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Abstract

Komlós [Tiling Turán theorems, Combinatorica, 20, 2 (2000), 203-218] determined the asymptotically optimal minimum degree condition for covering a given proportion of vertices of a host graph by vertex-disjoint copies of a fixed graph. We show that the minimum degree condition can be relaxed in the sense that we require only a given fraction of vertices to have the prescribed degree.

1 Introduction and Results

A classical question in extremal graph theory is to study which density condition forces a graph to contain a certain fixed subgraph. Let us cite Turán theorem [Tur41], which gives the average degree forcing a clique, or the Erdős–Stone theorem [ES46], which essentially determines the average degree condition guaranteeing the containment of a fixed non-bipartite graph H. This approach can be generalised by seeking the density condition forcing the containment of several copies of H. One might consider edge-disjoint copies of H, a so-called *packing* of H, or vertex-disjoint copies of H, known as an H-tiling. The latter concept generalises the one of a matching, where we take Hto be an edge, and this is the direction we follow in this paper.

Definition 1.1 (Tiling, perfect tiling, tiling number). Let H be a fixed graph. We say that a graph G has an H-tilling of size m, if there are m vertex-disjoint copies of H in G. An H-tiling is perfect if it has size $\frac{|V(G)|}{|V(H)|}$. The size of the largest H-tiling in G is called the H-tiling number (in G) (or just tiling number, when H is clear from the context).

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The concept is not new and researchers have obtain many results in this direction. The Hajnal-Szemerédi theorem on equitable colouring [HS70] (formulated for the complement of the host graph) determines the minimum degree condition guaranteeing a perfect clique tiling. One can deduce from their result the minimum degree condition needed to force a partial clique-tiling of any given size. For the corresponding question of determining the average degree condition forcing a clique tiling of a given size, the current state of knowledge is rather poor. The only two known cases for which the average degree condition has been determined correspond to K_2 -tilings and K_3 -tilings, given by Erdős and Gallai [EG59] and Allen et al. [ABHP15], respectively. The case of tiling cliques of higher order is wide open and no conjecture has even been formulated.

Generalising the concept of clique tilings to the one of tilings with an arbitrary graph H, Komlós [Kom00] extended the Hajnal-Szemerédi theorem.

Theorem 1.2 (Komlós tiling theorem). Let $x \in (0, \frac{1}{|V(H)|})$ and $r \ge 2$. If H is an r-colourable graph with colour class sizes $\ell_1 \ge \ldots \ge \ell_r > 0$ and G is an n-vertex graph with minimum degree at least $(r-2+x\ell_r)\frac{n}{r-1}$, then G contains an H-tiling of size at least (x-o(1))n.

As H may have many r-colourings, in order to apply Komlós' theorem with the weakest hypothesis, we need to fix an r-colouring of H which minimises the size of its smallest colour class. For the corresponding value of ℓ_r and for each x, Komlós constructs graphs with minimum degree $(r-2+x\ell_r)\frac{n}{r-1}$ and H-tiling number xn. This shows that his result is asymptotically optimal. Recently, Hladký, Hu, and Piguet [HHPa] proved stability of this result. Komlós tiling theorem was complemented by Kühn and Osthus [KO09], who established the optimal minimum degree condition forcing a perfect H-tiling up to an additive constant. Considering average degree rather than minimum degree, Grosu and Hladký [GH12] extended the Erdős–Gallai theorem, by asymptotically establishing the average degree condition forcing the containment of an H-tiling, for a fixed bipartite graph H.

A finer approach to extremal problems is to take into account more information encoded in the degree sequence of the host graph. Let us give an example in the area of tree containment. It is trivial to see that a minimum degree of k ensures a copy of any tree of size k. However, Loebl, Komlós, and Sós conjecture that only half of the vertices need to have this degree to guarantee the same assertion. Taking this perspective in the area of tilings, Treglown asymptotically determined an optimal degree sequence condition forcing a perfect H-tiling [Tre16] and very recently Hyde, Liu and Treglown [HLT] asymptotically determined the optimal degree sequence for a partial tiling. In this paper, we inquire what portion of vertices need to meet the degree bound in Komlós tiling theorem in order to guarantee an H-tiling of a given size without any requirement on the other vertices. Our main result is the following.

Theorem 1.3. Let H be an r-colourable graph with colour class sizes $\ell_1 \geq \ldots \geq \ell_r > 0$ and let $x \in (0, \frac{1}{|V(H)|})$. Then for any $\eta > 0$ there is an $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$ and for $\delta := (r - 2 + x\ell_r) \frac{n}{r-1}$ the following holds.

Any n-vertex graph with at least $(1+\eta)(r-2+x|V(H)|)\frac{n}{r-1}$ vertices of degree at least $(1+\eta)\delta$ contains an H-tiling of size at least xn.

Theorem 1.3 strengthens Komlós tiling theorem: the degree bound δ is the same but we do not require all the vertices of the host graph to meet this bound, but rather only roughly a $f = \frac{r-2+x|V(H)|}{r-1}$ fraction of them. Note that as x ranges from 0 to $\frac{1}{|V(H)|}$, f ranges from $\frac{r-2}{r-1}$ to 1.

