Linkages in Polytope Graphs

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Abstract

A graph is k-linked if any k disjoint vertex-pairs can be joined by k disjoint paths. We improve a lower bound on the linkedness of polytopes slightly, which results in exact values for the minimal linkedness of 7-, 10- and 13-dimensional polytopes.

We analyze in detail linkedness of polytopes on at most $(6d + 7)/5$ vertices. In that case, a sharp lower bound on minimal linkedness is derived, and examples meeting this lower bound are constructed. These examples contain a class of examples due to Gallivan.

1 Introduction

In the 2004 edition of the *Handbook of Discrete and Computational Geometry* the following question by Larman and Mani [10] was stated as an open problem:

> [9, Problem 20.2.6] Let G be the graph of a d-polytope and $k = \lfloor d/2 \rfloor$. Is it true that for every two disjoint sequences $(v_1, \ldots, v_k)$ and $(w_1, \ldots, w_k)$ of vertices of G there are k vertex-disjoint paths connecting v_i to w_i, $i = 1, \ldots, k$?

However, polytopes showing that this question has a negative answer in dimensions 8, 10, and $d \geq 12$ are not hard to construct. Even when k is chosen as $\lfloor 2(d + 4)/5 \rfloor$ such paths do not necessarily exist. Indeed, such polytopes were already discovered in the 1970s by Gallivan and later published in [11] and [4].

In customary graph theory language the above question can be rephrased as: Is the graph of every d-polytope $\lfloor d/2 \rfloor$-linked? Gallivan's examples show that this is not the case. We can then ask for the largest integer $k(d)$ such that every d-polytope is at least $k(d)$-linked. This is the question of determining minimal linkedness of polytope graphs. It is far from answered: Gallivan's examples show $k(d) \leq \lfloor (2d + 3)/5 \rfloor$, while a lower bound of $\lfloor (d+1)/3 \rfloor$ was proved by Larman and Mani [10]. We improve this lower bound slightly to $\lfloor (d + 2)/3 \rfloor$ in Section 2, which implies exact values for $k(d)$ in dimensions 7, 10, and 13.

For the class of simplicial polytopes minimal linkedness can be determined. Larman and Mani [10] have shown that every simplicial polytope is at least $\lfloor (d + 1)/2 \rfloor$-linked. The stacked polytopes show that this bound cannot be improved.

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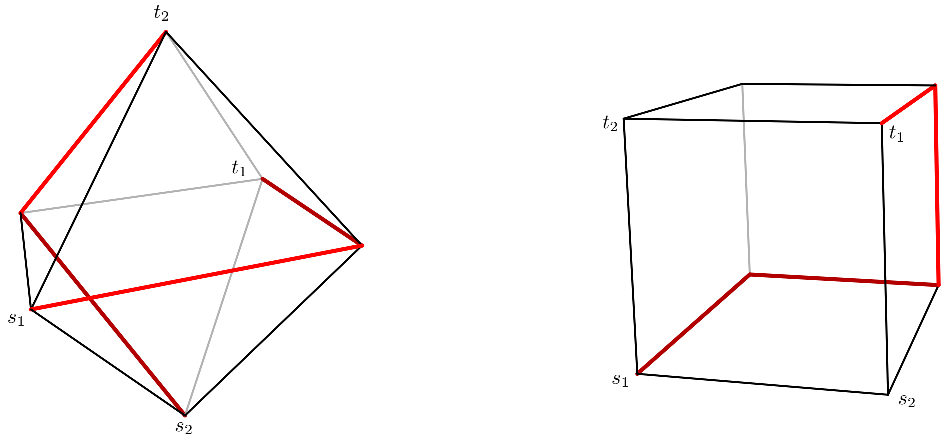

(a) Simplicial 3-polytopes are 2-linked. (b) Every path from s_1 to t_1 disconnects s_2 and t_2.

Figure 1: Simplicial polytopes and 3-dimensional polytopes.

Also, in dimensions $d \leq 5$ the values for $k(d)$ are known. While this is trivial in dimensions $d = 0, 1, 2$, in dimension 3 a polytope is 2-linked if and only if it is simplicial and otherwise 1-linked (see Figures 1(a) and 1(b)). Every 4-polytope and every 5-polytope is 2-linked—this follows from the characterization of 2-linked graphs in [13] or the results in [7]—and examples of polytopes that are not 3-linked are easy to find.

Analysis of Gallivan's examples made it apparent that minimal linkedness of d-polytopes on $d + \gamma + 1$ vertices does depend on γ, at least if γ is small. We introduce a new parameter $k(d, \gamma)$ that measures minimal linkedness of d-polytopes on $d + \gamma + 1$ vertices. We determine $k(d, \gamma)$ for polytopes on at most $(6d+7)/5$ vertices in Section 3 and analyze the combinatorial types of those polytopes with linkedness exactly $k(d, \gamma)$. Among the combinatorial types that meet the lower bound Gallivan's polytopes are in some sense the canonical ones, in some cases even unique: If $d - \gamma$ is even, there is only one combinatorial type with linkedness $k(d, \gamma)$ among all polytopes on $d + \gamma + 1$ vertices. This type is given by an iterated pyramid over a join of several quadrilaterals.

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Definitions and preliminaries

Throughout this paper we consider polytopes only up to combinatorial equivalence, that is, up to isomorphisms of their face lattices; none of the presented results depend on the geometry.

