# BIJECTIONS ON TWO VARIATIONS OF NONCROSSING PARTITIONS

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ABSTRACT. We find bijections on 2-distant noncrossing partitions, 12312-avoiding partitions, 3-Motzkin paths, UH-free Schröder paths and Schröder paths without peaks at even height. We also give a direct bijection between 2-distant noncrossing partitions and 12312-avoiding partitions.

### 1. INTRODUCTION

Noncrossing partitions were first introduced by Kreweras [6] in 1972. Recently, they have received great attention, and have been generalized in many different ways; for instance, see [1, 2, 3, 5, 7] and the references therein. In this paper we consider two variations of noncrossing partitions: k-distant noncrossing partitions and  $12 \cdots r12$ -avoiding partitions introduced by Drake and Kim [3], and Mansour and Severini [7] respectively, where they reduce to noncrossing partitions when k = 1 and r = 2.

A (set) partition of  $[n] = \{1, 2, ..., n\}$  is a collection of mutually disjoint nonempty subsets, called *blocks*, of [n] whose union is [n]. We will write a partition as a sequence of blocks  $(B_1, B_2, ..., B_k)$  such that  $\min(B_1) < \min(B_2) < \cdots < \min(B_k)$ . An *edge* of a partition is a pair (i, j) of integers contained in the same block that does not contain any integer t with i < t < j. The *standard representation* of a partition  $\pi$  of [n] is the diagram having n vertices labeled with 1, 2, ..., n, where iand j are connected by an arc if (i, j) is an edge of  $\pi$ ; see Figure 1. A noncrossing partition is a partition without any two crossing edges, i.e.  $(i_1, j_1)$  and  $(i_2, j_2)$  such that  $i_1 < i_2 < j_1 < j_2$ . It is well known the number of noncrossing partitions of [n]is the Catalan number  $\frac{1}{n+1} {2n \choose n}$ .

For a positive integer k, a k-distant noncrossing partition is a partition without any two edges  $(i_1, j_1)$  and  $(i_2, j_2)$  satisfying  $i_1 < i_2 < j_1 < j_2$  and  $j_1 - i_2 \ge k$ . Note that 1-distant noncrossing partitions are just noncrossing partitions. We denote by NC<sub>k</sub>(n) the set of k-distant noncrossing partitions of [n]. Drake and Kim [3] found the following generating function for the number of 2-distant noncrossing

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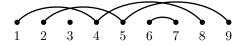


FIGURE 1. The standard representation of  $(\{1, 4, 8\}, \{2, 5, 9\}, \{3\}, \{6, 7\})$ .

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partitions:

(1) 
$$\sum_{n \ge 0} \# \operatorname{NC}_2(n) x^n = \frac{3 - 3x - \sqrt{1 - 6x + 5x^2}}{2(1 - x)}$$

The canonical word of a partition  $\pi = (B_1, B_2, \ldots, B_k)$  is the word  $a_1 a_2 \cdots a_n$ , where  $a_i = j$  if  $i \in B_j$ . For instance, the canonical word of the partition in Figure 1 is 123124412. In the literature canonical words are also called restricted growth functions. For a word  $\tau$ , a partition is called  $\tau$ -avoiding if its canonical word does not contain a subword which is order-isomorphic to  $\tau$ . It is easy to see that a partition is noncrossing if and only if it is 1212-avoiding. We denote by  $P_{\tau}(n)$  the set of  $\tau$ -avoiding partitions of [n].

Using the kernel method, Mansour and Severini [7] found the generating function for the number of  $12 \cdots r 12$ -avoiding partitions of [n]. Interestingly, as a special case of their result, the generating function for the number of 12312-avoiding partitions of [n] is the same as (1), which implies  $\#NC_2(n) = \#P_{12312}(n)$ . Moreover, this number also counts several kinds of lattice paths. The main purpose of this paper is to find bijections between  $NC_2(n)$  and  $P_{12312}(n)$  together with some lattices paths described below.