When ℓ_r is chosen according to the colouring minimising the size of the smallest colour class, Theorem 1.3 is asymptotically optimal for each value of x. To show this, we construct an *n*-vertex graph $G = G(x, |V(H)|, r, \ell_r, n)$ with $(r - 2 + x|V(H)|) \frac{n}{r-1}$ vertices of degree $\delta := (r - 2 + x\ell_r) \frac{n}{r-1}$ that does not contain an *H*-tiling of size more than xn. The vertex set of *G* is partitioned into four sets, $|V_1| = x\ell_r n, |V_2| = x(|V(H)| - \ell_r) \frac{n}{r-1}, |V_3| = (r-2)(1 - x\ell_r) \frac{n}{r-1}$, and $|S| = (1 - x|V(H)|) \frac{n}{r-1}$. The sets V_1, V_2 and *S* are independent, and the set V_3 induces a balanced complete (r - 2)-partite graph. There is a complete bipartite graph between V_3 and $V(G) \setminus V_3$ and a complete bipartite graph between V_1 and V_2 , and no other edge. The graph $G(x, |V(H)|, r, \ell_r, n)$ is depicted in Figure 1.1. Note that the set $L = V_1 \cup V_2 \cup V_3$ has size $|L| = (r - 2 + x|V(H)|) \frac{n}{r-1}$ and its vertices meet our degree assumption. Indeed, vertices in V_1 have degree $|L| - |V_1| \ge (r - 2 + x\ell_r) \frac{n}{r-1}$, as $|V(H)| \ge r\ell_r$, vertices in V_2 have degree $|L| - |V_2| = (r - 2 + x\ell_r) \frac{n}{r-1}$, and vertices V_3 have degree $n - \frac{|V_3|}{r-2} = (r - 1 - \frac{(r-2)(1-x\ell_r)}{r-2}) \frac{n}{r-1} = (r - 2 + x\ell_r) \frac{n}{r-1}$. We claim that each copy of *H* in *G* must have at least ℓ_r vertices in V_1 . Suppose it is not the case. We can then recolour *H* so that the smallest colour class has less than ℓ_r vertices, as follows. Give the vertices embedded in V_1 colour 1, the ones embedded in $V_2 \cup S$ colour 2, and distribute the r - 2 other colours to vertices embedded in V_3 . In this colouring, the smallest colour class has at most as many vertices as colour 1 has, which is less than ℓ_r . Since $|V_1| = x\ell_r n$, we conclude that there cannot be an *H*-tiling of size more than xn.

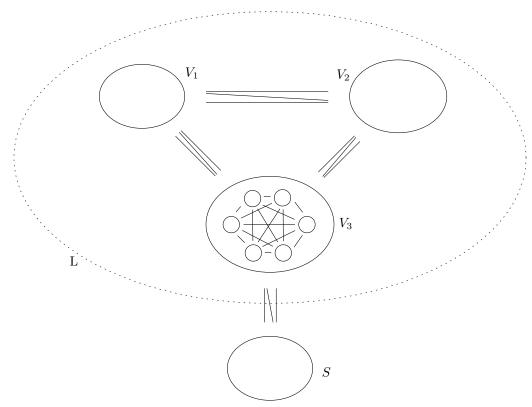


Figure 1.1: The extremal graph $G(x, |V(H)|, r, \ell_r, n)$. The sizes of the sets are as follows: $|V_1| = x\ell_r n, |V_2| = x(|V(H)| - \ell_r)\frac{n}{r-1}, |V_3| = (r-2)(1-x\ell_r)\frac{n}{r-1}, \text{ and } |S| = n - |L| = (r-2-x|V(H)|)\frac{n}{r-1}.$

The proof of Theorem 1.3 is postponed to Section 3 and it relies on the regularity method and on the generalisation of the LP-duality between fractional matching and fractional cover. Such a generalisation of the LP-duality has already been used in [MJ17] for the particular case of *fractional* clique-tiling. However, we think that this method deserves more attention and is exposed here in the more general form for fractional H-tiling¹. Similar use of LP-duality for graph-tiling was used in [DH, HHPa, HHPb] but in the context of graphons.

2 Notation and Preliminaries

2.1 The regularity lemma and the embedding lemma

Given two disjoint subsets $X, Y \subset V(H)$, we denote by e(X, Y) the number of edges between vertices in X and vertices in Y, that is,

$$e(X,Y) := |\{(x,y) \in E(H), x \in X \text{ and } y \in Y\}|$$
.

Definition 2.1 (Regular pairs, bipartite density). For a given $\varepsilon > 0$, a pair (U, W) of disjoint sets $U, W \subset V(H)$ is called an ε -regular pair if for every $U' \subset U$, $W' \subset W$ with $|U'| \ge \varepsilon |U|$, $|W'| \ge \varepsilon |W|$ we have that $|d(U, W) - d(U', W')| < \varepsilon$, where the (bipartite) density d between two disjoint sets X and Y is defined as $d(X, Y) := \frac{e(X,Y)}{|X| \cdot |Y|}$. If the pair (U, W) is not ε -regular, then we call it ε -irregular.

Lemma 2.2 (Slicing lemma, Fact 1.5 in [KS96]). Let G be a graph and (X, Y) be an ε -regular pair of density d in G. Let $X' \subseteq X$ and $Y' \subseteq Y$ such that $|X'| \ge \alpha |X|$ and $|Y'| \ge \alpha |Y|$. Then, (X', Y') is an ε' -regular pair of density at least $d - \varepsilon$, where $\varepsilon' = \max(\varepsilon/\alpha, 2\varepsilon)$.

For completeness, let us next state the fundamental regularity lemma.

Lemma 2.3 (Szemerédi's regularity lemma, Theorem 7.4.1 in [Die16]). For all $\varepsilon_R > 0$ and $\ell \in \mathbb{N}$ there exist $n_R, M \in \mathbb{N}$ such that for every $n \ge n_R$ and every n-vertex graph G there exists a partition U_0, U_1, \ldots, U_p of $V(G), \ \ell , with the following properties:$

1) For every $i, j \in [p]$ we have $|U_i| = |U_j| := m$, and $|U_0| < \varepsilon_R n$.

2) All but at most $\varepsilon_R p^2$ pairs (U_i, U_j) , $i, j \in [p]$, $i \neq j$, are ε_R -regular.

