Decomposition of bi-colored square arrays into balanced diagonals

Dani Kotlar and Ran Ziv

Department of Computer Science, Tel-Hai College, Upper Galilee, Israel

Abstract

Given an $n \times n$ array M ($n \ge 7$), where each cell is colored in one of two colors, we give a necessary and sufficient condition for the existence of a partition of M into n diagonals, each containing at least one cell of each color. As a consequence, it follows that if each color appears in at least 2n-1 cells, then such a partition exists. The proof uses results on completion of partial Latin squares.

1 Introduction

Let M be a $t \times n$ array with $t \leq n$. A diagonal in M is a subset of t cells of M such that no two cells are in the same row or in the same column. For a natural number k, such that $0 < k \leq n$, a k-coloring of M is an assignment of a color from a given set of k colors to each cell of M. Given a k-coloring of M, an l-transversal $(l \leq k)$ is a diagonal of M in which at least l distinct colors are represented. We call a diagonal D in a k-colored array M balanced if all k colors appear in D.

A known conjecture of Stein [12] asserts that for any n-coloring of an $n \times n$ array M, where each color appears in n cells, there exists an (n-1)-transversal. Stein's conjecture generalizes an earlier conjecture of Ryser and Brualdi [4, 10] which state that such a transversal exists for any n-coloring in which all colors in each row and each column are distinct.

A problem related to the Ryser-Brualdi-Stein Conjectures is the search for conditions allowing a decomposition of a k-colored $t \times n$ array into n disjoint t-transversals. For some conjectures and asymptotic results on the subject see [1, 2, 6, 7, 8].

In this paper we give a necessary and sufficient condition for a 2-colored $n \times n$ arrays to be partitioned into n disjoint balanced diagonals.

Definition 1.1. We call a subset A of cells in an $n \times n$ array *improper* if there exists $i, j \in [n]$ such that each cell in A lies either in row i or in column j but not in both. Otherwise, a set is called *proper*.

Figure 1 illustrates an improper set (marked with x's). Our main result is the following theorem:

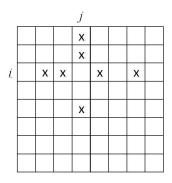


Figure 1

Theorem 1.2. Suppose $n \geq 7$ and let $M = (m_{ij})_{i,j=1}^n$ be an $n \times n$ array where each cell is colored red or blue. Then M can be partitioned into n balanced diagonals if and only if for each color there is a proper set of n cells colored with it.

The proof of Theorem 1.2 relies upon results on completion of partial Latin squares.

2 Completion of partial Latin squares

A Latin square of order n is an $n \times n$ array filled with the symbols $1, \ldots, n$ so that all symbols in each row and each column are distinct. A diagonal in a Latin square consisting of equal symbols is called a *symbol diagonal*. A partial Latin square of order n and size k is an $n \times n$ array in which exactly k cells are filled and no symbol appears more than once in a row or a column.

As a starting point for our discussion we quote the following well-known theorem, conjectured by Evans [5] and proved by Smetaniuk [11]. A different proof was given by Andersen and Hilton [3].

Theorem 2.1. A partial Latin square of order n and of size at most n-1 can be completed to a Latin square of order n.

Observation 2.2. Let M be an $n \times n$ array in which at least n-1 cells are colored blue. Then, there exists a partition of the cells of M into n disjoint diagonals, so that at least n-1 of them contain a blue cell.

Proof. We assign the symbols $1, \ldots, n-1$ to the n-1 blue cells and obtain a partial Latin square. By Theorem 2.1, we can complete it to a Latin square in which the symbol diagonals form a partition of M into diagonals, so that at least n-1 of them contains a blue cell.

In order to explore the case where a square array contains n blue cells we shall use the following theorem of Andersen and Hilton [3]:

Theorem 2.3. A partial Latin square of order n and of size n can be completed to a Latin square of order n, unless it can be brought by permuting rows and columns and possibly taking the transpose into one of the two forms depicted in Figure 2.

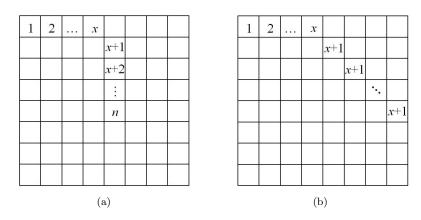


Figure 2

Observation 2.4. Let M be a colored $n \times n$ array containing a proper subset of n cells, which are all colored blue. Then, there is a partition of M into n diagonals, each containing a blue cell.

