# BIJECTIONS ON TWO VARIATIONS OF NONCROSSING PARTITIONS 

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#### Abstract

We find bijections on 2-distant noncrossing partitions, 12312-avoiding partitions, 3-Motzkin paths, UH-free Schröder paths and Schröder paths without peaks at even height. We also give a direct bijection between 2-distant noncrossing partitions and 12312-avoiding partitions.


## 1. Introduction

Noncrossing partitions were first introduced by Kreweras [6] in 1972. Recently, they have received great attention, and have been generalized in many different ways; for instance, see [1, 2, 3, 5, 7, and the references therein. In this paper we consider two variations of noncrossing partitions: $k$-distant noncrossing partitions and $12 \cdots r 12$-avoiding partitions introduced by Drake and Kim [3, and Mansour and Severini [7] respectively, where they reduce to noncrossing partitions when $k=1$ and $r=2$.

A (set) partition of $[n]=\{1,2, \ldots, n\}$ is a collection of mutually disjoint nonempty subsets, called blocks, of $[n]$ whose union is $[n]$. We will write a partition as a sequence of blocks $\left(B_{1}, B_{2}, \ldots, B_{k}\right)$ such that $\min \left(B_{1}\right)<\min \left(B_{2}\right)<\cdots<\min \left(B_{k}\right)$. An edge of a partition is a pair $(i, j)$ of integers contained in the same block that does not contain any integer $t$ with $i<t<j$. The standard representation of a partition $\pi$ of $[n]$ is the diagram having $n$ vertices labeled with $1,2, \ldots, n$, where $i$ and $j$ are connected by an arc if $(i, j)$ is an edge of $\pi$; see Figure 1 A noncrossing partition is a partition without any two crossing edges, i.e. $\left(i_{1}, j_{1}\right)$ and $\left(i_{2}, j_{2}\right)$ such that $i_{1}<i_{2}<j_{1}<j_{2}$. It is well known the number of noncrossing partitions of [ $n$ ] is the Catalan number $\frac{1}{n+1}\binom{2 n}{n}$.

For a positive integer $k$, a $k$-distant noncrossing partition is a partition without any two edges $\left(i_{1}, j_{1}\right)$ and $\left(i_{2}, j_{2}\right)$ satisfying $i_{1}<i_{2}<j_{1}<j_{2}$ and $j_{1}-i_{2} \geq k$. Note that 1 -distant noncrossing partitions are just noncrossing partitions. We denote by $\mathrm{NC}_{k}(n)$ the set of $k$-distant noncrossing partitions of [ $n$ ]. Drake and Kim [3] found the following generating function for the number of 2-distant noncrossing

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Figure 1. The standard representation of $(\{1,4,8\},\{2,5,9\},\{3\},\{6,7\})$.
partitions:

$$
\begin{equation*}
\sum_{n \geq 0} \# \mathrm{NC}_{2}(n) x^{n}=\frac{3-3 x-\sqrt{1-6 x+5 x^{2}}}{2(1-x)} . \tag{1}
\end{equation*}
$$

The canonical word of a partition $\pi=\left(B_{1}, B_{2}, \ldots, B_{k}\right)$ is the word $a_{1} a_{2} \cdots a_{n}$, where $a_{i}=j$ if $i \in B_{j}$. For instance, the canonical word of the partition in Figure 1 is 123124412 . In the literature canonical words are also called restricted growth functions. For a word $\tau$, a partition is called $\tau$-avoiding if its canonical word does not contain a subword which is order-isomorphic to $\tau$. It is easy to see that a partition is noncrossing if and only if it is 1212 -avoiding. We denote by $P_{\tau}(n)$ the set of $\tau$-avoiding partitions of $[n]$.

Using the kernel method, Mansour and Severini [7] found the generating function for the number of $12 \cdots r 12$-avoiding partitions of $[n]$. Interestingly, as a special case of their result, the generating function for the number of 12312-avoiding partitions of $[n]$ is the same as (11), which implies $\# \mathrm{NC}_{2}(n)=\# P_{12312}(n)$. Moreover, this number also counts several kinds of lattice paths. The main purpose of this paper is to find bijections between $\mathrm{NC}_{2}(n)$ and $P_{12312}(n)$ together with some lattices paths described below.