A lattice path of length n is a sequence of points in  $\mathbb{N} \times \mathbb{N}$  starting at (0,0)and ending at (n,0). For a lattice path  $L = ((x_0, y_0), (x_1, y_1), \dots, (x_k, y_k))$ , each  $S_i = (x_i - x_{i-1}, y_i - y_{i-1})$  is called a *step* of L. The *height* of the step  $S_i$  is defined to be  $y_{i-1}$ . Sometimes we will identify a lattice path L with the word  $S_1S_2 \dots S_k$ of its steps. Note that the number of steps is not necessarily equal to the length of the lattice path.

Let U, D and H denote an up step, a down step and a horizontal step respectively, i.e., U = (1, 1), D = (1, -1) and H = (1, 0).

A Schröder path is a lattice path consisting of steps U, D and  $H^2 = HH = (2, 0)$ . Let  $L = S_1 S_2 \cdots S_k$  be a Schröder path. A UH-pair of L is a pair  $(S_i, S_{i+1})$  of consecutive steps such that  $S_i = U$  and  $S_{i+1} = H^2$ . We say that L is UH-free if it does not have a UH-pair. A peak of L is a pair  $(S_i, S_{i+1})$  of consecutive steps such that  $S_i = U$  and  $S_{i+1} = D$ . The height of a peak  $(S_i, S_{i+1})$  is the height of  $S_{i+1} = D$ . We denote by  $SCH_{UH}(n)$  the set of UH-free Schröder paths of length 2n, and by  $SCH_{even}(n)$  (resp.  $SCH_{odd}(n)$ ) the set of Schröder paths of length 2n which have no peaks of even (resp. odd) height.

A labeled step is a step together with an integer label. Let  $D_i$  (resp.  $H_i$ ) denote a labeled down step (resp. a labeled horizontal step) with label *i*. We denote by  $CH_2(n)$  the set of lattice paths  $L = S_1 S_2 \cdots S_n$  of length *n* consisting of *U*,  $D_1$ ,  $D_2$ ,  $H_0$ ,  $H_1$  and  $H_2$  such that

- if  $S_i = H_\ell$  or  $S_i = D_\ell$ , then  $S_i$  is of height at least  $\ell$ ,
- if  $S_i = H_2$  or  $S_i = D_2$ , then  $i \ge 2$  and  $S_{i-1} \in \{U, H_1, H_2\}$ .

A 3-Motzkin path is a lattice path consisting of U, D,  $H_0$ ,  $H_1$  and  $H_2$ . We denote by MOT<sub>3</sub>(n) the set of 3-Motzkin paths of length n.

Drake and Kim [3] showed that the well known bijection  $\psi$  between partitions and Charlier diagrams, see [4, 5], yields a bijection  $\psi : \operatorname{NC}_2(n) \to \operatorname{CH}_2(n)$ . Yan [10] found a bijection  $\phi : \operatorname{SCH}_{\mathrm{UH}}(n-1) \to P_{12312}(n)$  and a bijection between  $\operatorname{SCH}_{\mathrm{UH}}(n)$ and  $\operatorname{SCH}_{\mathrm{even}}(n)$ . Thus all of  $\operatorname{NC}_2(n)$ ,  $\operatorname{CH}_2(n)$ ,  $\operatorname{SCH}_{\mathrm{even}}(n-1)$ ,  $\operatorname{SCH}_{\mathrm{UH}}(n-1)$  and  $P_{12312}(n)$  have the same cardinality, which is counted by sequence A007317 from

$$\begin{array}{ccc} \operatorname{NC}_{2}'(n) & \operatorname{SCH}_{\operatorname{even}}'(n-1) & P_{12312}'(n) \\ & \downarrow \psi & & \uparrow \iota & \uparrow \phi \\ \operatorname{CH}_{2}'(n) \xrightarrow{f} \operatorname{MOT}_{3}(n-2) \xrightarrow{g} \operatorname{SCH}_{\operatorname{odd}}(n-1) \xrightarrow{h} \operatorname{SCH}_{\operatorname{UH}}'(n-1) \end{array}$$

FIGURE 2. Main bijections for  $n \ge 2$ .