For $d \geq 0, \gamma \geq 0$ define the class

$$\mathcal{P}_d^\gamma := \{P : P \text{ is a } d\text{-polytope on } d + \gamma + 1 \text{ vertices}\}.$$

We denote the d-dimensional simplex by Δ_d, the d-dimensional crosspolytope by C_d^Δ (polar to the cube C_d), and the 2-dimensional quadrilateral by $\Box = C_2$. The interested reader will find plenty of information about polytopes in the books by Grünbaum [6] and Ziegler [14].

If P is a polytope we denote by $\dim(P)$ the dimension of P and by $f_0(P)$ the number of vertices. We define $\gamma(P) = f_0(P) - \dim(P) - 1$.

The *graph of a polytope* P is the graph $G(P)$ defined by the vertices and edges of P and their incidence relations.

Let $G = (V, E)$ be a graph and $k \in \mathbb{N}$. Then G is called *k-(vertex-)connected* if $|V| > k$ and if for each $C \subseteq V$ of cardinality $|C| < k$ the graph $G - C$ is connected. The graph G is *k-linked* if $|V| \geq 2k$, and if for every choice of $2k$ distinct vertices $s_1, \ldots, s_k, t_1, \ldots, t_k$ there exist k disjoint paths $L_1, \ldots, L_k$ such that L_i joins s_i and t_i for $i = 1, \ldots, k$. This implies that G is at least $(2k-1)$-connected. The paths $L_1, \ldots, L_k$ are called a *linkage* for $s_1, \ldots, s_k, t_1, \ldots, t_k$.

We say that a polytope P is k-linked if the graph $G(P)$ is k-linked and define the following parameters:

$$\begin{aligned} k(P) &:= \max\{k : \ P \text{ is } k\text{-linked}\} \\ k(d,\gamma) &:= \max\{k : \forall P \in \mathcal{P}_d^\gamma : k(P) \geq k\} \\ k(d) &:= \min_\gamma k(d,\gamma) \end{aligned}$$

A graph G' is a *subdivision of* G if G' is obtained from G by replacing each edge $uv \in E$ of G by a path M_{uv} with end-vertices u and v (possibly of length one). We call the set of all interior vertices of these paths the *subdividing vertices*, the other vertices the *branch vertices*. If there is a vertex $v \in V$ such that the set of branch vertices is $\{v\} \cup U$ with $U \subseteq N(v)$, we say that G' is a subdivision of G *rooted at* v.

If M and N are paths in a graph, we write MN for the union of M and N.

Many arguments in this paper crucially depend on the following theorems by Balinski and Grünbaum.

Balinski's Theorem (1961 [1])**.** Let P be a d-polytope and $G = G(P)$ be its graph. Then G is d-connected.

Grünbaum's Theorem (1965 [5] [6, Section 11.1, p. 200])**.** Let P be a d-polytope, $v \in V(P)$ a vertex of P, and $G = G(P)$ the graph of P. Then G contains a subdivision of K_{d+1} rooted at v.

The original wording of Grünbaum's theorem is different: It is not mentioned that the subdivision can be chosen rooted at a specified vertex. However, this extension is an obvious by-product of Grünbaum's proof.

Both theorems were proved by Barnette [2] for structures more general than polytopes.

2 Lower and upper bounds on minimal linkedness of polytopes

In this section we provide lower and upper bounds on $k(d)$ for general polytopes in arbitrary dimension d. We show that there is an upper bound on $k(d,\gamma)$ that is independent of γ.

2.1 A lower bound on minimal linkedness

Larman and Mani [10] have shown that every $2k$-connected graph that contains a K_{3k} subdivision is k-linked. This statement also follows from a more general result by Robertson and Seymour [12]. Together with Balinski's theorem and Grünbaum's theorem we conclude that every d-polytope is $\lfloor (d+1)/3 \rfloor$-linked. However, already in dimension 4 this bound is not tight. It is easy to see by a geometric argument and also follows from the characterization of 2-linked graphs in [13] or the results in [7] that every 4-polytope is 2-linked.

We improve Larman and Mani's bound slightly by taking a closer look at the graph structure of d-polytopes. The following argument is a variation of the proof of Larman and Mani's result given in [3, pp. 70–71].

Lemma 2.1. *Let $G = (V, E)$ be a $2k$-connected graph. Suppose that for every vertex v of G the graph G contains a subdivision of K_{3k-1} rooted at v. Then G is k-linked.*

Proof. See Figure 2 for an illustration of the proof.

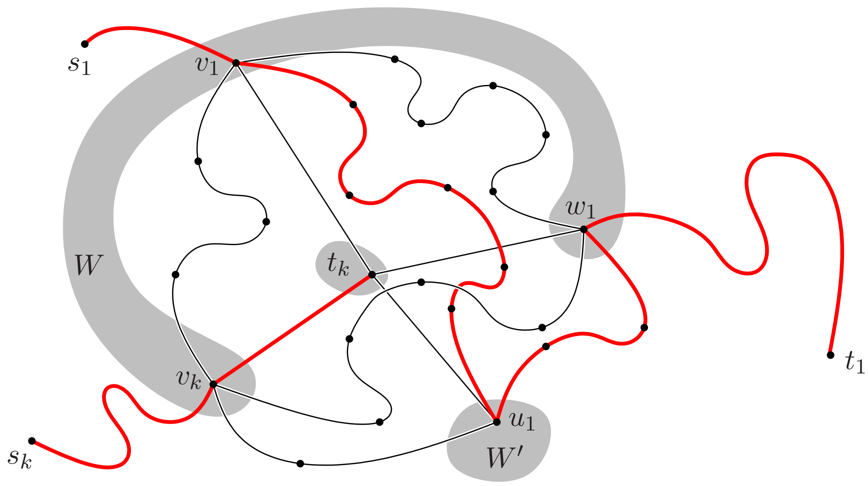


Figure 2: Illustration of the proof of Lemma 2.1 with $k = 2$.

