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On the bipartite vertex frustration of graphs

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Abstract

The bipartite vertex (resp. edge) frustration of a graph G , denoted by $\psi(G)$ (resp. $\varphi(G)$), is the smallest number of vertices (resp. edges) that have to be deleted from G to obtain a bipartite subgraph of G . A sharp lower bound of the bipartite vertex frustration of the line graph $L(G)$ of every graph G is given. In addition, the exact value of $\psi(L(G))$ is calculated when G is a forest.

Keywords: Bipartite vertex frustration, bipartite edge frustration, line graph, Hamiltonian graph, tree.

1 Introduction

Throughout this paper, all the graphs are simple, that is, with neither loops nor multiple edges. Notations and terminology not explicitly given here can be found in the book by Chartrand and Lesniak [1].

Given a graph G with vertex set $V(G)$ and edge set $E(G)$, a subset $F \subset V(G)$ such that $G - F$ is bipartite is called a vertex bipartization for G . The

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minimum cardinality of a vertex bipartization for G is called the *bipartite vertex frustration* of G and it is denoted by $\psi(G)$. An analogous definition of *bipartite edge frustration* $\varphi(G)$ of G is stated. Thus, the bipartite vertex (resp. edge) frustration of a graph G , denoted by $\psi(G)$ (resp. $\varphi(G)$), is the smallest number of vertices (resp. edges) that have to be deleted from G to obtain a bipartite subgraph of G .

These two parameters have interesting applications in different fields of science as, for instance, fullerene chemistry. Since its very beginning, the rapid development of fullerene chemistry has been paralleled by a similarly rapid build-up of interest and a flow of results on the graphs that serve as mathematical models of fullerene isomers. Very early it became clear that the fullerene stability is related to the absence of abutting pentagons in the corresponding graphs (see, for instance, [4,6,7]). It is well known that the bipartite graphs are characterized by the absence of cycles of odd length. Hence, one may think on the number of vertices or edges that need to be removed in order to make a bipartite graph as a measure of non-bipartivity of this graph. An idea to transplant a bipartivity measure into the context of fullerene chemistry is that the minimum number of vertices and/or edges which have to be deleted to make a graph bipartite may be related to the fullerene stability.

In Graph Theory is usual to study a lot of parameters in several families of graphs. One of the best known is line graphs (see for instance [5]). The bipartite vertex/edge frustration have been also studied in several families of graphs which model different typologies of networks (see [2,3,9,10,11,12]). In [9] Yarahmadi and Ashrafi study some extremal properties of the bipartite vertex frustration of graphs and provide the exact value for the corona product of two graphs and the line graph. For this last family, there is a step in the proof of Theorem 9 of [9] which is incorrect. In this paper we find the mistake and give a sharp lower bound of the bipartite vertex frustration of the line graph. Moreover, the exact value of $\psi(L(G))$ when G is a forest is determined.

2 Main results

The *line graph* $L(G)$ of a graph G has the edge set $E(G)$ as vertex set and two vertices in $V(L(G))$ are adjacent, whenever they are incident as edges in G . In an interesting paper [9], Yarahmadi and Ashrafi study the bipartite vertex frustration of the line graph $L(G)$ of any graph G . It is understood that the considered graphs are connected, since previously, they state the following lemma whose proof is straightforward.

Lemma 2.1 ([9]) *Let G be a graph with components G_1, G_2, \dots, G_k . Then $\psi(G) = \sum_{i=1}^k \psi(G_i)$.*

In Theorem 9 of [9] they prove that

$$(1) \quad \psi(L(G)) = \begin{cases} 2(|E(G)| - |V(G)|), & \text{if } G \text{ is not an odd cycle} \\ 1, & \text{otherwise.} \end{cases}$$

In the proof they apply Theorem 7.1.16 of [8], which says that $L(G)$ decomposes into complete subgraphs, with each vertex of $L(G)$ corresponding to an edge of G appearing in two of these complete subgraphs. Using this result and that $\psi(K_n) = n - 2$ for every integer $n \geq 2$, and denoting by d_i the degree of vertex $v_i \in V(G)$ for $i = 1, \dots, |V(G)|$, they deduce that $\psi(L(G)) = \sum_{i=1}^{|V(G)|} \psi(K_{d_i})$. This last step is incorrect, because the complete graphs K_{d_i} , $i = 1, \dots, |V(G)|$, are non-disjoint in $L(G)$ and therefore, Lemma 2.1 cannot be applied.