[8]. In order to find bijections between these objects, we introduce the following sets:

- $\operatorname{NC}_2'(n) = \{\pi \in \operatorname{NC}_2(n) : n \text{ is not a singleton}\}\$
- $\operatorname{CH}_2'(n) = \{L \in \operatorname{CH}_2(n) : \text{the last step of } L \text{ is } D_1\}$
- $SCH'_{even}(n) = \{L \in SCH_{even}(n) : \text{the first step of } L \text{ is } U\}$
- $\operatorname{SCH}'_{\operatorname{UH}}(n) = \{L \in \operatorname{SCH}_{\operatorname{UH}}(n) : \text{the first step of } L \text{ is } U\}$
- $P'_{12312}(n) = \{\pi \in P_{12312}(n) : 1 \text{ and } 2 \text{ are not in the same block}\}\$

Note that we can identify  $\pi \in \mathrm{NC}_2(n)$  with  $\pi' \in \mathrm{NC}_2'(k)$ , where k is the integer such that j is a singleton for all  $j \in \{k + 1, k + 2, \ldots, n\}$  and k is not a singleton in  $\pi$ , and  $\pi'$  is the partition obtained from  $\pi$  by deleting integers greater than k. We can also identify  $\pi \in P_{12312}(n)$  with  $\overline{\pi} \in P'_{12312}(k)$ , where k is the integer such that the number of consecutive 1's at the beginning of the canonical word of  $\pi$  is n - k + 1, and  $\overline{\pi}$  is the partition whose canonical word is obtained from that of  $\pi$  by deleting the first n - k 1's. Thus any bijection between  $\mathrm{NC}'_2(n)$  and  $P'_{12312}(n)$  naturally induces a bijection between  $\mathrm{NC}_2(n)$  and  $P_{12312}(n)$ . Similarly, any bijection between A'(n) and B'(n) naturally induces a bijection between A(n)and B(n) where A and B are any two of  $\mathrm{NC}_2$ ,  $\mathrm{CH}_2$ ,  $\mathrm{SCH}_{\mathrm{even}}$ ,  $\mathrm{SCH}_{\mathrm{UH}}$ , and  $P_{12312}$ . Thus in order to find a bijection between  $\mathrm{NC}_2(n)$  and  $P_{12312}(n)$ , it is enough to find a bijection between  $\mathrm{NC}'_2(n)$  and  $P'_{12312}(n)$ .

In this paper we find bijections between these objects. For the overview of our bijections see Figure 2, where  $\psi$  is the known bijection between partitions and Charlier diagrams [4, 5], and  $\phi$  is Yan's bijection [10]. We note that our bijection g in Figure 2 is also discovered by Shapiro and Wang [9]. We also provide a direct bijection between NC<sub>2</sub>(n) and  $P_{12312}(n)$  in Section 3.

# 2. Bijections

In this section we find the bijections f, g, h, and  $\iota$  in Figure 2.

2.1. The bijection  $f : CH'_2(n) \to MOT_3(n-2)$ . Recall that  $CH'_2(n)$  is the set of lattice paths  $L = S_1 S_2 \cdots S_n$  of length n consisting of  $U, D_1, D_2, H_0, H_1$  and  $H_2$  such that

- if  $S_i = H_\ell$  or  $S_i = D_\ell$ , then  $S_i$  is of height at least  $\ell$ ,
- if  $S_i = H_2$  or  $S_i = D_2$ , then  $i \ge 2$  and  $S_{i-1} \in \{U, H_1, H_2\}$ ,
- $S_n = D_1$ .

The second condition above is equivalent to the condition that the lattice path consists of the following combined steps for any  $k \ge 0$ :

(2) 
$$UH_2^k, UH_2^k D_2, H_1 H_2^k, H_1 H_2^k D_2, H_0, D_1.$$

Let A(n) denote the set of lattice paths of length n consisting of the combined steps in (2) such that  $H_2$  does not touch the x-axis. Let B(n) denote the set of

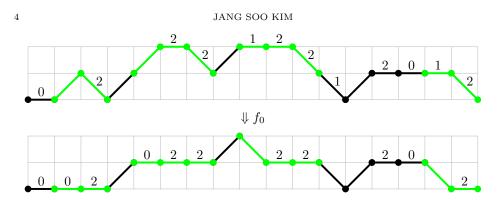


FIGURE 3. An example of  $f_0$ .

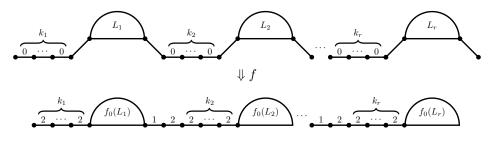


FIGURE 4. Definition of f.

