# Unexpected behaviour of crossing sequences 



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#### Abstract

The $n^{\text {th }}$ crossing number of a graph $G$, denoted $c r_{n}(G)$, is the minimum number of crossings in a drawing of $G$ on an orientable surface of genus $n$. We prove that for every $a>b>0$, there exists a graph $G$ for which $c r_{0}(G)=a, c r_{1}(G)=b$, and $c r_{2}(G)=0$. This provides support for a conjecture of Archdeacon et al. and resolves a problem of Salazar.


## 1 Introduction

Planarity is ubiquitous in the world of structural graph theory, and perhaps the two most obvious generalizations of this concept - crossing number, and embeddings in more complicated surfaces - are topics which have been thoroughly researched. Despite this, relatively little work has been done on the common generalization of these two: crossing numbers of graphs drawn on surfaces. This subject seems to have been introduced in [6], and studied further in [1]. Following these authors, we define for every nonnegative integer $i$ and every graph $G$, the $i^{\text {th }}$ crossing number, $c r_{i}(G)$, (and also the $i^{\text {th }}$ nonorientable crossing number, $\left.\tilde{c r}_{i}(G)\right)$ to be the minimum number of crossings in a drawing of $G$ on the orientable (nonorientable, respectively) surface of genus $i$. We consider drawings where each vertex $x$ of $G$ is represented by a point $\phi(x)$ of the suface, each edge $u v$ by a curve with ends at points $\phi(u)$ and $\phi(v)$ and with interior avoiding all points $\phi(x)$ for $x \in V(G)$. Moreover, we assume that no three edges are drawn so that they have an interior point in common. Observe that

[^0]$c r_{i}(G)=0$ (respectively, $\tilde{c r} r_{i}(G)=0$ ) if and only if $i$ is greater or equal to the genus (resp., nonorientable genus) of $G$. This gives, for every graph $G$, two finite sequences of integers, $\left(c r_{0}(G), c r_{1}(G), \ldots, 0\right)$ and $\left(\tilde{c r_{0}}(G), \tilde{c r}_{1}(G), \ldots, 0\right)$, both of which terminate with a single zero. The first of these is the orientable crossing sequence of $G$, the second the nonorientable crossing sequence of $G$.

A natural question is to characterize crossing sequences of graphs. This is the focus of both [6] and [1]. If we are given a drawing of a graph in a surface $\mathcal{S}$ with at least one crossing, then modifying our surface in the neighborhood of this crossing by either adding a crosscap or a handle gives rise to a drawing of $G$ in a higher genus surface with one crossing less. It follows from this that every orientable and nonorientable crossing sequence is strictly decreasing until it hits 0 . This necessary condition was conjectured to be sufficient in [1].

## Conjecture 1.1 (Archdeacon, Bonnington, and Širáñ)

If $\left(a_{1}, a_{2}, \ldots, 0\right)$ is a sequence of integers which strictly decreases until 0 , then there is a graph whose crossing sequence (nonorientable crossing sequence) is ( $a_{1}, a_{2}, \ldots, 0$ ).

To date, there has been very little progress on this appealing conjecture. For the special case of sequences of the form $(a, b, 0)$, Archdeacon, Bonnington, and Širáň [1] constructed some interesting examples for both the orientable and nonorientable cases. We shall postpone discussion of their examples for the oriented case until later, but let us highlight their result for the nonorientable case here.

Theorem 1.2 (Archdeacon, Bonnington, and Širáň) If $a$ and $b$ are integers with $a>$ $b>0$, then there exists a graph $G$ with nonorientable crossing sequence $(a, b, 0)$.

It has been believed by some that such a result cannot hold for the orientable case. For the most extreme special case $(N, N-1,0)$, where $N$ is a large integer, Salazar asked [5] if this sequence could really be the crossing sequence of a graph. The following quote of Dan Archdeacon illustrates why such crossing sequences are counterintuitive:

If $G$ has crossing sequence $(N, N-1,0)$, then adding one handle enables us to get rid of no more than a single crossing, but by adding the second handle, we get rid of many. So, why would we not rather add the second handle first?

Our main theorem is an analogue of Theorem 1.2 for the orientable case, and its special case $a=N, b=N-1$ resolves Salazar's question [5].

Theorem 1.3 If $a$ and $b$ are integers with $a>b>0$, then there exists a graph $G$ whose orientable crossing sequence is $(a, b, 0)$.

Quite little is known about constructions of graphs for more general crossing sequences. Next we shall discuss the only such construction we know of. Consider a sequence $\mathbf{a}=$ $\left(a_{0}, a_{1}, \ldots, a_{g}\right)$ and define the sequence $\left(d_{1}, \ldots, d_{g}\right)$ by the rule $d_{i}=a_{i-1}-a_{i}$. If $\mathbf{a}$ is the crossing sequence of a graph, then, roughly speaking, $d_{i}$ is the number of crossings which
can be saved by adding the $i^{\text {th }}$ handle. It seems intuitively clear that sequences for which $d_{1} \geq d_{2} \geq \cdots \geq d_{g}$ should be crossing sequences, since here we receive diminishing returns for each extra handle we use. Indeed, Širáň [6] constructed a graph with crossing sequence a whenever $d_{1} \geq d_{2} \geq \cdots \geq d_{g}$.

Constructing graphs for sequences which violate the above condition is rather more difficult. For instance, it was previously open whether there exist graphs with crossing sequence $(a, b, 0)$ where $a / b$ is arbitrarily close to 1 . The most extreme examples are due to Archdeacon, Bonnington and Širáň [1] and have $a / b$ approximately equal to $6 / 5$. Although our main theorem gives us a graph with every possible crossing sequence of the form $(a, b, 0)$, we don't know what happens for longer sequences. In particular, it would be nice to resolve the following problem which asks for graphs where the first $s$ handles save only an epsilon fraction of what is saved by the $s+1^{s t}$ handle.

Problem 1.4 For every positive integer $s$ and every $\varepsilon>0$, construct a graph $G$ for which $c r_{0}(G)-c r_{s}(G) \leq \varepsilon\left(c r_{s}(G)-c r_{s+1}(G)\right)$.

For graph embeddings, the genus of a disconnected graph is the sum of the genera of its connected components. For drawing, this situation is presently unclear. If we have a graph which is a disjoint union of $G_{1}$ and $G_{2}$, then we can always "use part of the surface for $G_{1}$ and the other part for $G_{2}$ ", leading to

$$
c r_{i}\left(G_{1} \cup G_{2}\right) \leq \min _{j}\left(c r_{j}\left(G_{1}\right)+c r_{i-j}\left(G_{2}\right)\right) .
$$

To the best of our knowledge, this inequality might always be an equality. More generally we shall pose the following problem.

Problem 1.5 Let $G$ be a disjoint union of the graphs $G_{1}$ and $G_{2}$, and let $\mathcal{S}$ be a (possibly nonorientable) surface. Is there an optimal drawing of $G$ on $\mathcal{S}$, such that no edge of $G_{1}$ crosses an edge of $G_{2}$ ?

This problem is trivially true when $\mathcal{S}$ is the plane, but it also holds when $\mathcal{S}$ is the projective plane:

Proposition 1.6 Let $G$ be a disjoint union of the graphs $G_{1}$ and $G_{2}$. Then

$$
\tilde{c}_{1}(G)=\min \left\{\tilde{c}_{1}\left(G_{1}\right)+c r_{0}\left(G_{2}\right), c r_{0}\left(G_{1}\right)+\tilde{c r}_{1}\left(G_{2}\right)\right\} .
$$

In other words, there is an optimal drawing of $G$ where planar drawing of $G_{2}$ is put into one of the regions defined by the drawing of $G_{1}$; or vice versa.