The partition U_0, U_1, \ldots, U_p of V(G) allows to define a *cluster graph* **G** (known as well as *reduced* graph) with parameters ε_R , m and d as follows: $V(\mathbf{G}) := \{U_1, \ldots, U_p\}$ and $U_i U_j \in E(\mathbf{G})$ if and only if (U_i, U_j) is an ε_R -regular pair such that $d(U_i, U_j) \ge d$.

Given a cluster graph \mathbf{G} , let the blow-up graph \mathbf{G}_s denote the graph defined as follows: Every vertex v of \mathbf{G} is replaced by a set of s vertices, which we call the *clone set* of v, and every edge by a complete bipartite graph between the corresponding clone-sets.

There is a natural association between the clusters of G, the vertices of \mathbf{G} , and the clone-sets of \mathbf{G}_s : The cluster U_i of G is associated to the vertex U_i of \mathbf{G} and, in \mathbf{G}_s , this vertex is replaced by the clone-set of U_i , which we denote by $U_i^{(s)}$

Lemma 2.4 (Lemma 7.5.2 in [Die16] (Embedding lemma)). For any $d \in (0, 1]$ and $\Delta \geq 1$, there exists an $\varepsilon_0 > 0$ such that the following holds. Let **G** be the cluster graph of a graph *G* with parameters $\varepsilon_R > 0$ for the regularity, $m \in \mathbb{N}$ for the size of its clusters and d > 0 for the minimal density of the regular pairs. If $\Delta(H) \leq \Delta$, $\varepsilon_R \leq \varepsilon_0$ and $m \geq 2s/d^{\Delta}$, then

$$H \subseteq \mathbf{G}_s \; \Rightarrow \; H \subseteq G$$

¹For a precise definition of fractional tiling, see Section 2.2.

From the proof of Lemma 2.4 given in [Die16], one can infer the following slightly stronger claim: If the hypothesis of the lemma are fulfilled and $H \subseteq \mathbf{G}_s$, then there exists an embedding of H in G such that, if a vertex of $H \subseteq \mathbf{G}_s$ is contained in clone-set $U_j^{(s)}$ of \mathbf{G}_s , then this vertex is contained in the cluster U_j of G.

2.2 Fractional hom_H-tiling, fractional hom_H-cover, and LP-duality

From Lemma 2.4 it is clear that a copy of H in a cluster graph \mathbf{G} implies a copy of H in the original graph G; actually using Lemma 2.4 it can be shown that it generates up to $(1 - o(1)) \frac{|V(G)|}{|V(\mathbf{G})|}$ many disjoint copies of H. Going through all (not-necessarily disjoint) copies of H in \mathbf{G} , we could use the notion of fractional tiling to find a large H-tiling in G. A fractional tiling of H in G is a function giving weights in [0, 1] to copies of H in G such that for each vertex $v \in V(G)$ the sum of the weights of all copies containing v is at most 1. Hence an H-tiling in G (Definition 1.1) is an integral tiling, i.e., a fractional tiling whose weights are either 0 or 1.

But not only copies of H in \mathbf{G} may generate copies of H in G. For example, if there is a triangle in \mathbf{G} , we know there are many copies of C_5 in G. Therefore, instead of looking for copies of Hin \mathbf{G} , we shall seek copies of H', where H' is some homomorphic image of H. A homomorphism $h: H \to G$ is a mapping from V(H) to V(G) such that $uv \in E(H)$ implies $h(u)h(v) \in E(G)$. Hence, we shall generalise the concept of fractional tiling of H in G, by enriching it by the homomorphism images of H.

 Set

 $G^H := \{h \mid h : H \to G \text{ is a homomorphism}\}.$

Definition 2.5 (Fractional hom_H-tiling). A function $f : \{h(H) \subseteq G : h \in G^H\} \to [0,1]$ is a fractional hom_H-tiling in G of size $\kappa \in \mathbb{R}^+$ if it satisfies the following two properties:

- 1. For any vertex $v \in V(G)$, we have $\sum f(h(H)) \cdot |h^{-1}(v)| \leq 1$, where the sum runs over all homomorphisms $h \in G^H$.
- 2. We have that $\sum f(h(H)) = \kappa$, where the sum runs through all homomorphisms $h \in G^H$.

So in a fractional hom_H-tiling we not only assign weights to isomorphic copies of H in G, but to homomorphic copies of H, as well.

In order to prove Theorem 1.3, we shall use LP-duality. So, similarly as above, we need to generalise the notion of fractional H-cover in a graph G to consider also homomorphic copies of H, which will be the dual notion of fractional hom_H-tiling (Definition 2.6). We shall prove in Proposition 2.8 that those two notions are indeed dual.

Definition 2.6 (Fractional hom_H-cover). A function $c : V(G) \to [0,1]$ is a fractional hom_H-cover in G of size $\kappa \in \mathbb{R}^+$ if it satisfies the following two properties:

- (i) For any subgraph $H' \subseteq G$ and any homomorphism $h: H \to H'$, we have $\sum c(v)|h^{-1}(v)| \ge 1$, where the sum runs through all vertices $v \in V(H')$.
- (ii) We have $\sum c(v) = \kappa$, where the sum runs through all vertices $v \in V(G)$.

Definition 2.7 (Fractional tiling/cover number). The fractional hom_H-tiling number of a graph G is the maximum of the sizes of all its fractional hom_H-tilings.

The fractional hom_H -cover number of a graph G is the minimum of the sizes of all its fractional hom_H -covers.

Proposition 2.8 (LP-duality for hom_H). For any graph G, its fractional hom_H -tiling number equals its fractional hom_H -cover number.

