Forcing clique immersions through chromatic number *

Gregory Gauthier [†], Tien-Nam Le [‡], and Paul Wollan [§]

Abstract

Building on recent work of Dvořák and Yepremyan, we show that every simple graph of minimum degree 7t + 7 contains K_t as an immersion and that every graph with chromatic number at least 3.54t + 4 contains K_t as an immersion. We also show that every graph on n vertices with no stable set of size three contains $K_{2\lfloor n/5 \rfloor}$ as an immersion.

Keywords: Graph immersion, Hadwiger conjecture, chromatic number.

1 Introduction

1.1 Hadwiger's conjecture

The graphs in this paper are simple and finite, while multigraphs may have loops and multiple edges. A fundamental question in graph theory is the relationship between the chromatic number of a graph G and the presence of certain structures in G. One of the most well-known specific example of this type of question is the Four Color Theorem, which states that every planar graph is 4-colorable. Hadwiger [7] in 1943 proposed a far-reaching generalization of the Four Color Theorem, which asserts that for all positive integers t, every graph of chromatic number t contains K_t , the clique on t vertices, as a minor. In 1937, Wagner [16] proved that the Hadwiger's conjecture for t = 5 is equivalent to the Four Color Theorem. Robertson, Seymour, and Thomas [13] settle the conjecture for t = 6, while the conjecture is still open for $t \ge 7$. On the other hand, it was independently proved in 1984 by Kostochka and Thomasson [8, 14] that a graph without a K_t -minor is $O(k\sqrt{\log k})$ -colorable for every $k \ge 1$, and there has been no improvement in the order $k\sqrt{\log k}$ since then.

For graphs with no stable set of size three (i.e. there do not exist three vertices, all pairwise nonadjacent), Duchet and Meyniel [5] proposed an analogous conjecture to the Hadwiger's conjecture that every graph with n vertices and no stable set of size

^{*}This work supported by the European Research Council under the European Union's Seventh Framework Programme (FP7/2007-2013)/ERC Grant Agreement no. 279558.

[†]Princeton University, Princeton, NJ, USA, gjg2@math.princeton.edu

[‡]Laboratoire d'Informatique du Parallélisme, École Normale Supérieure de Lyon, France, tien-nam.le@ens-lyon.fr

[§]Department of Computer Science, University of Rome, "La Sapienza", Rome, Italy, wollan@di.uniroma1.it

three contains a $K_{\lceil n/2\rceil}$ -minor and proved that such graphs contain $K_{\lceil n/3\rceil}$ as a minor, which remains the best bound to date. Plumber, Stiebitz, and Toft [11] showed that the conjecture of Duchet and Meyniel is indeed equivalent to the Hadwiger's conjecture for graphs with no stable set of size three.

1.2 Graph immersion

In this paper, we focus on the immersion relation on graphs, which is a variant of minor relation (see [12]). We follow the definitions in [17]. Given loopless multigraphs G, H, we say that G admits an *immersion* of H if there exists functions $\pi_1 : V(H) \to V(G)$ and π_2 mapping the edges of H to paths of G satisfying the following:

- the map π_1 is an injection;
- for every edge $e \in E(H)$ with endpoints x and y, $\pi_2(e)$ is a path with endpoints equal to $\pi_1(x)$ and $\pi_2(y)$; and
- for edges $e, e' \in E(H), e \neq e', \pi_2(e)$ and $\pi_2(e')$ have no edge in common.

We say that G admits a strong immersion of H if the following condition holds as well.

• For every edge $e \in E(H)$ with endpoints x and y, the path $\pi_2(e)$ intersects the set $\pi_1(V(H))$ only in its endpoints.

The vertices $\{\pi_1(x) : x \in V(H)\}$ are the branch vertices of the immersion. We will also say that G (strongly) immerses H or alternatively that G contains H as a (strong) immersion.

We can alternately define immersions as follows. Let e_1 and e_2 be distinct edges in G such that the endpoints of e_1 are x, y and the endpoints of e_2 are y, z. To *split* off the edges e_1 and e_2 , we delete the edges e_1 and e_2 from G and add a new edge ewith endpoints x and z (note that this might result in a multi-edge or a loop). Then Gcontains H as an immersion if and only if H can be obtained from a subgraph of G by repeatedly splitting off pairs of edges and deleting isolated vertices.

We consider a variant of Hadwiger's conjecture to graph immersions due to Lescure and Meynial [10] in 1989 and, independently, to Abu-Khzam and Langston [1] in 2003. The conjecture explicitly states the following.

Conjecture 1.1 ([1], [10]). For every positive integer t, every graph with no K_t immersion is properly colorable with at most t - 1 colors.

Conjecture 1.1 is trivial for $t \leq 4$, and was independently proved by Lescure and Meyniel [10] and DeVos et al. [4] for $5 \leq t \leq 7$. One can immediately show that a minimum counterexample to Conjecture 1.1 has minimum degree t - 1. Thus, the conjecture provides additional motivation for the natural question of what is the smallest minimum degree necessary to force a clique immersion. DeVos et al. [2] showed that minimum degree 200t suffices to force a K_t immersion in a simple graph. This implies that every graph without a K_t -immersion is 200t-colorable, providing the first linear bound for Conjecture 1.1, while, as we discussed above, the best known bound for the Hadwiger's conjecture is superlinear. The bound 200t was recently improved by Dvořák and Yepremyan [3] to 11t + 7.

Theorem 1.2 (Dvořák–Yepremyan, [3]). Every graph with minimum degree at least 11t + 7 contains an immersion of K_t .

We give a new result on clique immersions in dense graphs; we leave the exact statement for Section 2 below. As a consequence, it is possible to improve the analysis in [3] and obtain the following bound.

Theorem 1.3. Every graph with minimum degree at least 7t + 7 contains an immersion of K_t .

Conjecture 1.1 can be relaxed to consider the following question.

Problem 1.4. What is the smallest function f such that for all positive t and all graphs G with $\chi(G) \ge f(t)$, it holds that G contains K_t as an immersion.

As observed above, a minimum counterexample to Conjecture 1.1 has minimum degree t-1. Thus by Theorem 1.3, we get that chromatic number at least f(t) = 7t + 8 forces a K_t immersion. By combining our results for dense graphs with arguments based on analyzing Kempe chains in proper colorings of graphs, we obtain the following improved bound.

Theorem 1.5. Every graph with chromatic number at least 3.54t + 4 contains an immersion of K_t .

For graphs with no stable set of size three, Vegara [15] proposed a similar conjecture as that of Duchet and Meyniel that every graph with n vertices and no stable set of size three contains a strong $K_{\lceil n/2 \rceil}$ -immersion and proved that it is equivalent to Conjecture 1.1 for graphs with no stable set of size three. In the same paper, Vegara showed that a relaxation to $K_{\lceil n/3 \rceil}$ -immersion holds. We improve this to $K_{2 \lfloor n/5 \rfloor}$.

Theorem 1.6. For every integer $n \ge 1$, every graph G with n vertices and no stable set of size three has a strong immersion of $K_{2|n/5|}$.

An extended abstract presenting Theorems 1.3 and 1.5 appeared in 2016 [9].

1.3 Notation

Given a multigraph G and distinct vertices $u, v \in V(G)$, if there are $k \geq 2$ edges between u and v, we say that uv is a multi-edge with multiplicity k, and if u is not adjacent to v, we say that uv is a missing edge. We denote by $N_G(v)$ the (non-repeated) set of neighbors of v in G, and by $d_G(v)$ the degree of v in G (where a loop is counted 2 and a multi-edge with multiplicity k is counted k). We denote by $E_G(v)$ the multi-set of edges (loops are excluded) incident with v (if uv is a multi-edge of multicity k then there are k edges uv in $E_G(v)$). Given $X \subseteq V(G)$, we denote by $f_G(v|X)$ the number of vertices in $X \setminus \{v\}$ which are not adjacent to v in G, and we write $f_G(v) = f_G(v|V(G))$ for short. When it is clear in the context, we omit the subscript G in this notation. Note that if

G is simple, then d(v) = |N(v)| = |E(v)| = |V(G)| - f(v) - 1, but may not be the case if G is a multigraph.

Given a multigraph G and a subset M of V(G), let G[M] denote the subgraph of G induced by M. Given a path linking vertices u and v, to split off the path, we delete the edges of the path and add an edge uv to G. Given a vertex v with $|E_G(v)|$ even, to suppress v, we first match all edges of $E_G(v)$ into pairs; then we split off every pair $\{vu, vw\}$ of the matching, and finally delete v and its loops (if any). Note that after suppressing a vertex, the degree of other vertices are unchanged. Both operations (splitting off a path and suppressing a vertex) can be expressed as a sequence of splitting off pairs of edges. Given two multigraphs G and G', we define the union of G and G', denoted $G \cup G' = G^*$ to be the multigraph with vertex set $V(G) \cup V(G')$ and the following edge set. For every two vertices u and v in $V(G) \cup V(G')$, the number of edges uv in G^* is equal to the sum of the number of edges uv in G and G'.

The structure of the paper is as follows. In sections 2, we give some results on clique immersion in dense graphs, which are necessary for the proofs of Theorems 1.3 and 1.5. Then we prove Theorems 1.3, 1.5, and 1.6 in Sections 3, 4, and 5, respectively.

2 Clique immersion in dense graphs

In the following lemma, we show that if G contains a set M of t vertices where the total sum of "missing degree" is small, then G immerses a K_t on M.

Lemma 2.1. Let G = (V, E) be a graph with n vertices and M be a subset of V with t vertices. If

$$\sum_{v \in M} f_G(v) \le \left(n - t - \max_{v \in M} f_G(v)\right) t,$$
(2.1)

then G contains an immersion of K_t .

Proof. Let $\overline{M} = V \setminus M$ and let $b = \max_{v \in M} f_G(v)$. Suppose that there are distinct vertices $v, v' \in M$ and $w \in \overline{M}$ such that $vv' \notin E(G)$ and $vw, wv' \in E(G)$. By splitting off the path vwv', we obtain the edge vv' while f(v) and f(v') are unchanged, and so (2.1) still holds for the new graph. Thus by repeatedly finding such triples and splitting off, we obtain new graphs satisfying (2.1) while the number of edges strictly decreases after each step. Therefore the process must halt and return a graph $G_1 = (V, E_1)$ satisfying

$$\sum_{v \in M} f_{G_1}(v) \le (n-t-b)t, \text{ and}$$
(2.2)

(i) there are no $v, v' \in M$ and $w \in \overline{M}$ such that $vv' \notin E_1$ and $vw, wv' \in E_1$.

For the rest of the proof, we write f instead of f_{G_1} . Let r be the number of missing edges of G_1 with two endpoints in M, and X be the set of endpoints of these missing edges. If r = 0, then $G_1[M]$ is a copy of K_t , which proves the lemma. Hence we may suppose that $r \ge 1$.

