# Independent subsets of powers of paths, and Fibonacci cubes 

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#### Abstract

We provide a formula for the number of edges of the Hasse diagram of the independent subsets of the $h^{\text {th }}$ power of a path ordered by inclusion. For $h=1$ such a value is the number of edges of a Fibonacci cube. We show that, in general, the number of edges of the diagram is obtained by convolution of a Fibonacci-like sequence with itself.


Keywords: Independent subset, path, power of graph, Fibonacci cube.

## 1 Introduction

For a graph $\mathbf{G}$ we denote by $V(\mathbf{G})$ the set of its vertices, and by $E(\mathbf{G})$ the set of its edges.
Definition 1.1 For $n, h \geq 0$, the $h$-power of a path, denoted by $\mathbf{P}_{n}^{(h)}$, is a graph with $n$ vertices $v_{1}, v_{2}, \ldots, v_{n}$ such that, for $1 \leq i, j \leq n$, $i \neq j$, $\left(v_{i}, v_{j}\right) \in E\left(\mathbf{P}_{n}^{(h)}\right)$ if and only if $|j-i| \leq h$.

[^0]Thus, for instance, $\mathbf{P}_{n}^{(0)}$ is the graph made of $n$ isolated nodes, and $\mathbf{P}_{n}^{(1)}$ is the path with $n$ vertices.

Definition 1.2 An independent subset of a graph $\mathbf{G}$ is a subset of $V(\mathbf{G})$ not containing adjacent vertices.
Notation. (i) We denote by $p_{n}^{(h)}$ the number of independent subsets of $\mathbf{P}_{n}^{(h)}$. (ii) We denote by $\mathbf{H}_{n}^{(h)}$ the Hasse diagram of the poset of independent subsets of $\mathbf{P}_{n}^{(h)}$ ordered by inclusion, and by $H_{n}^{(h)}$ the number of edges of $\mathbf{H}_{n}^{(h)}$.

In this work we evaluate $p_{n}^{(h)}$, and $H_{n}^{(h)}$. Our main result (Theorem 3.4) is that, for $n, h \geq 0$, the sequence $H_{n}^{(h)}$ is obtained by convolving the sequence $\underbrace{1, \ldots, 1}_{h}, p_{0}^{(h)}, p_{1}^{(h)}, p_{2}^{(h)}, \ldots$ with itself.

Clearly, $\mathbf{H}_{n}^{(0)}$ is the $n$-dimensional cube. Thus, on one hand, our work generalizes the known formula $n 2^{n-1}$ for the number of edges of the Boolean lattice with $n$ atoms, obtained by the convolution of the sequence $\left\{2^{n}\right\}_{n \geq 0}$ with itself. From a different perspective, this work could be seen as yet another generalization of the notion of Fibonacci cube. Indeed, observe that every independent subset $S$ of $\mathbf{P}_{n}^{(h)}$ can be represented by a binary string $b_{1} b_{2} \cdots b_{n}$, where, for $i=1, \ldots, n, b_{i}=1$ if and only if $v_{i} \in S$. More specifically, each independent subset of $\mathbf{P}_{n}^{(h)}$ is associated with a binary string of length $n$ such that the distance between any two 1's of the string is greater than $h$. For $h=1$ the binary strings associated with independent subsets of $\mathbf{P}_{n}^{(h)}$ are Fibonacci strings of order $n$, and the Hasse diagram of the set of all such strings ordered bitwise is a Fibonacci cube of order $n$ (see [5,7]). Fibonacci cubes were introduced as an interconnection scheme for multicomputers in [3], and their combinatorial structure has been further investigated, e.g. in $[6,7]$. Several generalizations of the notion of Fibonacci cubes has been proposed (see, e.g., $[4,5]$ ). As far as we now, our generalization, described in terms of independent subsets of powers of paths ordered by inclusion, is a new one.