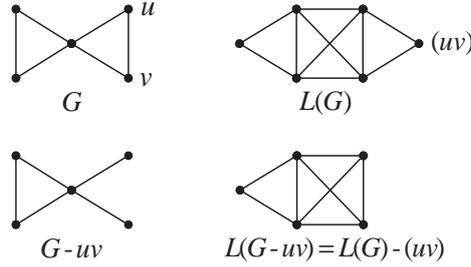


Fig. 1. An example which shows that $L(G - uv) = L(G) - (uv)$.

The following lemma help us to find an idea to get a lower bound of the bipartite vertex frustration of the line graph of every connected graph. Basically, if $W \subset E(G)$ is a set of edges and $F \subset V(L(G))$ is its corresponding set of vertices, then $L(G - W) \simeq L(G) - F$ (see Figure 1).

Lemma 2.2 *Let G be a connected graph with at least 4 vertices. Let $uv \in E(G)$ be any edge and denote by $(uv) \in V(L(G))$ the corresponding vertex of $L(G)$. Then the following assertions hold.*

- (i) $L(G - uv) \simeq L(G) - (uv)$.
- (ii) $L(G)$ contains a triangle if and only if $\Delta(G) \geq 3$.

Given a connected graph G of order at least 4, from Lemma 2.2 it follows that the deletion of a minimum vertex bipartization from $L(G)$ is equivalent

to find a minimum edge subset $W \subset E(G)$ such that $\Delta(G - W) \leq 2$ and $G - W$ contains no odd cycles. Thus, $\psi(L(G)) = \theta(G)$.

Next theorem shows a lower bound of the bipartite vertex frustration of the line graph $L(G)$ of any connected graph G containing cycles. Taking into account Lemma 2.1, the result can be extended to any non-necessarily connected graph by the sum of the bipartite vertex frustration of its components.

Theorem 2.3 *Let G be a connected graph other than a tree. Then*

$$\psi(L(G)) \geq |E(G)| - |V(G)| + \frac{1}{2} ((-1)^{|V(G)|+1} + 1).$$

The lower bound of Theorem 2.3 is an equality when the graph is Hamiltonian, as we can see in the following corollary.

Corollary 2.4 *Let G be a Hamiltonian graph. Then*

$$\psi(L(G)) = |E(G)| - |V(G)| + \frac{1}{2} ((-1)^{|V(G)|+1} + 1).$$

Theorem 2.3 does not consider the case when G is a tree. Given a tree T , as T contains no cycles, by Lemma 2.2, the problem of determining the minimum number of vertices whose deletion from $L(T)$ produces a bipartite graph is equivalent to find $\theta(T)$, that is, the cardinality of a minimum set $W \subset E(T)$ such that $T - W$ is formed by the union of vertex disjoint paths. Indeed, the number of disjoint paths is equal to $|W| + 1$.

Lemma 2.5 *Let G be a forest. Suppose that G contains an edge $uv \in E(G)$ such that $\min\{d_G(u), d_G(v)\} \geq 3$ and $d_G(w) \leq 2$ for every $w \in N_G(u) \setminus \{v\}$. Then $\theta(G) = 1 + \theta(G - uv)$.*

Remark 2.6 Given any tree T , if $\min\{d_T(w_0), d_T(z_0)\} \geq 3$ for some edge $w_0z_0 \in E(T)$, then there exists $uv \in E(T)$ such that $\min\{d_T(u), d_T(v)\} \geq 3$ and $d_T(w) \leq 2$ for every $w \in N_T(u) \setminus \{v\}$.

Let us consider the following algorithm.

Algorithm 1 *Let G_0 be a forest. Let us construct a subset $Z \subset E(G_0)$ following these steps:*

Step 1: If $\min\{d_{G_0}(u), d_{G_0}(v)\} \geq 3$ for some edge $uv \in E(G_0)$, then take $e_1 = u_1v_1 \in E(G_0)$ as in Remark 2.6 and denote by $G_1 = G_0 - e_1$ and $Z_1 = Z \cup \{e_1\}$. Otherwise, $Z = \emptyset$ and we finish the algorithm.