Let $s_1, \ldots, s_k, t_1, \ldots, t_k$ be distinct vertices of G. Let K be a subdivision of K_{3k-1} rooted at vertex t_k with branch vertices $U := \{t_k\} \cup U'$, for $U' \subseteq N(t_k)$.

Since $G \setminus \{t_k\}$ is $(2k-1)$-connected there exist $2k-1$ disjoint paths $S_1, \ldots, S_k, T_1, \ldots, T_{k-1}$ in G avoiding t_k such that S_i joins s_i to U', for $i = 1, \ldots, k$, and T_i joins t_i to U', for $i = 1, \ldots, k-1$. Moreover, we assume that the paths have been chosen such that they do not have interior vertices in U' (and thus also not in U) and that their total number of edges outside of $E(K)$ is minimal.

Let $W = \{v_1, \ldots, v_k, w_1, \ldots, w_{k-1}\}$ be the vertices of these paths in U', where v_i is in S_i and w_i is in T_i. We then have a partition of U into sets $\{t_k\}$, W and $W' := U' \setminus W$ with $|W'| = k - 1$. Let $u_1, \ldots, u_{k-1}$ be the vertices in $W' \subseteq U$. We call these vertices *free*.

Since the path S_k joins s_k to a neighbor of t_k the path $L_k := S_k t_k$ joins s_k and t_k.

Now fix some $i \in \{1, \ldots, k-1\}$ and let M_i be the path in K from the free vertex u_i to v_i and N_i be the path in K from u_i to w_i. Since the paths $S_1, \ldots, S_k, T_1, \ldots, T_{k-1}$ were chosen minimal with respect to their number of edges outside of K and u_i is a free vertex, the paths S_j are disjoint from M_i for $j \neq i$, and they are disjoint from N_i for all $j = 1, \ldots, k$. Similarly, the paths T_j are disjoint from N_i for $j \neq i$, and they are disjoint from M_i for all $j = 1, \ldots, k-1$. Hence we can join v_i to w_i via the free vertex u_i.

We get pairwise disjoint paths

$$L_i = \begin{cases} S_i M_i N_i T_i & , \quad 1 \leq i \leq k-1 \\ S_k t_k & , \quad i = k \end{cases}$$

such that L_i joins s_i and t_i, that is, a linkage for the vertices $s_1, \ldots, s_k, t_1, \ldots, t_k$. □

Theorem 2.2. *Every d-polytope is $\lfloor (d+2)/3 \rfloor$-linked. Thus, $k(d,\gamma) \geq k(d) \geq \lfloor (d+2)/3 \rfloor$ for all $\gamma \geq 0$ and $d \geq 1$.*

Proof. Let P be a d-polytope and $G = G(P)$ its graph. The statement is clearly true for $d = 1$. We set $k := \lfloor (d+2)/3 \rfloor$. For $d \geq 2$ we then have $d \geq 2k$ and $d+1 \geq 3k-1$. Therefore, by Grünbaum's theorem, the graph G contains a K_{3k-1} subdivision at every vertex and, by Balinski's theorem, G is $2k$-connected. By Lemma 2.1, the graph of P is k-linked. □

2.2 An upper bound on minimal linkedness

Theorem 2.3. *Let $d \geq 2$ and $\gamma \geq 1$. The minimal linkedness of d-polytopes on $d+\gamma+1$ vertices satisfies*

$$k(d,\gamma) \leq \lfloor d/2 \rfloor .$$

Proof. For $d = 2$ the assertion is trivially true.

Let $d \geq 3$ and $\gamma \geq 1$. To prove the statement we have to construct a d-polytope on $d+\gamma+1$ vertices with $k(P) \leq \lfloor d/2 \rfloor$.

For this let Q be a 3-polytope on $4+\gamma$ vertices that has a square facet. For instance, for $\gamma = 1$ take the pyramid over a square and for $\gamma > 1$ stack this pyramid $\gamma - 1$ times over triangular facets. Let $P := \mathsf{pyr}^{d-3}(Q)$, the $(d-3)$-fold pyramid over Q. Then P is a d-polytope and has $d+\gamma+1$ vertices. Additionally, P is not $(\lfloor d/2 \rfloor + 1)$-linked. To see this let s_1, t_1, s_2, t_2 be the vertices of a square facet of Q (in that order around the facet). Then, by planarity, these cannot be linked in $G(Q)$. Additionally, with $m = \lfloor (d-3)/2 \rfloor$ there are $2m$ vertices left in $V(P) \setminus V(Q)$ if d is odd and $2m+1$ if d is even. We choose distinct vertices $s_3, \ldots, s_{m+2}, t_3, \ldots, t_{m+2}$ arbitrarily from the set $V(P) \setminus V(Q)$ and, if d is even, s_{m+3} the last vertex left in $V(P) \setminus V(Q)$ and t_{m+3} arbitrarily from $V(Q) \setminus \{s_1, s_2, t_1, t_2\}$. This set of $\lfloor d/2 \rfloor + 1$ pairs of vertices cannot be linked in P. Therefore $k(P) \leq \lfloor d/2 \rfloor$. □

In the special case $\gamma = 0$ we trivially have $k(d,\gamma) = \lfloor (d+1)/2 \rfloor$, as the d-simplex is $\lfloor (d+1)/2 \rfloor$-linked.