For every $v \in X$, there is $v' \in M$ such that $vv' \notin E_1$. From (i) we have $f(v|\overline{M}) + f(v'|\overline{M}) \ge |\overline{M}| = n - t$; otherwise, there exists $w \in \overline{M}$ such that $vw, wv' \in E_1$. Hence

$$n - t \le f(v|\overline{M}) + f(v'|\overline{M}) \le f(v|\overline{M}) + f(v') \le f(v|\overline{M}) + b_{\overline{v}}$$

and so $f(v|\overline{M}) \ge n - t - b$ for every $v \in X$. This gives

$$\sum_{v \in X} f(v) = \sum_{v \in X} f(v|\overline{M}) + \sum_{v \in X} f(v|M) \ge (n - t - b)|X| + 2r.$$
(2.3)

We will construct a K_t immersion in G_1 as follows: for every non-adjacent pair of vertices v, v' in X, we will obtain the edge vv' by splitting off path vwuw'v' for some $u \in Y = M \setminus X$ and $w, w' \in \overline{M}$. As a first step to finding such 4-edge paths, for all $u \in Y$, define

$$h(u) = \max\left(0, \left\lfloor\frac{n-t-b-f(u)+1}{2}\right\rfloor\right).$$

It holds that $2h(u) \ge n - t - b - f(u)$. Hence

$$2\sum_{u \in Y} h(u) \ge (n - t - b)|Y| - \sum_{u \in Y} f(u).$$

Combining with (2.3), and then with (2.2) yields

$$\begin{split} 2\sum_{u\in Y} h(u) - 2r &\geq \left((n-t-b)|Y| - \sum_{u\in Y} f(u) \right) + \left((n-t-b)|X| - \sum_{v\in X} f(v) \right) \\ &\geq (n-t-b)(|X| + |Y|) - \sum_{v\in M} f(v) \\ &\geq (n-t-b)t - (n-b-t)t = 0. \end{split}$$

Hence $\sum_{u \in Y} h(u) \ge r$.

Choose arbitrarily two non-adjacent vertices v, v' in M (clearly $v, v' \in X$), and an arbitrary vertex $u \in Y$ such that $h(u) \ge 1$. Such a vertex u always exists as $\sum_{u \in Y} h(u) \ge r \ge 1$ and h(u) is an integer for every u. By definition of function h, we have

$$f(u|\overline{M}) \le f(u) \le n - t - b + 1 - 2h(u) \le n - t - b - 1.$$

From $f(v) \leq b$, we have

$$f(u|\overline{M}) + f(v|\overline{M}) \le (n-t-b-1) + f(v) \le n-t-1 < |\overline{M}|,$$

so u and v have a common neighbor $w \in \overline{M}$. Similarly u and v' have a common neighbor $w' \in \overline{M}$. If w = w' then $vw, wv' \in E_1$, contrary to (i).

By splitting off the path vwuw'v', we get the edge vv'. In doing so, we have that f(v) and f(v') remain unchanged while f(u) increases by 2, i.e., h(u) decreases by 1. Thus $\sum_{u \in Y} h(u)$ decreases by 1. However, the number of missing edges in $G_1[M]$ also decreases by 1, so we still have that $\sum_{u \in Y} h(u)$ is at least the number of missing edges in $G_1[M]$. We repeat the process above until we link all pairs of non-adjacent vertices in M, and so obtain a complete graph on M. Thus G_1 contains an immersion of K_t , and consequently, G contains K_t as an immersion as well. This proves the lemma.

As a corollary of Lemma 2.1, the following lemma provides a more general bound for clique immersion of a graph by its average "missing degree".

Lemma 2.2. Let G be a graph on n vertices, and let $\gamma = \sum_{v \in V(G)} f_G(v)/n$ be the average "missing degree" of G. If $\gamma \leq n/2$, then G contains an immersion of K_t where $t = \min(\lfloor n/2 \rfloor, \lfloor n - 2\gamma \rfloor)$.

Proof. Let M be a set of $t = \min(\lfloor n/2 \rfloor, \lfloor n-2\gamma \rfloor)$ vertices minimizing $\sum_{v \in M} f(v)$. Let $b = \max_{v \in M} f(v)$ and $\overline{M} = V(G) \backslash M$.

If $2b \leq n-t$, note that $f(v) \leq b$ for every $v \in M$, and so $\sum_{v \in M} f(v) \leq bt \leq (n-t-b)t$, and we apply Lemma 2.1 to complete the proof.

Otherwise, 2b > n - t. By the minimality of f on M, we have $f(w) \ge b$ for every $w \in \overline{M}$. Hence

$$\sum_{v \in M} f(v) = \sum_{v \in V(G)} f(v) - \sum_{w \in \overline{M}} f(w) \le \gamma n - b(n-t).$$

$$(2.4)$$

We now show that $\gamma n - b(n-t) \leq (n-t-b)t$. Indeed,

$$\gamma n - b(n-t) \leq (n-t-b)t$$

$$\iff 2(\gamma n - bn + bt) \leq 2(n-t-b)t$$

$$\iff 2\gamma n - n^2 + tn \leq 2b(n-2t) - (n-t)(n-2t)$$

$$\iff (2\gamma + t - n)n \leq (2b - n + t)(n - 2t).$$
(2.5)

Since $t = \min(\lfloor n/2 \rfloor, \lfloor n-2\gamma \rfloor)$, we have $2\gamma \le n-t$ and $2t \le n$. Combining with 2b > n-t yields

$$(2\gamma + t - n)n \le 0 \le (2b - n + t)(n - 2t).$$

Hence (2.5) holds, and so $\gamma n - b(n-t) \leq (n-t-b)t$. This, together with equality (2.4), implies that $\sum_{v \in M} f(v) \leq (n-t-b)t$, and we apply Lemma 2.1 to complete the proof.

In the case $n/4 \le \gamma \le n/2$, by tightening the analysis, we can slightly improve the bound in Lemma 2.2 to $t = \lfloor n - 2\gamma \rfloor + 1$, which is sharp even if γ is the maximum missing degree (see [6], Lemma 2.1). In the case $\gamma < n/4$, the above technique could yield $t = \max(\lfloor n/2 \rfloor, \lfloor n - \sqrt{2\gamma n} \rfloor)$; however, $t = \lfloor n/2 \rfloor$ is enough for our purpose.

3 Forcing a clique immersion via minimum degree

In this section, we show how the proof of Theorem 1.2 can be refined to give the proof of Theorem 1.3. The main idea is as follows. Suppose, to reach a contradiction, that there is a graph with high minimum degree which does not contain a K_t -immersion. We choose such a graph G with as few vertices as possible. If G is dense, then we can find a K_t immersion, a contradiction. Otherwise, G is sparse, and so we can supress a vertex to get a smaller graph, which still has high minimum degree and does not contain a K_t -immersion, a contradiction again. The main difficulty is how to suppress a vertex of G so that the new graph is still simple. We first state several results from [3].

Proposition 3.1 ([3], Lemma 6). Every complete multipartite graph of minimum degree at least t contains an immersion of K_t .

A graph on odd number of vertices is *hypomatchable* if deleting any vertex results in a graph with a perfect matching.

Proposition 3.2 ([3], Lemma 8). Fix t and let H be a graph not containing any complete multipartite subgraph with minimum degree at least t. Suppose that the complement graph \overline{H} of H neither has a perfect matching nor is hypomatchable. Then there exist disjoint subsets W, L of V(H) such that

- $|W| \le t 1$ and $|L| \ge |V(H)| 2|W|;$
- $f_H(v) \leq |W|$ for every $v \in W$; and
- $uv \in E(H)$ for every $u \in W$ and $v \in L$.

Given a multigraph G, we say that a vertex v of G can be well-suppressed (in G) if we can suppress v without creating any new loop or multi-edge in G. Precisely, v can be well-suppressed if there is a matching of edges of $E_G(v)$ such that

- for every pair $\{vu_1, vu_2\}$, we have $u_1 \neq u_2$ and $u_1u_2 \notin E(G)$, and
- for every two pairs $\{vu_1, vu_2\}$ and $\{vu'_1, vu'_2\}$ we have $\{u_1, u_2\} \neq \{u'_1, u'_2\}$.

A vertex v can be *nearly well-suppressed* if for all edges $e \in E_G(v)$, the vertex v can be well-suppressed after deleting e.

Given a simple graph G, it is straightforward that if a vertex v can be well-suppressed (nearly well-suppressed), then the complement graph of the induced subgraph G[N(v)]has a perfect matching (is hypomatchable, respectively). The situation is more complex when G is a multigraph. In the next lemma, we consider the case where some multi-edges are allowed.

Lemma 3.3. Fix $t \ge 1$ and let G' be a loopless multigraph with vertex set $V \cup \{z\}$ (where $z \notin V$) such that for every $v \in V$, zv is either an edge or a multi-edge with multiplicity 2. Let R be the set of vertices incident with z by a multi-edge. If

- $|V| 2|R| \ge 3t,$
- G := G'[V] is simple and does not contains K_t as an immersion, and
- z cannot be well-suppressed or nearly well-suppressed in G',

then there is a set $W \subseteq V$ such that $|W| \leq t-1$ and $f_G(v) \leq |W| + |R|$ for every $v \in W$.

Proof. We define an auxiliary (simple) graph H as follows. Beginning with G, for every vertex $v \in R$, we add a *clone* vertex v_c to H which has the following neighbors: all the vertices of R, all the neighbors of v in G, and every other clone vertex u_c . Explicitly, H has vertex set $V \cup \{v_c | v \in R\}$ and edge set

$$E(H) = E(G) \cup \{u_c v | u, v \in R\} \cup \{u_c v_c | u, v \in R\} \cup \{v_c x | v \in R, vx \in E(G)\}.$$

Each vertex in H indeed corresponds to an edge of $E_{G'}(z)$, where each clone vertex v_c represents the additional edge in the multi-edge zv.

Let \overline{H} be the complement graph of H. We will show that \overline{H} neither has a perfect matching nor is hypomatchable. If \overline{H} has a perfect matching, then by the construction of H, that perfect matching corresponds to a matching of edges in $E_{G'}(z)$ such that

- for every pair $\{zu_1, zu_2\}$, we have $u_1 \neq u_2$ and $u_1u_2 \notin E(G')$, and
- for every two pairs $\{zu_1, zu_2\}$ and $\{zu'_1, zu'_2\}$ we have $\{u_1, u_2\} \neq \{u'_1, u'_2\}$.

Thus we can we can well-suppress z in G', a contradiction to the third assumption of the lemma. If \overline{H} is hypomatchable, then for every $v \in V(H)$, there is a perfect matching of $V(H) \setminus \{v\}$ in \overline{H} . The same argument shows that z can be nearly well-suppressed in G', a contradiction. We conclude that \overline{H} neither has a perfect matching nor is it hypomatchable.