Proof: To see this, consider an optimal drawing of $G$ on the projective plane, and suppose (for a contradiction) that some edge of $G_{1}$ crosses an edge of $G_{2}$. If there is a crossing involving two edges in $G_{1}$, then by creating a new vertex at this crossing point, we obtain an optimal drawing of this new graph. Continuing in this manner, we may assume that both $G_{1}$
and $G_{2}$ are individually embedded in the projective plane. For $i=1,2$, let $a_{i}$ be the length of a shortest noncontractible cycle in the dual graph of the embedding of $G_{i}$. Note that $a_{i} \geq 2$ as otherwise $G_{i}$ embeds in the plane, so $G$ embeds in the projective plane. Assume (without loss) that $a_{1} \leq a_{2}$. Now, it follows from a theorem of Lins [2] that there exists a half-integral packing of noncontractible cycles in $G_{i}$ with total weight $a_{i}$ for $i=1,2$. Since any two noncontractible curves in the projective plane meet, it follows that the total number of crossings in this drawing is at least $a_{1} a_{2}$. However, we can draw $G$ in the projective plane by embedding $G_{2}$ and then drawing $G_{1}$ in a face of this embedding with a total of $\binom{a_{1}}{2}=\frac{1}{2} a_{1}\left(a_{1}-1\right)<a_{1} a_{2}$ crossings, a contradiction.

Our primary family of graphs used in proving Theorem 1.3 can be constructed with relatively little machinery, so we shall introduce them here. We will however use a couple of gadgets which are common in the study of crossing numbers ([1, 4). Let us pause here to define them precisely. A special graph is a graph $G$ together with a distinguished subset $T \subseteq E(G)$ of thick edges, a subset $U \subseteq V(G)$ of rigid vertices and a family $\left\{\pi_{u}\right\}_{u \in U}$ of prescribed local rotations for the rigid vertices. Here, $\pi_{u}$ describes the cyclic ordering of the ends of edges incident with $u$. A drawing of a special graph $G$ in a surface $\Sigma$ is a drawing of the underlying graph $G$ with the added property that for every $u \in U$, the local rotation of the edges incident with $u$ given by this drawing either in the local clockwise or counterclockwise order matches $\pi_{u}$. The crossing number of a drawing of the special graph $G$ is $\infty$ if there is an edge in $T$ which contains a crossing, and otherwise it is the same as the crossing number of the drawing of the underlying graph. We define the crossing number of a special graph $G$ in a surface $\Sigma$ to be the minimum crossing number of a drawing of $G$ in $\Sigma$, and $c r_{i}(G)$ to be the crossing number of $G$ in a surface of genus $i$. In the next section, we shall prove the following result.

Lemma 1.7 If $G$ is a special graph with crossing sequence a consisting of real numbers, then there exists an (ordinary) simple graph with crossing sequence $\mathbf{a}$.

This result permits us to use special graphs in our constructions. Indeed, starting in the third section, we shall consider special graphs on par with ordinary ones, and we shall drop the term special. When defining a (special) graph with a diagram, we shall use the convention that thick edges are drawn thicker, and vertices which are marked with a box instead of a circle have the distinguished rotation scheme as given by the figure. With this terminology, we can now introduce our principal family of graphs.

The $n^{\text {th }}$ hamburger graph $H_{n}$ is a special graph with $3 n+8$ vertices. Its thick edges form a cycle $C=q v_{1} \ldots v_{n} r r^{\prime} s^{\prime} s u_{n} \ldots u_{1} t t^{\prime} q^{\prime} q$ of length $2 n+8$ together with two additional thick edges $\tau_{0}=q r$ and $\tau_{1}=s t$. See Figure 1. In addition to these, $H_{n}$ has $n$ special vertices $u_{i}^{\prime}$ (for odd values of $i$ ) and $v_{i}^{\prime}$ (for even values of $i$ ) with rotation as shown in the figure. These vertices are of degree 4 and they lie on paths $r_{1}=q^{\prime} v_{2}^{\prime} v_{4}^{\prime} \ldots v_{m}^{\prime} r^{\prime}$ (where $m=n$ if $n$ is even and $m=n-1$ otherwise) and $r_{2}=t^{\prime} u_{1}^{\prime} u_{3}^{\prime} \ldots u_{l}^{\prime} s^{\prime}$ (where $l=n$ if $n$ is odd and $m=n-1$ otherwise). These two paths will be referred to as the rows of $H_{n}$. Each $u_{i}^{\prime}$ and each $v_{i}^{\prime}$ is adjacent to $u_{i}$ and $v_{i}$, and the 2-path $c_{i}=u_{i} u_{i}^{\prime} v_{i}$ (or $c_{i}=u_{i} v_{i}^{\prime} v_{i}$, depending on the parity of $i$ ) is called a column of $H_{n}, i=1, \ldots, n$.


Figure 1: The graph $H_{n}$ (for $n=6$ ).

We claim that the hamburger graph $H_{n}$ has crossing sequence $(n, n-1,0)$ whenever $n \geq 5$ (or $n=3$ ). Although this does not handle all possible sequences of the form $(a, b, 0)$, as discussed above, these are in some sense the most difficult and counterintuitive cases. Indeed, a rather trivial modification of these will be used to get all possible sequences.

Since it is quite easy to sketch proofs of $c r_{0}\left(H_{n}\right)=n$ and $c r_{2}\left(H_{n}\right)=0$, let us pause to do so here (rigorous arguments will be given later). The first of these equalities follows from the observation that every row must meet every column in any planar drawing in which thick edges are crossing-free. The second equality follows from the observation that $H_{n}$ minus the thick edges $\tau_{0}, \tau_{1}$ is a graph which can be embedded in the sphere. Using an extra handle for each of $\tau_{0}, \tau_{1}$ gives an embedding of the whole graph in a surface of genus 2. Of course, it is possible to draw $H_{n}$ in the torus with only $n-1$ crossings by starting with the drawing in the figure and then adding a handle to remove one crossing. In the third section we shall show that these are indeed optimal drawings (for $n=3$ and $n \geq 5$ ).

## 2 Gadgets

The goal of this section is to establish Lemma 1.7 which permits us to use special graphs in our constructions. Similar gadgets as used in our proof have been used previously, cf., e.g., Pelsmajer et al. 4] or Archdeacon et al. [1]. We include the constructions and proofs for reader's convenience.

## Thick edges

For every $e \in E(G)$ choose positive integer $w(e)$ and replace $e$ by a copy of $L_{w(e)}$ whenever $w(e)>1$. Let $G^{\prime}$ be the resulting graph. We claim, that the crossing number of $G^{\prime}$ is the same as the "weighted crossing number" of $G$ : each crossing of edges $e_{1}, e_{2}$ is counted $w\left(e_{1}\right) w\left(e_{2}\right)$-times. Obviously, $c r\left(G^{\prime}\right)$ is at most that, as we can draw each $L_{e}$ sufficiently close to where $e$ was drawn. Moreover, there is an optimal drawing of this form (which proves the


Figure 2: Putting weights on the edges (here $w(e)=5$ ).


Figure 3: Controlling the prescribed local rotations.
converse inequality): Given an optimal embedding of $G^{\prime}$, consider the subgraph $L_{e}$ and from the $w(e)$ paths of length 2 between its "end-points" pick the one, that is crossed the least number of times. We can draw the whole subgraph $L_{e}$ close to this path without increasing the number of crossings.

This shows that we can "simulate weighted crossing number" by crossing number of a modified graph. In particular, we can let $w(e)=1$ for each ordinary edge and $w(e)>c r(G)$ for each thick edge $e$ of $G$. This proves Lemma 1.7 for graphs with thick edges.

## Rigid vertices

Suppose that we are considering drawings in surfaces of Euler genus $\leq g$; put $n=3 g+2$. Let $G$ be a special graph with rigid vertices. We replace each rigid vertex $v$ by a copy of $V_{n, \operatorname{deg}(v)}$. That is, we add $n$ nested thick cycles of length $d=\operatorname{deg}(v)$ around $v$ as shown in Figure 3 for $d=6$ and $n=5$. When doing this, the cycles meet the edges incident with $v$ in the same order as requested by the local rotation $\pi_{v}$ around $v$. If an edge incident with $v$ is thick, then all edges in $G^{\prime}$ arising from it are thick too (as indicated in the figure for one of the edges). Call the resulting graph $G^{\prime}$.