A lattice path of length $n$ is a sequence of points in $\mathbb{N} \times \mathbb{N}$ starting at $(0,0)$ and ending at $(n, 0)$. For a lattice path $L=\left(\left(x_{0}, y_{0}\right),\left(x_{1}, y_{1}\right), \ldots,\left(x_{k}, y_{k}\right)\right)$, each $S_{i}=\left(x_{i}-x_{i-1}, y_{i}-y_{i-1}\right)$ is called a step of $L$. The height of the step $S_{i}$ is defined to be $y_{i-1}$. Sometimes we will identify a lattice path $L$ with the word $S_{1} S_{2} \ldots S_{k}$ of its steps. Note that the number of steps is not necessarily equal to the length of the lattice path.

Let $U, D$ and $H$ denote an up step, a down step and a horizontal step respectively, i.e., $U=(1,1), D=(1,-1)$ and $H=(1,0)$.

A Schröder path is a lattice path consisting of steps $U, D$ and $H^{2}=H H=(2,0)$. Let $L=S_{1} S_{2} \cdots S_{k}$ be a Schröder path. A UH-pair of $L$ is a pair $\left(S_{i}, S_{i+1}\right)$ of consecutive steps such that $S_{i}=U$ and $S_{i+1}=H^{2}$. We say that $L$ is $U H$-free if it does not have a UH-pair. A peak of $L$ is a pair $\left(S_{i}, S_{i+1}\right)$ of consecutive steps such that $S_{i}=U$ and $S_{i+1}=D$. The height of a peak $\left(S_{i}, S_{i+1}\right)$ is the height of $S_{i+1}=D$. We denote by $\operatorname{SCH}_{\mathrm{UH}}(n)$ the set of UH-free Schröder paths of length $2 n$, and by $\mathrm{SCH}_{\text {even }}(n)\left(\right.$ resp. $\mathrm{SCH}_{\text {odd }}(n)$ ) the set of Schröder paths of length $2 n$ which have no peaks of even (resp. odd) height.

A labeled step is a step together with an integer label. Let $D_{i}\left(\right.$ resp. $\left.H_{i}\right)$ denote a labeled down step (resp. a labeled horizontal step) with label $i$. We denote by $\mathrm{CH}_{2}(n)$ the set of lattice paths $L=S_{1} S_{2} \cdots S_{n}$ of length $n$ consisting of $U, D_{1}$, $D_{2}, H_{0}, H_{1}$ and $H_{2}$ such that

- if $S_{i}=H_{\ell}$ or $S_{i}=D_{\ell}$, then $S_{i}$ is of height at least $\ell$,
- if $S_{i}=H_{2}$ or $S_{i}=D_{2}$, then $i \geq 2$ and $S_{i-1} \in\left\{U, H_{1}, H_{2}\right\}$.

A 3-Motzkin path is a lattice path consisting of $U, D, H_{0}, H_{1}$ and $H_{2}$. We denote by $\operatorname{MOT}_{3}(n)$ the set of 3 -Motzkin paths of length $n$.

Drake and Kim 3] showed that the well known bijection $\psi$ between partitions and Charlier diagrams, see [4] [5], yields a bijection $\psi: \mathrm{NC}_{2}(n) \rightarrow \mathrm{CH}_{2}(n)$. Yan [10 found a bijection $\phi: \operatorname{SCH}_{\mathrm{UH}}(n-1) \rightarrow P_{12312}(n)$ and a bijection between $\mathrm{SCH}_{\mathrm{UH}}(n)$ and $\mathrm{SCH}_{\text {even }}(n)$. Thus all of $\mathrm{NC}_{2}(n), \mathrm{CH}_{2}(n), \mathrm{SCH}_{\text {even }}(n-1), \mathrm{SCH}_{\mathrm{UH}}(n-1)$ and $P_{12312}(n)$ have the same cardinality, which is counted by sequence A007317 from


Figure 2. Main bijections for $n \geq 2$.