Sketch of the proof. The proof is a straightforward generalisation of the LP-duality between the fractional matching and the fractional vertex-cover. Assume G is an n-vertex graph and assume the number of homomorphisms $h \in G^H$ of H in G is m (we have $m \leq n^{|V(H)|}$). To attain the fractional hom_H-tiling number is equivalent to the following instance of linear programming:

$$\begin{array}{ll} \text{maximise} & \mathbf{v}^T \mathbf{z} \\ \text{subject to} & A \mathbf{z} \leq \mathbf{u} \\ \text{and} & \mathbf{z} \geq \mathbf{0} \end{array},$$

where \mathbf{v} and \mathbf{u} are all-one vectors of length m and n, respectively, \mathbf{z} is the vector (of length m) of variables (to be determined) corresponding to a fractional hom_H-tiling, and A is an $n \times m$ matrix. Each entry of the vector \mathbf{z} corresponds to the weight given to the associated homomorphism. Each column of A corresponds to a homomorphism $h \in G^H$ with image H' = h(H), each row corresponds to a vertex v of G, and $A_{H',v} = |h^{-1}(v)|$. Since this optimisation problem clearly has an optimal solution, the Strong LP-duality theorem says that it is equivalent to the following dual problem:

$$\begin{array}{ll} \text{minimise} & \mathbf{u}^T \mathbf{y} \\ \text{subject to} & A^T \mathbf{y} \ge \mathbf{v} \\ \text{and} & \mathbf{y} \ge \mathbf{0} \end{array},$$

where **y** is the vector (of length n) of variables (to be determined) corresponding to a fractional hom_H-cover.

The core of the proof of Theorem 1.3 is the next proposition, which shows that a graph satisfying conditions similar to those in Theorem 1.3 has large fractional cover.

Proposition 2.9. Let H be an r-colourable graph with colour class sizes

$$\ell_1 \ge \ldots \ge \ell_r > 0 \tag{2.1}$$

and let $x \in (0, \frac{1}{|V(H)|})$. Set

$$\delta := (r - 2 + x\ell_r) \frac{n}{r - 1} .$$
(2.2)

Then any n-vertex graph G with at least $(r-2+x|V(H)|)\frac{n}{r-1}$ vertices of degree at least δ has fractional hom_H-cover number least xn.

3 Proof of Theorem 1.3

Before jumping into the proof of Theorem 1.3, we explain first its main ideas. By applying Szemerédi's regularity lemma to the input graph G, we obtain a corresponding cluster graph \mathbf{G} , which satisfies degree conditions similar to the ones of G. Proposition 2.9 implies that the cluster graph \mathbf{G} has large fractional hom_H-cover number. By the LP-duality for hom_H, the cluster graph \mathbf{G} has large fractional hom_H-tiling number. The embedding lemma yields a large H-tiling in the original input graph G.

Let *H* be a fixed *r*-colourable graph with colour classes $\ell_1 \geq \cdots \geq \ell_r$ and let $x \in (0, \frac{1}{|V(H)|})$ be fixed. Assume we are given $\eta > 0$. Notice that we can assume that $\eta < 1$. Set $d := \frac{1}{8} \left(\frac{\eta x \ell_r}{r-1} \right)^2$. Since $\eta < 1$ and $x < \frac{1}{|V(H)|} < \frac{1}{\ell_r}$, we have that d < 1. Lemma 2.4 with input $d_{L2.4} := d/2$ and $\Delta_{L2.4} := |V(H)|$ outputs an $\varepsilon_0 > 0$. We define

$$\varepsilon_R := \min\left\{\varepsilon_0, \frac{d}{6}, \left(\frac{\eta}{6}\right)^2, \frac{1}{2}\left(\frac{1}{3|V(H)|x} - \frac{1}{3}\right)^2\right\}.$$
(3.1)

Notice that $\left(\frac{1}{3|V(H)|x} - \frac{1}{3}\right) > 0$, which implies that $\sqrt{\varepsilon_R} < \frac{1}{3|V(H)|x} - \frac{1}{3}$. The regularity lemma (Lemma 2.3) with input ε_R and $\ell := \frac{1}{\varepsilon_R}$ outputs $n_R, M \in \mathbb{N}$. Set

$$n_0 := \max\left\{n_R, \frac{2^{\Delta_{L2.4}+1} \cdot |V(H)| \cdot M}{d^{\Delta_{L2.4}} \cdot \sqrt{\varepsilon_R} \cdot (1-\varepsilon_R)}\right\} .$$

$$(3.2)$$

Let $n \ge n_0$ and let G be a *n*-vertex graph with at least $(1 + \eta)(r - 2 + x|V(H)|)\frac{n}{r-1}$ vertices of degree at least $(1 + \eta)\delta$. We apply Lemma 2.3 on G (with parameters ε_R and ℓ) and obtain an equitable ε_R -regular partition U_0, U_1, \ldots, U_p with $\ell . We erase all edges within clusters <math>U_i$, in irregular pairs, in regular pairs of density smaller than d, and edges incident to the cluster U_0 . Slightly abusing notation, we still call this subgraph G. Let **G** be the corresponding cluster graph with parameters $\varepsilon_R, m := |U_i|$ ($i = 1, \ldots, p$), and d.

Claim 3.1. After erasing edges as described in the previous paragraph, G has at least $(1+\eta/2)(r-2+x|V(H)|)\frac{n}{r-1}$ vertices of degree at least $(1+\eta/2)\delta$.

