Quasiperfect Domination in Trees

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Abstract

A $k-$*quasiperfect dominating set* ($k \geq 1$) of a graph G is a vertex subset S such that every vertex not in S is adjacent to at least one and at most k vertices in S. The cardinality of a minimum $k-$quasiperfect dominating set of G is denoted by $\gamma_{1k}(G)$. Those sets were first introduced by Chellali et al. (2013) as a generalization of the perfect domination concept (which coincides with the case $k = 1$) and allow us to construct a decreasing chain of quasiperfect dominating parameters

$$\gamma_{11}(G) \geq \gamma_{12}(G) \geq \ldots \geq \gamma_{1,\Delta}(G) = \gamma(G), \tag{1}$$

in order to indicate how far is G from being perfectly dominated. In this work, we study general properties, tight bounds, existence and realization results involving the parameters of the so-called *QP-chain* (1), for trees.

Keywords: Domination, Perfect domination, Quasiperfect domination, Trees.

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1 Introduction

Recall that a *tree* is a connected acyclic graph. A *leaf* is a vertex of degree 1 and vertices of degree at least 2 are *interior* vertices. We denote by $L(T)$ the set of leaves of a tree T and by $\ell(T)$ the number of leaves of T. A *support vertex* is a vertex having at least a leaf in its neighborhood and a *strong support vertex* is a support vertex adjacent to at least two leaves.

Given a graph G, a subset S of its vertices is a *dominating set* of G if every vertex v not in S is adjacent to at least one vertex in S. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G, and a dominating set of cardinality $\gamma(G)$ is called a γ-*code* [9].

An extreme way of domination occurs when every vertex not in S is adjacent to exactly one vertex in S. In that case, S is called a *perfect dominating set* [2] and $\gamma_{11}(G)$, the minimum cardinality of a perfect dominating set of G, is the *perfect domination number*. A dominating set of cardinality $\gamma_{11}(G)$ is called a γ_{11}-*code*.

In a perfect dominating set what is gained from the point of view of accuracy is lost in size, comparing it with a dominating set. Between both notions there is a graduation of definitions: k-quasiperfect domination. A *k-quasiperfect dominating set* for $k \geq 1$ (γ_{1k}-*set* for short) [7,11] is a dominating set S where every vertex not in S is adjacent to at most k vertices of S. Again the *k-quasiperfect domination number* $\gamma_{1k}(G)$ is the minimum cardinality of a γ_{1k}-set of G and a γ_{1k}-*code* is a γ_{1k}-set of cardinality $\gamma_{1k}(G)$.

Given a graph G of order n and maximum degree Δ, $\gamma_{1\Delta}$-sets are precisely dominating sets. Thus, one can construct the following chain of quasiperfect domination parameters:

$$n \geq \gamma_{11}(G) \geq \gamma_{12}(G) \geq \ldots \geq \gamma_{1\Delta}(G) = \gamma(G), \tag{2}$$

known as the quasiperfect chain of G, or simply the *QP-chain* of G.

2 Known general results

In this section, we review some results founded in the literature about quasiperfect parameters. Table 2 summarizes the values of parameters under consideration for some simple families of graphs.

Theorem 2.1 *[7] If G is a graph of order n verifying at least one of the following conditions: (1) $\Delta(G) \geq n-3$; (2) $\Delta(G) \leq 2$; (3) G is a cograph; (4) G is a claw-free graph, then $\gamma_{12}(G) = \gamma(G)$.*

	paths	cycles	cliques	stars	bicliques	wheels
G	P_n	C_n	K_n	$K_{1,n-1}$	$K_{p,n-p}$	W_n
n	$n \geq 3$	$n \geq 4$	$n \geq 2$	$n \geq 4$	$2 \leq p \leq n-p$	$n \geq 3$
$\Delta(G)$	2	2	$n-1$	$n-1$	$n-p$	$n-1$
$\gamma_{11}(G)$	$\lceil \frac{n}{3} \rceil$	$\lceil \frac{2n}{3} \rceil - \lfloor \frac{n}{3} \rfloor$	1	1	2	1
$\gamma_{12}(G)$	$\lceil \frac{n}{3} \rceil$	$\lceil \frac{n}{3} \rceil$	1	1	2	1
$\gamma(G)$	$\lceil \frac{n}{3} \rceil$	$\lceil \frac{n}{3} \rceil$	1	1	2	1