8]. In order to find bijections between these objects, we introduce the following sets:

- $\mathrm{NC}_{2}^{\prime}(n)=\left\{\pi \in \mathrm{NC}_{2}(n): n\right.$ is not a singleton $\}$
- $\mathrm{CH}_{2}^{\prime}(n)=\left\{L \in \mathrm{CH}_{2}(n)\right.$ : the last step of $L$ is $\left.D_{1}\right\}$
- $\mathrm{SCH}_{\text {even }}^{\prime}(n)=\left\{L \in \mathrm{SCH}_{\text {even }}(n)\right.$ : the first step of $L$ is $\left.U\right\}$
- $\mathrm{SCH}_{\mathrm{UH}}^{\prime}(n)=\left\{L \in \operatorname{SCH}_{\mathrm{UH}}(n)\right.$ : the first step of $L$ is $\left.U\right\}$
- $P_{12312}^{\prime}(n)=\left\{\pi \in P_{12312}(n): 1\right.$ and 2 are not in the same block $\}$

Note that we can identify $\pi \in \mathrm{NC}_{2}(n)$ with $\pi^{\prime} \in \mathrm{NC}_{2}^{\prime}(k)$, where $k$ is the integer such that $j$ is a singleton for all $j \in\{k+1, k+2, \ldots, n\}$ and $k$ is not a singleton in $\pi$, and $\pi^{\prime}$ is the partition obtained from $\pi$ by deleting integers greater than $k$. We can also identify $\pi \in P_{12312}(n)$ with $\pi \in P_{12312}^{\prime}(k)$, where $k$ is the integer such that the number of consecutive 1's at the beginning of the canonical word of $\pi$ is $n-k+1$, and $\bar{\pi}$ is the partition whose canonical word is obtained from that of $\pi$ by deleting the first $n-k$ 's. Thus any bijection between $\mathrm{NC}_{2}^{\prime}(n)$ and $P_{12312}^{\prime}(n)$ naturally induces a bijection between $\mathrm{NC}_{2}(n)$ and $P_{12312}(n)$. Similarly, any bijection between $A^{\prime}(n)$ and $B^{\prime}(n)$ naturally induces a bijection between $A(n)$ and $B(n)$ where $A$ and $B$ are any two of $\mathrm{NC}_{2}, \mathrm{CH}_{2}, \mathrm{SCH}_{\text {even }}, \mathrm{SCH}_{\mathrm{UH}}$, and $P_{12312}$. Thus in order to find a bijection between $\mathrm{NC}_{2}(n)$ and $P_{12312}(n)$, it is enough to find a bijection between $\mathrm{NC}_{2}^{\prime}(n)$ and $P_{12312}^{\prime}(n)$.

In this paper we find bijections between these objects. For the overview of our bijections see Figure 2, where $\psi$ is the known bijection between partitions and Charlier diagrams [4, 5], and $\phi$ is Yan's bijection 10. We note that our bijection $g$ in Figure 2 is also discovered by Shapiro and Wang 9. We also provide a direct bijection between $\mathrm{NC}_{2}(n)$ and $P_{12312}(n)$ in Section 3.

## 2. Bijections

In this section we find the bijections $f, g, h$, and $\iota$ in Figure 2
2.1. The bijection $f: \mathrm{CH}_{2}^{\prime}(n) \rightarrow \operatorname{MOT}_{3}(n-2)$. Recall that $\mathrm{CH}_{2}^{\prime}(n)$ is the set of lattice paths $L=S_{1} S_{2} \cdots S_{n}$ of length $n$ consisting of $U, D_{1}, D_{2}, H_{0}, H_{1}$ and $H_{2}$ such that

- if $S_{i}=H_{\ell}$ or $S_{i}=D_{\ell}$, then $S_{i}$ is of height at least $\ell$,
- if $S_{i}=H_{2}$ or $S_{i}=D_{2}$, then $i \geq 2$ and $S_{i-1} \in\left\{U, H_{1}, H_{2}\right\}$,
- $S_{n}=D_{1}$.