Proof. By erasing all edges within clusters, we remove at most $\binom{m}{2}p$ edges. Notice that

$$\binom{m}{2}p \le \binom{n/p}{2}p \le \frac{n^2}{2p} \le \frac{\epsilon_R n^2}{2}.$$

By erasing all edges between irregular pairs, we remove at most $\epsilon_R p^2 m^2 \leq \epsilon_R n^2$ edges. By erasing all edges between regular pairs of density smaller than d, we remove at most $\binom{p}{2}dm^2 \leq \frac{d}{2}n^2$ edges. Finally, by erasing all edges incident to cluster U_0 , we remove at most $\epsilon_R n^2$ edges. In total, the number of edges that have been removed from G is at most

$$3\epsilon_R n^2 + \frac{d}{2}n^2 \le dn^2 = \frac{1}{8} \left(\frac{\eta x \ell_r}{r-1}\right)^2 n^2.$$

If the statement of the claim were not true, after the transformations, more than $\frac{\eta}{2}(r-2+x|V(H)|)\frac{n}{r-1}$ vertices of G would have had their degree decreased by more than $\frac{\eta}{2}\delta$. This would imply that the number of edges removed from G is at least

$$\frac{1}{2}\frac{\eta}{2}(r-2+x|V(H)|)\frac{n}{r-1}\frac{\eta}{2}(r-2+x\ell_r)\frac{n}{r-1} > \frac{1}{8}\left(\frac{\eta x\ell_r}{r-1}\right)^2 n^2 + \frac{1}{2}\left(\frac{\eta x\ell_r}{r-1}\right)^2 n^2 + \frac{1}{2}\left(\frac{\eta x\ell_r}{r-1}\right)^2 n^2 + \frac{1}{2}\left(\frac{\eta x\ell_r}{r-1}\right)^2 + \frac{1}{2}\left(\frac{\eta x\ell_r$$

which yields a contradiction.

Claim 3.2. The cluster graph **G** has at least $(1 + \eta/2)(r - 2 + x|V(H)|)\frac{p}{r-1}$ vertices of degree at least $(1 + \eta/2)(r - 2 + x\ell_r)\frac{p}{r-1}$.

Proof. For every vertex v of G, we denote by U_v the cluster of the regular partition containing v.

Let v a vertex of G with degree at least $(1 + \eta/2)\delta$. Given a distinct vertex w of G, we have that $vw \in E(G)$ implies that $U_v U_w \in E(\mathbf{G})$. Since v has at most m neighbours in every cluster U_w , we obtain that $U_v \in V(\mathbf{G})$ has degree at least

$$(1+\eta/2)\frac{\delta}{m} \ge (1+\eta/2)(r-2+x\ell_r)\frac{p}{r-1}$$
.

Similarly, let $U_v \in V(\mathbf{G})$ have degree at least $(1 + \eta/2)(r - 2 + x\ell_r)\frac{p}{r-1}$. At most *m* vertices of *G* of degree at least $(1 + \eta/2)\delta$ belong to cluster U_v . Therefore, the number of vertices of **G** with degree at least $(1 + \eta/2)(r - 2 + x\ell_r)\frac{p}{r-1}$ is at least

$$\frac{1}{m}(1+\eta/2)(r-2+x|V(H)|)\frac{n}{r-1} \ge (1+\eta/2)(r-2+x|V(H)|)\frac{p}{r-1}.$$

Claim 3.3. If the fractional hom_H-tiling number of **G** is at least $(1 + 3\sqrt{\varepsilon_R})xp$, then the H-tiling number of G is at least xn.

Proof. Let f be a fractional hom_H-tiling of **G** of maximum size. By hypothesis,

$$\sum f(h(H)) \ge (1 + 3\sqrt{\varepsilon_R}) xp,$$

where the sum runs through all homomorphisms $h \in \mathbf{G}^{H}$.

Using the embedding lemma (Lemma 2.4), we will sequentially find copies of H in G, as long as $\sqrt{\varepsilon_R}m$ unused vertices are left in each cluster. We denote by W the vertices of G used by the copies of H. At the beginning $W = \emptyset$. We define an auxiliary graph $G' \subseteq G$ as follows. Let $m' = \min\{|U_i \setminus W| : i = 1, \ldots, p\}$ be the smallest number of free vertices in a cluster. For each $i = 1, \ldots, p$ let U'_i be an arbitrary set of size m' in $U_i \setminus W$. The graph G' is the subgraph of Ginduced by $\bigcup_{i=1}^p U'_i$. Observe that as long as $m' \ge \sqrt{\varepsilon_R}m$, we have by Lemma 2.2 that the pairs (U'_i, U'_i) are ε' -regular with density at least d', where $\varepsilon' = \sqrt{\varepsilon_R}$ and d' = d/2.

For every $h \in \mathbf{G}^{H}$, we shall construct $mf(h(H))(1 - \sqrt{\varepsilon_R})$ vertex disjoint copies of H in G. Any such copy $g: H \to G$ will satisfy that, for every vertex $v \in V(H)$, its image g(v) is contained in the cluster of G corresponding to h(v). So assume that we have a homomorphism $h \in \mathbf{G}^{H}$ and let $h(H) \subseteq \mathbf{G}$. We first construct a copy of H in $\mathbf{G}_{|V(H)|}$ as follows: Let $U_i \in V(h(H))$; then each of the vertices in $h^{-1}(U_i)$ is embedded to a distinct vertex in the clone-set $U_i^{(s)}$ of $\mathbf{G}_{|V(H)|}$. Since h is a homomorphism, no two vertices of this group are adjacent in H. By repeating this procedure for all vertices in V(h(H)), we obtain an embedding of the vertices of H in $\mathbf{G}_{|V(H)|}$. By construction of $\mathbf{G}_{|V(H)|}$ and the fact that homomorphisms map edges of H to edges of h(H), we obtain that the edges of H are also correctly embedded in $\mathbf{G}_{|V(H)|}$. Due to the choices of ε_R and

 n_0 given in equations (3.1) and (3.2), we can use Lemma 2.4 with $d_{L2.4} = d'$, $\Delta_{L2.4} := |V(H)|$, $m_{L2.4} := m'$, and $s_{L2.4} := |V(H)|$ to find an embedding of H in G such that for $U_i \in V(\mathbf{G})$ all of the vertices in $h^{-1}(U_i)$ are embedded in the cluster U_i of G. We put the vertices involved in this copy of H in G in the set W. We repeat the process above $mf(h(H))(1 - \sqrt{\varepsilon_R})$ times, and obtain $mf(h(H))(1 - \sqrt{\varepsilon_R})$ vertex-disjoint copies of H in G. This can be done using Lemma 2.4 (with the same parameters as above) as long as $m' \geq \sqrt{\varepsilon_R}m$.