Proposition 2.2 *[3] Let $G = (V, E)$ a graph of order n.*

(i) *If $\gamma(G) \leq \Delta(G)$, then $\gamma_{1\gamma}(G) = \ldots = \gamma_{1\Delta}(G) = \gamma(G)$;*

(ii) *$\gamma_{1\delta}(G) < n$;*

(iii) *$\gamma_{11}(G) = 1$ if and only if $\Delta(G) = n - 1$.*

(iv) *$\gamma_{11}(G) \leq n - \ell(G)$ where $\ell(G)$ is the number of vertices of degree one.*

Theorem 2.3 *[3] Let k, n be positive integers such that $n \geq 6$ and $2 \leq k \leq n$. Then, there exists a graph G of order n such that $\Delta(G) = n-2$ and $\gamma_{11}(G) = k$.*

Theorem 2.4 *[3] Let (h, k, n) be a triple of integers such that $2 \leq h \leq 3$, $2 \leq k \leq n$ and $n \geq 9$. Then, there exists a graph G such that $|V(G)| = n$, $\Delta(G) = n - 3$, $\gamma(G) = h$ and $\gamma_{11}(G) = k$.*

Theorem 2.5 *[3] Let G be a graph of order n and $\Delta(G) = 3$, other than the bull graph. Then, $\gamma_{11}(G) \leq n - 3$.*

Proposition 2.6 *[3] Let G be either a cubic graph other than K_4, or a tree with order $n \geq 7$ and $\Delta(G) = 3$. Then, $\gamma_{11}(G) \leq n - 4$.*

The *join* $G = G_1 \vee G_2$ of graphs G_1 and G_2 is the graph such that $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1), v \in V(G_2)\}$.

Theorem 2.7 *[3] Let $G = G_1 \vee G_2$ be a join graph of order n. Then,*

(i) *$\gamma_{11}(G) = 1$ if and only if G_1 or G_2 have a universal vertex.*

(ii) *$\gamma_{11}(G) = 2$ if and only if both G_1 and G_2 have at least an isolated vertex.*

(iii) *$\gamma_{11}(G) = n$ in other case.*

Corollary 2.8 *[3] Let $G = G_1 \vee G_2$ be a connected cograph without universal vertices. Then, $\gamma_{11}(G) = 2$ if both G_1 and G_2 have at least an isolated vertex, and $\gamma_{11}(G) = n$ in any other case.*

Theorem 2.9 *[3] Let h, k, n be integers such that $4 \leq n$, $2 \leq h \leq k \leq n$ satisfying either $h + k \leq n$ or $3\,h + k + 1 \leq 2\,n$. Then, there exists a claw-free graph G of order n such that $\gamma(G) = h$ and $\gamma_{11}(G) = k$.*

The *corona* of a graph G, denoted by $\mathrm{cor}(G)$, is the graph obtained by attaching a leaf to each vertex of G.

Theorem 2.10 *[8,10] For any graph G the domination number satisfies $\gamma(G) \leq n/2$. And if G is a graph of even order n, then $\gamma(G) = n/2$ if and only if G is the cycle of order 4 or the corona of a connected graph.*

Graphs with odd order n and maximum domination number $\gamma(G) = \lfloor n/2 \rfloor$ are also completely characterized in [1], as a list of six graph classes.

Proposition 2.11 *[5] Let T be a tree of order $n \geq 3$. Then*

(i) *Every $\gamma - code$ of T contains all its strong support vertices.*

(ii) *Every $\gamma_{11} - code$ of T contains all its strong support vertices.*

(iii) $\gamma_{11}(T) \leq n/2$.

(iv) $\gamma_{11}(T) = n/2$ *if and only if* $\gamma(T) = n/2$ *if and only if* $T = cor(T')$ *for some tree* T'.

A tree for which removal of all its leaves results in a path is called a caterpillar.