Step 2: If $\min\{d_{G_1}(u), d_{G_1}(v)\} \geq 3$ for some edge $uv \in E(G_1)$, then take $e_2 = u_2v_2 \in E(G_1)$ as in Remark 2.6 and denote by $G_2 = G_1 - e_2$ and

$Z_2 = Z_1 \cup \{e_1\}$. Otherwise, $Z = Z_1$ and we finish the algorithm.

...

Step j : If $\min\{d_{G_{j-1}}(u), d_{G_{j-1}}(v)\} \geq 3$ for some edge $uv \in E(G_{j-1})$, then take $e_j = uv \in E(G_{j-1})$ as in Remark 2.6 and denote by $G_j = G_{j-1} - e_j$ and $Z_j = Z_{j-1} \cup \{e_j\}$. Otherwise, $Z = Z_{j-1}$ and we finish the algorithm.

...

Step $k+1$. $\min\{d_{G_k}(u), d_{G_k}(v)\} \leq 2$ for every $uv \in E(G_k)$, leading to $Z = Z_k$ and we finish the algorithm.

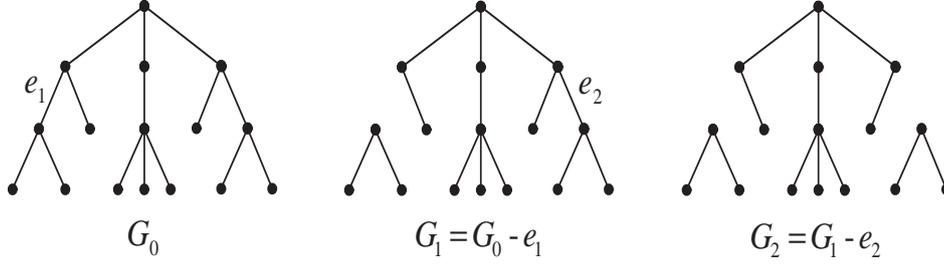


Fig. 2. An example which shows application of Algorithm 1 to obtain $Z = \{e_1, e_2\}$.

Figure 2 depicts a graph G_0 where Algorithm 1 is applied, obtaining $Z = \{e_1, e_2\}$ such that $\min\{d_{G-Z}(u), d_{G-Z}(v)\} \leq 2$ for every edge $uv \in E(G - Z)$.

Lemma 2.7 *Let r be a positive integer and let G_0 be a forest with r leaves such that*

$\min\{d_{G_0}(u), d_{G_0}(v)\} \geq 3$ for some edge $uv \in E(G_0)$. Then there exist an integer $k \geq 1$ and a set $Z = \{e_1, \dots, e_k\} \subset E(G_0)$, with $e_i = u_i v_i$ for $i = 1, \dots, k$, satisfying these conditions:

- (i) $G_i = G_{i-1} - e_i$, for $i = 1, \dots, k$.
- (ii) $\min\{d_{G_{i-1}}(u_i), d_{G_{i-1}}(v_i)\} \geq 3$ and $d_{G_{i-1}}(w) \leq 2$ for every $w \in N_{G_{i-1}}(u_i) \setminus \{v_i\}$, for $i = 1, \dots, k$.
- (iii) $\min\{d_{G_0-Z}(u), d_{G_0-Z}(v)\} \leq 2$, for every edge $uv \in E(G_0) \setminus Z$.
- (iv) The graph $G_0 - Z$ has r leaves.

Lemma 2.8 *Let m and r be two positive integers and let G be a forest with m components and r leaves such that $\min\{d_G(u), d_G(v)\} \leq 2$ for every $uv \in E(G)$. Then $\psi(L(G)) = r - 2m$.*

The following theorem gives the bipartite vertex frustration of the line graph of a forest.

Theorem 2.9 *Let m and r be two positive integers and let G be a forest with m components and r leaves. Let Z be a set of edges obtained by application of Algorithm 1 to G . Then $\psi(L(G)) = r - 2m - |Z|$.*

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