Observe that removing a vertex of a complete multipartite graph with minimum degree d results in a complete multipartite graph with minimum degree at least d-1. Hence suppose that H contains a multipartite subgraph of minimum degree at least |R|+t. By removing all clone vertices of H, we obtain G, which still contains a complete multipartite subgraph with minimum degree at least (|R|+t) - |R| = t. By Proposition 3.1, G contains K_t as an immersion, a contradiction. We conclude that H does not contains any multipartite subgraph of minimum degree at least |R| + t.

Applying Proposition 3.2 to H, we obtain disjoint subsets W', L' of V(H) such that

- (a) $|W'| \le |R| + t 1$ and $|L'| \ge |V(H)| 2|W'|$;
- (b) $f_H(v) \leq |W'|$ for every $v \in W'$; and
- (c) $uv \in E(H)$ for every $u \in W'$ and $v \in L'$.

Let R_c be the set of clone vertices of H and $W = W' \setminus R_c$ and $L = L' \setminus R_c$. We will show that W is a desired set. By (a) we have

$$|L'| \ge |V(H)| - 2|W'| > (|V| + |R|) - 2(|R| + t) \ge |V| - |R| - 2t.$$

Thus $|L'| - |R| \ge |V| - 2|R| - 2t$. Recall from the hypothesis that $|V| - 2|R| \ge 3t$, and so $|L| \ge |L'| - |R| \ge t$. Note that by (c), $uv \in E(H)$ for every $u \in W$ and $v \in L$, and hence $uv \in E(G)$ for every $u \in W$ and $v \in L$. If $|W| \ge t$, then $G[W \cup L]$ contains a complete bipartite graph with minimum degree at least t, and so contains K_t as an immersion by Proposition 3.1, a contradiction. Thus it holds that $|W| \le t - 1$.

Note that $f_G(v) \leq f_H(v)$ since G is an induced subgraph of H. It follows from (b) that $f_G(v) \leq f_H(v) \leq |W'| \leq |W| + |R|$ for every $v \in W$. This completes the proof of the lemma.

Given an integer t > 1, we call a graph t-deficient if it can be obtained from a graph with minimum degree t by removing a few edges. Precisely, a graph G is t-deficient if $\sum_{v \in V(G)} \max(0, t - d_G(v)) < t$.

Proposition 3.4 ([3], Lemma 13). If G is a graph of minimum degree at least 7t + 7 that does not contain an immersion of K_t , then G contains an immersion of some 7t-deficient eulerian graph G'.

Proposition 3.5 ([3], Lemma 15). Every 7t-deficient eulerian graph contains a vertex of degree at least 7t.

The main technical step in the proof of Theorem 1.3 is the following lemma. Dvořák and Yepremyan proved a similar result for 11t + 7-deficient eulerian graphs in [3].

Lemma 3.6. Every 7t-deficient eulerian graph contains an immersion of K_t .

Theorem 1.3 follows easily from Lemma 3.6. Suppose for a contradiction that there exists a graph G of minimum degree at least 7t + 7 and does not have an immersion of K_t . By Proposition 3.4, G contains an immersion of a 7t-deficient eulerian graph G'. By Lemma 3.6, G' contains an immersion of K_t , a contradiction.

Proof of Lemma 3.6. Suppose that there exists a 7t-deficient simple eulerian graph which does not contain an immersion of K_t . Let G = (V, E) be such a graph with as few vertices as possible. The idea of the proof is as follows. If G has few edges, we show it would be possible to well-suppress some vertex of G to get a smaller counterexample, a contradiction. Hence G has many edges. We are then able to find in G two disjoint sets of vertices A and B of size around t and 6t, respectively, such that there are very few missing edges between A and B. We apply Lemma 2.1 to obtain an immersion of K_t and so reach a contradiction.

Let z_1 be a vertex in G with $d(z_1) \ge 7t$, as guaranteed by Proposition 3.5. Let $1 \le p < t$ be the maximum integer such that there exists an ordered set $A = \{z_1, z_2, ..., z_p\}$ satisfying

$$f(z_i|B) \le p + i + r_i, \text{ for all } i \ge 2.$$

$$(3.1)$$

where $B = N(z_1) \setminus A$ and $r_i = |\{j \le i : z_j \notin N(z_1)\}|$ for every $i \ge 2$. Such number p clearly exists since (3.1) trivially holds for $A = \{z_1\}$. Since $|N(z_1) \cap A| = p - r_p$, we have

$$|B| = |N(z_1) \setminus A| = d(z_1) - |N(z_1) \cap A| \ge 7t - p + r_p.$$
(3.2)

Let $\overline{A} = V \setminus A$. Starting with $G_p = G$, we will attempt to sequentially split off the vertices of A in order $z_p, z_{p-1}, \ldots, z_1$ to create graphs $G_{p-1}, G_{p-2}, \ldots, G_0$. At each step, if we could find the complement of a perfect matching in $N_{G_i}(z_i)$, we could split off z_i to obtain G_{i-1} and maintaining the property that G_{i-1} is simple. However, the requirement that $N_{G_i}(z_i)$ have the complement of a perfect matching is too strong and so we will have to slightly relax it. In doing so, we will need to introduce parallel edges into the graphs G_i , but we will want to do so in a tightly controlled manner. This leads us to the following definition.

Fix $q, 0 \le q \le p$ and multigraphs $G_i, q \le i \le p$ which satisfy the following.

- (i) $G_p = G$ and for all $i, q \leq i < p, G_i$ is obtained from G_{i+1} by suppressing z_{i+1} .
- (ii) For all $i, q \leq i \leq p, G_i[\overline{A}]$ is simple.
- (iii) For all *i*, every multi-edge of G_i with an endpoint in \overline{A} has multiplicity 2.
- (iv) For all $j, 2 \leq j \leq q$, there are at most $r_p r_q$ multi-edges from z_j to vertices of \overline{A} in G_q , and there are at most p q multi-edges from z_1 to vertices of \overline{A} in G_q ,

- (v) There are at least $|\overline{A}| p + q$ vertices in \overline{A} not incident with any multi-edge in any $G_i, q \leq i \leq p$.
- (vi) Given $v \in \overline{A}$ and $z \in A$, if vz is a multi-edge in some $G_i, q \leq i \leq p$, then for every $z' \in A, z' \neq z$ and every $j, q \leq j \leq p, vz'$ is not a multi-edge in G_j .
- (vii) Subject to (i) (vi), we choose q and G_i , $q \le i \le p$ to minimize q.

Such a number q and multigraphs G_i , $q \le i \le p$ trivially exist, given the observation that q = p and $G_p = G$ satisfy (i) – (vi) as G is simple.

We begin with the observation that q > 0. Otherwise, the graph G_0 does not contain K_t as an immersion because G_0 itself immerses in G by construction. Moreover, G_0 is simple by (ii), and for all $v \in V(G_0)$, $d_{G_0}(v) = d_G(v)$. We conclude that G_0 is both eulerian and t-deficient, contrary to our choice of G to be a counterexample on a minimum number of vertices.

We now consider the graph G_q and keep in mind that by the minimality of q in (vii), we cannot supress z_q to obtain G_{q-1} which satisfies all (i) – (vi). Let $X = N_{G_q}(z_q) \cap \overline{A}$. We will show that $G' := G_q[X \cup \{z_q\}]$ satisfies all hypotheses of Lemma 3.3. From (ii) and (iii), we have G' is a loopless multigraph with vertex set $X \cup \{z_q\}$ such that for every $v \in X$, $z_q v$ is either an edge or a multi-edge with multiplicity 2. Let R be the set of vertices in X incident with z_q by a multi-edge. Then by (iv) we have

$$\begin{cases} |R| \le p - 1 & \text{if } q = 1, \\ |R| \le r_p - r_q & \text{if } q > 1. \end{cases}$$
(3.3)

Claim 3.7. G' satisfies all hypotheses of Lemma 3.3.

Proof. We verify the hypotheses one by one.

• $G'[X] = G_q[X]$ is simple and does not contains K_t as an immersion.

 $G_q[X]$ is simple by (ii), and does not contains K_t as an immersion by (i) and the assumption that G does not contains K_t as an immersion.

• $|X| - 2|R| \ge 3t.$

To prove $|X| - 2|R| \ge 3t$, note that $|B \setminus X|$ is the number of vertices in B not adjacent to z_q in G_q , which is at most the number of vertices in B not adjacent to z_q in Gsince no edge between z_q and B have been removed in suppressing z_p, \ldots, z_{q+1} . Thus $|B \setminus X| \le f_G(z_q|B)$. Combining with (3.1) we have

$$\begin{cases} |B \setminus X| = 0 & \text{if } q = 1, \\ |B \setminus X| \le p + q + r_q & \text{if } q > 1. \end{cases}$$
(3.4)

In the case q > 1, by (3.2),

$$|X| \ge |B| - |B \setminus X| \ge (7t - p + r_p) - (p + q + r_q).$$

From the fact that $t \ge \max(p, q, r_p)$ and (3.3), we have

$$|X| - 2|R| \ge 7t - 2p - q - r_p + r_q \ge 3t.$$

In the case q = 1, by (3.2),

$$|X| \ge |B| - |B \setminus X| \ge |B| \ge 7t - p + r_p.$$

Hence from (3.3) we have $|X| - 2|R| \ge 7t - 3p + r_p \ge 3t$.

• z_q cannot be well-suppressed or nearly well-suppressed in G'.

Suppose that z_q can be well-suppressed in G'. We first split off all edges from z_q to X by that matching. Then there are even number of edges incident with z_q remaining in G_q , all from z_q to A since $X = N_{G_q}(z_q) \cap \overline{A}$. We now suppress z_q in G_q arbitrarily to obtain G_{q-1} . Since we do not create any new edge between A and \overline{A} , (i) – (vi) hold trivially for G_{q-1} , which contradicts (vii).

As the second case, suppose that z_q can be nearly well-suppressed in G'. Pick a vertex $v \in X$ which is not incident with any multi-edge in G_i , for all $q \leq i \leq p$. Such vertex v exists since by (v), there was at most p - q distinct vertices of \overline{A} incident with some multi-edge over all G_i , $q \leq i \leq p$, while $|X| \geq 3t > p - q$ (as we show above that $|X| - 2|R| \geq 3t$).

Since z_q is can be nearly well-suppressed in G', if we remove the edge $z_q v$ in G', we can well-suppress z_q (in G'), and we do so. Since $d_{G_q}(z_q)$ is even, z_q must be adjacent to some vertex z_s with s < q. We choose such s as small as possible and split off $z_s z_q v$. We now suppress z_q in G_q arbitrarily to obtain G_{q-1} and will show that G_{q-1} satisfies (i) – (vi) and hence violates (vii). Properties (i) and (ii) hold trivially. The only possible new multi-edge that we have created is $z_s v$. Since v is not incident with any multi-edge in G_i for all $q \le i \le p$, (iii), (v) and (vi) hold for G_{q-1} . To prove (iv), first observe that (iv) clearly holds if $z_s = z_1$. If $z_s \ne z_1$, then z_1 is not incident with z_q by the choice of s, and so $r_{q-1} = r_q - 1$ by the definition of function r. Thus $r_p - r_{q-1} = r_p - r_q + 1$ and therefore (iv) holds.