Theorem 2.3 implies that $k(d) \leq \lfloor d/2 \rfloor$; in the next section this bound will be significantly improved.

3 Linkages in polytopes with few vertices

We now study linkedness of polytopes having rather few vertices compared to their dimension.

If we have $\gamma \leq (d+2)/5$, we can precisely determine the value of $k(d,\gamma)$. However, most statements in this section make sense for all $\gamma \leq d$ but not for $\gamma > d$. We therefore require $\gamma \leq d$ throughout the whole section.

The theory of polytopes with few vertices is closely linked to the theory of Gale diagrams. However, we will not use Gale diagrams and prove all statements combinatorially.

We need two basic operations that create new polytopes from given ones. By $P_1 * P_2$ we denote the *join* of two polytopes P_1 and P_2. For example, the join of a polytope P with an additional vertex v, that is, with a 0-dimensional polytope, results in the pyramid $\mathsf{pyr}\, P = P * v$ over P. Similarly, $P_1 \oplus P_2$ denotes the *sum* of the polytopes P_1 and P_2. A special case is the sum of a polytope P and an interval I, which yields the bipyramid $\mathsf{bipyr}\, P = P \oplus I$ over P.

3.1 A lower bound for polytopes with few vertices

Linkedness of a graph is a local property in the following sense: If a graph is highly connected, then a k-linked subgraph ensures k-linkedness for the whole graph. This is made precise in the following lemma.

Lemma 3.1. *Let $G = (V, E)$ be a $2k$-connected graph and G' a subgraph of G that is k-linked. Then G is k-linked.*

Proof. Let $s_1, \ldots, s_k, t_1, \ldots, t_k$ be a pairing of distinct vertices in G. Since G is $2k$-connected, there exist $2k$ vertex disjoint paths $S_1, \ldots, S_k, T_1, \ldots, T_k$ such that S_i connects s_i to G' and T_i connects t_i to G'. We choose the paths such that each contains only one vertex from G'. Let $\{s'_i\} = G' \cap S_i$ and $\{t'_i\} = G' \cap T_i$. Since G' is k-linked there exists a linkage $L'_1, \ldots, L'_k$ in G' for the distinct vertices $s'_1, \ldots, s'_k, t'_1, \ldots, t'_k$ and

$$L_i = S_i L'_i T_i \quad , \qquad 1 \leq i \leq k$$

is a linkage for $s_1, \ldots, s_k, t_1, \ldots, t_k$ in G. ☐

We obtain a lower bound on linkedness of polytopes with few vertices by finding a highly-linked subgraph in the graph of P. This highly-linked subgraph is a complete subgraph: the graph of a simplex face of high dimension.

Lemma 3.2 ([8])**.** *Let P be a d-polytope on $d + \gamma + 1$ vertices. Then P has a $(d - \gamma)$-face that is a simplex.*

Proof. It is easily checked that the statement is true for every 2-polytope on $3 + \gamma$ vertices, $\gamma \geq 0$.

Let P be a d-polytope, $d \geq 3$. Choose a facet F, which is of dimension $d' = d - 1$ and has $d' + \gamma' + 1$ vertices, where $0 \leq \gamma' \leq \gamma$. By induction, F has a simplex face S of dimension $\dim(S) = d' - \gamma' = d - 1 - \gamma' = d - (\gamma' + 1)$. If $\gamma \geq \gamma' + 1$, then $\dim(S) \geq d - \gamma$ and we are done. If $\gamma = \gamma'$, then $V(P) \setminus V(F) = \{v\}$ and $P = F * v$ is a pyramid over F. Hence $S * v$ is a face of P and a simplex of dimension $\dim S + 1 = d - \gamma$. ☐

Theorem 3.3. *Let $d \geq \gamma \geq 0$. Then*

$$k(d,\gamma) \;\geq\; \left\lfloor \frac{d-\gamma+1}{2} \right\rfloor .$$

Proof. For the special cases $d = 0$, $d = 1$ as well as $\gamma = 0$ (with arbitrary d) the assertion is trivially true. For $d \geq 2$, $\gamma \geq 1$ it follows directly from Lemmas 3.1 and 3.2, since $2\lfloor (d-\gamma+1)/2 \rfloor \leq d-\gamma+1 \leq d$, and the graph of a d-polytope is at least d-connected, by Balinski's theorem. $\square$

3.2 An upper bound for polytopes with few vertices

To prove a good upper bound on the number $k(d,\gamma)$ we have to find a polytope P on $d+\gamma+1$ vertices with small $k(P)$. For $\gamma \leq (d+2)/5$ the lower bound from Theorem 3.3 can be attained.

The class of examples we describe here was first discovered in this context by Gallivan [4], who constructed it using Gale diagrams.