Proof. Let B be a proper set of blue cells of size n. We assign the symbols $1, \ldots, n$ to the cells of B to obtain a partial Latin square L. Since B is proper and properness is preserved under permutation of rows and columns and taking the transpose, it follows from Theorem 2.3 that L can be completed to a Latin square. The symbol diagonals of this Latin square form a partition of M into diagonals, each containing a blue cell.

Example 2.5. The array in Figure 3 shows that 2n-2 blue cells may not ensure the existence of a decomposition into diagonals, each containing a blue cell. Note that any diagonal containing the cell marked with an 'x' cannot contain a blue cell.

Since any set of 2n-1 cells is proper, and thus contains a proper subset of size n, we have the following observation:

Observation 2.6. Let M be a $n \times n$ array in which at least 2n-1 cells are colored blue. Then, there is a partition of M into n diagonals, each containing a blue cell.

3 Proof of the main result

For the proof of Theorem 1.2 we shall need the following theorem of Ryser [9]:

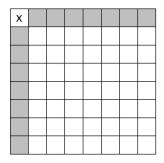


Figure 3

Theorem 3.1. Let 0 < r, s < n and let A be a partial Latin square of order n in which cell (i, j) in A is filled if and only if $i \le r$ and $j \le s$. Then A can be completed to a Latin square if and only if $N(i) \ge r + s - n$ for i = 1, ..., n, where N(i) is the number of cells in A that are filled with i.

Proof of Theorem 1.2. Let M_b and M_r be the subsets of M consisting of blue and red cells, respectively. Without loss of generality we may assume that $|M_b| \leq |M_r|$. If $|M_b| < n$, then clearly there is no decomposition of M into balanced diagonals. Hence, we may assume that $|M_b| \geq n$. If M_b does not contain a proper subset of size n, then M_b is improper. Suppose M_b is contained in row i and column j, then for any partition of M into diagonals, the diagonal through m_{ij} will be contained in M_r . Thus, the condition is necessary.

We now show that the condition of the theorem is sufficient. For contradiction, we make the following assumption:

Assumption 1. A decomposition of M into balanced diagonals does not exist.

The proof consists of a sequence of claims.

Claim 1. M_b contains two diagonals T_1 and T_2 such that $|T_1 \cap T_2| = 1$.

Proof of Claim 1. By Observation 2.4, there exists a decomposition of M into diagonals, each containing a blue cell. By Assumption 1, at least one of these diagonals is contained in M_b . We denote this diagonal by T_1 . If we put all the symbols $\{1,\ldots,n\}$ in T_1 , then clearly we have a partial Latin square that can be completed to a Latin square L'. By Assumption 1, at least one of the symbol diagonals of L' is contained in M_b . Let T_2 be one such diagonal. We have $T_1 \cup T_2 \subset M_b$ and $|T_1 \cap T_2| = 1$, since T_1 contains the distinct symbols $1,\ldots,n$ and T_2 contains the same symbol in all its cells.

Claim 2. Let $T_1 \cap T_2 = \{m_{ij}\}$. Then, there exists a cell in $M_b \setminus (T_1 \cup T_2)$ which is not in row i and not in column j.

Proof of Claim 2. Note that we can choose $\lceil n/2 \rceil$ columns $C_1, \ldots C_{\lceil n/2 \rceil}$ of M, such that $D = (\bigcup_{i=1}^{\lceil n/2 \rceil} C_i) \cap (T_1 \cup T_2)$ has size n (if n is even we take any set of n/2 columns that does not include column j. If n is odd we take any set of $\lceil n/2 \rceil$ columns that includes column j). If we put the symbols $1, \ldots, n$ in the cells of D, we can complete to a Latin square L'', by Theorem 2.3. By Assumption 1, L'' must have a symbol diagonal T_3 which is contained in M_b . Note that $|T_3 \cap D| = 1$ since D contains distinct symbols and T_3 has the same symbol in all its cells. We look at the set $T_3 \cap (\bigcup_{i=1}^{\lceil n/2 \rceil} C_i)$, which is of size $\lceil n/2 \rceil$. It contains one cell of D, possibly one cell from row i and possibly one cell from column j. Thus, the set of cells $T_3 \cap (\bigcup_{i=1}^{\lceil n/2 \rceil} C_i)$ contains at least $\lceil n/2 \rceil - 3$ cells which are neither in $T_1 \cup T_2$ nor in column i nor in column i. This number is positive since we assumed $n \geq 7$.

