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Some results on caps and codes related to orthogonal Grassmannians — a preview

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Abstract

In this note we offer a short summary of some recent results, to be contained in a forthcoming paper [4], on projective caps and linear error correcting codes arising from the Grassmann embedding ε_k^{gr} of an orthogonal Grassmannian Δ_k. More precisely, we consider the codes arising from the projective system determined by $\varepsilon_k^{gr}(\Delta_k)$ and determine some of their parameters. We also investigate special sets of points of Δ_k which are met by any line of Δ_k in at most 2 points proving that their image under the Grassmann embedding is a projective cap.

Keywords: Polar spaces, orthogonal Grassmannians, Dual polar spaces, embeddings, caps, error correcting codes.

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1 Introduction

This note contains a quick preview of some new results on caps and linear error correcting codes related to the Grassmann embedding ε_k^{gr} of orthogonal Grassmannians. These results will be presented in full in a forthcoming paper [4]. In Section 2 we provide some preliminaries on the topic; in particular, Subsection 2.1 recalls some properties of orthogonal Grassmannians, while codes arising from projective systems are discussed in Subsection 2.2. Our results are outlined in Section 3.

2 Preliminaries

2.1 Orthogonal Grassmannians and their embeddings

Let $V := V(2n+1, q)$ be a $(2n+1)$–dimensional vector space over a finite field $\mathbb{F}_q$ endowed with a non–singular quadratic form η of Witt index n. For $1 \leq k \leq n$ denote by $\mathcal{G}_k$ the k–Grassmannian of $\mathrm{PG}(V)$ and by Δ_k its k–polar Grassmannian. Recall that the k–polar Grassmannian Δ_k is the proper subgeometry of $\mathcal{G}_k$ whose points are the k–subspaces of V which are totally singular for η; the lines of Δ_k are

- for $k < n$: $\ell_{X,Y} := \{Z \mid X \subset Z \subset Y,\ \dim(Z) = k\}$, with $\dim X = k-1$, $\dim Y = k+1$ and Y totally singular;
- for $k = n$: $\ell_X := \{Z \mid X \subset Z \subset X^\perp,\ \dim(Z) = n,\ Z \text{ totally singular}\}$, with X a totally singular $(n-1)$–subspace of V and $X^\perp$ its orthogonal with respect to η.

When $k = n$ the points of ℓ_X form a conic in the projective plane $\mathrm{PG}(X^\perp/X)$. Clearly, Δ_1 is just the orthogonal polar space of rank n associated to η; the geometry Δ_n can be regarded as the dual of Δ_1 and is thus called *orthogonal dual polar space* of rank n.

Given a point–line geometry $\Gamma = (\mathcal{P}, \mathcal{L})$ we say that an injective map $e\colon \mathcal{P} \to \mathrm{PG}(V)$ is a *projective embedding* of Γ if the following two conditions hold:

(1) $\langle e(\mathcal{P}) \rangle = \mathrm{PG}(V)$;

(2) e maps any line of Γ onto a projective line.

Following [20], (see also [5]), when condition (2) is replaced by

(2') e maps any line of Γ onto a non–singular conic of $\mathrm{PG}(V)$,

we say that e is a *Veronese embedding* of Γ.

Let now $W_k := \bigwedge^k V$. The *Grassmann* embedding $e_k^{gr} : \mathcal{G}_k \to \mathrm{PG}(W_k)$ maps the arbitrary k–subspace $\langle v_1, v_2, \ldots, v_k\rangle$ of V (that is a point of $\mathcal{G}_k$) to the point $\langle v_1 \wedge v_2 \wedge \cdots \wedge v_k\rangle$ of $\mathrm{PG}(W_k)$. Let $\varepsilon_k^{gr} := e_k^{gr}|_{\Delta_k}$ be the restriction of e_k^{gr} to Δ_k. For $k < n$, the mapping ε_k^{gr} is a projective embedding of Δ_k in the subspace $\mathrm{PG}(W_k^{gr}) := \langle \varepsilon_k^{gr}(\Delta_k)\rangle$ of $\mathrm{PG}(W_k)$ spanned by $\varepsilon_k^{gr}(\Delta_k)$. We call ε_k^{gr} the *Grassmann embedding of* Δ_k.

If $k = n$, then ε_n^{gr} is a Veronese embedding and maps the lines of Δ_n onto non–singular conics of $\mathrm{PG}(W_n)$. The dual polar space Δ_n affords also a projective embedding of dimension 2^n, namely the spin embedding ε_n^{spin}.

Suppose ν_{2^n} to be the usual quadratic Veronese map $\nu_{2^n} \colon V(2^n, \mathbb{F}) \to V(\binom{2^n+1}{2}, \mathbb{F})$. It is well known that ν_{2^n} defines a Veronese embedding of the point–line geometry $\mathrm{PG}(2^n - 1, \mathbb{F})$ in $\mathrm{PG}(\binom{2^n+1}{2} - 1, \mathbb{F})$, which will also be denoted by ν_{2^n}. The composition $\varepsilon_n^{\mathrm{vs}} := \nu_{2^n} \cdot \varepsilon_n^{spin}$ is a Veronese embedding of Δ_n in a subspace $\mathrm{PG}(W_n^{vs})$ of $\mathrm{PG}(\binom{2^n+1}{2} - 1, \mathbb{F})$: it is called the *Veronese–spin embedding* of Δ_n. Properties of Grassmann and Veronese–spin embedding, fundamental in order to obtain our results, are extensively investigated in [5] and [6].