Proposition 2.12 *[7] If T is a caterpillar, then $\gamma(T) = \gamma_{12}(T)$.*

3 Our results on Trees

Theorem 3.1 *[4] Let T be a tree. Then, $\gamma_{1k}(T) \leq \gamma(T) + \lceil \frac{\gamma(T)}{k} \rceil - 1$, for every integer $k \in \{1, \ldots, \Delta(T)\}$.*

Corollary 3.2 *For every tree T, $\gamma_{11}(T) \leq 2\gamma(T) - 1$.*

Remark 3.3 *This bound is not true for general graphs and the difference between both parameters can be as large as desired. For example, the graph displayed in Figure 1 satisfies $\gamma(G) = 2$ and $\gamma_{11}(G) = |V(G)| > 2\gamma(G) - 1$.*

Next, we present a realization theorem for the short chain $\gamma \leq \gamma_{11}(T) \leq 2\gamma - 1$. Note that, for every caterpillar T of order $n \geq 3$, Proposition 2.12

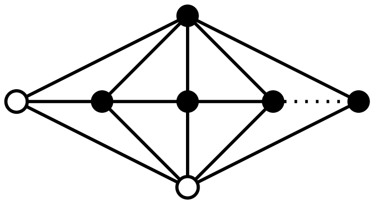

Fig. 1. The pair of white vertices form a γ-code.

and Corollary 3.2 just allow two possible situations, namely, either $\gamma(T) = \gamma_{11}(T) \le n/2$ or $\gamma(T) < \gamma_{11}(T) < n/2$. In the following result, we show that both of them are feasible and that parameters γ and γ_{11} can take every possible value in each case.

Proposition 3.4 *[4] Let a, b, n be positive integers.*

(i) *If $2 \le 2a \le n$, then there exists a caterpillar T of order n such that $\gamma(T) = \gamma_{11}(T) = a$.*

(ii) *If $2 \le a < b \le 2a - 1$ and $n > 2b$, then there exists a caterpillar T of order n such that $\gamma(T) = a$ and $\gamma_{11}(T) = b$.*

Proposition 3.5 *[4] A caterpillar T satisfies $\gamma_{11}(T) = 2\gamma(T) - 1$ if and only if belongs to the family shown in Figure 2.*

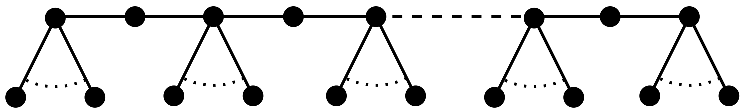

Fig. 2. Caterpillar with $\gamma_{11}(T) = 2\gamma(T) - 1$.

Let T a tree with maximum degree $\Delta \ge 3$. Next theorem shows that for each inequality of the QP-chain, both possibilities, the equality and the strict inequality, are feasible.

Theorem 3.6 *[4] There exists a tree with maximum degree $\Delta \ge 3$, satisfying each one of the $2^{\Delta-1}$ possible combinations of the inequalities of the QP- chain.*

Finally, we present the general form of the QP-chain in the case of k-ary trees, that has just two different terms.

Proposition 3.7 *[4] Let $T = T(k, h)$ the full k-ary tree of order $n = \dfrac{k^{h+1} - 1}{k - 1}$, where all leaves are at distance $h - 1$ from the root, with $k \ge 2$, $h \ge 3$. Then*

$$n - \ell(T) = \gamma_{11}(T) = \gamma_{12}(T) = \ldots = \gamma_{1,k-1}(T) > \gamma_{1,k}(T) = \gamma_{1,k+1}(T) = \gamma(T)$$

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Appendix

Proof of Theorem 3.1

Remark 1 Let T be a tree and S a dominating set. Then, since T has no cyles, every vertex not in S has at most one neighbor at each connected component of the subgraph $T[S]$.
Remark 2 Let T be a tree and S a dominating set such that the subgraph $T[S]$ has at most k connected components. Then, S is a γ_{1k}-set.

Let S be a γ-code of T. If S is also a γ_{1k}-set, then the inequality stated in the theorem holds.

Suppose on the contrary that S is not a γ_{1k}-set.

We construct a γ_{1k}-set S^* containing S and satisfying the inequality stated in the theorem. Let r be the number of connected components of the subgraph induced by S, denoted by $T[S]$. Then, $\gamma(T) \geq r$ and, by **Remark 2**, $r > k$.