We claim that the crossing number of $G^{\prime}$ (graph with thick edges but no rigid vertices) is the same as that of $G$. Any drawing of $G$ that respects the rotations at each rigid vertex can be extended to a drawing of $G^{\prime}$ without any new crossing; in this drawing all $n$ thick cycles in each $V_{v}$ are contractible and $v$ is contained in the disc that any of them is bounding. We will show, that there is an optimal drawing of $G^{\prime}$ of this "canonical" type.

Let us consider an optimal drawing (respecting thick edges) of $G^{\prime}$ in $S$ (of genus $\leq g$ ). Let $v$ be a rigid vertex of $G$, and consider the inner $n-1$ out of the $n$ thick cycles in $V_{v}$. No edge of these cycles is crossed; so by [3, Proposition 4.2.6], either one of these cycles is contractible in $S$, or two of them are homotopic.

Suppose first, that one of the cycles, $Q$, is contractible. Since $Q$ separates the graph into two connected components, either the disk $D$ bounded by $Q$ or its exterior contains no vertex or edge of $G^{\prime}$ apart from some cycles and edges of $V_{v}$. Let us assume that this is the interior of $D$. Now delete the drawing of all thick cycles in $V_{v}$ except $Q$, and delete the drawing of all $\operatorname{deg}(v)$ paths from $Q$ to $v$. Now think of $Q$ as the outermost cycle of $V_{v}$ and draw the rest on $V_{v}$ inside $D$ without crossings.

Suppose next, that two of the cycles, $Q_{1}$ and $Q_{2}$ are homotopic (and that $Q_{1}$ is closer to $v$ in $\left.G^{\prime}\right)$. We cut $S$ along $Q_{1}$, and patch the two holes with a disc. This simplifies the surface, so if we can draw $G^{\prime}$ on it without new crossings, we get a contradiction. Such drawing of $G^{\prime}$ indeed exists, as we may delete the drawing of all of $V_{v}$ that is "inside" $Q_{1}$ and draw it in one of the new discs.

By performing such a change to each rigid vertex, we obtain an optimum drawing of $G^{\prime}$ which is canonical. Consequently, it gives rise to a legitimate drawing of the special graph $G$, and which is also optimal for $G$. This shows that Lemma 1.7 holds also when there are special vertices.

## 3 Hamburgers

The goal of this section is to prove Theorem 1.3, showing the existence of a graph with crossing sequence $(a, b, 0)$ for every $a>b>0$. The hamburger graphs $H_{n}$ (defined in the introduction) have all of the key features of interest. These are actually special graphs, but thanks to Lemma 1.7 it is enough to consider crossing sequences of special graphs. Indeed, in the remainder of the paper we will omit the term 'special'.

We have redrawn $H_{n}$ (for $n=5$ ) again in Figure 4 where we have given names to numerous subgraphs of it. We have previously defined the rows $r_{1}, r_{2}$ and columns $c_{1}, \ldots, c_{n}$. For convenience we add rows $r_{0}$ and $r_{3}$ and columns $c_{0}$ and $c_{n+1}$ (see Figure 4). The cycle $C$ (consisting of $c_{0}, r_{0}, c_{n+1}$, and $r_{3}$ ) has two trivial bridges (the thick edges $\tau_{0}$ and $\tau_{1}$ ) and two other bridges. The first, denoted by $B_{1}$, consists of the row $r_{1}$ together with all columns $c_{i}$ with $i$ even (and, of course, $1 \leq i \leq n$ ). The second one is denoted by $B_{2}$ and consists of the row $r_{2}$ and columns $c_{i}$ with $i$ odd (and, again, $1 \leq i \leq n$ ).

To get every possible crossing sequence ( $a, b, 0$ ), we will also require a slightly more general class of graphs. For every $n, k \in \mathbb{N}$ with $n \geq 3$, we define the graph $H_{n, k}$, which is obtained from $H_{n}$ by adding $k$ duplicates of the second column $c_{2}$ as shown in Figure 5 for the case of $n=4$ and $k=3$. Note that $H_{n} \cong H_{n, 0}$.

We shall denote by $\mathbb{S}_{g}(g \geq 0)$ the orientable surface of genus $g$.
Lemma $3.1 c r_{2}\left(H_{n, k}\right)=0$ for every $n, k \in \mathbb{N}$ with $n \geq 3$.

Proof: To draw $H_{n}$ in the double torus $\mathbb{S}_{2}$, start by embedding $H_{n}-\tau_{0}-\tau_{1}$ in the sphere $\mathbb{S}_{0}$. Now, use one handle to route the edge $\tau_{0}$, and another handle for $\tau_{1}$.

Lemma $3.2 c r_{0}\left(H_{n, k}\right)=n+k$ for every $n, k \in \mathbb{N}$ with $n \geq 3$.


Figure 4: Main constituents of the graph $H_{n}$ (for $n=5$ ).


Figure 5: The graph $H_{n, k}$ (for $n=4$ and $k=3$ ).

Proof: Consider a drawing of $H_{n, k}$ in the sphere. If this drawing has finite crossing number, the cycle $C$ must be embedded as a simple closed curve which separates the surface into two discs $D_{1}, D_{2}$ and is not crossed by any edge. Moreover, both thick edges $\tau_{0}$ and $\tau_{1}$ are drawn in the same disc, say $D_{2}$. Now every column of $B_{1}$ crosses the row $r_{2}$ and every column of $B_{2}$ crosses the row $r_{1}$, so we have at least $n+k$ crossings. Since $H_{n, k}$ is drawn in $\mathbb{S}_{0}$ with $n+k$ crossings in Figure [5, we conclude that $c r_{0}\left(H_{n, k}\right)=n+k$ as required.

Not surprisingly, the situation when drawing our graphs $H_{n}$ on the torus is considerably more complicated to analyze. By drawing $H_{n}$ in the plane with $n$ crossings and then using a handle to remove one crossing, we see that $c r_{1}\left(H_{n}\right) \leq n-1$ for all $n \geq 3$ (even $c r_{1}\left(H_{n, k}\right) \leq$ $n-1$ for all $n \geq 3$ and $k \geq 0$ ). For $n \geq 5$, we shall prove that this is the best which can be achieved. For $n \leq 4$, however, there is some exceptional behavior (cf. Lemma 3.7).

Lemma 3.3 For every optimal drawing of $H_{n}$ (in some surface), each column $c_{i}(1 \leq i \leq n)$ is a simple curve.

Proof: It is easy to see that in every optimal drawing, every edge is represented by a simple curve. Let us now consider a column $c_{i}=v_{i} v_{i}^{\prime} u_{i}$ (or similarly for $v_{i} u_{i}^{\prime} u_{i}$ ) and suppose that the edges $e=v_{i} v_{i}^{\prime}$ and $f=u_{i} v_{i}^{\prime}$ cross. Suppose that $e$ is represented by the simple curve $\alpha(t), 0 \leq t \leq 1$, where $\alpha(0)=v_{i}$ and $\alpha(1)=v_{i}^{\prime}$. Similarly, let $f$ be represented by the simple curve $\beta(t), 0 \leq t \leq 1$, where $\beta(0)=u_{i}$ and $\beta(1)=v_{i}^{\prime}$. Let $\alpha\left(t^{\prime}\right)=\beta\left(t^{\prime}\right)\left(0<t^{\prime}<1\right)$ be where they cross. Now let $\tilde{\alpha}(t)=\alpha(t)$ for $t \leq t^{\prime}$ and $\tilde{\alpha}(t)=\beta(t)$ for $t \geq t^{\prime}$. Change similarly $\beta$ to $\tilde{\beta}$. Then the crossing becomes a touching of the two curves, which can be eliminated yielding a drawing with fewer crossings. Observe that the local rotation at the special vertex $v_{i}^{\prime}$ changes from clockwise to anticlockwise but this is still consistent with the requirement for this special vertex. Therefore the new drawing contradicts the optimality of the original one.

At several occasions in the proof we will use the following well-known fact about closed curves on the torus.