We next repeat this process for the remaining homomorphisms $h \in \mathbf{G}^{H}$. For this to be possible, we need to argue that $m' \geq \sqrt{\varepsilon_R}m$. Let U_i be a vertex in $V(\mathbf{G})$. The number of vertices in $U_i \setminus U'_i$ is at most the number of vertices used in copies of H in G so far. This is always at most $\sum mf(h(H))(1-\sqrt{\varepsilon_R})|(h)^{-1}(v)|$, where the sum runs through all homomorphisms $h \in \mathbf{G}^{H}$. In the next formula, the sums always run through all homomorphisms $h \in \mathbf{G}^{H}$:

$$\sum mf(h(H))(1-\sqrt{\varepsilon_R})|(h)^{-1}(v)| = m(1-\sqrt{\varepsilon_R})\left(\sum f(h(H))|(h)^{-1}(v)|\right)$$

$$\leq m(1-\sqrt{\varepsilon_R}).$$

Consequently, the number of vertices in U'_i is at least $m - m(1 - \sqrt{\varepsilon_R}) = \sqrt{\varepsilon_R}m$.

It only remains to bound the size of the *H*-tiling of *G* obtained by the above procedure. As we have seen, each homomorphism $h \in \mathbf{G}^H$ yields $mf(h(H))(1 - \sqrt{\varepsilon_R})$ vertex-disjoint copies of *H* in *G*. Hence, the total size of the tiling is

$$\sum mf(h(H))(1-\sqrt{\varepsilon_R}) \ge m\left(1+3\sqrt{\varepsilon_R}\right)xp\left(1-\sqrt{\varepsilon_R}\right) \ge xn\left(1-\varepsilon_R\right)\left(1+3\sqrt{\varepsilon_R}\right)\left(1-\sqrt{\varepsilon_R}\right) \ge xn,$$

where the sum runs through all homomorphisms $h \in \mathbf{G}^H$, and where we have used that $\varepsilon_R \leq 1/5$. Notice that this holds because $\varepsilon_R \leq d/6 < 1/6$.

From Proposition 2.8, we infer that in order to prove Theorem 1.3 it is enough to show that the fractional hom_H-cover number of **G** is at least $(1 + 3\sqrt{\varepsilon_R})xp$. We apply Proposition 2.9 to the cluster graph **G**, where the value $x_{P2.9}$ used in the proposition is $x_{P2.9} := (1 + 3\sqrt{\varepsilon_R})x$. For this to be valid, we need to argue that $(1 + 3\sqrt{\varepsilon_R})x \in (0, \frac{1}{|V(H)|})$, and that **G** has the appropriate number of vertices of the appropriate degree. Since $\varepsilon_R < \left(\frac{1}{3|V(H)|x} - \frac{1}{3}\right)^2$ (see (3.1) and the paragraph afterwards), we have that $(1 + 3\sqrt{\varepsilon_R})x < \frac{1}{|V(H)|}$. Regarding the degree condition, by Claim 3.2, **G** has at least

$$(1+\eta/2)(r-2+x|V(H)|)\frac{p}{r-1} \stackrel{(3.1)}{\geq} (1+3\sqrt{\varepsilon_R})(r-2+x|V(H)|)\frac{p}{r-1}$$

vertices of degree at least

$$(1+\eta/2)(r-2+x\ell_r)\frac{p}{r-1} \stackrel{(3.1)}{\geq} (1+3\sqrt{\varepsilon_R})(r-2+x\ell_r)\frac{p}{r-1}.$$

Thus, we can use Proposition 2.9 with the desired parameters, which ensures a fractional hom_Hcover number of at least $(1 + 3\sqrt{\varepsilon_R})n$ in **G**. Proposition 2.8 (LP-duality for hom_H) then implies that there is a fractional hom_H-tiling of size at least $(1 + 3\sqrt{\varepsilon_R})n$ in **G**. Finally, Claim 3.3 ensures an H-tiling in G of size at least xn.

4 Proof of Proposition 2.9

Let c be any fractional hom_H-cover of G, and let ||c|| denote its size. In order to prove the proposition, we want to show that $||c|| \ge xn$. In the proof, we show that there is a clique of size r in G, which is a homomorphic image of H. This implies a lower bound on the total cover of the vertices of the clique. Among all possible such cliques, we consider one with minimum possible cover on its vertices with respect to the lexicographical order. From this we derive a lower bound for the cover of particular subsets of V(G), allowing us to compute a lower-bound for ||c||.

Let L denote the set of vertices of G having degree at least δ , and let S denote the set containing the remaining vertices of G. By hypothesis,

$$|L| \ge (r - 2 + x|V(H)|) \frac{n}{r - 1}.$$
(4.1)

For any vertex v of G, we denote the set of neighbours of v in L and S by $N_L(v)$ and $N_S(v)$, respectively.

Claim 4.1. There is a clique K_{r-1} of size r-1 in L and the vertices of any clique of size r-1 in L have a non-empty common neighbourhood in V(G).

