

Definition 5.6. *Let $g : G \to H$ be a map in a category $\mathcal{C}$. Recall that f is a retract of g if there is a commutative diagram of the following form,*

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "f"'] & G \arrow[r, "\gamma"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ Y \arrow[r, "\beta"'] & H \arrow[r, "\delta"'] & Y \end{tikzcd}$$

where the horizontal composites are identities.

Lemma 5.7. *Homotopy equivalences of graphs are closed under retracts.*

Proof. Suppose g is a homotopy equivalence. Then for every graph T we have the diagram,

$$\begin{tikzcd} \mathrm{Hom}(T,X) \arrow[r, "\alpha_T"] \arrow[d, "f_T"'] & \mathrm{Hom}(T,G) \arrow[r, "\gamma_T"] \arrow[d, "g_T"', "\simeq"] & \mathrm{Hom}(T,X) \arrow[d, "f_T"] \\ \mathrm{Hom}(T,Y) \arrow[r, "\beta_T"'] & \mathrm{Hom}(T,H) \arrow[r, "\delta_T"'] & \mathrm{Hom}(T,Y) \end{tikzcd}$$

with $g_T : \mathrm{Hom}(T,G) \to \mathrm{Hom}(T,H)$ a homotopy equivalence. We consider the induced maps on homotopy groups. Since $\gamma_T \alpha_T = \mathrm{id}$, we have that $(\alpha_T)_*$ is injective and hence so is $(f_T)_*$, since $(\beta_T)_*(f_T)_* = (g_T)_*(\alpha_T)_*$ is injective. Similarly, since $\delta_T \beta_T = \mathrm{id}$, we have that $(\delta_T)_*$ is surjective and hence so is $(f_T)_*$. We conclude that f_T induces an isomorphism on all homotopy groups and hence f_T is a homotopy equivalence on the CW-type Hom complexes. $\square$

6 Foldings, stiff graphs, and dismantlable graphs

In this section we investigate some further properties and consequences of $\times$-homotopy of graphs. The relevant operation in this context will that of a graph *folding*, which we will see is closely related to $\times$-homotopy.

Definition 6.1. *Let u and v be vertices of a graph G satisfying $N(v) \subseteq N(u)$. Then the map $f : G \to G\backslash v$ given by $f(x) = x$, $x \neq v$, and $f(v) = u$, is called a folding of G at the vertex v. Similarly, the inclusion $i : G\backslash v \to G$ is called an unfolding (see Figure 10).*

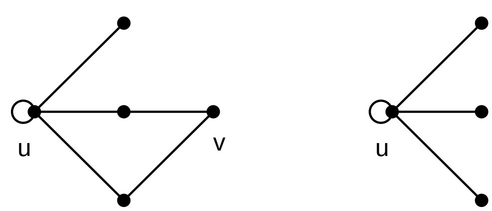

Figure 10: The graph G and the folded graph $G\backslash v$

In the original papers regarding Hom complexes (see for example [2]), it was shown that foldings in the first coordinate of the Hom complex preserved homotopy type. For some time it was an open question whether the same was true in the second coordinate of the Hom complex. Kozlov investigated this question in the papers [16] and [14], and showed that indeed this was the case.

Proposition 6.2 (Kozlov)**.** *If G and H are graphs, and u and v are vertices of G such that $N(v) \subseteq N(u)$, then the folding and unfolding maps induce inclusions of strong deformation retracts*

$$\mathrm{Hom}(G\backslash v, H) \overset{f^H}{\underset{\simeq}{\hookrightarrow}} \mathrm{Hom}(G, H), \qquad \mathrm{Hom}(H, G\backslash v) \overset{i_H}{\underset{\simeq}{\hookrightarrow}} \mathrm{Hom}(H, G).$$

In fact, Kozlov exhibits these deformation retracts as closure maps on the levels of the posets, which he shows preserve the *simple* homotopy type of the associated simplicial complex (we refer to [15] for necessary definitions). We note that although Kozlov deals only with the situation of finite H, his proof extends to the case of arbitrary H. In Sections 5 and 6 of this paper we see the further importance of folds in the context of the Hom complex.