Lemma 3.4 ([3, Proposition 4.2.6]) Let $\varphi, \psi$ be two simple closed noncontractible curves on the torus that are not freely homotopic. Then $\varphi$ and $\psi$ cross each other.

The following is well-known (cf., e.g., [7]).
Lemma 3.5 Let $\varphi, \psi$ be two closed curves on some surface; assume $\psi$ is contractible. The curves may intersect themselves and each other, but we assume that

1. the total number of intersections is finite, and
2. each point of intersection is a crossing (the curves do not touch and there are no more than two arcs that run through the point).

Then, the number of intersections of $\varphi$ with $\psi$ is even.


Figure 6: Illustration for the proof of Lemma 3.5.

Proof (hint): Let us transform $\psi$ continuously to a trivial curve. The number of intersections of $\varphi$ with $\psi$ stays the same, or changes by 2 when we modify $\psi$ as in Figure 6.

It will be convenient for us to classify different types of drawings of $H_{n}$ in the torus depending on the drawing of the thick subgraph $C+\tau_{0}+\tau_{1}$. In Figure 7 we have listed nine possible embeddings of $C+\tau_{0}+\tau_{1}$ in $\mathbb{S}_{1}$, where $\tau_{0}$ and $\tau_{1}$ are drawn with dashed lines. We shall say that a drawing of $H_{n}$ is of type $A, B, C, C^{\prime}, D, E, E^{\prime}, E^{\prime \prime}$, or $E^{\prime \prime \prime}$ if the induced drawing of $C+\tau_{0}+\tau_{1}$ is as in the corresponding part of Figure 7. Although there are other possible drawings of $C+\tau_{0}+\tau_{1}$ in the torus, our next lemma shows that the only ones which extend to finite crossing number drawings of $H_{n}$ have one of these types.

Lemma 3.6 Every drawing of $H_{n}$ for $n \geq 3$ on a torus $\mathcal{S}$ with crossing number less than $n$ has type $A, B, C, C^{\prime}, D, E, E^{\prime}, E^{\prime \prime}$, or $E^{\prime \prime \prime}$.

Proof: Let $\mathcal{S}^{\prime}$ be the bordered surface obtained from $\mathcal{S}$ by cutting along the cycle $C$. First suppose that $C$ is contractible. Then $\mathcal{S}^{\prime}$ is disconnected, with one component a disc $D$, and the other component $\mathcal{S}^{\prime \prime}$ homeomorphic to $\mathbb{S}_{1}$ minus a disc. If both $B_{1}$ and $B_{2}$ are drawn in $D$, then we have at least $n$ crossings (as in Lemma 3.2). If only one of $B_{1}$ or $B_{2}$, say $B_{1}$ is drawn in $D$, then $B_{2}$ and the edges $\tau_{0}$ and $\tau_{1}$ are drawn in $\mathcal{S}^{\prime \prime}$ (else the crossing number is infinite). Consider the curves $\tau_{0} \cup r_{0}$ and $\tau_{1} \cup r_{3}$ in $\mathcal{S}^{\prime \prime}$. If either of these is contractible, then $B_{2}$ must cross it (yielding infinite crossing number). Otherwise (using the Lemma 3.4) they must be freely homotopic noncontractible curves in $\mathcal{S}^{\prime \prime}$, so $\tau_{0} \cup c_{0} \cup \tau_{1} \cup c_{n+1}$ is a contractible curve. Therefore $B_{2}$ must cross it, yielding again infinitely many crossings. Thus, we may assume that both $\tau_{0}$ and $\tau_{1}$ are drawn in the disc $D$ and $B_{1}$ and $B_{2}$ are drawn in $\mathcal{S}^{\prime \prime}$ so our drawing is of type $A$.

Next suppose that $C$ is not contractible. In this case, the surface $\mathcal{S}^{\prime}$ is a cylinder bounded by two copies of the cycle $C$. If both $\tau_{0}$ and $\tau_{1}$ have all of their ends on the same copy of $C$, we must have a drawing of type $B, C$, or $C^{\prime}$. If one has both ends on one copy of $C$, and the other has both ends on the other copy of $C$, then there are infinitely many crossings, unless the drawing is of type $D$. Finally, if one of $\tau_{0}$, $\tau_{1}$, has its ends on distinct copies of $C$, then the crossing number will be infinite unless the other one of $\tau_{0}, \tau_{1}$, has both ends on the same copy of $C$ giving us a drawing of type $E, E^{\prime}, E^{\prime \prime}$, or $E^{\prime \prime \prime}$.

If $G$ is a graph drawn on a surface and $A, B \subseteq G$, then we shall denote by $C r(A \mid B)$ the total number of crossings of an edge from $A$ with an edge from $B$, where crossings of an


Figure 7: Nine special types of embedding of the thick subgraph $C+\tau_{0}+\tau_{1}$ in the torus. In types $B-E^{\prime \prime \prime}$, the cycle $C$ is drawn on the top and bottom sides of the square.


Figure 8: Exceptional drawings of $H_{3}$.


Figure 9: Exceptional type $B$ drawing of $H_{4}$.
edge $e \in E(A \cap B)$ with another edge $f \in E(A \cap B)$ are counted only once. In particular, the total number of crossings of graph $G$ is equal to $\operatorname{Cr}(G \mid G)$.

Lemma $3.7 c r_{1}\left(H_{n}\right)=n-1$ if $n=3$ or $n \geq 5$, while $c r_{1}\left(H_{4}\right)=2$. Furthermore, Figure $8(a)-\left(c\right.$ ') shows the only drawings of $H_{3}$ in the torus with two crossings and the added property that $\operatorname{Cr}\left(r_{2} \mid G\right)=0$. Figure 9 displays the unique drawing of $H_{4}$ in the torus with two crossings.

Proof: We proceed by induction on $n$. Consider a drawing $\mathcal{D}$ of $H_{n}$ in a surface $\mathcal{S}$ homeomorphic to the torus, such that $\mathcal{D}$ yields minimum crossing number. We shall frequently use the inductive assumption for $n-1$ and $n-2$, since by deleting the edges of the column $c_{1}$, the column $c_{n}$, or two consecutive columns $c_{i}$ and $c_{i+1}$ we obtain a new graph which is a subdivision of $H_{n-1}$ or $H_{n-2}$ (assuming $n \geq 3$ ). This technique will be used throughout the
proof. It is also worth noting that after applying this operation to $\mathcal{D}$, the drawing of the smaller hamburger graph is of the same type as the drawing $\mathcal{D}$.

The cycle $C$ is not crossed in $\mathcal{D}$, so we may cut our surface along this curve. This leaves us with a drawing of $H_{n}$ in a closed bordered surface - which we shall denote $\mathcal{S}^{\prime}$-where each edge of $C$ appears twice on the boundary. We shall use $C^{1}$ and $C^{2}$ to denote these copies.

Essential to our proof is an analysis of the homotopy behavior of the rows and columns. To make this precise, let us now choose a point $N$ in the interior of the row $r_{0}, S$ in the interior of $r_{3}, W$ in the interior of $c_{0}$ and $E$ in the interior of $c_{n+1}$. (Actually, for each of these points we have two copies: $N^{1}$ and $N^{2}$, etc. But we will avoid distinguishing these if there is no danger of confusion). For each column $c_{i}(0 \leq i \leq n+1)$ let $c_{i}^{+}$be a simple curve in $\mathcal{S}^{\prime}$ obtained by extending $c_{i}$ along the appropriate copies of the rows $r_{0}$ and $r_{3}$ so that it has ends $N$ and $S$. Similarly, for each row $r_{i}(0 \leq i \leq 3)$ let $r_{i}^{+}$be a curve in $\mathcal{S}^{\prime}$ obtained by extending $r_{i}$ along the appropriate copies of the columns $c_{0}$ and $c_{n+1}$ so that it has ends $E$ and $W$. We shall focus our attention on the homotopy types in $\mathcal{S}^{\prime}$ of the curves $c_{i}^{+}$where $N$ and $S$ are the fixed end points (and similarly $r_{i}^{+}$where $E$ and $W$ are fixed): we say that $c_{i}^{+}$and $c_{j}^{+}$are homotopic if $c_{i}^{+}$may be continuously deformed to $c_{j}^{+}$in the surface $\mathcal{S}^{\prime}$, while keeping their endpoints fixed. Note that $c_{i}^{+}$and $c_{j}^{+}$can only be homotopic if $c_{i}$ and $c_{j}$ are connecting the same copies of $N$ and $S$-that is they attach on the same side of $C$ in the original surface $\mathcal{S}$. Also note, that for $i=0$ or $i=n+1$ we actually have two copies of $c_{i}$, so we should be speaking of, e.g., $c_{0}^{+1}$ and $c_{0}^{+2}$. We will refrain from this distinction whenever possible to keep the notation clearer - so when saying $c_{0}^{+}$and $c_{1}^{+}$are homotopic we will actually mean that $c_{1}^{+}$is homotopic to $c_{0}^{+s}$ for some $s \in\{1,2\}$.