The second condition above is equivalent to the condition that the lattice path consists of the following combined steps for any $k \geq 0$ :

$$
\begin{equation*}
U H_{2}^{k}, U H_{2}^{k} D_{2}, H_{1} H_{2}^{k}, H_{1} H_{2}^{k} D_{2}, H_{0}, D_{1} \tag{2}
\end{equation*}
$$

Let $A(n)$ denote the set of lattice paths of length $n$ consisting of the combined steps in (2) such that $H_{2}$ does not touch the $x$-axis. Let $B(n)$ denote the set of


Figure 3. An example of $f_{0}$.


Figure 4. Definition of $f$.

3-Motzkin paths of length $n$ such that each $H_{2}$ touching the $x$-axis must occur after $D, H_{0}$ or $H_{2}$.

We define $f_{0}: A(n) \rightarrow B(n)$ as follows. Let $L \in A(n)$. Then $f_{0}(L)$ is defined to be the lattice path obtained from $L$ by changing $U H_{2}^{k} D_{2}$ to $H_{0} H_{2}^{k+1}, H_{1} H_{2}^{k} D_{2}$ to $D H_{2}^{k+1}$ and $D_{1}$ to $D$. It is easy to see that $f_{0}(L) \in B$ and $f_{0}$ is invertible. See Figure 3.

Now we define $f: \mathrm{CH}_{2}^{\prime}(n) \rightarrow \operatorname{MOT}_{3}(n-2)$ as follows. Let $L \in \mathrm{CH}_{2}^{\prime}(n)$. Then $L$ is decomposed uniquely as

$$
H_{0}^{k_{1}}\left(U L_{1} D_{1}\right) H_{0}^{k_{2}}\left(U L_{2} D_{1}\right) \cdots H_{0}^{k_{r}}\left(U L_{r} D_{1}\right)
$$

where $L_{i} \in A\left(n_{i}\right)$ for some $k_{i}, n_{i} \geq 0$ and $r \geq 1$. Then define $f(L)$ to be

$$
H_{2}^{k_{1}} f_{0}\left(L_{1}\right)\left(H_{1} H_{2}^{k_{2}+1} f_{0}\left(L_{2}\right)\right)\left(H_{1} H_{2}^{k_{3}+1} f_{0}\left(L_{3}\right)\right) \cdots\left(H_{1} H_{2}^{k_{r}+1} f_{0}\left(L_{r}\right)\right)
$$

See Figure 4
Theorem 2.1. The map $f: \mathrm{CH}_{2}^{\prime}(n) \rightarrow \operatorname{MOT}_{3}(n-2)$ is a bijection.
Proof. Each $L \in \operatorname{MOT}_{3}(n-2)$ is uniquely decomposed as

$$
H_{2}^{k_{1}} L_{1}\left(H_{1} H_{2}^{k_{2}+1} L_{2}\right)\left(H_{1} H_{2}^{k_{3}+1} L_{3}\right) \cdots\left(H_{1} H_{2}^{k_{r}+1} L_{r}\right),
$$

where $L_{i} \in B\left(n_{i}\right)$ for some $k_{i}, n_{i} \geq 0$ and $r \geq 1$. Thus we have the inverse $f^{-1}(L)$ which is decomposed as

$$
H_{0}^{k_{1}}\left(U f_{0}^{-1}\left(L_{1}\right) D_{1}\right) H_{0}^{k_{2}}\left(U f_{0}^{-1}\left(L_{2}\right) D_{1}\right) \cdots H_{0}^{k_{r}}\left(U f_{0}^{-1}\left(L_{r}\right) D_{1}\right)
$$