Remark 6.3. *We can apply Theorem 5.2 to obtain the following alternate proof of one part of Proposition 6.2. As we mentioned, it was previously known that if $G \to H = G\backslash\{v\}$ is a folding, then $f^T : \mathrm{Hom}(H, T) \to \mathrm{Hom}(G, T)$ is a homotopy equivalence for all T. We can then apply (6) $\Rightarrow$ (2) in Theorem 5.2 to conclude that $f_T : \mathrm{Hom}(T, G) \to \mathrm{Hom}(T, H)$ is also a homotopy equivalence, and hence 'folds in the second coordinate' also preserve homotopy type of* Hom *complexes. Our theorem also provides some insight into the symmetry involved in the two entries of the* Hom *complex.*

6.1 Stiff graphs

If $f : G \to \tilde{G}$ is a map realized by a sequence of foldings and unfoldings, then $f_T : \mathrm{Hom}(T, G) \to \mathrm{Hom}(T, \tilde{G})$ is a homotopy equivalence for all T, and hence G and $\tilde{G}$ are homotopy equivalent. One can then consider the case when G has no more foldings available. From [11] we have the following notion.

Definition 6.4. *A graph G is called stiff if there does not exist a pair of distinct vertices $u, v \in V(G)$ such that $N(v) \subseteq N(u)$.*

Lemma 6.5. *Suppose G is a stiff graph. Then the identity map id_G is an isolated point in the realization of $\mathrm{Hom}(G, G)$.*

Proof. If not, then we have some $\alpha \in \mathrm{Hom}(G, G)$ such that $x \in \alpha(x)$ for all $x \in V(G)$, and such that $\{v, w\} \subseteq \alpha(v)$ for some $v \neq w$. Since G is stiff we have some vertex $x \in V(G)$ such that $x \in N(v)\backslash N(w)$. But then since $x \in \alpha(x)$ we need x to be adjacent to w (to satisfy the conditions of Hom), a contradiction. □

Proposition 6.6. *If G and H are both stiff graphs, then G and H are homotopy equivalent if and only if they are isomorphic.*

Proof. Sufficiency is clear. For the other direction, suppose $f : G \to H$ is a homotopy equivalence with inverse $g : H \to G$. Then gf is $\times$-homotopic to the identity id_G, so that gf and id_G are in the same component of $\mathrm{Hom}(G, G)$ by Proposition 4.7. But then $gf = \mathrm{id}_G$ since G is stiff. Similarly we get $fg = \mathrm{id}_H$, so that f is an isomorphism. □

From this it follows that if G and H are finite graphs and $f : G \to H$ is a homotopy equivalence, then one can fold both graphs to their unique (up to isomorphism) stiff subgraphs $\tilde{G}$ and $\tilde{H}$ and get an isomorphism $\tilde{G} = \tilde{H}$. However, in general one cannot make these foldings commute with the map f, as the next example illustrates.

Example 6.7. *Let G be the graph with 5 vertices $V(G) = \{1, 2, 3, 4, 5\}$ and edges $E(G) = \{11, 12, 15, 22, 23, 33, 35, 34, 44, 45\}$ (see Figure 11). Let $f : \mathbf{1} \to G$ be the map that maps $\mathbf{1} \mapsto 1$.*

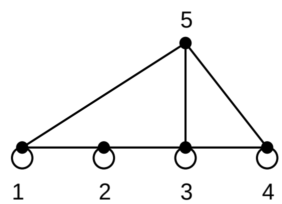

Figure 11: The graph G

We note that G is foldable to a looped vertex $\mathbf{1}$ (and hence homotopy equivalent to G), but cannot be folded to $\mathrm{im}(f)$ by a sequence of foldings and unfoldings.