Consider a vertex $x_0 \in V(T) \setminus S$ with at least $k+1$ neighbors in S and let $S_1 = S \cup \{x_0\}$. By **Remark 1**, all the neighbors of x_0 in S lie in different connected components of $T[S]$, therefore S_1 is a dominating set inducing a subgraph $T[S_1]$ with at most $r-k$ connected components. If S_1 is a γ_{1k}-set, let $S^* = S_1$.

Otherwise, consider a vertex $x_1 \in V(T) \setminus S_1$ having at least $k+1$ neighbors in S_1 and let $S_2 = S_1 \cup \{x_1\}$. By **Remark 1**, all the neighbors of x_1 in S_1 lie in different connected components of $T[S_1]$, therefore S_2 is a dominating set inducing a subgraph $T[S_2]$ with at most $(r-k)-k = r-2k$ connected components. If S_2 is a γ_{1k}-set, let $S^* = S_2$.

Otherwise, we repeat this procedure until we obtain a γ_{1k}-set. Observe that this procedure will end since the number of connected components induced by the sets $S_1, S_2, \ldots$ is strictly decreasing. Moreover, since $T[S_i]$ has at most $r-ik$ connected components, by **Remark 2**, S_i is a γ_{1k}-set whenever $r-ik \leq k$. Therefore, the number of steps needed in order to obtain that S_i is a γ_{1k}-set, is at most $i = \lceil \frac{r-k}{k} \rceil$.

Let $S^* = S_j$ be a γ_{1k}-set obtained in this way, where $j \leq \lceil \frac{r-k}{k} \rceil$. Then,

$$\gamma_{1k}(T) \leq |S^*| = |S| + j \leq \gamma(T) + \left\lceil \frac{r-k}{k} \right\rceil \leq \gamma(T) + \left\lceil \frac{\gamma(T)-k}{k} \right\rceil = \gamma(T) + \left\lceil \frac{\gamma(T)}{k} \right\rceil - 1.$$

Proof of Proposition 3.4

(i) Consider the caterpillar obtained by attaching a leaf to each of the first $a-1$ vertices of a path of order a and $n-2a+1 \geq 1$ leaves to the last vertex of the path (see Figure 3). Then the vertices of the path is both a γ-code and a γ_{11}-code, and $\gamma(T) = \gamma_{11}(T) = a$.

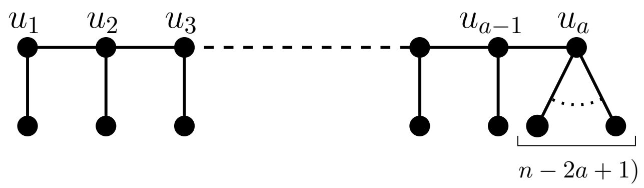

Fig. 3. T has order n, $\gamma(T) = \gamma_{11}(T) = a$.

(ii) Note that $\gamma(T) = 1$ implies $\gamma_{11}(T) = 1$, so if both parameter do not agree them $\gamma(T) \geq 2$.

Using that $1 \leq b-a \leq a-1$, let P be the path of order b with consecutive vertices labeled with

$$u_1, v_1, \ldots, u_{b-a}, v_{b-a}, u_{b-a+1}, u_{b-a+2}, \ldots, u_a$$

and consider the caterpillar obtained by attaching two leaves to each of the vertices $u_1, u_2, \ldots, u_{b-a}$, one leaf to each of the vertices $u_{b-a+2}, u_{b-a+3}, \ldots, u_a$ and $n-2b+1$ leaves to vertex u_{b-a+1} (see Figure 4). Since $n-2b+1 \geq 2$ we obtain that $\{u_1, u_2, \ldots, u_a\}$ is a γ-code with a vertices and $\{u_1, u_2, \ldots, u_a\} \cup \{v_1, \ldots, v_{b-a}\}$ is a γ_{11}-code with b vertices.