Hence G' satisfies the hypotheses of Lemma 3.3, and so there is a set $W \subseteq X$ such that $|W| \leq t - 1$ and $f_{G_q[X]}(v) \leq |W| + |R|$ for every $v \in W$.

We next show that $|W| \ge t - p$. To do so, we need the following claim.

Claim 3.8. $f_G(v|B) \leq |W| + 2p + r_p$ for every $v \in W$.

Proof. We first show that $f_{G_q}(v|B) \leq |W| + p + q + r_p$. Note that $f_{G_q}(v|X) = f_{G_q[X]}(v)$ for every $v \in X$, and so

$$f_{G_q}(v|B) \le f_{G_q}(v|X) + f_{G_q}(v|B \setminus X) \le (|W| + |R|) + |B \setminus X|.$$

If q > 1, recall that $|B \setminus X| \le p + q + r_q$ from (3.4) and $|R| \le r_p - r_q$ from (3.3). Hence we have

$$f_{G_q}(v|B) \le (|W| + r_p - r_q) + (p + q + r_q) \le |W| + p + q + r_p.$$

If q = 1, recall that $|B \setminus X| = 0$ from (3.4) and $|R| \le p - 1$ from (3.3). Thus we have

$$f_{G_q}(v|B) \le |W| + p < |W| + p + q + r_p.$$

We conclude that $f_{G_q}(v|B) \leq |W| + p + q + r_p$ in all cases. To complete the claim, it suffices to show that

$$f_G(v|B) \le f_{G_q}(v|B) + (p-q)$$

for every $v \in X$. Fix $v \in X$. By property (vi), there exists a value s such that $z_i v$ is not a multi-edge in G_i for every $i \neq s, q \leq i \leq p$. Thus for every $i \neq s, q < i \leq p$, there is at most one edge $z_i v$ in G_i , and so when we supress z_i in G_i to obtain G_{i-1} , we add at most one edge between v and B into G_{i-1} . If s > q, note that there are at most two edges $z_s v$ in G_s by property (iii). Hence when we supress z_s in G_s to obtain G_{s-1} , we add at most two edges between v and B into G_{s-1} . Thus from $G = G_p$, when we supress $z_p, ..., z_{q+1}$ to get G_q , we add in total at most p - q - 1 + 1 = p - q edges from v to B, and so

$$f_G(v|B) = f_{G_p}(v|B) \le f_{G_q}(v|B) + (p-q).$$

 \Diamond

This proves the claim.

Claim 3.9. $|W| \ge t - p$.

Proof. Suppose for a contradiction that $|W| + p = p^* < t$. Let $A^* = A \cup W$ where elements in W are enumerated $z_{p+1}, ..., z_{p^*}$, and let $B^* = N(z_1) \setminus A^* = B \setminus W$. Then

- $f_G(z_i|B^*) \le f_G(z_i|B) \le p + i + r_i \le p^* + i + r_i$ for every $i, 2 \le i \le p$.
- $f_G(z_i|B^*) \leq f_G(z_j|B) \leq |W| + 2p + r_p \leq p^* + i + r_i$ for every i > p (note that $r_i \geq r_p$ since by definition r is a non-decreasing function).

Hence (3.1) holds for p^* and A^* , contrary to the maximality of p. Thus $|W| \ge t - p$.

Let \hat{A} be an arbitrary set of t - p vertices in W and enumerate them $z_{p+1}, ..., z_t$. Let $M = A \cup \hat{A}$ and $\overline{M} = B \setminus \hat{A}$. Let $U = M \cup \overline{M}$ and H = G[U]. We will apply Lemma 2.1 to H and deduce that H must contain an immersion of K_t , which contradicts the assumption that G does not contains an immersion of K_t and so complete the proof of Lemma 3.6. We first give some bounds for function f in H. Observe that $f_H(z_i|\overline{M}) = f_G(z_i|\overline{M}) \leq f_G(z_i|B)$ for every $i, 1 \leq i \leq p$. Note also that $f_G(z_1|B) = 0$, and $f_G(z_i|B) \leq 2p + i$ for every $i, 1 < i \leq p$, and by Claim 3.8,

$$f_G(z_i|B) \le |W| + 2p + r_p \le t + 2p + r_p$$

for every $i, p < i \leq t$ (recall that $|W| \leq t$). Thus

$$\begin{cases} f_H(z_i|\overline{M}) \le 2p+i & \text{if } i \le p, \\ f_H(z_i|\overline{M}) \le t+2p+r_p & \text{if } i > p. \end{cases}$$
(3.5)

Also note that |M| = t, and from (3.2),

$$|\overline{M}| \ge |B| - |\hat{A}| \ge 7t - p + r_p - (t - p) = 6t + r_p.$$

Claim 3.10. *H* contains an immersion of K_t .

Proof. We consider two cases.

Case 1: $p \leq t/2$. We have $f_H(z_i|M) \leq |M| \leq t$ for every z_i , and so

$$\sum_{z_i \in M} f_H(z_i) \leq \sum_{z_i \in M} f_H(z_i|M) + \sum_{z_i \in M} f_H(z_i|\overline{M})$$
$$\leq t^2 + \sum_{1 \leq i \leq p} f_H(z_i|\overline{M}) + \sum_{p < i \leq t} f_H(z_i|\overline{M})$$
$$\leq t^2 + \sum_{i \leq p} (2p+i) + \sum_{p < i \leq t} (t+2p+r_p)$$
$$\leq t^2 + 3p^2 + (t-p)(t+3p)$$
$$\leq 2t^2 + 2tp \leq 3t^2.$$

Since $2p \leq t$, we have

$$\max_{z_i \in M} f_H(z_i) \le t + \max_{z_i \in M} f_H(z_i|\overline{M}) \le t + (t + 2p + r_p) \le 3t + r_p.$$

Note that $|U| = |M| + |\overline{M}| = 7t + r_p$. Hence

$$\sum_{z_i \in M} f_H(z_i) \le 3t^2 \le \left(|U| - t - \max_{z_i \in M} f(z_i)\right)t.$$

Apply Lemma 2.1 to obtain an immersion of K_t on H.

Case 2: p > t/2. Set $q = |\hat{A}| = t - p$, and so p > q. The analysis of this case is more involved. Even though $\sum_{z_i \in M} f_H(z_i)$ is small, $\max_{z_i \in M} f_H(z_i)$ could be very large, and so we cannot apply Lemma 2.1 directly. However, we can still use a similar argument to that in the proof of Lemma 2.1. We present the argument as an algorithm to explicitly find a series of splitting off of edges to yield a K_t immersion by finding edge disjoint paths of length two or four linking the desired pairs of vertices.

Consider an arbitrary loopless multigraph H' with vertex set U and distinct vertices $z_i, z_j \in M$. We first define a subroutine called $\text{LINK}(H', z_i, z_j)$: the algorithm finds $w \in \overline{M}$ such that $z_i w, w z_j \in E(H')$ and then split off the path $z_i w z_j$ to obtain an edge $z_i z_j$. The algorithm then returns H' after splitting off the path. Such a w can be found by checking all possible choices for w. In the case that multiple choices exist for w, the algorithm arbitrarily chooses one.

In order to successfully run, the algorithm $LINK(H', z_i, z_j)$ assumes that the input satisfies:

$$f_{H'}(z_i|\overline{M}) + f_{H'}(z_j|\overline{M}) < 6t + r_p \le |\overline{M}|, \tag{3.6}$$

Under assumption (3.6), such a $w \in \overline{M}$ must exist and therefore, the algorithm correctly terminates. Note also that z_i, z_j are adjacent after performing LINK (H', z_i, z_j) , and that the input H' contains the output graph as an immersion.

We now present the main algorithm to split off edges of H to obtain a complete graph on $M = A \cup \hat{A}$. Set H' := H. The algorithm proceeds in stages. In stage 1, we link all vertices between $\{z_{q+1}, ..., z_p\}$ and \hat{A} . In stage 2, we link each pair of vertices between $\{z_1, ..., z_q\}$ and \hat{A} with multi-edges of order two. Thus after stages 1 and 2, we obtain two edge-disjoint complete bipartite subgraphs, one between A and \hat{A} and another between $\{z_1, ..., z_q\}$ and \hat{A} (the latter will be used later to obtain a complete graph on \hat{A}). In stage 3, we link all vertices inside A, and then obtain a complete graph on M.
$$\begin{split} \text{MAIN}(H') \\ \text{1. Start with } s &:= p \text{ and repeat the following whenever } s > q. \\ \text{Start with } i &:= p + 1 \text{ and repeat the following whenever } i \leq t. \\ \text{LINK}(H', z_s, z_i), i &:= i + 1. \\ s &:= s - 1. \end{split}$$
2. Start with s &:= q and repeat the following whenever $s \geq 1$. Start with i &:= p + 1 and repeat the following whenever $i \leq t. \\ \text{LINK}(H', z_s, z_i), \text{LINK}(H', z_s, z_i), i &:= i + 1. \\ s &:= s - 1. \end{aligned}$ 3. Start with s &:= p and repeat the following whenever $s \geq 1$. Start with i &:= s - 1 and repeat the following whenever $i \leq 1$. Start with s &:= p and repeat the following whenever $s \geq 1$. Start with s &:= p and repeat the following whenever $s \geq 1$. Start with i &:= s - 1 and repeat the following whenever $i \geq 1$. LINK $(H', z_s, z_i), i &:= i - 1. \\ s &:= s - 1. \end{aligned}$ 4. Return H'.

Suppose that we have performed MAIN(H') successfully. The output H' contains two edge-disjoint complete bipartite subgraphs, H_1 from A to \hat{A} , and H_2 from $\{z_1, ..., z_q\}$ to \hat{A} , and a complete graph H_3 on A. We now show how to obtain from H_2 a complete graph H_4 on \hat{A} . Since $|\hat{A}| = q$, by Vizing Theorem, we can color the edges of an imagined complete graph on \hat{A} by q colors $\{1, 2, ..., q\}$ so that any two incident edges have different color. Now for every $z_i, z_j \in \hat{A}$, if the edge $z_i z_j$ in that imagined graph has color s, then we split off edges $z_i z_s z_j$ in the complete bipartite graph H_2 to get an edge $z_i z_j$, and so obtain a complete graph H_4 on \hat{A} . Hence $H_1 \cup H_3 \cup H_4$ is a complete graph on M. Thus the output H' contains K_t as an immersion, which implies that H contains K_t as an immersion.

It only remains to show that we can perform MAIN(H') successfully, which is equivalent to verifying that for each call to the subroutine $LINK(H', z_i, z_j)$ we have that (3.6) is satisfied. We omit the subscript H' of f in the rest of this proof. Observe that after performing $LINK(H', z_i, z_j)$, $f(z_i|\overline{M})$ and $f(z_j|\overline{M})$ each increases by at most 1.