We will use frequently the following fact that connects the homotopy types of columns and their crossing behaviour with respect to the rows (and vice versa). We will refer to this statement as to "the Claim".

Claim: If $c_{i}^{+}$and $c_{i+1}^{+}$are homotopic $(1 \leq i<n)$, then $\operatorname{Cr}\left(r_{j} \mid c_{i} \cup c_{i+1}\right) \geq 1$ for $j=1,2$. Similarly, if $r_{1}^{+}$and $r_{2}^{+}$are homotopic, then $C r\left(r_{1} \cup r_{2} \mid c_{i}\right) \geq 1$ for every $1 \leq i \leq n$.

To see this, let us observe that the closed curve obtained by following $c_{i}^{+}$from $S$ to $N$ and then $c_{i+1}^{+}$from $N$ to $S$ is contractible, after deleting part of its intersection with the cycle $C$, we get a contractible curve $\psi$ that intersects itself only at finitely many points. The row $r_{j}$ must cross either $c_{i}^{+}$or $c_{i+1}^{+}$(depending on the parity) in their common vertex (it cannot only touch it as their common vertex has prescribed local rotation). We may extend $r_{j}^{+}$into a closed curve $\varphi$ by following closely along the cycle $C$. This way we are adding two (or zero) intersections with $\psi$. By Lemma 3.5 curves $\varphi$ and $\psi$ have an even number of intersection, thus $r_{j}$ must have another crossing with $\psi$ and we are done. The same argument holds when the rows and columns exchange their roles.

Corollary: If $r_{1}^{+}$and $r_{2}^{+}$are homotopic, we are done, as there are at least $n$ intersections.

In light of Lemma 3.6 we may assume that our drawing is of type $A, B, C, C^{\prime}, D, E, E^{\prime}$,
$E^{\prime \prime}$, or $E^{\prime \prime \prime}$, and we now split our argument into these nine cases.
Case 1: Type $A$.
Let us first suppose that $n \geq 4$. If there exists $1 \leq i \leq n$ so that $c_{i}^{+}$is homotopic to $c_{0}^{+}$, then either $c_{1}$ crosses $c_{i}$, or $c_{1}^{+}$is homotopic to $c_{0}^{+}$. In the latter case, $c_{1}$ crosses $r_{1}$. So, in short, $\operatorname{Cr}\left(c_{1} \mid H_{n}\right) \geq 1$ and by removing this column and applying induction, we deduce that there are at least $n-1$ crossings in our drawing. Note here that the resulting drawing of $H_{n-1}$ is still of type $A$, so it must have at least $(n-1)-1$ crossings, even if $n=5$. Thus, we may assume that $c_{i}^{+}$is not homotopic to $c_{0}^{+}$for any $1 \leq i \leq n$. By a similar argument, $c_{i}^{+}$is not homotopic to $c_{n+1}^{+}$. If there exist $i, j \in\{1, \ldots, n\}$ with $c_{i}^{+}$not homotopic to $c_{j}^{+}$, then $c_{i}^{+}$and $c_{j}^{+}$cross (Lemma (3.4), and further, $\operatorname{Cr}\left(c_{k} \mid c_{i} \cup c_{j}\right) \geq 1$ for every $k \in\{1, \ldots, n\}$ with $k \neq i, j$. This implies that we have at least $n-1$ crossings, as desired. The only other possibility is that $c_{i}^{+}$and $c_{j}^{+}$are homotopic for every $i, j \in\{1, \ldots, n\}$. In this case, it follows from the Claim (applied to $c_{1}^{+}$and $c_{2}^{+}, c_{3}^{+}$and $c_{4}^{+}, \ldots$ ) that there are at least $n-1$ crossings.

Suppose now that $n=3$. If $c_{2}^{+}$is homotopic to $c_{1}^{+}$or $c_{3}^{+}$, then it follows from the Claim that each row has at least one crossing, and we are done. Thus, we may assume that $c_{2}^{+}$has distinct homotopy type from that of $c_{1}^{+}$and from that of $c_{3}^{+}$. If $c_{2}^{+}$is homotopic to $c_{0}^{+}$, then $\operatorname{Cr}\left(c_{2} \mid r_{2}\right) \geq 1$ and $\operatorname{Cr}\left(c_{2} \mid c_{1}\right) \geq 2$ (since $c_{1}^{+}$is not homotopic to $c_{2}^{+}$) giving us too many crossings. Thus, $c_{2}^{+}$is not homotopic to $c_{0}^{+}$, and by a similar argument, we find that $c_{2}^{+}$is not homotopic to $c_{4}^{+}$. Now, either $c_{1}^{+}$is homotopic to $c_{0}^{+}$(in which case $\operatorname{Cr}\left(c_{1} \mid r_{1}\right) \geq 1$ ) or $c_{1}^{+}$is not homotopic to $c_{0}^{+}$(in which case $\operatorname{Cr}\left(c_{1} \mid c_{2}\right) \geq 1$ ). So, in short $\operatorname{Cr}\left(c_{1} \mid r_{1} \cup c_{2}\right) \geq 1$. By a similar argument, $\operatorname{Cr}\left(c_{3} \mid r_{1} \cup c_{2}\right) \geq 1$. Since there are at most two crossings, we must have $\operatorname{Cr}\left(c_{1} \cup c_{3} \mid r_{1} \cup c_{2}\right)=2$ and this accounts for all of our crossings. In particular, this implies that $r_{1}$ and $r_{2}$ are simple curves. Since $\operatorname{Cr}\left(r_{2} \mid G\right)=0$, it follows that $r_{2}^{+}$is not homotopic to $r_{0}^{+}$or $r_{3}^{+}$. By the Claim, $r_{1}^{+}$is not homotopic to $r_{2}^{+}$, and this together with $\operatorname{Cr}\left(r_{1} \mid r_{2}\right)=0$ implies that $r_{1}^{+}$is homotopic to $r_{0}^{+}$. It follows from this that $\operatorname{Cr}\left(r_{1} \mid c_{i}\right)=1$ for $i=1,3$ and this accounts for all of the crossings. Such a drawing is possible, but must be equivalent with that in Figure 8(a).

In all the remaining cases, we have that $\mathcal{S}^{\prime}$ is a cylinder, and in our figures we have drawn $\mathcal{S}^{\prime}$ with the boundary component $C^{1}$ on the top and $C^{2}$ on the bottom.

Case 2: Type $B$.
Here all of the column curves $c_{i}^{+}$have ends $N^{2}$ and $S^{2}$. Recall that these are copies of $N$ and $S$ drawn at the "bottom copy" $C^{2}$ of $C$. Since all of these curves are simple, it follows that for every $1 \leq i \leq n$, the curve $c_{i}^{+}$is either homotopic to the simple curve $N^{2}-W^{2}-S^{2}$ in $C^{2}$ (we shall call this homotopy type $\ell$ ), or to the simple curve $N^{2}-E^{2}-S^{2}$ in $C^{2}$ (homotopy type $r$ ). Let $\mathbf{a}=a_{1} a_{2} \ldots a_{n}$ be the word given by the rule that $a_{i}$ is the homotopy type of $c_{i}^{+}$. We now have the following simple crossing property.