Question 6.8. *Suppose G and H are (finite) graphs and $f : G \to H$ is a homotopy equivalence. Under what circumstances can f be factored as a sequence of foldings and unfoldings?*

Note that an answer to this question would yield another characterization of homotopy equivalence to the list in Theorem 5.2, under the relevant conditions on G and H.

(8) The graph map $f : G \to H$ can be factored as a sequence of foldings and unfoldings.

6.2 Dismantlable graphs

As in [11], a finite graph G is called *dismantlable* if it can be folded down to $\mathbf{1}$. Note that G is dismantlable if *any* sequence of foldings of G down to its stiff subgraph results in the looped vertex $\mathbf{1}$. Dismantlable graphs have gained some attention in the recent papers of Brightwell and Winkler (see [6] and [7]), where they are related to the uniqueness of Gibbs measure on the set of homomorphisms between two graphs. We can apply the results of Theorem 5.2 to obtain the following characterizations of dismantlable graphs.

Proposition 6.9. *Suppose G is a finite graph, and let $f : G \to \mathbf{1}$ be the unique map. Then the following are equivalent:*

(0) G is dismantlable.

(1) There exists a map $g : \mathbf{1} \to G$ such that $fg \simeq_\times \mathrm{id}_{\mathbf{1}}$ and $gf \simeq_\times \mathrm{id}_G$.

(2) For every graph T, the map $f_T : \mathrm{Hom}(T, G) \to \mathrm{Hom}(T, \mathbf{1})$ is a homotopy equivalence.

(2a) For every graph T, $\mathrm{Hom}(T,G)$ *is contractible.*

(3) For every graph T, $\mathrm{Hom}(T,G)$ *is connected.*

(4) For every graph T, the set $[T,G]_\times$ consists of a single homotopy class.

(5) G has at least one looped vertex and $\mathrm{Hom}(G,G)$ *is connected.*

(6) The map $f^G : \mathrm{Hom}(\mathbf{1},G) \to \mathrm{Hom}(G,G)$ *induces an isomorphism on path components.*

Proof. (1) $\Rightarrow$ (2) is a special case of Theorem 5.2 (with $H = \mathbf{1}$), and (2) $\Rightarrow$ (2a) since $\mathrm{Hom}(T,\mathbf{1})$ is contractible for all T.

(2a) $\Rightarrow$ (3) is clear, and the sequence of equivalences (3) $\iff$ (4) $\iff$ (1) is again a special case of Theorem 5.2.

The implication (3) $\Rightarrow$ (5) is clear. For (5) $\Rightarrow$ (1), we assume that $v \in V(G)$ is a looped vertex, and that $\mathrm{Hom}(G,G)$ is connected. Let $g : \mathbf{1} \to G$ be the graph map given by $\mathbf{1} \to v$. We claim that g satisfies the conditions of (1). First, we have $fg = \mathrm{id}_{\mathbf{1}}$. Also, since $\mathrm{Hom}(G,G)$ is path connected, we have that $gf : G \to G$ is in the same path component as the identity id_G. Hence $gf \simeq_\times \mathrm{id}_G$, and so g is the desired graph map.

Finally, (6) $\iff$ (1) is another special case of Theorem 5.2. Here note that $f^{\mathbf{1}}$: $\mathrm{Hom}(\mathbf{1},\mathbf{1}) \to \mathrm{Hom}(G,\mathbf{1})$ is always an isomorphism.

It only remains to show (0) $\iff$ (3). If G is foldable to a looped vertex then Proposition 6.2 implies that $\mathrm{Hom}(T,G) \simeq \mathrm{Hom}(T,\mathbf{1})$; the latter space is a point (and hence connected) for all T. For the other direction, we suppose $\mathrm{Hom}(T,G)$ is connected for all graphs T. The unique map $G \to \mathbf{1}$ gives a bijection $\pi_0\big((\mathrm{Hom}(T,G)\big) \to \pi_0\big(\mathrm{Hom}(T,\mathbf{1})\big)$ for all T, and hence G and $\mathbf{1}$ are homotopy equivalent. So then if G is stiff, we have that G is isomorphic to $\mathbf{1}$ by Proposition 6.6. Otherwise we perform folds to reduce the number of vertices and use induction on $|V(G)|$. $\square$

7 Other internal homs and A-theory

In this last section we investigate other notions of graph homotopy that arise under considerations of different internal hom structures. One such homotopy theory (associated to the cartesian product) recovers the *A-theory* of graphs as defined in [4].