Consider step (s, i) of stage 1. The vertex z_s has been linked i - p - 1 times and so from (3.5) we have $f(z_s|\overline{M}) , and <math>z_i$ has been linked p - s - 1 times and so $f(z_i|\overline{M}) < t + 3p + r_p - s$. Then

$$f(z_s|\overline{M}) + f(z_i|\overline{M}) < t + 4p + i + r_p \le 6t + r_p$$

and so (3.6) holds for every step (s, i) of stage 1. From (3.5) and the definition of the algorithm, we have that at the end of stage 1:

$$\begin{aligned} f(z_s|M) &\leq 2p + s & \text{if } s \leq q, \\ f(z_s|\overline{M}) &\leq (2p + s) + q & \text{if } q < s \leq p \\ f(z_i|\overline{M}) &\leq (t + 2p + r_p) + (p - q) & \text{if } i > p. \end{aligned}$$

Consider step (s,i) of stage 2. The vertex z_s has been linked 2(i-p-1) times during stage 2 and so $f(z_s|\overline{M}) \leq 2p+s+2q-2=2t+s-2$ (since t=p+q), and z_i has been linked 2(q-s-1) times. Thus $f(z_i|\overline{M}) \leq 2t+2p+r_p-2s-2$. It follows that

$$f(z_s|\overline{M}) + f(z_i|\overline{M}) \le 4t + 2p + r_p - s - 4 \le 6t + r_p - 4.$$

We can perform $LINK(H', z_s, z_i)$ twice. At the end of stage 2, we have

$$f(z_s|\overline{M}) \le (2p+s) + 2q \le 2t + s \quad \text{if } s \le q,$$

$$f(z_s|\overline{M}) \le (2p+s) + q \le 2t + s \quad \text{if } q < s \le p.$$

Consider step (s, i) of stage 3. The vertex z_s has been linked (p-s)+(s-i-1) times during stage 3 (in which p-s times with $z_r, s < r \le p$ and s-i-1 with $z_j, i < j \le s$) and so $f(z_s|\overline{M}) < (2t+s) + p-i$, and z_i has been linked p-s-1 times and so $f(z_i|\overline{M}) < (2t+i) + p-s$. Then

$$f(z_s|\overline{M}) + f(z_i|\overline{M}) < 4t + 2p \le 6t + r_p,$$

and so (3.6) holds for every step (s, i) of stage 3. Claim 3.10 now follows.

This proves Lemma 3.6, and so prove Theorem 1.3.

4 Forcing a clique immersion via the chromatic number

In this section we shall prove Theorem 1.5. Recall that given $\ell \geq 1$, a graph G is ℓ -critical if the chromatic number of G is ℓ , and deleting any vertex of G results in a subgraph with chromatic number $\ell - 1$. A well-known property of critical graphs is that if G is a graph with chromatic number ℓ , then G contains an ℓ -critical subgraph. Let us restate Theorem 1.5.

Theorem 4.1. Every graph with chromatic number at least 3.54t + 4 contains an immersion of K_t .

Proof. Assume the theorem is false, and suppose that there exists a graph of chromatic number $\ell \geq 3.54t + 4$ but which does not immerse K_t . Let G^* be an ℓ -critical subgraph of that graph. Let v_0 be a vertex of G^* with minimum degree. By Theorem 1.3, $d_{G^*}(v_0) \leq$ 7t + 6. Let $N = N_{G^*}(v_0)$, and let G be the graph obtained from G^* by deleting v_0 . It follows that G does not immerse K_t . The graph G^* is ℓ -critical, so G has chromatic number $\ell - 1$. Furthermore, for any coloring of G, N always has at least one vertex of each of the $\ell - 1$ colors, otherwise we could color G^* with $\ell - 1$ colors. The proof uses Kempe-chains, introduced by Alfred Kempe in an 1879 attempt to prove that planar graphs are 4-colorable, to build a clique immersion with branch vertices in N.

Given a coloring of G, we call a vertex $v \in N$ a singleton if v is the unique vertex in N with its color. Two vertices $v, v' \in N$ of the same color form a doubleton if they are the only two vertices with a given color in N. Let C be an $\ell - 1$ coloring of G which maximizes the number of singletons. Let $X = \{x_1, ..., x_\alpha\}$ and $Y = \{y_1, y'_1, ..., y_\beta, y'_\beta\}$ be the sets of singletons and doubletons, respectively, where x_i has color a_i and y_i, y'_i share

 \Diamond

color b_i . All other colors appear at least 3 times in N. Thus, $\ell - 1$, the number of colors in N, is at most

$$\alpha + \beta + \frac{|N| - \alpha - 2\beta}{3} = \frac{|N| + 2\alpha + \beta}{3}.$$

Since $|N| = d_{G^*}(a) \le 7t + 6$, we have

$$3.54t + 3 \le \ell - 1 \le \frac{7t + 6 + 2\alpha + \beta}{3}$$
$$\implies 2\alpha + \beta \ge 2.62t + 3. \tag{4.1}$$

Given colors a, b, an(a, b)-chain is a path with vertices colored alternately by colors a and b. Clearly, if $\{a, b\} \neq \{a', b'\}$, then any (a, b)-chain and (a', b')-chain are edgedisjoint. The idea is as follows. We first show that there are many chains with endpoints in $X \cup Y$. Since these chains are edge-disjoint, we can split them off to get a dense graph on $X \cup Y$, then apply Lemma 2.2 to obtain a K_t immersion, which leads to the contradiction.

Claim 4.2. The following hold.

- (a) For all pairs of distinct colors a_i, a_j , there is an (a_i, a_j) -chain from x_i to x_j .
- (b) For any colors a_i, b_j , there is an (a_i, b_j) -chain from x_i to y_j , or from x_i to y'_j .
- (c) For all pairs of distinct colors b_i, b_j , one of the following holds:
 - (i) there exist two edge-disjoint (b_i, b_j) -chains linking y_i to y_j and y'_i to y'_j ;
 - (ii) there exist two edge-disjoint (b_i, b_j) -chains linking y_i to y'_j and y'_i to y_j ;
 - (iii) there exist (b_i, b_j) -chains from any of y_i, y'_i to any of y_j, y'_j but they cannot be chosen edge-disjoint.

Proof. For every color a, let $V_a \subseteq V(G)$ be the set of all vertices of color a in \mathcal{C} .

To prove (a), suppose that there exist two distinct colors a_i, a_j such that there is no (a_i, a_j) -chain from x_i to x_j . Then x_i, x_j are disconnected in $G[V_{a_i} \cup V_{a_j}]$. Let U be the connected component containing x_i in $G[V_{a_i} \cup V_{a_j}]$. We exchange the color of all vertices in U from color a_i to a_j and vice versa and obtain a new coloring \mathcal{C}' in G. Clearly \mathcal{C}' is a proper coloring in $G[V_{a_i} \cup V_{a_j}]$, and so is a proper coloring in G. Now both x_i and x_j has color a_j , so \mathcal{C}' has no vertex of color a_i , contrary to the fact that N has all colors for every $(\ell - 1)$ -coloring of G.

To prove (b), the same argument works. Suppose that there exist two distinct colors a_i, b_j such that there is no (a_i, b_j) -chain from x_i to $\{y_j, y'_j\}$. Then x_i is disconnected with $\{y_j, y'_j\}$ in $G[V_{a_i} \cup V_{b_j}]$. Let U be the connected component containing x_i in $G[V_{a_i} \cup V_{b_j}]$. We exchange the color of all vertices in U from color a_i to b_j and vice versa and obtain a new coloring \mathcal{C}' in G. Then \mathcal{C}' is a proper coloring in G and has smaller number of colors on N than \mathcal{C} , contrary to the fact that N has all colors for every $(\ell - 1)$ -coloring of G.

To prove (c), we first prove that

(d) for every pair of distinct colors b_i, b_j , there is a (b_i, b_j) -chain from y_i to y_j , or from y_i to y'_j .

Suppose that there exist two distinct colors b_i, b_j such that there is no (b_i, b_j) -chain from y_i to $\{y_j, y'_j\}$. Then y_i is disconnected with $\{y_j, y'_j\}$ in $G[V_{b_i} \cup V_{b_j}]$. Let U be the connected component containing y_i in $G[V_{y_i} \cup V_{b_j}]$. We exchange the color of all vertices in U from color b_i to b_j and vice versa and obtain a new coloring \mathcal{C}' in G. Then \mathcal{C}' is a proper coloring in G. If $y'_i \in U$, then \mathcal{C}' has no vertex of color b_i in N, contrary to the fact that N has all colors for every $(\ell - 1)$ -coloring of G. If $y'_i \notin U$, then \mathcal{C}' has exactly one vertex of color b_i in N, and so has more singletons than \mathcal{C}' , which contradicts our choice of \mathcal{C} to maximize the number of singletons.

We now show how (d) implies (c). From (d), every pair of distinct colors b_i, b_j , there is a (b_i, b_j) -chain from y_i to $\{y_j, y'_j\}$ and another (b_i, b_j) -chain from y'_i to $\{y_j, y'_j\}$. If one chain go to y_j and another go to y'_j , then there are three possibilities. First, these chains are edge-disjoint and between y_i, y_j and y'_i, y'_j , then (i) holds. Second, these chains are edge-disjoint and between y_i, y'_j and y'_i, y_j , then (ii) holds. Third, they are not edge-disjoint, then all $\{y_i, y'_i, y_j, y'_j\}$ are connected by these two chains, and (iii) holds. Otherwise, say these chains both go from y_i, y'_i to y_j . Then by (d), there is a (b_i, b_j) -chain from y'_j to either y_i or y'_i . Hence all $\{y_i, y'_i, y_j, y'_j\}$ are connected by some (b_i, b_j) -chains, and (iii) holds.

For every pair of colors, we fix a subgraph based on the appropriate outcome of Claim 4.2. For every $i, j, 1 \leq i < j \leq \alpha$, fix $C_a(i, j)$ to be an (a_i, a_j) -chain from x_i to x_j . For all $i, j, 1 \leq i \leq \alpha, 1 \leq j \leq \beta$, fix $C_b(i, j)$ to be an (a_i, b_j) -chain from x_i to either y_j or y'_j . Let i, j be such that $1 \leq i < j \leq \beta$; one of (i) - (iii) holds for the colors b_i and b_j . If either (i) or (ii) holds, fix $C_c(i, j)$ to be the subgraph consisting of two edge disjoint (b_i, b_j) -chains linking $\{y_i, y'_i\}$ and $\{y_j, y'_j\}$. If (iii) holds, fix C(i, j) to be an edge minimal subgraph containing (b_i, b_j) -chains linking each of y_i, y'_i to each of y_j, y'_j . For $i, j, 1 \leq i < j \leq \beta$, we say that $C_c(i, j)$ has one of 3 types, namely (i), (ii), or (iii), depending on which outcome of (c) holds. Note that $C_a(i, j), C_b(i, j)$, and $C_c(i, j)$ are all pairwise edge disjoint.