Proof. We find vertices $v_1, v_2, \ldots, v_{r-1} \in L$ of the clique K_{r-1} in the following way. As L is nonempty, we pick v_1 arbitrarily in L. If r = 2, $v_1 = v_{r-1}$ and we are done. If r > 2, suppose that we have already selected i < r - 1 vertices v_1, \ldots, v_i in L. We define:

$$N_L(i) = \bigcap_{j=1}^{i} N_L(v_j),$$

$$N_S(i) = \bigcap_{j=1}^{i} N_S(v_j).$$

We show that $|N_L(i)| > 0$, as long as i < r - 1. It is well-known that, given a set U of size m and i subsets A_j of U,

$$\left|\bigcap_{j=1}^{i} A_{j}\right| \geq \sum_{j=1}^{i} |A_{j}| - (i-1) \cdot m.$$

Consequently,

$$|N_{L}(i)| + |N_{S}(i)| \geq i \cdot \delta - (i-1) \cdot n = i \cdot \left((r-2+x\ell_{r})\frac{n}{r-1} \right) - (i-1) \cdot n$$

= $\left(\frac{r-1-i}{r-1} \right) \cdot n + \frac{ix\ell_{r}n}{r-1}$
= $(r-1-i+ix\ell_{r})\frac{n}{r-1}$. (4.2)

Note that Inequality (4.2) holds also in the case when i = r - 1. Since

$$|N_S(i)| \leq |S| = n - |L| \stackrel{(4.1)}{\leq} (1 - x|V(H)|) \frac{n}{r - 1}$$

and i < r-1, we have that $|N_L(i)| > 0$ and we can pick $v_{i+1} \in N_L(i)$. Continuing in this way, we obtain K_{r-1} in L.

Observe that if i = r - 1, we obtain that v_1, \ldots, v_{r-1} have common neighbourhood of size at least $x\ell_r n > 0$ (and hence there is a clique of size r in G). However, this common neighbourhood may lie completely in S.

Among all possible cliques of size r-1 in L, pick one with the smallest cover with respect to the lexicographical order and let v_1, \ldots, v_{r-1} be its vertices (in lexicographical order w.r.t. the cover). We set $u \in N_L(r-1)$ to be a vertex with the smallest c(u), and $w \in N_S(r-1)$ one with the smallest c(w). It may happen that u or w does not exist, but at least one of them does. For $i = 1, \ldots, r-1$, we define $\alpha_i = c(v_i)$. Let $\alpha_L = c(u)$, if u exists, and set $\alpha_L = 1$, otherwise. Similarly, let $\alpha_S = c(w)$, if w exists, and set $\alpha_S = 1$, otherwise. Let $\alpha_r = \min{\{\alpha_L, \alpha_S\}}$. We then have:

$$\alpha_1 \le \alpha_2 \le \dots \le \alpha_{r-1} \le \alpha_L \,. \tag{4.3}$$

As *H* is *r*-colourable, there exists a homomorphism $h : H \to G$ that sends all vertices in the ℓ_i -sized color class of *H* to v_i (for i = 1, ..., r - 1), and all vertices in the ℓ_r -sized color class of *H* to u or w. Since *c* is a fractional hom_{*H*}-cover of *G*, we obtain the following relation:

$$\ell_1 \alpha_1 + \ell_2 \alpha_2 + \dots + \ell_r \alpha_r \ge 1. \tag{4.4}$$

As $x < \frac{1}{|V(H)|}$, we have

$$n > x \left| V(H) \right| n \,. \tag{4.5}$$

To obtain a lower bound for ||c||, we will use the following auxiliary inequality:

Claim 4.2. $|N_L(r-1)|(\alpha_L - \alpha_{r-1}) + |N_S(r-1)| \cdot \alpha_r \ge \alpha_r \cdot x \cdot \ell_r \cdot n - \alpha_{r-1}[(r-1)\delta - (r-2)n].$

Proof. In order to prove the claim, we need to consider the two following cases:

CASE 1:
$$\alpha_L \ge \alpha_r + \alpha_{r-1}$$

CASE 2: $\alpha_L < \alpha_r + \alpha_{r-1}$

In CASE 1, we have

$$|N_L(r-1)| (\alpha_L - \alpha_{r-1}) + |N_S(r-1)| \cdot \alpha_r \ge |N_L(r-1)| \cdot \alpha_r + |N_S(r-1)| \cdot \alpha_r$$

$$\stackrel{(4.2)}{\ge} \alpha_r \cdot x \cdot \ell_r \cdot n$$

$$\ge \alpha_r \cdot x \cdot \ell_r \cdot n + \alpha_{r-1} \left((r-2)n - (r-1)\delta \right).$$

In CASE 2, we get

$$|N_{L}(r-1)| (\alpha_{L} - \alpha_{r-1}) + |N_{S}(r-1)| \cdot \alpha_{r}$$

$$\geq |N_{L}(r-1)| (\alpha_{L} - \alpha_{r-1}) + |N_{S}(r-1)| (\alpha_{L} - \alpha_{r-1})$$

$$\stackrel{(4.2)}{\geq} ((r-1)\delta - (r-2)n) (\alpha_{L} - \alpha_{r-1})$$

$$= \alpha_{r-1} ((r-2)n - (r-1)\delta) + \alpha_{L} \cdot x \cdot \ell_{r} \cdot n$$

$$\geq \alpha_{r-1} ((r-2)n - (r-1)\delta) + \alpha_{r} \cdot x \cdot \ell_{r} \cdot n.$$

We now have the necessary ingredients to prove that $||c|| \ge xn$. We lower-bound ||c|| by computing the cover of particular neighbourhoods of v_1, \ldots, v_{r-1} .