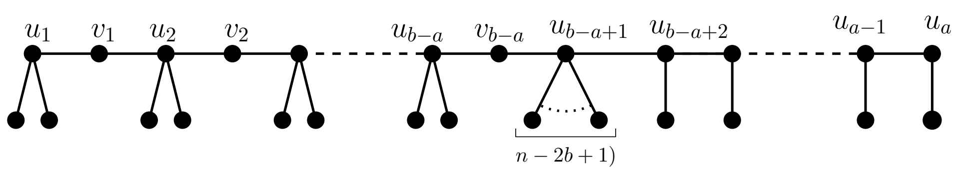

Fig. 4. T has order $n > 2b$, $a = \gamma(T) < \gamma_{11}(T) = b \leq 2a-1$.

Proof of Theorem 3.6

Remark 1 If u is a vertex of a graph G with at least d leaves in its neighborhood, then u is in every $\gamma_{1,h}$-code, for any $h \in \{1, \ldots, d-1\}$.
Remark 2 If G is a graph with maximum degree Δ and u is a vertex with at least $\Delta - 1$ leaves in its neighborhood, then u is in every $\gamma_{1,h}$-code, for any $h \in \{1, \ldots, \Delta-2\}$.
Remark 3 Let T be a tree with maximum degree Δ and s support vertices. Then $\gamma_{1,\Delta}(T) = \gamma(T) \geq s$.

Let $\Delta \geq 3$. For all $i \in \{1, \ldots, \Delta-1\}$, we write $\circledast_i$ for the symbol '=' or '>' in $\gamma_{1,i}(T) \geq \gamma_{1,i+1}(T)$.

(i) Case 1. If $\circledast_i$ is '=' for all $i \in \{1, \ldots, \Delta-2\}$. We distinguish two subcases.

(a) Case 1.1. If $\circledast_{\Delta-1}$ is '='. The complete bipartite graph $T = K_{1,\Delta}$ is a tree with maximum degree Δ satisfying:

$$\gamma_{11}(T) = \gamma_{12}(T) = \ldots = \gamma_{1,\Delta-1}(T) = \gamma_{1,\Delta}(T) = \gamma(T) = 1.$$

(b) Case 1.2. If $\circledast_{\Delta-1}$ is '>'. We consider the following tree T with maximum degree Δ: let u be a vertex of degree Δ adjacent to vertices $x_1, x_2, \ldots, x_\Delta$, and attach $\Delta - 1$ leaves to each x_i, $1 \leq i \leq \Delta$. Then, we easily derive from **Remark 2** that $\{x_1, \ldots, x_\Delta\}$ is a γ-code and $\{u, x_1, \ldots, x_\Delta\}$ is a $\gamma_{1,i}$-code for any i such that $i < \Delta$. Therefore, T satisfies

$$\Delta+1 = \gamma_{11}(T) = \gamma_{12}(T) = \ldots = \gamma_{1,\Delta-1}(T) > \gamma_{1,\Delta}(T) = \gamma(T) = \Delta.$$

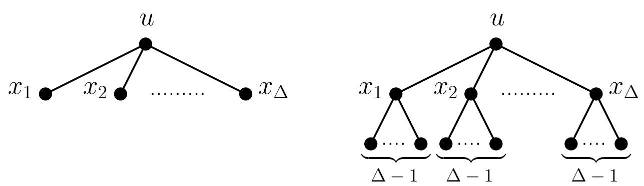

Fig. 5. Trees illustrating Case 1. of Theorem 3.6.

(ii) Case 2. If $\circledast_i$ is '>' for some $i \in \{1, \ldots, \Delta-2\}$.

If $\Delta = 3$, consider the graphs showed in Figure 6. The tree T on the left side satisfies $6 = \gamma_{11}(T) > \gamma_{12}(T) = \gamma_{1,3}(T) = \gamma(T) = 4$, since support vertices form a γ-code (and also a γ_{12}-code and a γ_{13}-code), and all vertices but the leaves form a γ_{11}-code. The tree T on the right side satisfies $\gamma_{11}(T) = 18 > \gamma_{12}(T) = 12 > \gamma_{1,3}(T) = \gamma(T) = 11$, since support vertices together with vertex u form a γ-code (and also a γ_{13}-code), support vertices together with vertices u and v form a γ_{12}-code, and all vertices but the leaves form a γ_{11}-code.