Definition. For integers $n, m \geq 0$ and $j_1, k_1, \ldots, j_m, k_m \geq 1$ define

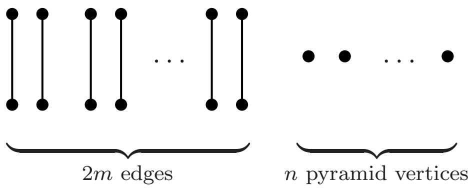
$$P\,(n, j_1, k_1, \ldots, j_m, k_m) := \Delta_{n-1} * (\Delta_{j_1} \oplus \Delta_{k_1}) * (\Delta_{j_2} \oplus \Delta_{k_2}) * \cdots * (\Delta_{j_m} \oplus \Delta_{k_m})$$

and

$$P\,(n,m) := P(n, \underbrace{1, \ldots, 1}_{2m \text{ times}}) = \Delta_{n-1} * \underbrace{\square * \cdots * \square}_{m \text{ times}}.$$

We consider the complement graph of $G(P\,(n,m))$ to examine the linkedness of the polytopes $P\,(n,m)$:

$\overline{G}(P\,(n,m))$:



Roughly speaking, the reason for the low linkedness of $P\,(n,m)$ is that there are few vertices that can be used on a “detour” for a linkage between the m pairs that are not connected by an edge.

The parameters d and γ for $P\,(n,m)$ can be determined by observing that $P\,(n,m)$ has $4m+n$ vertices, so $d+\gamma+1 = 4m+n$, and dimension $\dim(P\,(n,m)) = n-1+3m$. Therefore we have

$$d \;=\; n-1+3m, \tag{1}$$

$$\gamma \;=\; m. \tag{2}$$

Lemma 3.4. *Let $n, m \geq 0$ be integers. The linkedness of $P(n,m)$ is given by*

$$k(P(n,m)) = \begin{cases} \left\lfloor \frac{4m+n}{3} \right\rfloor, & n \leq 2m-1 \\ \left\lfloor \frac{2m+n}{2} \right\rfloor, & n \geq 2m-1. \end{cases}$$

If we use substitutions (1) *and* (2)*, this evaluates to*

$$k(P(n,m)) = \begin{cases} \left\lfloor \frac{d+\gamma+1}{3} \right\rfloor, & d \leq 5\gamma-2 \\ \left\lfloor \frac{d-\gamma+1}{2} \right\rfloor, & d \geq 5\gamma-2. \end{cases}$$

Proof. To prove the upper bound on $k(P(n,m))$ we exhibit a "worst possible" pairing of the vertices of $P(n,m)$. It is easy to see that for the given example we have to pair as many vertices defining an edge in the complement graph $\overline{G}(P(n,m))$ as possible. Those pairs will necessarily block a third vertex when they are connected by a path in $G(P(n,m))$.

If $n \leq 2m-1$, we choose $\lfloor (4m+n)/3+1 \rfloor$ edges of $\overline{G}(P(n,m))$ as pairs. This many edges exist in $\overline{G}(P(n,m))$, and to connect one of these pairs in $G(P(n,m))$ we have to use one additional vertex. However, there are only $4m+n$ vertices altogether, which is not enough to connect all $\lfloor (4m+n)/3+1 \rfloor$ pairs. This shows that $k(P(n,m)) \leq \lfloor (4m+n)/3 \rfloor$ if $n \leq 2m-1$.

To show the reverse inequality we have to find a linkage for $\lfloor (4m+n)/3 \rfloor$ pairs of vertices. Note that every pair can be connected by a path using at most one other vertex. Also, each of the $4m+n-2\lfloor (4m+n)/3 \rfloor$ vertices not in the $\lfloor (4m+n)/3 \rfloor$ pairs can be used as such a "detour vertex." As we have the inequalities

$$4m+n-2\left\lfloor \frac{4m+n}{3} \right\rfloor \geq 3\left\lfloor \frac{4m+n}{3} \right\rfloor - 2\left\lfloor \frac{4m+n}{3} \right\rfloor = \left\lfloor \frac{4m+n}{3} \right\rfloor,$$

these are enough to connect all pairs in $G(P(n,m))$ by disjoint paths.

If $n \geq 2m$, we choose as pairs all $2m$ edges in $\overline{G}(P(n,m))$ and additionally as many of the remaining $n-2m$ isolated vertices as possible. This leaves us with

$$2m + \left\lfloor \frac{n-2m}{2} \right\rfloor = \left\lfloor \frac{2m+n}{2} \right\rfloor$$

pairs that can be linked with at most one more vertex remaining.

In the case $n = 2m-1$ the construction described earlier applies, but still both formulas provide the same value for $k(P(n,m))$:

$$\left\lfloor \frac{4m+n}{3} \right\rfloor = \left\lfloor \frac{6m-1}{3} \right\rfloor = 2m-1 = \left\lfloor \frac{4m-1}{2} \right\rfloor = \left\lfloor \frac{2m+n}{2} \right\rfloor.$$

The lower bound for $n \geq 2m-1$ follows from Theorem 3.3 and Equations (1) and (2). $\square$

Example. *Let $d = 8$ and $\gamma = 2$. Then $n = 3$ and $m = 2$ and we obtain the 8-polytope*

$$P := P(3,2) = \Delta_2 * \square * \square = \mathsf{pyr}^3(\square * \square).$$

The complement of the graph of P consists of 4 disjoint edges and 3 isolated vertices. Obviously, $G(P)$ is not 4-linked.