## 2 The independent subsets of powers of paths

We denote by $p_{n, k}^{(h)}$ the number of independent $k$-subsets of $\mathbf{P}_{n}^{(h)}$.
Lemma 2.1 For $n, h, k \geq 0, p_{n, k}^{(h)}=\binom{n-h k+h}{k}$.
Proof. See [2, Theorem 1], and [1], where we establish a bijection between independent $k$-subset of $\mathbf{P}_{n}^{(h)}$ and $k$-subsets of a set with $(n-h k+h)$ elements.

For $n, h \geq 0$, the number of all independent subsets of $\mathbf{P}_{n}^{(h)}$ is

$$
p_{n}^{(h)}=\sum_{k=0}^{[n /(h+1)\rceil} p_{n, k}^{(h)}=\sum_{k=0}^{\lceil n /(h+1)\rceil}\binom{n-h k+h}{k} .
$$

Remark 2.2 Denote by $F_{n}$ the $n^{\text {th }}$ element of the Fibonacci sequence $F_{1}=1$, $F_{2}=1$, and $F_{i}=F_{i-1}+F_{i-2}$, for $i>2$. Then, $p_{n}^{(1)}=F_{n+2}$.
Lemma 2.3 For $n, h \geq 0, p_{n}^{(h)}= \begin{cases}n+1 & \text { if } n \leq h+1, \\ p_{n-1}^{(h)}+p_{n-h-1}^{(h)} & \text { if } n>h+1 .\end{cases}$
Proof. See the first part of [2, Proof of Theorem 1], or [1].

## 3 The poset of independent subsets of powers of paths

Figure 1 shows a few Hasse diagrams $\mathbf{H}_{n}^{(h)}$. Notice that, as mentioned in the introduction, for each $n, \mathbf{H}_{n}^{(1)}$ is a Fibonacci cube.


Fig. 1. Some $\mathbf{H}_{n}^{(h)}$.
Since in $\mathbf{H}_{n}^{(h)}$ each non-empty independent $k$-subset covers exactly $k$ independent $(k-1)$-subsets, we can write

$$
\begin{equation*}
H_{n}^{(h)}=\sum_{k=1}^{\lceil n /(h+1)\rceil} k p_{n, k}^{(h)}=\sum_{k=1}^{\lceil n /(h+1)\rceil} k\binom{n-h k+h}{k} . \tag{1}
\end{equation*}
$$

Let now $T_{k, i}^{(n, h)}$ be the number of independent $k$-subsets of $\mathbf{P}_{n}^{(h)}$ containing the vertex $v_{i}$, and let, for $h, k \geq 0, n \in \mathbb{Z}, \bar{p}_{n, k}^{(h)}= \begin{cases}p_{0, k}^{(h)} & \text { if } n<0, \\ p_{n, k}^{(h)} & \text { if } n \geq 0 .\end{cases}$
Lemma 3.1 For $n, h, k \geq 0$, and $1 \leq i \leq n$,

$$
T_{k, i}^{(n, h)}=\sum_{r=0}^{k-1} \bar{p}_{i-h-1, r}^{(h)} \bar{p}_{n-i-h, k-1-r}^{(h)} .
$$

Proof. No independent subset of $\mathbf{P}_{n}^{(h)}$ containing $v_{i}$ contains any of the elements $v_{i-h}, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{i+h}$. Let $r$ and $s$ be non-negative integers whose
sum is $k-1$. Each independent $k$-subset of $\mathbf{P}_{n}^{(h)}$ containing $v_{i}$ can be obtained by adding $v_{i}$ to a ( $k-1$ )-subset $R \cup S$ such that
(a) $R \subseteq\left\{v_{1}, \ldots, v_{i-h-1}\right\}$ is an independent $r$-subset of $\mathbf{P}_{n}^{(h)}$;
(b) $S \subseteq\left\{v_{i+h+1}, \ldots, v_{n}\right\}$ is an independent $s$-subset of $\mathbf{P}_{n}^{(h)}$.

Viceversa, one can obtain each of this pairs of subsets by removing $v_{i}$ from an independent $k$-subset of $\mathbf{P}_{n}^{(h)}$ containing $v_{i}$. Thus, $T_{k, i}^{(n, h)}$ is obtained by counting independently the subsets of type (a) and (b). Noting that the subsets of type (b) are in bijection with the independent s-subsets of $\mathbf{P}_{n-i-h}^{(h)}$, the lemma is proved.