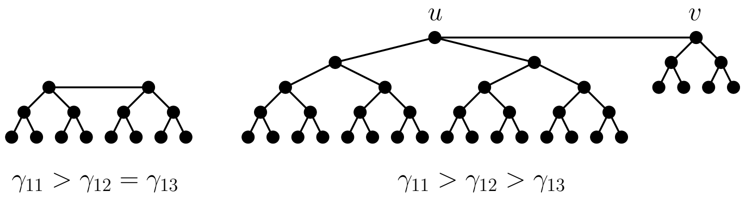

Fig. 6. Trees illustrating Case 2 of Theorem 3.6 when $\Delta = 3$.

Now suppose $\Delta \geq 4$. Let

$$\{i_1, i_2, \ldots, i_k\} = \{j \,:\, \gamma_{1,j}(T) > \gamma_{1,j+1}(T)\,,\, j \leq \Delta - 2\},$$

where $k \geq 1$ by hypotheses, and assume $1 \leq i_1 < \ldots < i_k \leq \Delta - 2$. We distinguish two subcases.

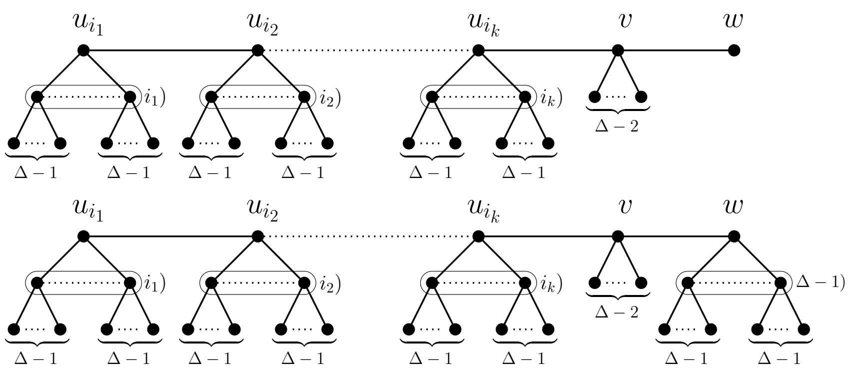

Fig. 7. Trees illustrating Case 2.1 (above) and Case 2.2 (bottom).

(a) Case 2.1. If $\circledast_{\Delta-1}$ is '$=$'.

Consider a path P of length $k+2$ with consecutive vertices labeled $u_{i_1}, \ldots, u_{i_k}, v, w$. Attach i_j new vertices to u_{i_j} and $\Delta - 1$ leaves to each one of those new vertices. Attach also $\Delta - 2$ leaves to vertex v.

For each vertex x of the path P, let $N'(x)$ be the set of vertices of $N(x)$ not belonging to the path P. Let $A = \cup_{j=1}^{k} N'(u_{i_j})$.

It is not hard to verify that $A \cup \{v\}$ is a γ-code of T, and also a $\gamma_{1,\Delta-1}$-code. Moreover, $A \cup \{v\} \cup \{u_{i_j} : h \leq j \leq k\}$ is a γ_{1i}-code if $i_{h-1} < i \leq i_h$.

(b) Case 2.2. If $\circledast_{\Delta-1}$ is '>'.

Consider the tree constructed in case 2.1 and attach $\Delta - 1$ new vertices to w and $\Delta - 1$ leaves to each one of those new vertices.

With the same notations as in Case 2.1, it is easy to verify that $A \cup \{v\} \cup N'(w)$ is a γ-code of T and $A \cup \{v, w\} \cup N'(w)$ is a $\gamma_{1,\Delta-1}$-code. Moreover, $A \cup \{v, w\} \cup N'(w) \cup \{u_{i_j} : h \leq j \leq k\}$ is a γ_{1i}-code if $i_{h-1} < i \leq i_h$.