In combination with Theorem 3.3 we obtain the following result.

Theorem 3.5. *Let $d \geq 0$ and $(d+2)/5 \geq \gamma \geq 0$. Then:*

$$k(d,\gamma) = \left\lfloor \frac{d-\gamma+1}{2} \right\rfloor .$$

Choosing $\gamma = \lfloor (d+2)/5 \rfloor$, we obtain Gallivan's examples, and the bound of the last theorem implies the following bound on $k(d)$ first given in [4].

Corollary 3.6. *For minimal linkedness of d-polytopes we have*

$$k(d) \leq \lfloor (2d+3)/5 \rfloor .$$

3.3 Analysis of polytopes meeting the lower bound

Lemma 3.7. *Let P be a d-polytope with the following property: Every facet F of P satisfies $|V(P) \setminus V(F)| \leq 2$. Then P is of the form*

$$P\left(n, j_1, k_1, \ldots, j_m, k_m\right) = \Delta_{n-1} * \left(\Delta_{j_1} \oplus \Delta_{k_1}\right) * \left(\Delta_{j_2} \oplus \Delta_{k_2}\right) * \cdots * \left(\Delta_{j_m} \oplus \Delta_{k_m}\right)$$

where $k_1 \ldots, k_m, j_1, \ldots, j_m \geq 1$ and $d = n - 1 + j_1 + k_1 + \ldots + j_m + k_m + m$.

Proof. The property $|V(P) \setminus V(F)| \leq 2$ implies that the hypergraph of facet-complements, that is, the hypergraph

$$G_{\text{cofacet}}(P) := (V(P), \{W \subseteq V(P) : V(P) \setminus W \text{ is vertex set of a facet of } P\})$$

is a graph (with no parallel edges, but possibly with loops). The edges of $G_{\text{cofacet}}(P)$ are in bijection with the facets of P. Since the combinatorial type of a polytope is determined by the vertex-facet incidences, the combinatorial type of $G_{\text{cofacet}}(P)$ determines the combinatorial type of P.

For $Q = P(n, j_1, k_1, \ldots, j_m, k_m)$ the graph $G_{\text{cofacet}}(Q)$ is a disjoint union of n copies of the graph that consists of one single vertex and one single loop, and complete bipartite graphs $K_{j_1,k_1}, \ldots, K_{j_m,k_m}$. Thus, we have to show that $G_{\text{cofacet}}(P)$ is of this type. It is easy to see that loops can only occur at isolated vertices, and that there are no vertices of degree 1 in $G_{\text{cofacet}}(P)$ (we follow the convention that loops contribute two edges to the degree count). Then it suffices to check the following two properties of $G_{\text{cofacet}}(P)$:

(i) The graph $G_{\text{cofacet}}(P)$ does not have odd cycles.

(ii) Whenever there is a path $v_1v_2v_3v_4$ of length 3 in $G_{\text{cofacet}}(P)$, then $\{v_1, v_4\}$ is also an edge of $G_{\text{cofacet}}(P)$.

In fact, Property (i) follows from Property (ii) and the non-existence of triangles, as any larger odd cycle (together with Property (ii)) implies existence of a triangle.

We now show that $G_{\text{cofacet}}(P)$ does not have triangles. Suppose there is a triangle with vertices v_1, v_2, v_3 and edges corresponding to facets F_1, F_2, F_3 with $V(F_1) = V(P) \setminus \{v_2, v_3\}$, $V(F_2) = V(P) \setminus \{v_1, v_3\}$, and $V(F_3) = V(P) \setminus \{v_1, v_2\}$. Let F' be the face $F_1 \cap F_2 = F_1 \cap F_3 =$

$F_2 \cap F_3$. Then clearly $F_1 = F' * v_1$, $F_2 = F' * v_2$, and $F_3 = F' * v_3$. Thus $\dim F' = d - 2$ and P/F' is a 1-polytope on 3-vertices, a contradiction.

Finally, we show that a path $v_1v_2v_3v_4$ of length 3 implies the existence of the edge $\{v_1, v_4\}$. Let the edges of the path $v_1v_2v_3v_4$ correspond to facets F_1, F_2, and F_3 with $V(F_1) = V(P) \setminus \{v_1, v_2\}$, $V(F_2) = V(P) \setminus \{v_2, v_3\}$, and $V(F_3) = V(P) \setminus \{v_3, v_4\}$. Let $F' = F_1 \cap F_2 \cap F_3$. Then clearly F' has dimension $d-3$. Since F_1, F_2 and F_3 are of dimension $d-1$ and each of them contains exactly two more vertices than F', we conclude that $F' * v_1$, $F' * v_2$, $F' * v_3$, and $F' * v_4$ are all faces of P. Thus, P/F' is a 2-polytope on 4 vertices, which implies that $F_4 := (F' * v_2) * v_3$ is also a facet of P with $V(F_4) = V(P) \setminus \{v_1, v_4\}$. □

Theorem 3.8. *Let P be a d-polytope on $d+\gamma+1$ vertices. Then the following are equivalent:*

(i) *Every facet F of P satisfies $|V(P) \setminus V(F)| \leq 2$.*

(ii) *P is of the form $P\,(n, j_1, k_1, \ldots, j_m, k_m)$.*

(iii) *P does not have a simplex face of dimension $d - \gamma + 1$.*

Proof. If $|V(P) \setminus V(F)| \leq 2$ for every facet F of P, then by Lemma 3.7 P is of the form $P\,(n, j_1, k_1, \ldots, j_m, k_m)$.