3-Motzkin paths of length n such that each  $H_2$  touching the x-axis must occur after  $D, H_0$  or  $H_2$ .

We define  $f_0 : A(n) \to B(n)$  as follows. Let  $L \in A(n)$ . Then  $f_0(L)$  is defined to be the lattice path obtained from L by changing  $UH_2^kD_2$  to  $H_0H_2^{k+1}$ ,  $H_1H_2^kD_2$ to  $DH_2^{k+1}$  and  $D_1$  to D. It is easy to see that  $f_0(L) \in B$  and  $f_0$  is invertible. See Figure 3.

Now we define  $f : CH'_2(n) \to MOT_3(n-2)$  as follows. Let  $L \in CH'_2(n)$ . Then L is decomposed uniquely as

 $H_0^{k_1}(UL_1D_1)H_0^{k_2}(UL_2D_1)\cdots H_0^{k_r}(UL_rD_1),$ 

where  $L_i \in A(n_i)$  for some  $k_i, n_i \ge 0$  and  $r \ge 1$ . Then define f(L) to be

$$H_2^{k_1} f_0(L_1)(H_1 H_2^{k_2+1} f_0(L_2))(H_1 H_2^{k_3+1} f_0(L_3)) \cdots (H_1 H_2^{k_r+1} f_0(L_r)).$$

See Figure 4.

**Theorem 2.1.** The map  $f : CH'_2(n) \to MOT_3(n-2)$  is a bijection.

*Proof.* Each  $L \in MOT_3(n-2)$  is uniquely decomposed as

$$H_2^{k_1}L_1(H_1H_2^{k_2+1}L_2)(H_1H_2^{k_3+1}L_3)\cdots(H_1H_2^{k_r+1}L_r),$$

where  $L_i \in B(n_i)$  for some  $k_i, n_i \ge 0$  and  $r \ge 1$ . Thus we have the inverse  $f^{-1}(L)$  which is decomposed as

$$H_0^{k_1}(Uf_0^{-1}(L_1)D_1)H_0^{k_2}(Uf_0^{-1}(L_2)D_1)\cdots H_0^{k_r}(Uf_0^{-1}(L_r)D_1).$$

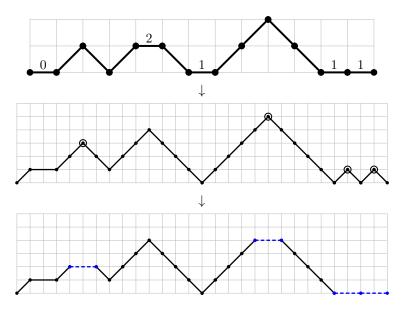


FIGURE 5. An example of g. Odd peaks are circled. The horizontal steps of even height are dashed and colored blue.

2.2. The bijection  $g : MOT_3(n) \to SCH_{odd}(n+1)$ . We define  $g : MOT_3(n) \to SCH_{odd}(n+1)$  as follows. Let  $L \in MOT_3(n)$ . Then g(L) is the lattice path obtained from L by doing the following.

- (1) Change U to UU, D to DD,  $H_0$  to  $H^2$ ,  $H_1$  to DU, and  $H_2$  to UD.
- (2) Add U at the beginning and D at the end.
- (3) Change all the consecutive steps UD which form a peak of odd height to  $H^2$ .

See Figure 5 for an example of g.

**Theorem 2.2.** The map  $g : MOT_3(n) \to SCH_{odd}(n+1)$  is a bijection.

*Proof.* Clearly the first and the second steps in the construction of g are invertible. The third step is also invertible because every step  $H^2$  of even height always comes from a peak of odd height. Thus g is invertible.

2.3. The bijection  $h : \text{SCH}_{\text{odd}}(n) \to \text{SCH}'_{\text{UH}}(n)$ . Let  $L = S_1 S_2 \cdots S_k$  be a Schröder path. For any up step  $S_i = U$  of L, there is a unique down step  $S_j = D$  such that i < j and  $S_{i+1}S_{i+2}\cdots S_{j-1}$  is a (possibly empty) lattice path. We call such  $S_j$  the down step corresponding to  $S_i$ . We also call  $S_i$  the up step corresponding to  $S_j$ .