If we split off all the possible edge disjoint paths contained in subgraphs from the previous paragraph, it will not necessarily be the case that we will have sufficient edges on $X \cup Y$ to apply Lemma 2.2. To get around this problem, we focus instead on the vertex set $X \cup \{y_1, \ldots, y_\beta\}$. The subgraphs $C_c(i, j)$ of type (i) or type (ii) contain a path which can be split off to yield the edge $y_i y_j$. Moreover, if we flip the labels y_i and y'_i , every $C_c(i, j)$ subgraph of type (ii) becomes a $C_c(i, j)$ subgraph of type (i) (and vice versa). Thus, we can increase the density of the resulting graph on $X \cup \{y_1, \ldots, y_\beta\}$ by flipping the appropriate pairs of labeles y_i, y'_i .

Unfortunately, this greedy approach will still not yield enough edges on $X \cup \{y_1, \ldots, y_\beta\}$ to apply Lemma 2.2. To further increase the final edge density, we will group together multiple $C_c(i, j)$ subgraphs of type (ii) to split off paths and add further edges to the set $\{y_1, \ldots, y_\beta\}$. The remainder of the argument carefully orders how the subgraphs are grouped together so that when we split them off and get as dense a subgraph as possible on the vertex set $X \cup \{y_1, \ldots, y_\beta\}$.

We begin by defining the subgraphs G_1 , G_2 and the auxiliary graph H as follows. Split off all paths of the form $C_a(i, j)$, $C_b(i, j)$, and the two edge disjoint $\{y_i y'_i\} - \{y_j, y'_j\}$ -paths contained in the subgraphs $C_c(i, j)$ of type (i) and (ii). Let G_1 the graph with vertex set V(G) and edge set the set of all new edges arising from splitting off these paths. Let G_2 be the subgraph of G with vertex set V(G) edge set the union of $E(C_c(i,j))$ for all subgraphs $C_c(i,j)$ of type (iii). Observe that $G_1 \cup G_2$ is an immersion of G and therefore does not immerse K_t . Clearly, $G_1[X]$ is a complete graph obtained from splitting off all the subgraphs $C_a(i,j)$, and so

$$\alpha = |X| \le t - 1. \tag{4.2}$$

We define an auxiliary graph H by replacing each pair of vertices y_i, y'_i with a single vertex z_i , and we color edges of incident with z_i to describe the behavior of y_i, y'_i . Precisely, let H be a graph with vertex set $X \cup Z$ where $Z = \{z_1, ..., z_\beta\}$ and edge set

$$E(H) = \{x_i z_j : 1 \le i \le \alpha, 1 \le j \le \beta\} \cup \cup \{z_i z_j : 1 \le i < j \le \beta \text{ and } C_c(i, j) \text{ is not of type (iii)}\}.$$

The edges of H are improperly colored by two colors *odd*, *even* as follows:

- $x_i z_j$ is even if $x_i y_j \in E(G_1)$, and is odd if $x_i y'_j \in E(G_1)$.
- $z_i z_j$ is even if $y_i y_j, y'_i y'_j \in E(G_1)$, and is odd if $y_i y'_j, y'_i y_j \in E(G_1)$.

To perform a *swap* at a vertex z_i , we exchange the colors of all edges incident with z_i in H; a swap is equivalent to switching the labels of y_i and y'_i in $G_1 \cup G_2$. To *swap* a set $S \subseteq Z$, we swap vertices in S sequentially in an arbitrarily chosen order. One can easily show that to swap a set S is equivalent to switching the color of every edges between Sand $V(H) \setminus S$.

A triangle in H is odd if it has odd number of odd-edges. A key property of oddtriangles is that an odd-triangle is still odd after any swap. Each odd-triangle either has 3 vertices in Z or exactly two vertices in Z – call them type 1 and type 2 odd-triangles, respectively. Given a type 1 odd-triangle $z_i z_j z_k$, the set of edges in G_1 with endpoints in $\{y_i, y'_i, y_j, y'_j, y_k, y'_k\}$ are called the *corresponding edges* of $z_i z_j z_k$. Similarly, given a type 2 odd-triangle $x_i z_j z_k$, the set of edges in G_1 with endpoints in $\{x_i, y_j, y'_j, y_k, y'_k\}$ are called the *corresponding edges* of $x_i z_j z_k$. Clearly, the set of corresponding edges of two edge-disjoint odd-triangles are disjoint. In Figure 1, we describe all possibilities (up to permutation of indices) of the set of corresponding edges of a type 1 odd-triangle (upper figures) and of a type 2 odd-triangle (lower figures).

Looking at Figure 1, we can easily verify the following.

- (A) If $z_i z_j z_k$ is an odd-triangle of type 1, we can split off its corresponding edges to obtain edges $y_i y_j, y_j y_k, y_k y_i$.
- (B) If $x_i z_j z_k$ is an odd-triangle of type 2, we can split off its corresponding edges to obtain the edge $y_j y_k$.
- (C) If $x_i z_j z_k$ is an odd-triangle of type 2, we can alternatively split off its corresponding edges to obtain two edges from the set $\{x_i y_j, y_j y_k, y_k x_i\}$ (exactly which two edges depends on which case from Figure 1 we find ourselves in).

Let H_1 be a graph obtained from H by removing an (inclusion-wise) maximal set \mathcal{T}_1 of pairwise edge-disjoint odd-triangles of type 1, and let H_2 be a graph obtained from H_1



Figure 1: Possibilities of corresponding edges of odd-triangles.

by removing an (inclusion-wise) maximal set \mathcal{T}_2 of pairwise edge-disjoint odd-triangles of type 2. In the following claims, we employ the assumption that G does not contain a K_t -immersion to bound the degree of vertices in $H_1[Z]$ and H_2 .

Claim 4.3. $d_{H_1[Z]}(z) < t$ for every $z \in Z$.

Proof. Suppose for a contradiction that there exists $z \in Z$ such that $d_{H_1[Z]}(z) \geq t$. Let $M_o(M_e)$ the sets of vertices adjacent to z in $H_1[Z]$ by an odd-edge (by an even-edge, respectively). Then $|M_o| + |M_e| = d_{H_1[Z]}(z) \geq t$. Every edge uv in $H_1[Z]$ with $u, v \in M_o$ $(u, v \in M_e$, respectively) must be even; otherwise, uvz is an odd-triangle of type 1, contradicting the maximality assumption on \mathcal{T}_1 . Similarly, every edge uv in $H_1[Z]$ with $u \in M_o$ and $v \in M_e$ must be odd.

We now swap M_o , and then the new graph $H_1[M_o \cup M_e]$ contains only even-edges. Let $M = \{y_i : z_i \in M_o \cup M_e\}$. Then $|M| = |M_o| + |M_e| \ge t$. For every odd-triangle in \mathcal{T}_1 , we split off corresponding edges in G_1 by method (A) to get $y_i y_j, y_j y_k, y_k y_i$. Then for any distinct vertices $y_i, y_j \in M$, we have

- if $z_i z_j \in H_1$, then $z_i z_j$ is even, and hence $y_i y_j \in G_1$.
- if $z_i z_j \in H \setminus H_1$, then $z_i z_j$ belongs to some odd-triangle in \mathcal{T}_1 , and we showed above that we can obtain $y_i y_j$ by splitting off edges of G_1 by method (A).
- if $z_i z_j \notin H$, then $C_c(i, j)$ is of type (iii) and so there exists a $y_i y_j$ path in $C_c(i, j)$ which can be split off to yield the edge $y_i y_j$.

We end up with a complete graph on M, and so conclude that $G_1 \cup G_2$ contains K_t as an immersion (since $|M| \ge t$), which is a contradiction.

Claim 4.4. $d_{H_2}(x) < t$ for every $x \in X$.

Proof. The proof is quite similar to the proof of Claim 4.3. We suppose that there exists $x \in X$ such that $d_{H_2}(x) \geq t$. Note that X is a stable set in H, and so all neighbors of x in H_2 are in Z. Let $M_o(M_e)$ be the set of vertices adjacent to x in H_2 by an odd-edge (by an even-edge, respectively). Then $M_o \cup M_e \subseteq Z$ and $|M_o| + |M_e| \geq t$. Every edge uv in H_2 with $u, v \in M_o$ ($u, v \in M_e$, respectively) must be even; otherwise, uvx is an odd-triangle of type 2, contradicting the maximality assumption of \mathcal{T}_2 . Similarly, every edge uv in H_2 with $u \in M_o$ and $v \in M_e$ must be odd.

We now swap M_o . The new graph $H_2[M_o \cup M_e]$ contains only even-edges. Let $M = \{y_i : z_i \in M_o \cup M_e\}$. Then $|M| = |M_o| + |M_e| \ge t$. For every odd-triangle in \mathcal{T}_1 , we split off corresponding edges in G_1 by method (A) to get $y_i y_j, y_j y_k, y_k y_i$. For every odd-triangle in \mathcal{T}_2 , we split off corresponding edges in G_1 by method (B) to get $y_i y_j$. Then for any distinct vertices $y_i, y_j \in M$, we have

- if $z_i z_j \in H_2$, then $z_i z_j$ is even, and hence $y_i y_j \in G_1$.
- if $z_i z_j \in H_1 E(H_2)$, then $z_i z_j$ belongs to some odd-triangle in \mathcal{T}_2 , and we showed above that we can obtain $y_i y_j$ by splitting off edges of G_1 by method (B).
- if $z_i z_j \in H E(H_1)$, then $z_i z_j$ belongs to some odd-triangle in \mathcal{T}_1 , and we showed above that we can obtain $y_i y_j$ by splitting off edges of G_1 by method (A).
- if $z_i z_j \notin H$, we split off a $y_i y_j$ path in $C_c(i, j)$ in G_2 to obtain $y_i y_j$.

We end up with a complete on M, and so $G_1 \cup G_2$ contains K_t as an immersion (since $|M| \ge t$), which is a contradiction.

The next claim guarantees that at least half of edges in H_2 are even.

Claim 4.5. There exists a subset S of vertices such that after swapping S in H_2 , the number of even-edges in H_2 is at least the number of odd-edges.

Proof. We first show that there exits a sequence of swaps resulting in the number of even-edges in $H_2[Z]$ being at least the number of odd-edges $H_2[Z]$. If there is $z \in Z$ such that z is incident with more odd-edges than even-edges in $H_2[Z]$, we swap z, then repeat. The process will halt since the number of even-edges in $H_2[Z]$ strictly increases after each swap. When the process halts, every $z \in Z$ is incident with at least as many even-edges as with odd-edges in $H_2[Z]$, and so in total, the number of even-edges in $H_2[Z]$ at least the number of odd-edges $H_2[Z]$.