Now, suppose P is an iterated pyramid over a join of sums of simplices. Let S be a simplex face of P of maximal dimension. Then S is the join of Δ_{n-1} with facets from each factor $\Delta_{j_i} \oplus \Delta_{k_i}$. A facet of this sum in turn is obtained by leaving out a vertex from each of the two simplices. Hence, S has

$$n + j_1 + k_1 + \ldots + j_m + k_m \;=\; d - m + 1 \;=\; d - \gamma + 1$$

vertices and therefore dimension $d - \gamma$.

Finally, if P does not have a simplex face of dimension $d - \gamma + 1$, then $|V(P) \setminus V(F)| \leq 2$ for every facet F. Otherwise, suppose there is a facet F with $|V(P) \setminus V(F)| \geq 3$. $\gamma(F) \leq \gamma - 2$, and by Lemma 3.2 the facet F has a simplex face of dimension

$$(d-1) - \gamma(F) = d - (\gamma(F) + 1) \geq d - \gamma + 1.$$

□

Theorem 3.8 contains the classification of polytopes on $d+2$ vertices, compare [6, pp. 97–101]: No d-polytope on $d+2$ vertices contains a simplex d-face. Thus, all polytopes on $d+2$ vertices are of type $P\,(n, j_1, k_1, \ldots, j_m, k_m)$ with $m = \gamma = 1$.

Lemma 3.9. *Let P be a d-polytope on $d+\gamma+1$ vertices. Suppose that the graph $G(P)$ does not have a $K_{d-\gamma+2}$-subgraph. Then P is of the form*

$$P\,(n, m) = \Delta_{n-1} * \underbrace{\square * \cdots * \square}_{m \text{ times}},$$

with $n = d - 3\gamma + 1$ and $m = \gamma$.

Proof. Since P does not have a $K_{d-\gamma+2}$-subgraph, P does not have a simplex face of dimension $d-\gamma+1$. Thus, by Theorem 3.8, P is of the form $P(n, j_1, k_1, \ldots, j_m, k_m)$.

To show that $j_1 = k_1 = \ldots = j_m = k_m = 1$ observe that the graph

$$G(\Delta_j \oplus \Delta_k) \begin{cases} \text{is the complete graph } K_{j+k+2} & \text{if } j,k \geq 2 \\ \text{contains a } K_{j+k+1} & \text{if } j \geq 2, k = 1 \text{ or } j = 1, k \geq 2 \\ \text{is a 4-cycle} & \text{if } j = k = 1. \end{cases}$$

Furthermore, in a join $P*Q$ every vertex of P defines an edge with every vertex of Q. Suppose now that $j_i \geq 2$ or $k_i \geq 2$ for some i. Then $G(P)$ contains a complete graph on

$$n + j_1 + k_1 + \ldots + j_i + k_i + 1 + \ldots + j_m + k_m \;=\; d - m + 2 \;=\; d - \gamma + 2$$

vertices, but this contradicts the hypothesis. $\square$

Theorem 3.10. *Let P be a d-polytope on $d+\gamma+1$ vertices with $k(P) = \lfloor (d-\gamma+1)/2 \rfloor$ and $n = d - 3\gamma + 1$, $m = \gamma$.*

If $d-\gamma$ is even, then

$$P = P\,(n,m) = \Delta_{n-1} * \underbrace{\Box * \cdots * \Box}_{m \text{ times}}.$$

If $d-\gamma$ is odd, there are three possibilities:

(i) $P = P\,(n,m)$, *or*

(ii) $P = P(n-1, \underbrace{1, \ldots, 1}_{2m-1 \text{ times}}, 2)$, *or*

(iii) *P has a facet F with*

$$F = \Delta_{n-2} * \underbrace{\Box * \cdots * \Box}_{m-2 \text{ times}}.$$

In particular, $k(F) = k(P)$.

Proof. Let $d-\gamma$ be even. If $k(P) = \lfloor (d-\gamma+1)/2 \rfloor$, then P cannot have a $K_{d-\gamma+2}$-subgraph, and by Lemma 3.9 we have $P = P\,(n,m)$ with $n = d-3\gamma+1$ and $m = \gamma$.

Let $d-\gamma$ be odd. If P does not have a $K_{d-\gamma+2}$ subgraph, then again $P = P\,(n,m)$. So suppose that P does have a $K_{d-\gamma+2}$ subgraph, but not a $K_{d-\gamma+3}$-subgraph. Thus P does not have a $d-\gamma+2$ simplex face. If P also does not have a $d-\gamma+1$ simplex face, then

$$P = P(n-1, \underbrace{1, \ldots, 1}_{2m-1 \text{ times}}, 2).$$

Consider now the case that P does have a $d-\gamma+1$ simplex face but not a $d-\gamma+2$ simplex face. Then for every facet F of P we have $|V(P) \setminus V(F)| \leq 3$. By Theorem 3.8 there has to be a facet F with $|V(P) \setminus V(F)| = 3$, and this facet must satisfy $|V(F) \setminus V(F')| \leq 2$ for every facet F' of F. Otherwise there is a ridge F' of P with a simplex face of dimension at least $d-\gamma+2$. Then F is of the form $P\,(n', m')$ with $n' = d - 3\gamma$ and $m' = \gamma - 2$ since $d - 1 - \gamma(F) = d - 1 - (\gamma - 2) = d - \gamma + 1$ is even and $k(F) = k(P)$. $\square$

The last theorem has interesting consequences. It implies that for $\gamma > (d+2)/5$ polytopes meeting the lower bound of $\lfloor (d-\gamma+1)/2 \rfloor$ do not exist. Polytopes that appear in the theorem are all at least $\lfloor (d+\gamma+1)/3 \rfloor$-linked if $\gamma > (d+2)/5$. But in that case this value is strictly larger than the lower bound.