Recall that in our construction of $\times$-homotopy, we relied on the fact that the categorical product has the looped vertex at its unit, and also possesses an internal hom (exponential) construction. This meant that graph maps from G to H were encoded by the looped vertices in the graph H^G, and two maps $f, g : G \to H$ were considered $\times$-homotopic if one could walk from f to g along a path composed of other graph maps.

Hence, in the general set-up we will be interested in monoidal category structures on the category of graphs that have the looped vertex as the unit element (this just means that we have an associative bifunctor $\otimes : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$), together with an internal hom for that structure. Recall that *having an internal hom* means that the set valued functor $T \mapsto \mathcal{G}(T \otimes G, H)$ is representable by an object of $\mathcal{G}$, which we will denote by H^G. We then have $T \mapsto \mathcal{G}(T \otimes G, H) = \mathcal{G}(T, H^G)$. Since we require the looped vertex (which we denote by $\mathbf{1}$) to be the unit we also get $\mathcal{G}(G, H) = \mathcal{G}(\mathbf{1} \otimes G, H) = \mathcal{G}(\mathbf{1}, H^G)$, so that H^G is a graph with the looped vertices as precisely the set of graph maps $G \to H$. A pair of graph maps f and g will then be considered homotopic in this context if, once again, we can find a (finite) path from f to g along looped vertices.

One such product of interest is the *cartesian product*; we recall its definition below.

Definition 7.1. *For graphs A and B, the cartesian product $A \square B$ is the graph with vertex set $V(A) \times V(B)$ and adjacency given by $(a, b) \sim (a', b')$ if either $a \sim a'$ and $b = b'$, or $a = a'$ and $b \sim b'$ (see Figure 12).*

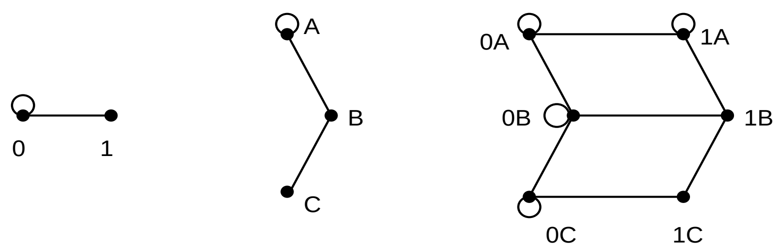

Figure 12: The graphs A, B, and $A \square B$

One can check that the cartesian product gives the category of graphs the structure of a monoidal category with a (unlooped) vertex as the unit element. We next claim that the cartesian product also has an internal hom; we first define the functor that will serve as its right adjoint.

Definition 7.2. *For graphs A and B, the cartesian exponential graph B^A is the graph with vertex set $\{f : A \to B\}$ the set of all graph maps, with adjacency given by $f \sim f'$ if $f(a) \sim f'(a)$ for all $a \in A$ (see Figure 13).*

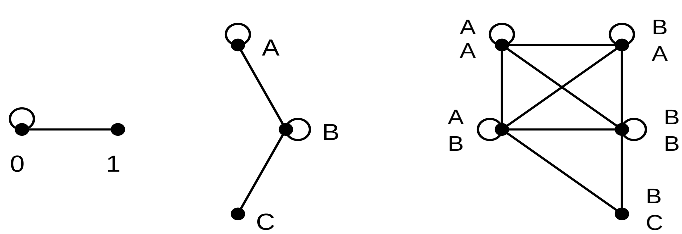

Figure 13: The graphs A, B, and B^A

Our next result shows that this exponential construction indeed provides the right adjoint for the cartesian product defined above.