Claim 3. The array M contains an $s \times t$ sub-rectangle R_1 , such that s+t=n, $s-1 \le t \le s+1$ and $|R_1 \cap M_b| \ge n$.

Proof of Claim 3. Let $T_1 \cap T_2 = \{m_{ij}\}$ and let $m_{kl} \in M_b \setminus (T_1 \cup T_2)$, as in Claim 2, that is $k \neq i$ and $l \neq j$. We regard T_1 and T_2 as two perfect matchings in $K_{n,n}$ and consider the subgraph G of $K_{n,n}$ consisting of the edges in $(T_1 \cup T_2) \setminus \{m_{ij}\}$. Since $T_1 \cap T_2 = \{m_{ij}\}$, it follows that G is the disjoint union of simple even cycles, each of length at least 4. We make the following two observations:

Observation 3.2. For any $k \leq 2n-2$, every induced subgraph of G with k edges, consisting of the union of cycles and possibly one path, has at most k+1 vertices.

Observation 3.3. For any two vertices u and v in G, there exists an induced subgraph H of G containing u and v, consisting of the union of cycles and possibly one path, such that |E(H)| = n - 1.

The proof of Observation 3.2 is left to the reader. We prove Observation 3.3.

Proof of Observation 3.3. First assume u and v are in the same simple cycle C of G. If $|C| \leq n-1$ we take C and add cycles and possibly one path (contained in a simple cycle) in G to obtain H as required. If |C| > n-1 we can take H to be a path in C containing u and v. Since |E(G)| = 2n-2, such a path H with n-1 edges always exists.

Now, assume u and v lie in disjoint cycles C_u and C_v of G, respectively. Since |E(G)| = 2n - 2, we may assume, without loss of generality, that C_u has size at most n-1. If the size of $C_u \cup C_v$ is greater than n-1 we take C_u and a path containing v from C_v to obtain H as required (in the case that C_u has size exactly n-1 we just add the path of length 0 consisting of v). In case the size of $C_u \cup C_v$ is less than n-1 we take $C_u \cup C_v$ and add possibly more cycles and possibly one path from G to obtain H.

Let e be the edge corresponding to m_{kl} . Since $k \neq i$ and $l \neq j$ and G consists of the edges in $(T_1 \cup T_2) \setminus \{m_{ij}\}$, the endpoints of e are in V(G). Let u and v be the endpoints of e. By Observations 3.2 and 3.3, there is an induced subgraph

H of G containing u and v, such that |E(H)| = n - 1 and $n - 1 \le |V(H)| \le n$. Let s and t be the sizes of the two sides of H. We have $n - 1 \le s + t \le n$ and, since H is the union of cycles and possibly one path, we must have that s and t differ by at most 1. The graph $H \cup \{e\}$ corresponds to a sub-rectangle of M of size $s \times t$. If s + t = n - 1 we augment this rectangle by a row or a column to satisfy s + t = n and $s - 1 \le t \le s + 1$.

Claim 4. M contains a $p \times q$ sub-rectangle R, such that p+q=n+1, $p-2 \le q \le p+2$, $|R \cap M_b| \ge n$ and $|R \cap M_r| \ge n$.

Proof of Claim 4. Let R_1 be a sub-rectangle of M as in Claim 3. If R_1 contains also n red cells, then we are done. Otherwise, since $|M_b| \leq |M_r|$, the square M must contain another $s \times t$ sub-rectangle R_2 with a majority of red cells and, since $n \geq 7$, $|R_2 \cap M_r| \geq n$. If R_2 contains n blue cells we are done, so we assume it does not. We can travel from R_1 to R_2 using an $s \times t$ sliding window (Figure 4), so that in each step we either drop a row and add a row or drop a column and add a column. Clearly, at some stage, by exchanging a single row or a single column we shall move from a rectangle R' containing a majority of blue cells to a rectangle R'' containing a majority of red cells. Let $R = R' \cup R''$. Clearly, R is a rectangle containing n blue cells and n red cells and its size is $p \times q$ satisfying p + q = s + t + 1 = n + 1 and p and q differ at most by 2.

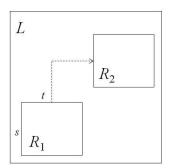


Figure 4

Claim 5. Let R be a sub-rectangle of M as in Claim 4. Then, we can fill n blue cells and n red cells of R with the numbers $1, \ldots, n$ so that each number appears once in a blue cell and once in a red cell, to form a partial Latin square.