Furthermore, for $\gamma \leq (d+2)/5$ and $d-\gamma$ even, polytopes meeting the lower bound are unique. Thus, they are characterized by Theorem 3.10.

However, if $d-\gamma$ is odd, such polytopes are not characterized by the three possibilities given. While the polytopes in Possibility (i) and (ii) are $\lfloor (d-\gamma+1)/2 \rfloor$-linked, polytopes as in Possibility (iii) can be higher linked. We find different examples of type (iii) by replacing certain factors of the join in $P\,(n,m)$.

If we replace the 5-dimensional polytope $Q := \Box * \Box$ by the two-fold pyramid over the 3-dimensional crosspolytope, that is,

$$\Box * \Box \quad \rightsquigarrow \quad \Delta_1 * C_3^{\Delta},$$

we obtain a polytope P with $k(P) = k(P\,(n,m))$: The polytope $\Delta_1 * C_3^{\Delta}$ is 5-dimensional and has the same number of vertices as Q. In the complement of the graph $G(P\,(n,m))$ four isolated edges are replaced by three isolated edges and two isolated vertices. This change does not increase linkedness, as the condition that $d-\gamma$ is odd in terms of n and m translates to the condition that n is even. Hence, we have $k(P) = k(P\,(n,m))$.

Similar observations show that if we replace Q by the two-fold pyramid over a triangular prism we also obtain a polytope P with $k(P) = k(P\,(n,m))$.

However, it is possible for a polytope to have a facet as in Possibility (iii) and nevertheless to be higher linked than $P\,(n,m)$. We obtain such a polytope P for instance if we replace the factor Q in $P\,(n,m)$ by a two-fold pyramid over a twice stacked 3-simplex.

4 Conclusions and open problems

Theorem 2.2 and Corollary 3.6 imply the values for $k(d)$ as displayed in Table 1.

d	$k(d)$	d	$k(d)$	d	$k(d)$
1	1	6	2,3	11	4,5
2	1	7	3	12	4,5
3	1	8	3	13	5
4	2	9	3,4	14	5,6
5	2	10	4	15	5,6,7

Table 1: Possible values of $k(d)$.

In particular, we get exact values in dimensions 7, 10, and 13. The value $k(8) = 3$ follows from Larman and Mani's old lower bound [10] and Gallivan's upper bound [4].

The first open value is $k(6)$ and it seems to be a difficult problem to determine it. Our analysis of polytopes with few vertices (Theorem 3.10) shows that $k(6,0) = k(6,1) = k(6,2) = 3$. We have also verified enumeratively that $k(6,3) = 3$; beyond that we do not know anything.

Problem 1. *Determine $k(6)$: Either show that all 6-polytopes are 3-linked, or give an example of a 6-polytope P with $k(P) = 2$.*

One can construct polytopes with $f_0 = 3\lfloor d/2 \rfloor - 1$ vertices that are not $\lfloor d/2 \rfloor$-linked, which is the bound in the original question by Larman and Mani. If d is even let

$$P \ := \ \Delta_2 * \Box * \Box * \underbrace{C_3^\Delta * \cdots * C_3^\Delta}_{m \text{ times}} .$$

Then $d = 4m + 8$, $f_0 = 6m + 11$ and $k(P) = 2m + 3$.

For d odd let

$$P \ := \ \Delta_4 * \Box * \Box * \Box * \underbrace{C_3^\Delta * \cdots * C_3^\Delta}_{m \text{ times}} .$$

Then $d = 4m + 13$, $f_0 = 6m + 17$ and $k(P) = 2m + 5$.

Problem 2. *Are all d-polytopes on at least $3\lfloor d/2 \rfloor$ vertices $\lfloor d/2 \rfloor$-linked? Weaker: Is there some $N(d)$, such that every d-polytope on at least $N(d)$ vertices is $\lfloor d/2 \rfloor$-linked?*

Only one obstruction for d-polytopes to not be $\lfloor d/2 \rfloor$-linked is known, the obstruction exploited by Gallivan: The polytopes have many missing edges and not enough vertices to route all paths around the missing edges. If a polytope has $3\lfloor d/2 \rfloor$ or more vertices, there has to be a different obstruction if it is not $\lfloor d/2 \rfloor$-linked. Regarding Problem 2, it would be interesting to know if the graph in Figure 3, which is not 4-linked, is a subgraph of the complement graph of an 8-polytope on 12 vertices. The complement of this graph is 8-connected, and at every vertex it has a subdivision of K_9 rooted at that vertex.

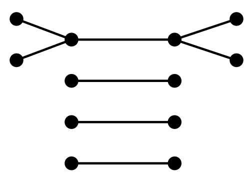


Figure 3: Is this a subgraph of the complement graph of some 8-polytope on 12 vertices?

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