If the number of even-edges from Z to X in H_2 is less than the number of odd-edges from Z to X in H_2 , we swap the set Z. After the switch, the number of even-edges from Z to X in H_2 is at least the number of odd-edges from Z to X in H_2 . Moreover, edges in $H_2[Z]$ are not affected by swapping Z. Finally, note that there is no edge in $H_2[X]$.

Thus at the end of this series of swaps, the number of even-edges in H_2 is at least the number of odd-edges in H_2 , proving the claim.

By Claim 4.5, we may assume that at least half of edges in H_2 are even. We now split off edges in $G_1 \cup G_2$ to obtain a dense graph on $X \cup \{y_1, ..., y_\beta\}$ as follows. For every odd-triangle in \mathcal{T}_1 , we split off its corresponding edges in G_1 by method (A). For every odd-triangle in \mathcal{T}_2 , we split off its corresponding edges in G_1 by method (C), which implies that we obtain two of three edges in the set $\{x_iy_j, y_jy_k, y_kx_i\}$. For every pairs b_i, b_j in case (iii), we also split off a path in $C_c(i, j)$ to get the edge y_iy_j as guaranteed by (iii). We denote by \hat{G} the induced subgraph of the new graph on $X \cup \{y_1, ..., y_\beta\}$. Note that \hat{G} is an immersion of $G_1 \cup G_2$, and so does not contain an immersion of K_t .

We will show that \hat{G} is dense, specifically by counting the number of non-edges in \hat{G} . We first observe that by construction, $\hat{G}[X]$ is complete. Thus, all non-edges in \hat{G} arise from odd-edges of H for which the corresponding edge of \hat{G} cannot be reconstructed through odd-triangles.

Observe that $H = (H - E(H_1)) \cup (H_1 - E(H_2)) \cup H_2$. We consider each of the subgraphs $H - E(H_1)$, $H_1 - E(H_2)$, and H_2 and how they can contribute non-edges to \hat{G} separately.

- Each odd-triangle $z_i z_j z_k \in \mathcal{T}_1$ contributes zero missing edge to \hat{G} since we obtain $y_i y_j, y_j y_k, y_k y_i$ by method (A). Hence $H E(H_1)$ (the union of odd-triangles in \mathcal{T}_1) contributes zero missing edge to \hat{G} .
- Each odd-triangle $x_i z_j z_k \in \mathcal{T}_2$ contributes exactly one missing edge to \hat{G} since we obtain two edges among $x_i y_j, y_j y_k, y_k x_i$ by method (C). Hence $H_1 E(H_2)$ (the union of odd-triangles in \mathcal{T}_2) contributes $|\mathcal{T}_2|$ missing edge to \hat{G} .
- Each odd-edge (even-edge) in H_2 contributes exactly one (zero, respectively) missing edge to \hat{G} . Hence H_2 contributes at most $|E(H_2)|/2$ missing edges to \hat{G} by Claim 4.5.

We conclude that the number of missing edges in \hat{G} is at most $|E(H_2)|/2 + |\mathcal{T}_2|$. We next give an explicit bound for the number of missing edges in \hat{G} .

Claim 4.6. The number of missing edges in \hat{G} is at most $(\alpha\beta + \alpha t + \beta t)/4$.

Proof. Let $p = |E(H_2)|/2$ and $q = |\mathcal{T}_2|$. Then the number of missing edges in \hat{G} is at most p + q. By claim 4.3, $H_1[Z]$ has β vertices and minimum degree less than t, and so $E(H_1[Z]) < \beta t/2$. Hence

$$2p + 3q \le |E(H_2)| + |E(H_1 - E(H_2))|$$

= |E(H_1)|
= |X||Z| + |E(H_1[Z])|
 $\le \alpha\beta + \beta t/2.$

Claim 4.4 states that every $x \in X$ is adjacent to at most t vertices of Z in H_2 , and so is adjacent to at most |Z| - t vertices of Z in $H_1 - E(H_2)$. This implies that for every $x \in X$, there are at least $(|Z| - t)/2 = (\beta - t)/2$ odd-triangles in \mathcal{T}_2 containing x (since $H_1 - E(H_2)$ is the union of odd-triangles in \mathcal{T}_2). This means that

$$q = |\mathcal{T}_2| \ge |X|(\beta - t)/2 = \alpha(\beta - t)/2.$$

Hence

$$p+q = \frac{(2p+3q)-q}{2} \le \frac{(\alpha\beta+\beta t/2)-\alpha(\beta-t)/2}{2} = \frac{\alpha\beta+\alpha t+\beta t}{4}$$

Hence the number of missing edges in \hat{G} is at most $(\alpha\beta + \alpha t + \beta t)/4$.

 \Diamond

We next show that if $|V(\hat{G})| = \alpha + \beta$ is large, then we can apply Lemma 2.2 to yield a contradiction that \hat{G} contains an immersion of K_t . Hence $\alpha + \beta$ is small, which contradicts (4.1), and the proof of Theorem 1.5 is complete.

Claim 4.7. $\alpha + \beta < 2.62(t+1)$.

Proof. Let $n = |V(\hat{G})| = \alpha + \beta$ and suppose for a contradiction that $n \ge 2.62(t+1)$. Let $\gamma = \frac{1}{n} \sum_{v \in \hat{G}} f_{\hat{G}}(v)$. Then $n\gamma/2$ is the number of missing edges in \hat{G} , and so by Claim 4.6 we have

$$2\gamma \le \frac{\alpha\beta + \alpha t + \beta t}{n} = \frac{\alpha\beta}{n} + t.$$
(4.3)

If $\gamma < n/4$, then by Lemma 2.2, \hat{G} contains an immersion of $K_{t'}$, where $t' = \lfloor n/2 \rfloor \ge t$. Hence \hat{G} contains an immersion of K_t , a contradiction.

Otherwise, since $\alpha\beta \leq (\alpha + \beta)^2/4 = n^2/4$, we have $2\gamma < n/4 + t < n$. Thus by applying Lemma 2.2, \hat{G} contains an immersion of $K_{t'}$, where $t' = \lfloor n - 2\gamma \rfloor > n - 2\gamma - 1$. We conclude that $n - 2\gamma - 1 < t$ since \hat{G} does not contain K_t as an immersion.

Recall that by (4.2), we have that $\alpha < n/2$. For every x such that $\alpha < x < n/2$, we have

$$\alpha\beta = \alpha(n-\alpha) < x(n-x)$$

Since $\alpha < t + 1 < n/2$, we can choose x := t + 1, and so

$$(n-2\gamma-1) - t \ge n - \left(\frac{\alpha\beta}{n} + t\right) - t - 1$$
$$\ge n - \frac{\alpha\beta}{n} - 2x$$
$$> \frac{n^2 - 3nx + x^2}{n}.$$

The assumption of the claim is that $n \ge 2.62x$, and hence $n^2 - 3nx + x^2 \ge 0$ (by solving the quadratic equation). This gives $n - 2\gamma - 1 \ge t$, which contradicts what we obtained above that $n - 2\gamma - 1 < t$. This prove the claim.

Combining Claim 4.7 with (4.2), we obtain $2\alpha + \beta < 3.62t + 3$, which contradicts (4.1). This completes the proof of Theorem 1.5.

5 Immersion in graphs with no stable set of size 3

We begin by reformulating Theorem 1.6.

Theorem 5.1. For all $t \ge 1$, every graph G with at least 5t vertices and no stable set of size three has a strong immersion of K_{2t} .

Proof. Assume that the theorem is false, and pick a counterexample G which minimizes |V(G)| + |E(G)|. Assume that G has at least 5t + 5 vertices and no strong immersion of K_{2t+2} . Since every graph on at least 5 vertices with no independent set of size three contains an edge, we may assume that $t \ge 1$. By minimality, we may assume that n = |V(G)| = 5t + 5. Furthermore, as G - e does not contain a strong immersion of

 K_{2t+2} for all edges e, by minimality it follows that deleting any edge results in a stable set of size three. All index arithmetic in the following proof is done mod 5.

Claim 5.2. G contains an induced cycle of length 5.

Proof. If G were the disjoint union of cliques, since it contains no stable set of size three then it must be a disjoint union of at most two cliques. One of the two cliques has at least $\lceil n/2 \rceil \ge 2t + 2$ vertices, and so G contains a strong K_{2t+2} -immersion, a contradiction.

Thus G is not a disjoint union of cliques, and there exist two adjacent vertices a_1, a_2 such that $N(a_1) \neq N(a_2)$. Without loss of generality, we may suppose that $N(a_2) \setminus N(a_1) \neq \emptyset$ and let $a_3 \in N_G(a_2) \setminus N_G(a_1)$. This gives $a_1a_3 \notin E$ and $a_2a_3 \in E$. Observe that there is a_4 with $a_1a_4, a_2a_4 \notin E$; otherwise, we can remove the edge a_1a_2 without creating any stable set of size three, which contradicts the minimality of G. By the same argument, there is a_5 with $a_2a_5, a_3a_5 \notin E$. Note that G does not contain any stable set of size three and $a_1a_4, a_1a_3 \notin E$, and so $a_3a_4 \in E$. Similarly, $a_1a_5, a_4a_5 \in E$. Thus $a_1a_2a_3a_4a_5$ forms an induced cycle of length 5 in G.

Let $C = \{a_i : 1 \le i \le 5\}$ induce a cycle of length five, and let $U = V \setminus C$. Then G[U] has n-5 = 5t vertices and G[U] contains no stable set of size three. By minimality of G, G[U] contains a strong immersion of K_{2t} with with some set of branch vertices M. Let $Q = U \setminus M$, and for every $i, 1 \le i \le 5$, let M_i be the set of vertices in M not adjacent to a_i .

In the following claim, we show that if there are two large disjoint sets X_1, X_3 in Q with some desired property, then for every $v \in M$, we can split off paths a_1xv or a_1xa_iv (of length 2 or 3) with $x \in X_1, i \in \{2, 4, 5\}$ to get the edge a_1v , and similarly to get the edge a_3v , and so get a strong clique immersion of size 2t + 2 on $M \cup \{a_1, a_3\}$, which is a contradiction.

Claim 5.3. Suppose that there are disjoint sets $X_1, X_3 \subseteq Q$ satisfying

- (i) $|X_1| \ge |M_1|$, and $|X_3| \ge |M_3|$;
- (ii) for every $x \in X_1$, we have $xa_1, xa_5 \in E$ and either $xa_2 \in E$ or $xa_4 \in E$; and
- (iii) for every $x \in X_3$, we have $xa_3, xa_4 \in E$ and either $xa_2 \in E$ or $xa_5 \in E$.

Then G has a strong immersion of K_{2t} , where the set of branch vertices is $M \cup \{a_1, a_3\}$.

Proof. Let E_1 be the set of edges in G from C to $M_1 \cup X_1$. We wish to split off paths in E_1 to obtain edges from a_1 to every vertex in M_1 . The process of splitting off is as follows.