$$\|c\| = \sum_{v \in V(G)} c(v)$$

$$\geq |L| \cdot \alpha_1 + \sum_{i=1}^{r-2} |N_L(i)| \cdot (\alpha_{i+1} - \alpha_i) + |N_L(r-1)| \cdot (\alpha_L - \alpha_{r-1}) + |N_S(r-1)| \cdot \alpha_S. \quad (4.6)$$

By (4.3), we have that $(\alpha_{i+1} - \alpha_i) \ge 0$, for $i = 1, \ldots, r-2$ and that $\alpha_L - \alpha_{r-1} \ge 0$. Therefore,

$$(4.6) \stackrel{(4.2)}{\geq} |L| \cdot \alpha_{1} + \sum_{i=1}^{r-2} [i \cdot \delta - (i-1) \cdot n - |N_{S}(i)|] \cdot (\alpha_{i+1} - \alpha_{i}) + |N_{L}(r-1)| \cdot (\alpha_{L} - \alpha_{r-1}) + |N_{S}(r-1)| \cdot \alpha_{S} \geq |L| \cdot \alpha_{1} + \sum_{i=1}^{r-2} [i \cdot \delta - (i-1) \cdot n - |S|] \cdot (\alpha_{i+1} - \alpha_{i}) + |N_{L}(r-1)| \cdot (\alpha_{L} - \alpha_{r-1}) + |N_{S}(r-1)| \cdot \alpha_{r} = \alpha_{1}(|L| + |S| - \delta) + \sum_{i=2}^{r-2} (\alpha_{i} \cdot (n-\delta)) + \alpha_{r-1} (n - (r-2)(n-\delta) - |S|) + |N_{L}(r-1)| \cdot (\alpha_{L} - \alpha_{r-1}) + |N_{S}(r-1)| \cdot \alpha_{r}.$$

$$(4.7)$$

Using the auxiliary inequality from Claim 4.2, we get

$$\begin{split} (4.7) &\geq & \alpha_1 \cdot (|L| + |S| - \delta) + \sum_{i=2}^{r-2} (\alpha_i \cdot (n - \delta)) + \alpha_{r-1} \cdot (n - \delta - |S|) + \alpha_r \cdot x \cdot \ell_r \cdot n \\ &\stackrel{(4.4)}{\geq} \alpha_1 \cdot (n - \delta) + \sum_{i=2}^{r-2} (\alpha_i \cdot (n - \delta)) + \alpha_{r-1} \cdot (|L| - \delta) + \frac{1}{\ell_r} \cdot \left(1 - \sum_{i=1}^{r-1} \ell_i \cdot \alpha_i\right) \cdot x \cdot \ell_r \cdot n \\ &\stackrel{(2.2),(4.1)}{\geq} \sum_{i=1}^{r-2} (\alpha_i \cdot (n - \delta - x \cdot \ell_i \cdot n)) + \alpha_{r-1} \cdot \left(x \left(|V(H)| - \ell_r\right) \frac{n}{r-1} - x \cdot \ell_{r-1} \cdot n\right) + x \cdot n \\ &\stackrel{(2.2)}{=} \sum_{i=1}^{r-2} \left(\alpha_i \cdot \left((1 - x \cdot \ell_r) \frac{n}{r-1} - x \cdot \ell_i \cdot n\right)\right) + \alpha_{r-1} \cdot \left(x \left(|V(H)| - \ell_r\right) \frac{n}{r-1} - x \cdot \ell_{r-1} \cdot n\right) + x \cdot n \\ &\stackrel{(4.5)}{=} x \cdot n + \sum_{i=1}^{r-1} \left(\frac{\alpha_i}{r-1} \left(\sum_{j=1}^{r-1} (x \cdot \ell_j \cdot n) - (r-1) \cdot x \cdot \ell_i \cdot n\right)\right) \right) \\ &= x \cdot n + \frac{x \cdot n}{r-1} \cdot \sum_{i=1}^{r-1} \left(\alpha_i \cdot \left(\sum_{j=1}^{r-1} (\ell_j - \ell_i)\right)\right) \\ &= x \cdot n + \frac{x \cdot n}{r-1} \cdot \sum_{i=1}^{r-1} \sum_{j=1}^{i-1} ((\ell_j - \ell_i) \cdot (\alpha_i - \alpha_j)) \end{aligned}$$

Hence the fractional \hom_H -cover number of G is at least xn.

5 Final remarks

The approximation in Theorem 1.3 cannot be totally removed, as witnessed by the following graphs. Assume that the graph $H = K_{3,3,3}$ we want to tile is the balanced complete tripartite graph on nine vertices. The vertex set of the host graph G is partitioned into four sets V_1, V_2, V_3, V_4 , with $|V_1| = 3xn - 2$, $|V_2| = 3xn + 2$, $|V_3| = (1 - 3x)n/2$ and $|V_4| = (1 - 9x)n/2$, where the sets V_1, V_3 , and V_4 are independent, the graph induced by V_2 is a (spanning) cycle, and all the edges between V_3 and $V_1 \cup V_2 \cup V_4$, and between V_1 and V_2 are present. The graph G has $(r-2+x|V(H)|)\frac{n}{r-1}$ vertices of degree at least $\delta := (r-2+x\ell_r)\frac{n}{r-1}$ but does not contain xn vertex disjoint copies of $K_{3,3,3}$, for x > 1/n.

It would be interesting however to know what approximation is really needed, i.e., given a graph H and $x \in (0, \frac{1}{|V(H)|})$, what functions f_1 and f_2 guarantee that any *n*-vertex graph with at least $\left(r-2+x\frac{1}{|V(H)|}\right)\frac{n}{r-1}+f_1(n)$ vertices of degree at least $\delta+f_2(n)$ contains an H-tiling of size at least xn? In particular, do $f_i(n)$, i = 1, 2 depend on n?

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