P1. If $a_{i}=r$ and $a_{j}=\ell$ where $1 \leq i<j \leq n$, then $\operatorname{Cr}\left(c_{i} \mid c_{j}\right) \geq 2$.
If there exists an $i(1 \leq i \leq n)$ so that $C r\left(c_{i} \mid H_{n}\right) \geq 4$, then $n \geq 5$ (otherwise the drawing is not optimal), and by removing $c_{i}$ and either $c_{i-1}$ or $c_{i+1}$ and applying the theorem
inductively to the resulting graph, we deduce that there are at least $4+c r_{1}\left(H_{n-2}\right) \geq n$ crossings in our drawing, a contradiction. It follows from this and P1, that either $\mathbf{a}=\ell^{i} r^{n-i}$ or $\mathbf{a}=\ell^{i} r \ell r^{n-i-2}$. We now split into subcases depending on $n$.

Suppose first that $n=3$. If $a_{1}=a_{2}=\ell$ or $a_{2}=a_{3}=r$, then it follows from the Claim that $\operatorname{Cr}\left(r_{j} \mid c_{1} \cup c_{2} \cup c_{3}\right) \geq 1$ for $j=1,2$ and we are finished. Otherwise, a must be $\ell r \ell$ or $r \ell r$ and $\operatorname{Cr}\left(c_{2} \mid c_{1} \cup c_{3}\right) \geq 2$. These configurations are possible, but require that our drawing is equivalent with the one in Figure $8(b)$-this comes from $\mathbf{a}=\ell r \ell$, if $\mathbf{a}=r \ell r$ we get a mirror image.

Next we consider the case when $n=4$ and $\mathbf{a}=\ell^{i} r^{4-i}$. Applying the Claim for the columns $c_{1}, c_{2}$ and $c_{3}, c_{4}$ resolves the cases when a is one of $\ell^{4}, r^{4}$, or $\ell^{2} r^{2}$ (each gives at least four crossings-a contradiction). Suppose that $\mathbf{a}=\ell^{3} r$ (or, with the same argument, $\left.\mathbf{a}=\ell r^{3}\right)$. It follows from the Claim that $C r\left(c_{1} \cup c_{2} \mid r_{1} \cup r_{2}\right) \geq 2$ and $C r\left(c_{2} \cup c_{3} \mid r_{1} \cup r_{2}\right) \geq 2$, so the only possibility for fewer than three crossings is that our drawing has 2 crossings, both of which are between $c_{2}$ and the rows $r_{1}$ and $r_{2}$. But then $c_{2}$ does not cross $c_{1}$ or $c_{3}$, so $c_{2}$ is separated from $c_{0}$ by $c_{1}^{+} \cup c_{3}^{+}$, so $\operatorname{Cr}\left(r_{1} \mid c_{1} \cup c_{3}\right)>0$, a contradiction.

Next suppose that $n=4$ and $\mathbf{a}=\ell^{i} r \ell r^{2-i}$. If $\mathbf{a}=\ell^{2} r \ell$, then it follows from P1 that $\operatorname{Cr}\left(c_{3} \mid c_{4}\right) \geq 2$ and from the Claim that $\operatorname{Cr}\left(c_{1} \cup c_{2} \mid r_{1} \cup r_{2}\right) \geq 2$, so we have at least four crossings-a contradiction. Similarly $\mathbf{a}=r \ell r^{2}$ is impossible. The only remaining possibility is $\mathbf{a}=\ell r \ell r$. In this case, we have $C r\left(c_{2} \mid c_{3}\right) \geq 2$, so the only possibility is that there are exactly two crossings, both between $c_{2}$ and $c_{3}$. This case can be realized, but requires that our drawing is equivalent to that of Figure 9 ,

Lastly, suppose that $n \geq 5$. Since $\mathbf{a} \in\left\{\ell^{i} r^{n-i}, \ell^{i} r \ell r^{n-i-2}\right\}$, either $a_{1}=a_{2}=\ell$ or $a_{n-1}=a_{n}=r$. As these arguments are similar, we shall consider only the former case. Now, it follows from the Claim that $C r\left(c_{1} \cup c_{2} \mid r_{1} \cup r_{2}\right) \geq 2$, so removing the first two columns gives us a drawing of $H_{n-2}$ with at least two crossings less than in our present drawing of $H_{n}$. By applying our theorem inductively to this new drawing, we find that the only possibility for less than $n-1$ crossings is that $n=6$ and $\mathbf{a}=\ell^{3} r \ell r$. In this case, we have $\operatorname{Cr}\left(c_{4} \mid c_{5}\right) \geq 2$, so we may eliminate two crossings by removing columns 4 and 5 . This leaves us with a drawing of a graph isomorphic to $H_{4}$ as above with the pattern $\ell^{3} r$. It follows from our earlier analysis, that this drawing has at least three crossings. This completes the proof of this case.

## Case 3: Type C.

Now each column curve has one end on the segment of $C^{2}$ between $q^{2}$ and $r^{2}$. As above, every curve $c_{i}^{+}$with both ends on $C^{2}$ must be homotopic with either the simple curve $N^{2}-$ $W^{2}-S^{2}$ in $C^{2}$ (denoted by $\ell$ ), or with the simple curve $N^{2}-E^{2}-S^{2}$ in $C^{2}$ (homotopy type $r$ ). Each row has both its ends on $C^{2}$.

The homotopy types of the other column curves will be represented by integers. Since $\mathcal{S}^{\prime}$ is a cylinder, we may choose a continuous deformation $\Psi$ of $\mathcal{S}^{\prime}$ onto the circle $\mathbb{S}^{1}$ with the property that $C^{1}$ and $C^{2}$ map bijectively to $\mathbb{S}^{1}$, and $N^{2}$ and $S^{1}$ map to the same point $x \in \mathbb{S}^{1}$. Now, each curve $c_{i}^{+}$maps to a closed curve in $\mathbb{S}^{1}$ from $x$ to $x$, and for an integer $\alpha \in \mathbb{Z}$, we say that $c_{i}^{+}$has homotopy type $\alpha$ if the corresponding curve in $\mathbb{S}^{1}$ has (counterclockwise) winding number $\alpha$. It follows that $c_{i}^{+}$and $c_{j}^{+}$are homotopic if and only if they have the same
homotopy type. As before, we let $\mathbf{a}=a_{1} a_{2} \ldots a_{n}$ be the word given by the rule that $a_{i}$ is the homotopy type of $c_{i}^{+}$. We now have the following crossing properties (for the appropriate choice of "clockwise" direction), whenever $1 \leq i<j \leq n$ :

P1. $C r\left(c_{i} \mid c_{j}\right) \geq\left|a_{i}-a_{j}-1\right|$ if $a_{i}, a_{j} \in \mathbb{Z}$.
P2. $C r\left(c_{i} \mid c_{j}\right) \geq 2$ if $a_{i}=r$ and $a_{j}=\ell$.
P3. $C r\left(c_{i} \mid c_{j}\right) \geq 1$ if either $a_{i}=r$ and $a_{j} \in \mathbb{Z}$ or $a_{i} \in \mathbb{Z}$ and $a_{j}=\ell$.
By choosing $\Psi$ appropriately, we may further assume that the smallest integer $1 \leq i \leq n$ for which $a_{i} \in \mathbb{Z}$ (if such $i$ exists) satisfies $a_{i}=0$. Again, we split into subcases depending on $n$.