For a UH-pair  $(S_i, S_{i+1})$ , i.e.  $S_i = U$  and  $S_{i+1} = H^2$ , we define the function  $\xi$  as follows.

$$\xi(S_i, S_{i+1}) = \begin{cases} i, & \text{if } S_{i+1} \text{ is of even height;} \\ j, & \text{if } S_{i+1} \text{ is of odd height,} \end{cases}$$

where j is the integer such that  $S_j$  is the down step corresponding to  $S_i$ . If L is not UH-free, we define the  $\xi$ -maximal UH-pair of L to be the UH-pair  $(S_i, S_{i+1})$ with the largest  $\xi$  value.

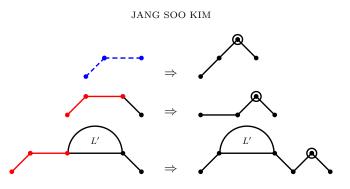


FIGURE 6. The essence of  $h_0$ . Red (resp. Dashed blue) color is for UH-pairs whose horizontal step is of odd (resp. even) height. Odd peaks are circled. The lattice path L' is not empty.

Now let  $L = S_1 S_2 \cdots S_k \in \text{SCH}_{\text{odd}}(n)$ . If L is not UH-free, we define  $h_0(L)$  as follows. Suppose  $(S_i, S_{i+1})$  is the  $\xi$ -maximal UH-pair of L, and  $S_j$  is the down step corresponding to  $S_i$ .

- (1) If  $S_{i+1}$  is of even height, then  $h_0(L)$  is the lattice path obtained from L by replacing  $S_i S_{i+1}$  with UUD.
- (2) If  $S_{i+1}$  is of odd height, then let  $L' = S_{i+2}S_{i+3}\cdots S_{j-1}$ .
  - (a) If L' is empty, i.e., j = i + 2, then  $h_0(L)$  is the lattice path obtained from L by replacing  $S_i S_{i+1} S_{i+2}$  with  $H^2 UD$ .
    - (b) If L' is not empty, then  $h_0(L)$  is the lattice path obtained from L by replacing  $S_i S_{i+1} \cdots S_j$  with UL'DUD.

See Figure 6.

Now we define  $h : \text{SCH}_{\text{odd}}(n) \to \text{SCH}'_{\text{UH}}(n)$  as follows. Let  $L \in \text{SCH}_{\text{odd}}(n)$  and  $L_0 = L$ . Then we define  $L_i = h_0(L_{i-1})$  for  $i \ge 1$  if  $L_{i-1}$  is not UH-free. Since the number of UH-free pairs of  $L_i$  is one less than that of  $L_{i-1}$ , or they are the same and

 $\xi$ (the maximal UH-pair of  $L_i$ ) <  $\xi$ (the maximal UH-pair of  $L_{i-1}$ ),

we always get  $L_r$  which is UH-free for some r. We define h(L) to be  $L_r$  if  $L_r$  does not start with  $H^2$ ; and the lattice path obtained from  $L_r$  by replacing  $H^2$  with UDotherwise. For an example, see Figure 7.

**Theorem 2.3.** The map  $h : SCH_{odd}(n) \to SCH'_{UH}(n)$  is a bijection.

*Proof.* In the procedure of h, the odd peaks are constructed from right to left. Since  $h_0$  is invertible, so is h.

2.4. The bijection  $\iota$ : SCH<sub>odd</sub> $(n) \to$  SCH'<sub>even</sub>(n). For  $L = S_1 S_2 \cdots S_k \in$  SCH<sub>odd</sub>(n), we define  $\iota(L)$  as follows.

- (1) If  $S_k = H^2$ , then  $\iota(L) = US_1 \cdots S_{k-1}D$ .
- (2) If  $S_k = D$ , then let  $S_i$  be the up step corresponding to  $S_k$  and we define  $\iota(L) = US_1 \cdots S_{i-1}DS_{i+1} \cdots S_{k-1}$ .

See Figure 8.

Then  $\iota(L) \in \operatorname{SCH}'_{\operatorname{even}}(n)$ . Clearly,  $\iota : \operatorname{SCH}_{\operatorname{odd}}(n) \to \operatorname{SCH}'_{\operatorname{even}}(n)$  is a bijection.