2.2 Projective systems and Codes

Error correcting codes are an essential component to any efficient communication system, as they can be used in order to guarantee arbitrarily low probability of mistake in the reception of messages without requiring noise–free operation; see [13]. An $[N, K, d]_q$ projective system Ω is a set of N points in $\mathrm{PG}(K - 1, q)$ such that for any hyperplane Σ of $\mathrm{PG}(K - 1, q)$, we have $|\Omega \setminus \Sigma| \geq d$. Existence of $[N, K, d]_q$ projective systems is equivalent to that of projective linear codes with the same parameters. Indeed, given a projective system $\Omega = \{P_1, \ldots, P_N\}$, fix a reference system $\mathfrak{B}$ in $\mathrm{PG}(K - 1, q)$ and consider the matrix G whose columns are the coordinates of the points of Ω with respect to $\mathfrak{B}$. Then, G is the generator matrix of an $[N, K, d]$ code over $\mathbb{F}_q$, say $\mathcal{C} = \mathcal{C}(\Omega)$, uniquely defined up to code equivalence. Furthermore, as any word of $\mathcal{C}(\Omega)$ is of the form $c = mG$ for some row vector $m \in \mathbb{F}_q^K$, it is straightforward to see that the number of zeroes in c is the same as the number of points x of Ω lying on the hyperplane of equation $m \cdot x = 0$ where $m \cdot x = \sum_{i=1}^{K} m_i x_i$ and $m = (m_i)_1^K$, $x = (x_i)_1^K$. In particular, the minimum distance of $\mathcal{C}$ turns out to be $d = min\{|\Omega| - |\Omega \cap \Sigma| \colon \Sigma \text{ is a hyperplane of } \mathrm{PG}(\mathrm{k} - 1, \mathrm{q})\}$. This provides a geometric interpretation of the meaning of minimum distance.

The link between incidence structures $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ and codes is deep and it dates at least to [15]; we refer the interested reader to [1,3] and [19] for more details. Traditionally, two basic approaches have proved to be most fruitful: either to consider the incidence matrix of a structure as a generator matrix for a binary code, see for instance [12], or to consider an embedding of $\mathcal{S}$ in a projective space and study the code arising from the projective system thus determined or its dual; see e.g. [2,11,7] for codes related to the Segre embedding [16].

Codes based on projective Grassmannians have been first introduced in [17] as generalisations of Reed–Muller codes of the first order; see also [18]. We refer to [14,9,10] for some recent developments.

3 Main results

We investigate linear codes associated with the projective system $\varepsilon_k^{gr}(\Delta_k)$ determined by the image of the Grassmann embedding ε_k^{gr} of Δ_k obtaining the following parameters.

Theorem 3.1 *Let $\mathcal{C}_{k,n}$ be the code determined by the projective system of $\varepsilon_k^{gr}(\Delta_k)$ for $1 \leq k < n$. Then, the parameters of $\mathcal{C}_{k,n}$ are*

$$N = \frac{(q^{n-k+1}+1)(q^{n-k+2}+1)\dots(q^n+1)(q^{n-k+1}-1)(q^{n-k+2}-1)\dots(q^n-1)}{(q-1)(q^2-1)\dots(q^k-1)},$$

$$K = \begin{cases} \binom{2n+1}{k} & \text{for } q \text{ odd} \\ \binom{2n+1}{k} - \binom{2n+1}{k-2} & \text{for } q \text{ even} \end{cases}, \quad d \geq 2q^{k(n-k)} - 1.$$

As for the codes arising from dual polar spaces of small rank, we have the following result where the minimum distance is precisely computed.

Theorem 3.2 *(i) The code $\mathcal{C}_{2,2}$ arising from a dual polar space of rank 2 has parameters*

$$N = (q^2+1)(q+1), \qquad K = \begin{cases} 10 & \text{for } q \text{ odd} \\ 9 & \text{for } q \text{ even} \end{cases}, \qquad d = q^2(q-1).$$

(ii) The code $\mathcal{C}_{3,3}$ arising from a dual polar space of rank 3 has parameters

$$N = (q^3+1)(q^2+1)(q+1), \;\; K = 35, \;\; d = q^2(q-1)(q^3-1) \quad \text{for } q \text{ odd}$$

and

$$N = (q^3+1)(q^2+1)(q+1), \;\; K = 28, \;\; d = q^5(q-1) \quad \text{for } q \text{ even.}$$

In order to define polar m–caps of Δ_k, the general notion of (m, v)–set of a partial linear space has been introduced. It has been shown that, under the Grassmann embedding, the points of a polar m–cap (a set having the property that it is met by any line of Δ_k it in at most 2 points) are mapped onto the points of a projective cap; see also [8] for other projective caps contained in Grassmannians.

Theorem 3.3 *Suppose $1 \leq k \leq n$. Then,*

(i) *for any polar m-cap $\mathcal{C}$ of Δ_k, its image $\varepsilon_k^{gr}(\mathcal{C})$ is a projective cap of $\mathrm{PG}(W_k)$;*

(ii) *the set $\varepsilon_n^{gr}(\Delta_n)$ is a projective cap.*

We were also able to explicitly construct polar caps of Δ_k for $k \leq n$ and build a related design, as shown in the following.

Theorem 3.4 *For any $r \leq \lfloor k/2 \rfloor$, the polar Grassmannian Δ_k contains a polar 2^r-cap. This cap is explicitly determined and it is shown that it can be suitably represented by means of a Hadamard matrix in Sylvester form.*

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