Suppose first that $n=3$. Note that every column of type $r$ or $\ell$ separates the segment $q^{2} t^{2}$ on $C^{2}$ from $r^{2} s^{2}$. Consequently, $C r\left(r_{1} \cup r_{2} \mid c_{i}\right) \geq 1$ whenever $a_{i} \in\{\ell, r\}$. Next we shall consider the homotopy types of our rows. If $r_{1}^{+}$is not homotopic to $r_{0}^{+}$or $r_{3}^{+}$, then $C r\left(r_{1} \mid r_{1}\right) \geq 1$ and further $C r\left(r_{1} \mid c_{1} \cup c_{3}\right) \geq 2$ (as in this case, $r_{1}$ separates $C^{2}$ from $C^{1}$ and also segment $q^{2} r^{2}$ from $s^{2} t^{2}$ ) which gives us too many crossings. If $r_{2}^{+}$is not homotopic to $r_{0}^{+}$or $r_{3}^{+}$, then $\operatorname{Cr}\left(r_{2} \mid r_{2}\right) \geq 1$ and $\operatorname{Cr}\left(r_{2} \mid c_{2}\right) \geq 1$, and we have nothing left to prove. Thus, we may assume that $r_{1}^{+}$(and also $r_{2}^{+}$) is homotopic to one of $r_{0}^{+}, r_{3}^{+}$. If $r_{1}^{+}$and $r_{2}^{+}$are homotopic, then the Claim implies that there are at least three crossings. Hence, we may assume that $r_{1}^{+}$is homotopic to $r_{0}^{+}$and $r_{2}^{+}$to $r_{3}^{+}$(the other possibility yields two crossings and each row crossed). It now follows from our assumptions that $\operatorname{Cr}\left(r_{1} \mid c_{i}\right) \geq 1$ for $i=1,3$, so assuming we have at most two crossings, our only crossings are between $r_{1}$ and $c_{1}$ and between $r_{1}$ and $c_{3}$. If $a_{i} \in \mathbb{Z}$ for $i \in\{1,3\}$, then $c_{i}$ also crosses $r_{2}$ because of the requirements concerning local rotations at the special vertices $u_{1}^{\prime}$ and $u_{3}^{\prime}$. It follows that there are at least three crossings unless $\mathbf{a}=\ell 0 \ell, \ell 0 r, r 0 \ell$, or $r 0 r$. Each of these, except $\ell 0 r$ gives at least three crossings by (P3). The remaining case is possible, but only as it appears in Figure 8(c).

Suppose now that $n \geq 4$. If either $c_{1}$ or $c_{n}$ is crossed, then we delete it and use the induction hypothesis. If neither has a crossing, then both $a_{1}$ and $a_{n}$ are integers (otherwise $\operatorname{Cr}\left(c_{1} \cup c_{n} \mid r_{1} \cup r_{2}\right) \geq 1$ as above). It follows that $a_{1}=0$, and $a_{n}=-1$ (otherwise $c_{1}$ and $c_{n}$ cross). Now there is no value for $a_{2}$ to avoid crossing with either $c_{1}$ or $c_{n}$. Hence one of $c_{1}$, and $c_{n}$ is crossed, after all, and we may use induction. This completes the proof of Case 3 .

Case 4: Type $C^{\prime}$.
This case is nearly identical to the previous one. We may define the homotopy types for the columns to be $r$, $\ell$, or an integer, exactly as before, so that the same homotopy properties are satisfied. Then the analysis for $n \geq 4$ is identical, and the only difference is the case when $n=3$. As before, if $r_{1}^{+}$is not homotopic to $r_{0}^{+}$or $r_{3}^{+}$, then $\operatorname{Cr}\left(r_{1} \mid r_{1}\right) \geq 1$ and $\operatorname{Cr}\left(r_{1} \mid c_{1} \cup c_{3}\right) \geq 2$ giving us too many crossings. Similarly, if $r_{2}^{+}$is not homotopic to $r_{0}^{+}$or $r_{3}^{+}$, then $\operatorname{Cr}\left(r_{2} \mid r_{2}\right) \geq 1$ and $\operatorname{Cr}\left(r_{2} \mid c_{2}\right) \geq 1$ and there is nothing left to prove. Now, using the Claim, we deduce that $r_{1}^{+}$is homotopic to $r_{0}^{+}$and $r_{2}^{+}$is homotopic to $r_{3}^{+}$. It follows from this that $\operatorname{Cr}\left(c_{2} \mid r_{2}\right) \geq 1$. If $a_{2} \in \mathbb{Z}$ then, as the vertex $v_{2}^{\prime}$ is rigid, it follows that $\operatorname{Cr}\left(c_{2} \mid r_{1}\right) \geq 1$ and we have nothing left to prove. Thus, we may assume that $a_{2} \in\{\ell, r\}$. If


Figure 10: Part of a type $D$ drawing of $H_{3}$.
$a_{i} \in\{\ell, r\}$ for $i=1$ or $i=3$, then $c_{i}$ crosses $r_{1}$ and we are done. Thus, we may assume that $a_{1}, a_{3} \in \mathbb{Z}$. It now follows that $\operatorname{Cr}\left(c_{2} \mid c_{1} \cup c_{3}\right) \geq 1$. This can be realized with exactly two crossings, but row $r_{2}$ must be crossed.

## Case 5: Type $D$.

In this case, every column has one end on $r_{0}^{2}$ and one end on $r_{3}^{1}$. We define the homotopy types of curves $c_{i}^{+}$using integers as in the previous case. Again, $c_{i}^{+}$and $c_{j}^{+}$are homotopic if and only if they have the same homotopy type. As before, we let $\mathbf{a}=a_{1} a_{2} \ldots a_{n}$ be the word given by the rule that $a_{i}$ is the homotopy type of $c_{i}^{+}$. And as before, we have the following useful crossing property:

$$
\text { P1. } C r\left(c_{i} \mid c_{j}\right) \geq\left|a_{i}-a_{j}-1\right| \text { if } 1 \leq i<j \leq n \text {. }
$$

Suppose first that $n \geq 4$. If the first column $c_{1}$ does not cross any other columns, then $\mathbf{a}=0(-1)^{n-1}$. Similarly, if the last column does not cross any other columns, then $\mathbf{a}=0^{n-1}(-1)$. Since these cases are mutually exclusive for $n \geq 4$, either the first, or the last column contains a crossing. Then we may remove it and apply induction.

If $n=3$, we proceed as follows. Using P1 (and the convention $a_{1}=0$ ) we get that the number of crossings between the columns is at least $\left|a_{2}+1\right|+\left|a_{3}+1\right|+\left|a_{2}-a_{3}-1\right| \geq$ $\left|a_{2}+1\right|+\left|a_{2}\right|$ (using the triangle inequality). Symmetrically, we get another lower bound for the number of crossings: $\left|a_{3}+1\right|+\left|a_{3}+2\right|$. If any of these bounds is at least 3 , we are done. It follows that $a_{2} \in\{0,-1\}$ and $a_{3} \in\{-1,-2\}$. Now, if there are two consecutive columns with the same homotopy type, then each row will cross some of these columns, and we are done. Consequently $\mathbf{a}=0,-1,-2$. It follows that $\operatorname{Cr}\left(c_{1} \mid c_{3}\right) \geq 1$. If $c_{2}$ crossed either $c_{1}$ or $c_{3}$, then it would have to cross the column twice - which would yield too many crossings. Similarly, if $C r\left(c_{1} \mid c_{3}\right)>1$, then $C r\left(c_{1} \mid c_{3}\right) \geq 3$ and we would have too many crossings. It follows that the three columns $c_{1}, c_{2}, c_{3}$ are drawn as in Figure 10. Now we have that $c_{1}$ and $c_{3}$ separate $c_{2}$ from $c_{0}^{1}, c_{0}^{2}, c_{n+1}^{1}$, and $c_{n+1}^{2}$. It follows that $\operatorname{Cr}\left(r_{1} \mid c_{1} \cup c_{3}\right) \geq 2$ giving us too many crossings.