Proof of Claim 5. Let X be a set of n blue cells in R and let Y be a set of n red cells in R. We form a bipartite graph G whose sides are the sets X and Y and we draw an edge between two vertices if and only if the corresponding cells are neither in the same row nor in the same column. We apply Hall's theorem to show that there is a perfect matching in G. Let $S \subset X$ such that |S| = k. Assume first that all the cells of S are in the same row or in the same column.

Since the largest side of R is of size at most $\frac{n+3}{2}$, then, $|N(S)| \geq n - (\frac{n+3}{2} - k)$. If we assume, for contradiction, that |N(S)| < k, then we get $k > n - (\frac{n+3}{2} - k)$ which leads to n < 3. Thus, we may assume that the cells of S are not all in the same row or in the same column. Note that in this case, |N(S)| = n unless the cells of S form an improper set, in which case |N(S)| = n - 1. But, the size of an improper set in R is at most (p-1) + (q-1) = n - 1. Thus, Hall's condition holds and the desired pairing exists.

We can now conclude the proof of Theorem 1.2. Let R be a sub-rectangle of M as in Claim 4. We fill n blue cells with $1, \ldots n$ and n red cells with $1, \ldots n$, as in Claim 5. Since for each cell in R there are (p-1)+(q-1)=n-1 other cells in R which are in the same row or in the same column and there are n symbols, all the cells in R can be filled to yield a partial Latin square L. We have p+q-n=1 and each symbol appears at least twice in R. By Theorem 3.1, L can be completed to a Latin square L'. Since each of the symbols $1, \ldots, n$ appears in a blue cell and in a red cell of R, all the symbol diagonals of L' are balanced. This completes the proof of Theorem 1.2.

Since any set of 2n-1 cells contains a proper subset of size n we have the following corollary:

Corollary 3.4. Let M be a 2-colored $n \times n$ array with $n \geq 7$. If each color appears in at least 2n-1 cells, then M can be partitioned into n balanced diagonals.

The results in this paper originated from questions on edge colorings of the complete bipartite graph $K_{n,n}$. Thus, we formulate Corollary 3.4 in these terms.

Definition 3.5. Let $f: E(K_{n,n}) \to \{1,2\}$ be a coloring. A matching in $P \subset E(K_{n,n})$ is called balanced if $f^{-1}(i) \cap P \neq \emptyset$ for i = 1, 2.

Theorem 3.6. Let $n \geq 7$ and let $f: E(K_{n,n}) \to \{1,2\}$ be a coloring. If $f^{-1}(i) \geq 2n-1$ for i=1,2, then there exists a partition of $E(K_{n,n})$ into n disjoint balanced matchings.

References

- [1] N. Alon and J. H. Kim, On the degree, size, and chromatic index of a uniform hypergraph, Journal of Combinatorial Theory, Series A 77 (1997), no. 1, 165–170.
- [2] N. Alon, J. Spencer, and P. Tetali, *Covering with latin transversals*, Discrete applied mathematics **57** (1995), no. 1, 1–10.
- [3] L. D. Andersen and A. J. W. Hilton, *Thank evans!*, Proceedings of the London Mathematical Society **3** (1983), no. 3, 507–522.

- [4] R. A. Brualdi and H. J. Ryser, Combinatorial matrix theory, Cambridge University Press, 1991.
- [5] T. Evans, *Embedding incomplete latin squares*, American Mathematical Monthly (1960), 958–961.
- [6] R. Häggkvist, and A. Johansson, Orthogonal Latin rectangles, Combinatorics, Probability and Computing 17 (2008), no. 04, 519–536.
- [7] A. J. W. Hilton, *Problem bcc* 13.20, Discrete Math. **125** (1994), 407–417.
- [8] N. Pippenger and J. Spencer, Asymptotic behavior of the chromatic index for hypergraphs, Journal of Combinatorial Theory, Series A 51 (1989), 24– 42.
- [9] H. J. Ryser, A combinatorial theorem with an application to Latin rectangles, Proceedings of the American Mathematical Society 2 (1951), no. 4, 550–552.
- [10] H. J. Ryser, Neuere probleme der kombinatorik, Vorträge über Kombinatorik, Oberwolfach, Matematisches Forschungsinstitute (Oberwolfach, Germany), July 1967, pp. 69–91.
- [11] B. Smetaniuk, A new construction on Latin squares i: A proof of the evans conjecture, Ars Combin 11 (1981), 155–172.
- [12] S. K. Stein, Transversals of Latin squares and their generalizations, Pacific J. Math 59 (1975), no. 2, 567–575.