Lemma 3.8 *Let T be a tree of order $n \geq k+1$ ($k \geq 2$) with all interior vertices of degree at least $k+1$, except at most one vertex of degree k, then $\gamma_{1,k-1}(T) = n - \ell(T)$.*

Proof. Notice that $V(T) \setminus L(T)$ is a $\gamma_{1,k-1}$-set for all $k \geq 2$. Suppose that S is a $\gamma_{1,k-1}$-code such that $S \neq V(T) \setminus L(T)$. If $V(T) \setminus L(T) \subset S$, then $|S| > |V(T) \setminus L(T)|$ which is a contradiction. Therefore, there exists a vertex $u_0 \in V(T) \setminus L(T)$ such that $u_0 \notin S$. Consider the connected component T_0 of u_0 in $T \setminus S$. Notice that T_0 is a tree of order $n_0 \geq 1$. If T_0 has only the vertex $u_0 \notin L(T)$, then u_0 is adjacent to at least k vertices of S, which is a contradiction. If T_0 has at least two vertices, T_0 has at least two leaves in T_0. Observe that a leaf w of T_0 can not be a leaf of T, otherwise the only neighbor of w is not in S, contradicting the fact that S is a dominating set. Therefore, T_0 has a leaf w_0 that is a vertex of degree al least $k+1$, implying that $\geq k$ neighbors of w_0 are in S, which is again a contradiction. $\square$

Proof of Proposition 3.7

The set of interior vertices of a tree is a $\gamma_{1,i}$-set for any $i \geq 1$. Therefore, by Lemma 3.8, $n - \ell(T) = \gamma_{11}(T) = \gamma_{12}(T) = \ldots = \gamma_{1,k-1}(T)$. On the other hand, for any $h \geq 3$ consider the set S described as follows:

$$S = \bigcup_{0 \leq i \leq r-1} L_{2+3i}, \text{ if } h = 3r,\ r \geq 1;$$

$$S = \{z\} \cup \bigcup_{1 \leq i \leq r} L_{3i}, \text{ where } z \in L_2, \text{ if } h = 3r+1,\ r \geq 1;$$

$$S = \bigcup_{0 \leq i \leq r} L_{1+3i}, \text{ if } h = 3r+2,\ r \geq 1.$$

Notice that S contains exactly the vertices of one of each three consecutive levels, taking into account that S must contain the strong support vertices, i.e., the vertices of level $h-1$, and in the case $h = 3r+1$ we have to add a vertex z of level 2 to dominate the root (see in Figure 8 an illustration of case $k = 2$).

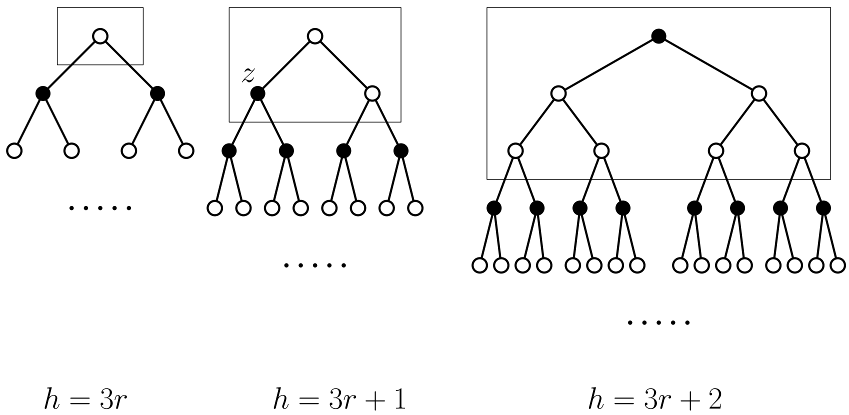

Fig. 8. If we add new groups of three levels in each case, being black vertices those of the middle level, the set of black vertices is a dominating code of $T(2,h)$, $h \geq 3$.

By construction, it is obvious that S is a $\gamma_{1,k}$-set and a $\gamma_{1,k+1}$-set, since a vertex not in S has at most k neighbors in S. We claim that S is a dominating code and consequently a $\gamma_{1,k}$-code and a $\gamma_{1,k+1}$-code. Let S be a dominating code of $T(k,h)$, $k \geq 2$, $h \geq 3$. We know that S contains all its strong support vertices, L_{h-1}, and these vertices dominate vertices of levels h, $h-1$ and $h-2$. So, we may assume that S does not contain any vertex of level $h-2$, otherwise we can change a vertex $x \in S \cap L_{h-2}$ by its neighbor in level $h-3$ obtaining also a dominating code. Therefore, S is obtained by adding a dominating code of the tree $T(k,h-3)$. Reasoning recursively, we deduce that S is a dominating code.