Lemma 7.3. *For graphs A, B, C, there is a natural bijection $\Phi : \mathcal{G}(A \Box B, C) \to \mathcal{G}(A, C^B)$ given by the cartesian exponential graph.*

Proof. Given $f \in \mathcal{G}(A \Box B, C)$, and $a \in V(A)$, $b \in V(B)$, we define $\Phi(f)(a)(b) = f(a, b)$. We first verify that $\Phi(f)(a)$ is a graph map, so that $\Phi(f)(a) \in C^B$. For this, suppose $b \sim b'$ are adjacent vertices of B. Then we have $(a, b) \sim (a, b')$ in $A \Box B$ and hence $f(a, b) \sim f(a, b')$ as desired.

Next we verify that $\Phi(f)$ is a graph map. For this suppose $a \sim a'$ are adjacent vertices of A. Then, once again, $(a, b) \sim (a', b)$ in $A \Box B$ for all $b \in V(B)$. Hence $\Phi(f)(a)(b) = f(a, b)$ is adjacent to $\Phi(f)(a')(b) = f(a', b)$ for all $b \in V(B)$, so that $\Phi(f)(a) \sim \Phi(f)(a')$.

To see that Φ is a bijection, we construct an inverse $\Psi : \mathcal{G}(A, C^B) \to \mathcal{G}(A \times B, C)$ via $\Psi(g)(a, b) = g(a)(b)(g)$ for all $g \in \mathcal{G}(A, C^B)$. One checks that Ψ is well defined and an inverse to Φ. $\square$

Recall that a *reflexive* graph is a graph with loops on each vertex, and that a map between reflexive graphs is just a map of the underlying graphs. The cartesian product of two reflexive graphs is once again reflexive, and hence the cartesian product gives the category $\mathcal{G}^\circ$ of reflexive graphs the structure of a monoidal category with the looped vertex $\mathbf{1}$ as the unit element.

Also, if A and B are both reflexive, then all vertices of B^A are looped (so that B^A is indeed a reflexive graph). Hence we have a graph B^A whose looped vertices are precisely the graph maps $B \to A$. The map Φ described above then gives a bijection $\mathcal{G}^\circ(A \Box B, C) \simeq \mathcal{G}^\circ(A, C^B)$.

In some recent papers (see for example [1] and [4]), a homotopy theory called A-theory has been developed as a way to capture 'combinatorial holes' in simplicial complexes. The definition can be reduced to a construction in graph theory, applied to a certain graph associated to the simplicial complex in question. It turns out that A-theory of graphs fits nicely into the set-up that we have described, where the homotopy theory is associated to the cartesian product in the category of reflexive graphs. We recall the definition of A-homotopy of graph maps and A-homotopy equivalence of graphs (as in [1]).

Definition 7.4. *Let $f, g : (G, x) \to (H, y)$ be a pair of based maps of reflexive graphs. Then f and g are said to be A-homotopic, denoted $f \simeq_A g$, if there is an integer $n \geq 1$ and a graph map $\varphi : G \Box I_n \to H$ such that $\varphi(?, 0) = f$ and $\varphi(?, n) = g$, and such that $\varphi(x, i) = y$ for all i.*

We call (G, x) and (H, y) A-homotopy equivalent if there exist based maps $f : G \to H$ and $g : H \to G$ such that $gf \simeq_A \mathrm{id}_G$ and $fg \simeq_A \mathrm{id}_H$.

Using the adjunction of Lemma 7.3, we see that an A-homotopy between two based maps of reflexive graphs $f, g : G \to H$ is the same thing as a map $\tilde{\varphi} : I_n \to H^G$ with $\tilde{\varphi}(0) = f$ and $\tilde{\varphi}(n) = g$, or in other words a path from f to g along looped vertices in the based version of the (cartesian) exponential graph H^G. This places the A-theory of graphs into the general set-up described above.

In [1] the authors seek a topological space whose (ordinary) homotopy groups recover the A-theory groups of a given graph, and the analogous question in the context of $\times$-homotopy is investigated in [8].

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