Figure 5. An example of $g$. Odd peaks are circled. The horizontal steps of even height are dashed and colored blue.
2.2. The bijection $g: \operatorname{MOT}_{3}(n) \rightarrow \operatorname{SCH}_{\text {odd }}(n+1)$. We define $g: \operatorname{MOT}_{3}(n) \rightarrow$ $\mathrm{SCH}_{\text {odd }}(n+1)$ as follows. Let $L \in \operatorname{MOT}_{3}(n)$. Then $g(L)$ is the lattice path obtained from $L$ by doing the following.
(1) Change $U$ to $U U, D$ to $D D, H_{0}$ to $H^{2}, H_{1}$ to $D U$, and $H_{2}$ to $U D$.
(2) Add $U$ at the beginning and $D$ at the end.
(3) Change all the consecutive steps $U D$ which form a peak of odd height to $H^{2}$.
See Figure 5 for an example of $g$.
Theorem 2.2. The map $g: \operatorname{MOT}_{3}(n) \rightarrow \mathrm{SCH}_{\text {odd }}(n+1)$ is a bijection.
Proof. Clearly the first and the second steps in the construction of $g$ are invertible. The third step is also invertible because every step $H^{2}$ of even height always comes from a peak of odd height. Thus $g$ is invertible.
2.3. The bijection $h: \mathrm{SCH}_{\text {odd }}(n) \rightarrow \mathrm{SCH}_{\mathrm{UH}}^{\prime}(n)$. Let $L=S_{1} S_{2} \cdots S_{k}$ be a Schröder path. For any up step $S_{i}=U$ of $L$, there is a unique down step $S_{j}=D$ such that $i<j$ and $S_{i+1} S_{i+2} \cdots S_{j-1}$ is a (possibly empty) lattice path. We call such $S_{j}$ the down step corresponding to $S_{i}$. We also call $S_{i}$ the up step corresponding to $S_{j}$.

For a UH-pair $\left(S_{i}, S_{i+1}\right)$, i.e. $S_{i}=U$ and $S_{i+1}=H^{2}$, we define the function $\xi$ as follows.

$$
\xi\left(S_{i}, S_{i+1}\right)= \begin{cases}i, & \text { if } S_{i+1} \text { is of even height } \\ j, & \text { if } S_{i+1} \text { is of odd height }\end{cases}
$$

where $j$ is the integer such that $S_{j}$ is the down step corresponding to $S_{i}$. If $L$ is not UH-free, we define the $\xi$-maximal UH-pair of $L$ to be the UH-pair ( $S_{i}, S_{i+1}$ ) with the largest $\xi$ value.


Figure 6. The essence of $h_{0}$. Red (resp. Dashed blue) color is for UH-pairs whose horizontal step is of odd (resp. even) height. Odd peaks are circled. The lattice path $L^{\prime}$ is not empty.

Now let $L=S_{1} S_{2} \cdots S_{k} \in \mathrm{SCH}_{\text {odd }}(n)$. If $L$ is not UH-free, we define $h_{0}(L)$ as follows. Suppose $\left(S_{i}, S_{i+1}\right)$ is the $\xi$-maximal UH-pair of $L$, and $S_{j}$ is the down step corresponding to $S_{i}$.
(1) If $S_{i+1}$ is of even height, then $h_{0}(L)$ is the lattice path obtained from $L$ by replacing $S_{i} S_{i+1}$ with $U U D$.
(2) If $S_{i+1}$ is of odd height, then let $L^{\prime}=S_{i+2} S_{i+3} \cdots S_{j-1}$.
(a) If $L^{\prime}$ is empty, i.e., $j=i+2$, then $h_{0}(L)$ is the lattice path obtained from $L$ by replacing $S_{i} S_{i+1} S_{i+2}$ with $H^{2} U D$.
(b) If $L^{\prime}$ is not empty, then $h_{0}(L)$ is the lattice path obtained from $L$ by replacing $S_{i} S_{i+1} \cdots S_{j}$ with $U L^{\prime} D U D$.
See Figure 6 .
Now we define $h: \mathrm{SCH}_{\text {odd }}(n) \rightarrow \mathrm{SCH}_{\mathrm{UH}}^{\prime}(n)$ as follows. Let $L \in \mathrm{SCH}_{\text {odd }}(n)$ and $L_{0}=L$. Then we define $L_{i}=h_{0}\left(L_{i-1}\right)$ for $i \geq 1$ if $L_{i-1}$ is not UH-free. Since the number of UH-free pairs of $L_{i}$ is one less than that of $L_{i-1}$, or they are the same and

$$
\xi\left(\text { the maximal UH-pair of } L_{i}\right)<\xi\left(\text { the maximal UH-pair of } L_{i-1}\right),
$$

we always get $L_{r}$ which is UH-free for some $r$. We define $h(L)$ to be $L_{r}$ if $L_{r}$ does not start with $H^{2}$; and the lattice path obtained from $L_{r}$ by replacing $H^{2}$ with $U D$ otherwise. For an example, see Figure 7 .