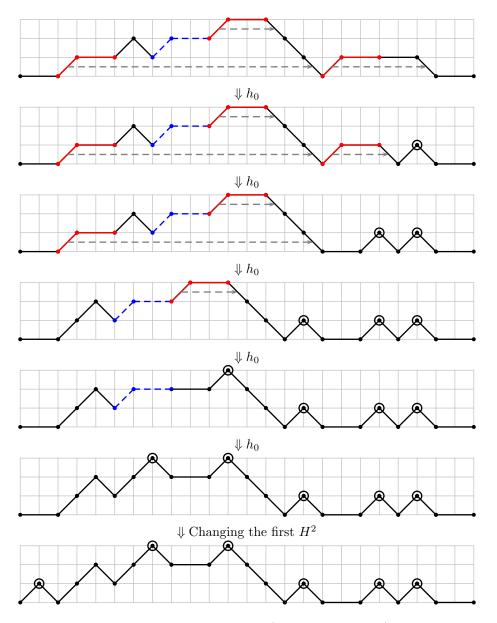


FIGURE 7. An example of h. Red (resp. Dashed blue) color is for UH-pairs whose horizontal step is of odd (resp. even) height. Odd peaks are circled. Dashed arrows indicate the down steps corresponding to the up steps.

# 3. A direct bijection between $NC_2(n)$ and $P_{12312}(n)$

Now we have a bijection  $\phi \circ h \circ g \circ f \circ \psi : \operatorname{NC}_2'(n) \to P'_{12312}(n)$ , see Figure 2. As noted in the introduction, this induces a bijection between  $\operatorname{NC}_2(n)$  and  $P_{12312}(n)$ . Since both  $\operatorname{NC}_2(n)$  and  $P_{12312}(n)$  are partitions with some conditions, it is natural to ask a direct bijection between them. In this section we find such a direct bijection.

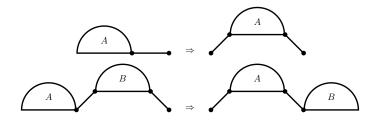


FIGURE 8. The map  $\iota$ .

From now on, we will identify a partition in  $P_{12312}(n)$  with its canonical word. A marked partition is a partition in which each part may be marked. Similarly a marked word is a word in which each letter may be marked.

Let  $\pi \in \mathrm{NC}_2(n)$ . For  $i \in [n]$ , let  $T_i$  be the marked partition of [i] obtained from  $\pi$  by removing all the integers greater than i and by marking integers which are connected to an integer greater than i. Using the sequence  $\emptyset = T_0, T_1, T_2, \ldots, T_n = \pi$  of marked partitions, we define a sequence of marked words  $\mathbf{w}_0, \mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n$  as follows. Here, if (i, j) is an edge we say that j is connected to i.

Let  $\mathbf{w}_0$  be the empty word. For  $1 \leq i \leq n$ ,  $\mathbf{w}_i$  is defined as follows.

- (1) If *i* is not connected to any integer in  $T_i$ , then  $\mathbf{w}_i = \mathbf{w}_{i-1}m$ , where  $m = \max(\mathbf{w}_{i-1}) + 1$ . Otherwise, *i* is connected to either the largest marked integer or the second largest marked integer of  $T_{i-1}$ .
  - If *i* is connected to the largest marked integer of  $T_{i-1}$ , then let  $\mathbf{w}_i = \mathbf{w}_{i-1}a_1$ , where  $a_1$  is the rightmost marked letter of  $\mathbf{w}_{i-1}$ . And then we make the marked letter  $a_1$  unmarked.
  - If *i* is connected to the second largest marked integer of  $T_{i-1}$ , then let  $\mathbf{w}_i = \mathbf{w}_{i-1}a_2$ , where  $a_2$  is the second rightmost marked letter of  $\mathbf{w}_{i-1}$ . The second rightmost marked letter of  $\mathbf{w}_{i-1}$  remains marked, however, we make the rightmost marked letter of  $\mathbf{w}_{i-1}$  unmarked in  $\mathbf{w}_i$ .
- (2) If *i* is marked in  $T_i$ , then we find the largest letters in  $\mathbf{w}_i$  and make the leftmost letter among them marked.