- Arbitrarily pair each vertex $v \in M_1$ with a vertex $x_v \in X_1$ such that $x_v \neq x_{v'}$ for every $v \neq v'$ (such a choice of x_v exists by (i)).
- For every $v \in M_1$, note that $va_4 \in E$ (otherwise, $\{a_1, v, a_4\}$ is a stable set of size three) and either $va_2 \in E$ or $va_5 \in E$ (otherwise, $\{a_2, v, a_5\}$ is a stable set of size three). If $va_5 \in E$, we split off the path $va_5x_va_1$ to get an edge va_1 .

• Otherwise, $va_4 \in E$ and $va_2 \in E$. By (ii), either $x_va_2 \in E$ or $x_va_4 \in E$. If $x_va_2 \in E$, we split off the path $va_2x_va_1$ to get the edge va_1 . Otherwise, we split off the path $va_4x_va_1$ to get the edge va_1 .

Note that in this process we only use edges of E_1 and at the end we obtain all edges from a_1 to M_1 , and so obtain all edges from a_1 to M. Let E_3 be the set of edges in Gfrom C to $M_1 \cup X_3$. Note that $E_1 \cap E_3 = \emptyset$, and hence we can split off paths in E_3 in the same manner to obtain all edges from a_3 to M_3 , and so obtain all edges from a_3 to M.

By minimality, we can split off edges of G[U] to obtain a K_{2t} on M. Note that E_1 , E_3 and E(G[U]) are pairwise disjoint, so we never split off an edge twice. By splitting off a_1a_2, a_2a_3 , we obtain a_1a_3 , and hence obtain a complete graph on $M \cup \{a_1, a_3\}$. Clearly, all split off paths are internally edge-disjoint from $M \cup \{a_1, a_3\}$. Hence G contains a strong immersion of K_{2t+2} , where the set of branch vertices is $M \cup \{a_1, a_3\}$, a contradiction. \diamond

To reach the contradiction, it only remains to show that such sets X_1, X_3 exist up to shifting indices. For every $i, 1 \le i \le 5$, let A_i be the set of non-neighbors of a_i in G[U]. Note that $A_i \cup \{a_{i-2}, a_{i+2}\}$ is a clique, and so $|A_i| \le 2t$ since G does not contain any K_{2t+2} -immersion. Also note that since G contains no stable set of size three, and hence $A_i \cap A_{i+2} = \emptyset$ for every i.

As discussed above, we wish to find sets X_1, X_3 satisfying Claim 5.3. One might hope to choose $X_1 := A_3 \cap Q$ and $X_3 := A_1 \cap Q$; these sets indeed satisfy (ii) and (iii) but may fail to meet (i) in the case either $|A_1|$ or $|A_3|$ is small. This problem can be avoided by enlarging A_1 and A_3 . This leads to the following definition of $A'_1..., A'_5$.

Let $A'_1, ..., A'_5$ be subsets of U such that $\sum_{i=1}^5 |A'_i|$ is as large as possible, and

$$\begin{cases}
A_i \subseteq A'_i, \\
|A'_i| \le 2t, \\
A'_i \cap A'_{i+2} = \emptyset,
\end{cases} \quad \forall 1 \le i \le 5. \tag{5.1}$$

Claim 5.4. There exists i such that $|A'_i| = |A'_{i+2}| = 2t$.

Proof. Assume the claim is false. Then there exists j such that $|A'_j|, |A'_{j+1}|, |A'_{j+2}| < 2t$. Without loss of generality, assume $|A'_1|, |A'_2|, |A'_3| < 2t$.

For every *i*, let $B_{i,i+1} = A'_i \cap A'_{i+1}$, and $D_i = A'_i \setminus (A_{i-1} \cup A'_{i+1})$. Then all 10 sets $D_i, B_{i,i+1}$ are pairwise disjoint, and $A'_i = B_{i-1,i} \cup D_i \cup B_{i,i+1}$. Note also that $D_i \cap A'_{i+1} = \emptyset$ and $D_i \cap A'_{i-1} = \emptyset$ for every *i*.

Suppose that there exists $v \in U$ such that $v \notin \bigcup_{i=1}^{5} A'_{i}$. Then $A'_{1} \cup \{v\}, A'_{2}, ..., A'_{5}$ satisfy (5.1), while the sum of their cardinalities is larger, a contradiction. This gives $\bigcup_{i=1}^{5} A'_{i} = U$. In other words,

$$\left(\bigcup_{i=1}^{5} D_i\right) \cup \left(\bigcup_{i=1}^{5} B_{i,i+1}\right) = U.$$

Since all these sets are pairwise disjoint, we have

$$\sum_{i=1}^{5} |D_i| + \sum_{i=1}^{5} |B_{i,i+1}| = |U| \ge 5t.$$
(5.2)

Observe that if $|A'_i| < 2t$ and there exists $v \in D_{i-1} \cup D_{i+1}$, then $(A'_i \cup \{v\}) \cap A'_{i+2} = \emptyset$ and $(A'_i \cup \{v\}) \cap A'_{i-2} = \emptyset$. Hence $A'_i \cup \{v\}, A'_{i+1}, \dots, A'_{i+4}$ satisfy (5.1), violating our choice to maximize the sum of their cardinalities. Hence if $|A'_i| < 2t$, then $D_{i-1} = \emptyset$ and $D_{i+1} = \emptyset$.

Recall the assumption that $|A'_1|, |A'_2|, |A'_3| < 2t$. By the observation in the previous paragraph, we have $D_j = \emptyset$ for every j. Hence from (5.2) we have $\sum_{i=1}^5 |B_{i,i+1}| \ge 5t$. Also note that $|A'_i| = |B_{i,i-1}| + |D_i| + |B_{i,i+1}| = |B_{i,i-1}| + |B_{i,i+1}|$ for every i. This gives

$$10t \le 2\sum_{i=1}^{5} |B_{i,i+1}| = \sum_{i=1}^{5} |A'_i| < 10t,$$

a contradiction. This proves the claim.

Without loss of generality, we may suppose that $|A'_1| = |A'_3| = 2t$.

Claim 5.5. Let $X_1 = A'_3 \cap Q$ and $X_3 = A'_1 \cap Q$. Then X_1, X_3 satisfy conditions in Claim 5.3.

Proof. We first show that (iii) holds for X_3 . Recall that $A_3 \subseteq A'_3$, $A_4 \subseteq A'_4$ and $A'_1 \cap (A'_3 \cup A'_4) = \emptyset$. Then $A'_1 \cap (A_3 \cup A_4) = \emptyset$, and so $X_3 \cap (A_3 \cup A_4) = \emptyset$ since $X_3 \subseteq A'_1$. Hence for every $v \in X_3$, we have $va_3 \in E$ and $va_4 \in E$ (otherwise, G contains a stable set of size three). Note that $B_{5,1} \cap B_{1,2} = \emptyset$. Hence for every $v \in X_3$, either $v \notin B_{5,1}$ or $v \notin B_{1,2}$. If $v \notin B_{5,1}$ then $v \notin A'_5$ (since $v \in A'_1$), and so $v \notin A_5$. This means that v is adjacent to a_5 . Otherwise, $v \notin B_{1,2}$, and by the same argument, v is adjacent to a_2 . Hence, (iii) holds for X_3 .

We now show that (i) holds for X_3 . Let $M'_1 = A'_1 \cap M$ and $M'_3 = A'_3 \cap M$. Then by (5.1), we have $M_1 \subseteq M'_1$, $M_3 \subseteq M'_3$, and $M'_1 \cap M'_3 = \emptyset$. Besides, $X_3 \cap M'_1 \subseteq Q \cap M = \emptyset$ and

$$X_3 \cup M'_1 = (A'_1 \cap Q) \cup (A'_1 \cap M) = A'_1 \cap U = A'_1.$$

This gives $|X_3| + |M'_1| = |A'_1| = 2t$, and so

$$|X_3| = 2t - |M_1'| = |M| - |M_1'| = |M \setminus M_1'| \ge |M_3'| \ge |M_3|.$$

Hence (i) holds for X_3 .

By the same arguments, (i) and (ii) hold for X_1 . This proves the claim.

Claims 5.3 and 5.5 complete the proof of Theorem 1.6.

 \Diamond

References

- F. Abu-Khzam and M. Langston, Graph coloring and the immersion order, Lecture Notes in Computer Science 2697 (2003), 394–403.
- [2] M. Devos, Z. Dvořák, J. Fox, J. McDonald, B. Mohar, and D. Scheide, Minimum degree condition forcing complete graph immersion, Combinatorica 34 (2014), 279– 298.

 \Diamond

- [3] Z. Dvořák and L. Yepremyan, Comptete graph immersions and minimum degree, preprint, arXiv:1512.00513.
- [4] M. Devos, K. Kawarabayashi, B. Mohar, and H. Okamura, *Immersing small com*plete graphs, Ars Math. Contemp. 3 (2010), 139–146.
- [5] P. Duchet and H. Meyniel, On Hadwigers number and the stability number, Ann. Discrete Math. 13 (1982), 71–74.
- [6] J. Fox and F. Wei, On the number of cliques in graphs with a forbidden subdivision or immersion, preprint, arXiv:1606.06810.
- [7] H. Hadwiger, Uber eine Klassifikation der Streckenkomplexe, Vierteljschr. Naturforsch. Ges. Zurich 88 (1943), 133–143.
- [8] A. Kostochka, Lower bound of the Hadwiger number of graphs by their average degree, Combinatorica 4 (1984), 307–316.
- [9] T.-N. Le and P. Wollan, Forcing clique immersions through chromatic number, Electronic Notes Disc. Math. 54(1) (2016), 121–126.
- [10] F. Lescure and H. Meynial, On a problem upon configurations contained in graphs with given chromatic number, Ann. Discrete Math. 41 (1989), 325–331. Graph theory in memory of G.A. Dirac, Sandbjerg (1985).
- [11] M. D. Plummer, M. Stiebitz, and B. Toft, On a special case of Hadwigers conjecture. Discuss. Math. Graph Theory 23 (2005), 333–363.
- [12] N. Robertson, P.D. Seymour, Graph minors XXIII, Nash-Williams immersion conjecture, J. Combin. Theory Ser. B 100 (2010), 181–205.
- [13] N. Robertson, P.D. Seymour, R. Thomas, Hadwigers conjecture for K6-free graphs, Combinatorica 13 (1993), 279–361.
- [14] A. Thomason, An extremal function for contractions of graphs, Math. Proc. Cambridge Philos. Soc. 95 (1984), 261–265.
- [15] S. Vergara, Complete graph immersions in dense graphs, Discrete Math. 340 (5) (2017), 1019–1027.
- [16] K. Wagner, Uber eine Eigenschaft der ebenen Komplexe, Math. Ann. 114 (1937), 570–590.
- [17] P. Wollan, The structure of graphs not admitting a fixed immersion, J. Combin. Theory Ser. B 110 (2015), 47–66.