Theorem 2.3. The map $h: \mathrm{SCH}_{\text {odd }}(n) \rightarrow \mathrm{SCH}_{\mathrm{UH}}^{\prime}(n)$ is a bijection.
Proof. In the procedure of $h$, the odd peaks are constructed from right to left. Since $h_{0}$ is invertible, so is $h$.
2.4. The bijection $\iota: \mathrm{SCH}_{\text {odd }}(n) \rightarrow \mathrm{SCH}_{\text {even }}^{\prime}(n)$. For $L=S_{1} S_{2} \cdots S_{k} \in \mathrm{SCH}_{\text {odd }}(n)$, we define $\iota(L)$ as follows.
(1) If $S_{k}=H^{2}$, then $\iota(L)=U S_{1} \cdots S_{k-1} D$.
(2) If $S_{k}=D$, then let $S_{i}$ be the up step corresponding to $S_{k}$ and we define $\iota(L)=U S_{1} \cdots S_{i-1} D S_{i+1} \cdots S_{k-1}$.
See Figure 8 .
Then $\iota(L) \in \mathrm{SCH}_{\text {even }}^{\prime}(n)$. Clearly, $\iota: \mathrm{SCH}_{\text {odd }}(n) \rightarrow \mathrm{SCH}_{\text {even }}^{\prime}(n)$ is a bijection.

$\Downarrow$ Changing the first $H^{2}$


Figure 7. An example of $h$. Red (resp. Dashed blue) color is for UH-pairs whose horizontal step is of odd (resp. even) height. Odd peaks are circled. Dashed arrows indicate the down steps corresponding to the up steps.

## 3. A direct bijection between $\mathrm{NC}_{2}(n)$ and $P_{12312}(n)$

Now we have a bijection $\phi \circ h \circ g \circ f \circ \psi: \mathrm{NC}_{2}^{\prime}(n) \rightarrow P_{12312}^{\prime}(n)$, see Figure 2, As noted in the introduction, this induces a bijection between $\mathrm{NC}_{2}(n)$ and $P_{12312}(n)$. Since both $\mathrm{NC}_{2}(n)$ and $P_{12312}(n)$ are partitions with some conditions, it is natural to ask a direct bijection between them. In this section we find such a direct bijection.


Figure 8. The map $\iota$.
From now on, we will identify a partition in $P_{12312}(n)$ with its canonical word.
A marked partition is a partition in which each part may be marked. Similarly a marked word is a word in which each letter may be marked.

Let $\pi \in \mathrm{NC}_{2}(n)$. For $i \in[n]$, let $T_{i}$ be the marked partition of $[i]$ obtained from $\pi$ by removing all the integers greater than $i$ and by marking integers which are connected to an integer greater than $i$. Using the sequence $\emptyset=T_{0}, T_{1}, T_{2}, \ldots, T_{n}=$ $\pi$ of marked partitions, we define a sequence of marked words $\mathbf{w}_{0}, \mathbf{w}_{1}, \mathbf{w}_{2}, \ldots, \mathbf{w}_{n}$ as follows. Here, if $(i, j)$ is an edge we say that $j$ is connected to $i$.

Let $\mathbf{w}_{0}$ be the empty word. For $1 \leq i \leq n, \mathbf{w}_{i}$ is defined as follows.
(1) If $i$ is not connected to any integer in $T_{i}$, then $\mathbf{w}_{i}=\mathbf{w}_{i-1} m$, where $m=$ $\max \left(\mathbf{w}_{i-1}\right)+1$. Otherwise, $i$ is connected to either the largest marked integer or the second largest marked integer of $T_{i-1}$.