In order to obtain our main result, we prepare a lemma.
Lemma 3.2 For positive n,

$$
H_{n}^{(h)}=\sum_{k=1}^{\lceil n /(h+1)\rceil} \sum_{i=1}^{n} T_{k, i}^{(n, h)} .
$$

Proof. The inner sum counts the number of $k$-subsets exactly $k$ times, one for each element of the subset. That is, $\sum_{i=1}^{n} T_{k, i}^{(n, h)}=k p_{n, k}^{(h)}$. The lemma follows directly from Equation (1).

Next we introduce a family of Fibonacci-like sequences.
Definition 3.3 For $h \geq 0$, and $n \geq 1$, the $h$-Fibonacci sequence $\mathcal{F}^{(h)}=$ $\left\{F_{n}^{(h)}\right\}_{n \geq 1}$ is the sequence whose elements are

$$
F_{n}^{(h)}= \begin{cases}1 & \text { if } n \leq h+1 \\ F_{n-1}^{(h)}+F_{n-h-1}^{(h)} & \text { if } n>h+1 .\end{cases}
$$

From Lemma 2.3, and setting for $h \geq 0$, and $n \in \mathbb{Z}, \bar{p}_{n}^{(h)}= \begin{cases}p_{0}^{(h)} & \text { if } n<0, \\ p_{n}^{(h)} & \text { if } n \geq 0,\end{cases}$ we have that,

$$
\begin{equation*}
F_{i}^{(h)}=\bar{p}_{i-h-1}^{(h)}, \text { for each } i \geq 1 . \tag{2}
\end{equation*}
$$

Thus, we can write $\mathcal{F}^{(h)}=\underbrace{1, \ldots, 1}_{h}, p_{0}^{(h)}, p_{1}^{(h)}, p_{2}^{(h)}, \ldots$.
In the following, we use the discrete convolution operation $*$, as follows.

$$
\begin{equation*}
\left(\mathcal{F}^{(h)} * \mathcal{F}^{(h)}\right)(n) \doteq \sum_{i=1}^{n} F_{i}^{(h)} F_{n-i+1}^{(h)} . \tag{3}
\end{equation*}
$$

Theorem 3.4 For $n, h \geq 0$, the following holds.

$$
H_{n}^{(h)}=\left(\mathcal{F}^{(h)} * \mathcal{F}^{(h)}\right)(n)
$$

Proof. The sum $\sum_{k=1}^{[n /(h+1)\rceil} T_{k, i}^{(n, h)}$ counts the number of independent subsets of $\mathbf{P}_{n}^{(k)}$ containing $v_{i}$. We can also obtain such a value by counting the independent subsets of both $\left\{v_{1}, \ldots, v_{i-h-1}\right\}$, and $\left\{v_{i+h+1}, \ldots, v_{n}\right\}$. Thus, we have:

$$
\sum_{k=1}^{[n /(h+1)\rceil} T_{k, i}^{(n, h)}=\bar{p}_{i-h-1}^{(h)} \bar{p}_{n-h-i}^{(h)} .
$$

Using Lemma 3.2 we can write
$H_{n}^{(h)}=\sum_{k=1}^{[n /(h+1)\rceil} \sum_{i=1}^{n} T_{k, i}^{(n, h)}=\sum_{i=1}^{n} \sum_{k=1}^{[n /(h+1)\rceil} T_{k, i}^{(n, h)}=\sum_{i=1}^{n} \bar{p}_{i-h-1}^{(h)} \bar{p}_{n-h-i}^{(h)}$.
By Equation (2) we have $\sum_{i=1}^{n} \bar{p}_{i-h-1}^{(h)} \bar{p}_{n-h-i}^{(h)}=\sum_{i=1}^{n} F_{i}^{(h)} F_{n-i+1}^{(h)}$. By (3), the theorem is proved.

Further properties of coefficients $H_{n}^{(h)}$, and $p_{n}^{(h)}$ are discussed in [1]. Moreover, in [1] we investigate the case of powers of cycles, and its connection with Lucas cubes.

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