Case 6: Type $E$.
In this case, every curve $c_{i}^{+}$must have one end in $r_{3}^{2}$ and the other end in either $r_{0}^{1}$ or $r_{0}^{2}$. In the first case, we say that $c_{i}^{+}$has homotopy type 0 and in the second we say it has type


Figure 11: Towards type $E$ drawings of $H_{3}$.
$\ell$. It is immediate that any two such curves of the same type are homotopic. As usual, we let $\mathbf{a}=a_{1} a_{2} \ldots a_{n}$ be the word given by the rule that $a_{i}$ is the homotopy type of $c_{i}^{+}$. The following rule indicates some forced crossing behavior.

$$
\text { P1. } C r\left(c_{i} \mid c_{j}\right) \geq 1 \text { if } a_{i}=0 \text { and } 1 \leq i<j \leq n \text {. }
$$

Let us first treat the case when $n \geq 4$. If the last column $c_{n}$ contains at least one crossing, then we may remove it and apply induction. Otherwise, (P1) implies that $\mathbf{a}=\ell^{n}$ or $\mathbf{a}=\ell^{n-1} 0$. It follows from the Claim that $\operatorname{Cr}\left(c_{1} \cup c_{2} \mid r_{1} \cup r_{2}\right) \geq 2$. Thus, if $n \geq 5$, we may remove the first two columns and apply induction. If $n=4$ and $\mathbf{a}=\ell^{4}$, then the Claim gives us at least four crossings - a contradiction with the minimality of our drawing. It remains to check $\mathbf{a}=\ell^{3} 0$. If there are fewer than three crossings, then (again by applying the Claim twice) there are exactly two, and both occur on $c_{2}$. However, in this case $\operatorname{Cr}\left(r_{1} \mid c_{3}\right)=0$. As $c_{3}$ separates $c_{2}$ from both $r^{1} s^{1}$ and $r^{2} s^{2}$ and $r_{1}$ has a common vertex with $c_{2}$, we get a contradiction.

Finally, suppose that $n=3$. If there are two consecutive columns with the same homotopy type, then we are finished (by the Claim), so we may assume $\mathbf{a}=0 \ell 0$ or $\mathbf{a}=\ell 0 \ell$. In the former case, we have $\operatorname{Cr}\left(c_{1} \mid c_{2} \cup c_{3}\right) \geq 2$, so we may assume that there are exactly two crossings, and the columns must be drawn as in Figure 11(a). However, it is impossible to complete this drawing to a drawing of $H_{3}$ with fewer than three crossings.

In the case $\mathbf{a}=\ell \ell \ell$ we have $\operatorname{Cr}\left(c_{2} \mid c_{3}\right) \geq 1$ (see Figure 11(b)) and the total number of crossings is at most two. If $r_{2}$ is crossed, then the drawing is not exceptional and we are done. There is a unique way to add $r_{2}$ to Figure 11(b) without creating any new crossing. Then there is no way to add $r_{1}$ without crossing $r_{2}$.

Case 7: Type $E^{\prime}$.
This case is very close to the previous one. A similar analysis reduces the problem to the case when $n=3$. This case is actually identical to the above: By reflecting both the torus pictured in $E^{\prime}$ and the standard drawing of $H_{3}$ (as in Figure 1) about a vertical symmetry axis we find ourselves in this previous case.

Case 8: Type $E^{\prime \prime}$.
This case is somewhat similar to that of Type $E$. We may define the homotopy types for the columns $0, \ell$ exactly as before, so that the crossing property ( P 1 ) from Type $E$ is
satisfied. Then the analysis for $n \geq 4$ is identical, and the only difference is the case when $n=3$. As before, if there are two consecutive columns with the same homotopy type, we are finished. Thus we may assume that $\mathbf{a}=0 \ell 0$ or $\mathbf{a}=\ell 0 \ell$. Then we get another drawing of $H_{3}$ with two crossings, but again, in this case $r_{1}$ and $r_{2}$ cross each other.

Case 9: Type $E^{\prime \prime \prime}$.
This case is essentially the same as the previous one, in the same way as type $E^{\prime}$ was related to $E$. This completes the proof of Lemma 3.7.

Next we bootstrap to the following Lemma.
Lemma 3.8 The graph $H_{n, k}$ has crossing sequence $(n+k, n-1,0)$ for every $n \geq 3$ and $k \geq 0$ with the exception of $n=4$ and $k=0$.

Proof: Lemmas 3.1 and 3.2 show that $c r_{0}\left(H_{n, k}\right)=n+k$ and $c r_{2}\left(H_{n, k}\right)=0$. We can draw $H_{n, k}$ in the torus with $n-1$ crossings by adding a handle to the drawing from Figure 5. It remains to show that $c r_{1}\left(H_{n, k}\right) \geq n-1$ (for $n \geq 3$, unless $n=4$ and $k=0$ ). Take a drawing of $H_{n, k}$ in the torus. By removing the $k$ extra columns we obtain a drawing of $H_{n, 0}$ in the torus, which (by Lemma 3.7) has $\geq n-1$ crossings, unless $n=4$. This completes the proof in all cases except when $n=4$.

If $n=4$, the same argument as above shows that $c r_{1}\left(H_{4, k}\right) \geq c r_{1}\left(H_{4,1}\right)$; we shall prove now that $c r_{1}\left(H_{4,1}\right) \geq 3$. Suppose this is false, and consider a drawing of $H_{4,1}$ in the torus with at most two crossings. By removing the added column, we obtain a drawing of $H_{4}$ in the torus with at most two crossings. It follows from Lemma 3.7 that this drawing is equivalent to that in Figure 9. Since this drawing does not extend to a drawing of $H_{4,1}$ with $\leq 2$ crossings, this gives us a contradiction.

Thus $H_{n, k}$ (for $\left.(n, k) \neq(4,0)\right)$, has crossing sequence $(n+k, n-1,0)$ as claimed.
Next we introduce one additional graph to get the crossing sequence $(4,3,0)$. We define the graph $H_{3}^{+}$in the same way as $H_{3}$ except that we have three rows instead of two. See Figure 12.

Lemma 3.9 The graph $H_{3}^{+}$has crossing sequence $(4,3,0)$
Proof: It follows from an argument as in Lemma 3.2 that $c r_{0}\left(H_{3}^{+}\right)=4$. Since $H_{3}^{+}-\tau_{0}-\tau_{1}$ is planar, it follows that $c r_{2}\left(H_{3}^{+}\right)=0$. It remains to show that $c r_{1}\left(H_{3}^{+}\right)=3$. Since $c r_{1}\left(H_{3}^{+}\right) \leq 3$, we need only to show the reverse inequality. Consider an optimal drawing of $H_{3}^{+}$in the torus, and suppose (for a contradiction) that it has fewer than three crossings. If the first row contains a crossing, then by removing its edges, we obtain a drawing of a subdivision of $H_{3}$ in the torus with at most one crossing - a contradiction. Thus, the first row must not have a crossing, and by a similar argument, the third row must not have a crossing. Now, we again remove the first row. This leaves us with a drawing of a subdivision of $H_{3}$ in the torus with at most two crossings, and with the added property that one row ( $r_{2}$ in this $H_{3}$ )


Figure 12: The special graph $H_{3}^{+}$.
has no crossings. By Lemma 3.7 this must be a drawing as in Figure 8. A routine check of these drawings shows that none of them can be extended to a drawing of $H_{3}^{+}$with fewer than 3 crossings.

We require one added Lemma for some simple crossing sequences.
Lemma 3.10 For every $a>1$ there is a graph with crossing sequence ( $a, 1,0$ ).

Proof: Let $G_{1}$ be a copy of $K_{5}$, let $G_{2}$ be the graph obtained from a copy of $K_{5}$ by replacing each edge, except for one of them, with $a-1$ parallel edges joining the same pair of vertices. Let $G$ be the disjoint union of $G_{1}$ and $G_{2}$. It is immediate that $c r_{0}(G)=a, c r_{2}(G)=0$, and $c r_{1}(G) \geq 1$. A drawing of $G$ in $\mathbb{S}_{1}$ with this crossing number is easy to obtain by embedding $G_{2}$ in the torus, and then drawing $G_{1}$ disjoint from $G_{2}$ with one crossing. Thus, $G$ has crossing sequence $(a, 1,0)$ as required.

Proof of Theorem 1.3: Let $(a, b, 0)$ be given with integers $a>b>0$. If $b=1$, then the previous lemma shows that there is a graph with crossing sequence $(a, b, 0)$. If $(a, b, 0)=(4,3,0)$ then Lemma 3.9 provides such a graph. Otherwise, Lemma 3.8 shows that the graph $H_{b+1, a-b-1}$ has crossing sequence ( $a, b, 0$ ).

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