- If $i$ is connected to the largest marked integer of $T_{i-1}$, then let $\mathbf{w}_{i}=$ $\mathbf{w}_{i-1} a_{1}$, where $a_{1}$ is the rightmost marked letter of $\mathbf{w}_{i-1}$. And then we make the marked letter $a_{1}$ unmarked.
- If $i$ is connected to the second largest marked integer of $T_{i-1}$, then let $\mathbf{w}_{i}=\mathbf{w}_{i-1} a_{2}$, where $a_{2}$ is the second rightmost marked letter of $\mathbf{w}_{i-1}$. The second rightmost marked letter of $\mathbf{w}_{i-1}$ remains marked, however, we make the rightmost marked letter of $\mathbf{w}_{i-1}$ unmarked in $\mathbf{w}_{i}$.
(2) If $i$ is marked in $T_{i}$, then we find the largest letters in $\mathbf{w}_{i}$ and make the leftmost letter among them marked.
For an example, see Figure 9
Lemma 3.1. The word $\mathbf{w}_{n}$ obtained above is 12312-avoiding.
Proof. Suppose $\mathbf{w}_{n}$ has a subsequence $a b c a b$ where $a<b<c$. When the second $b$ is added the first $b$ must have been marked. Moreover, the first $b$ must have been marked before adding the second $a$ because an unmarked integer becomes marked only if it is the largest integer (in this case at least $c$ ) in the sequence. Thus when the second $a$ is added, the first $a$ and $b$ have been marked. Since the first $a$ is the second rightmost marked integer at this moment, we must unmark the rightmost marked integer, the first $b$, and mark the largest integer which is at least $c$. Thus after this process, $b$ cannot be marked and we cannot have the second $b$, which is a contradiction.

If we know $\mathbf{w}_{n}$, we can reverse this procedure. For $1 \leq i \leq n, \mathbf{w}_{i-1}$ is obtained from $\mathbf{w}_{i}$ as follows. Suppose $m=\max \left(\mathbf{w}_{i}\right)$ and $t$ is the last letter of $\mathbf{w}_{i}$.
(1) If the leftmost $m$ is marked in $\mathbf{w}_{i}$, then make it unmarked.


Figure 9. $T_{i}$ 's and corresponding $\mathrm{w}_{i}$ 's. Marked integers and marked letters are circled.
(2) If $t$ appears only once in $\mathbf{w}_{i}$ (equivalently $t$ is greater than any other letters in $\mathbf{w}_{i}$ ), then we simply remove $t$. Otherwise, find the leftmost $t$ in $\mathbf{w}_{i}$.

- If the leftmost $t$ is unmarked, then we remove the last letter $t$ and make the leftmost $t$ marked.
- If the leftmost $t$ is marked, then we must have $t<m$ since we have made the leftmost $m$ unmarked. In this case we remove the last $t$, and make the leftmost $t$ still marked and the leftmost $m$ marked.

Now we construct $T_{0}, T_{1}, \ldots, T_{n}$ as follows. Let $T_{0}=\emptyset$. For $1 \leq i \leq n, T_{i}$ is obtained as follows.
(1) First, let $T_{i}$ be the marked partition obtained from $T_{i-1}$ by adding $i$.
(2) If the last letter of $\mathbf{w}_{i}$ is equal to the rightmost (resp. the second rightmost) marked letter of $\mathbf{w}_{i-1}$, then connect $i$ to the largest (resp. the second largest) marked integer, say $j$, of $T_{i-1}$, and make $j$ unmarked.
(3) Let $m=\max \left(\mathbf{w}_{i}\right)$. If the leftmost $m$ is marked in $\mathbf{w}_{i}$, then make $i$ marked in $T_{i}$.
It is easy to check that this is the inverse map. Thus we get the following theorem.

Theorem 3.2. For $\pi \in \mathrm{NC}_{2}(n)$, the map $\pi \mapsto \mathbf{w}_{n}$ is a bijection from $\mathrm{NC}_{2}(n)$ to $P_{12312}(n)$.

The bijection $\pi \mapsto \mathbf{w}_{n}$ is different from the composition $\phi \circ h \circ g \circ f \circ \psi$. For instance, if $\pi=(\{1,3\},\{2\})$, then $\mathbf{w}_{3}=121$ but $(\phi \circ h \circ g \circ f \circ \psi)(\pi)=112$.

Note that both $\mathrm{NC}_{2}(n)$ and $P_{12312}(n)$ contain noncrossing partitions. It would be interesting to find a bijection between $\mathrm{NC}_{2}(n)$ and $P_{12312}(n)$ which sends noncrossings partitions to noncrossings partitions.

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