For an example, see Figure 9.

## **Lemma 3.1.** The word $\mathbf{w}_n$ obtained above is 12312-avoiding.

*Proof.* Suppose  $\mathbf{w}_n$  has a subsequence *abcab* where a < b < c. When the second *b* is added the first *b* must have been marked. Moreover, the first *b* must have been marked before adding the second *a* because an unmarked integer becomes marked only if it is the largest integer (in this case at least *c*) in the sequence. Thus when the second *a* is added, the first *a* and *b* have been marked. Since the first *a* is the second rightmost marked integer at this moment, we must unmark the rightmost marked integer, the first *b*, and mark the largest integer which is at least *c*. Thus after this process, *b* cannot be marked and we cannot have the second *b*, which is a contradiction.

If we know  $\mathbf{w}_n$ , we can reverse this procedure. For  $1 \le i \le n$ ,  $\mathbf{w}_{i-1}$  is obtained from  $\mathbf{w}_i$  as follows. Suppose  $m = \max(\mathbf{w}_i)$  and t is the last letter of  $\mathbf{w}_i$ .

(1) If the leftmost m is marked in  $\mathbf{w}_i$ , then make it unmarked.

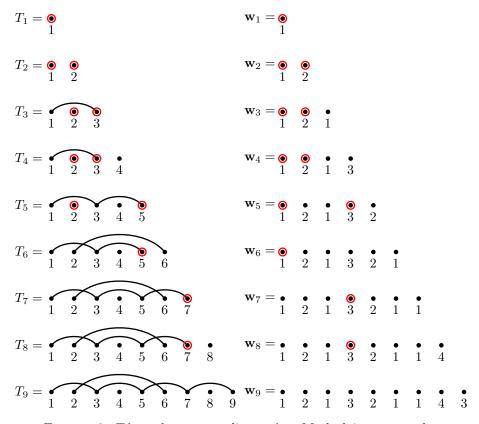


FIGURE 9.  $T_i$ 's and corresponding  $\mathbf{w}_i$ 's. Marked integers and marked letters are circled.

- (2) If t appears only once in  $\mathbf{w}_i$  (equivalently t is greater than any other letters in  $\mathbf{w}_i$ ), then we simply remove t. Otherwise, find the leftmost t in  $\mathbf{w}_i$ .
  - If the leftmost t is unmarked, then we remove the last letter t and make the leftmost t marked.
  - If the leftmost t is marked, then we must have t < m since we have made the leftmost m unmarked. In this case we remove the last t, and make the leftmost t still marked and the leftmost m marked.

Now we construct  $T_0, T_1, \ldots, T_n$  as follows. Let  $T_0 = \emptyset$ . For  $1 \le i \le n$ ,  $T_i$  is obtained as follows.

- (1) First, let  $T_i$  be the marked partition obtained from  $T_{i-1}$  by adding *i*.
- (2) If the last letter of  $\mathbf{w}_i$  is equal to the rightmost (resp. the second rightmost) marked letter of  $\mathbf{w}_{i-1}$ , then connect *i* to the largest (resp. the second largest) marked integer, say *j*, of  $T_{i-1}$ , and make *j* unmarked.
- (3) Let  $m = \max(\mathbf{w}_i)$ . If the leftmost m is marked in  $\mathbf{w}_i$ , then make i marked in  $T_i$ .

It is easy to check that this is the inverse map. Thus we get the following theorem.

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**Theorem 3.2.** For  $\pi \in NC_2(n)$ , the map  $\pi \mapsto \mathbf{w}_n$  is a bijection from  $NC_2(n)$  to  $P_{12312}(n)$ .

The bijection  $\pi \mapsto \mathbf{w}_n$  is different from the composition  $\phi \circ h \circ g \circ f \circ \psi$ . For instance, if  $\pi = (\{1,3\}, \{2\})$ , then  $\mathbf{w}_3 = 121$  but  $(\phi \circ h \circ g \circ f \circ \psi)(\pi) = 112$ .

Note that both  $NC_2(n)$  and  $P_{12312}(n)$  contain noncrossing partitions. It would be interesting to find a bijection between  $NC_2(n)$  and  $P_{12312}(n)$  which sends noncrossings partitions to noncrossings partitions.

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