# PAIRWISE INTERSECTING HOMOTHETS OF A CONVEX BODY

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ABSTRACT. We show that the maximum number of pairwise intersecting positive homothets of a *d*-dimensional centrally symmetric convex body, none of which contains the center of another in its interior, is at most  $3^{d+1}$ . Also, we improve upper bounds for cardinalities of *k*-distance sets in Minkowski spaces.

## 1. INTRODUCTION

A convex body K in the d-dimensional Euclidean space  $\mathbb{R}^d$  is a compact convex set with non-empty interior, and it is *o-symmetric* if K = -K. A homothet of K is a set of the form  $\mathbf{v} + \lambda K := {\mathbf{v} + \lambda \mathbf{k} : \mathbf{k} \in K}$ , where  $\lambda \in \mathbb{R}$  is the homothety ratio, and  $\mathbf{v} \in \mathbb{R}^d$  is a translation vector. A homothet of K is called *positive* if its homothety ratio is positive. We will consider only positive homothets of *o*-symmetric bodies here, and thus we will omit the word "positive" most of the time. Also, we write [n] for the set  $\{1, 2, \ldots, n\}$ ,  $\operatorname{dist}(h_1, h_2)$  for the Euclidean distance between two parallel hyperplanes  $h_1$  and  $h_2$ ,  $\operatorname{dim}(h)$ for the dimension of a flat h.  $\operatorname{conv}(A)$ ,  $\operatorname{aff}(A)$ ,  $\operatorname{vol}(A)$  and  $\partial A$  stand for the convex hull, the affine hull, the volume and the boundary of a set  $A \subset \mathbb{R}^d$  respectively.

A Minkowski arrangement of an o-symmetric convex body K is called a family  $\{\mathbf{v}_i + \lambda_i K\}$  of positive homothets of K such that none of the homothets contains the center of any other homothet in its interior (see [7]). We write  $\kappa(K)$  for the largest number of homothets that a pairwise intersecting Minkowski arrangement of K can have. Z. Füredi and P.A. Loeb [2] proved that  $\kappa(K) \leq 5^d$ . Recently, M. Naszódi, J. Pach and K. Swanepoel [4] improved this result to  $\kappa(K) \leq O(3^d d \log d)$ . The authors of [4] noted that it is obvious that for the d-dimensional cube  $C^d$  we have  $\kappa(C^d) = 3^d$ . We prove the following upper bound for  $\kappa(K)$ , which is sharp up to the constant factor.

**Theorem 1.** For any d-dimensional o-symmetric convex body K,

$$\kappa(K) \le 3^{d+1}.$$

Also, some generalization of a Minkowski arrangement for non-symmetric bodies (the role of the center is played by an arbitrary interior point) was studied in [4]. Unfortunately, it is impossible to generalize our approach for non-symmetric bodies.

We call a subset S of a metric space a k-distance set if the set of non-zero distances occurring between points of S is of size at most k. A 1-distance set is called an *equilateral* set. For d-dimensional Minkowski spaces it is well known that the maximal cardinality of an equilateral (that is, a 1-distance) set is  $2^d$  with equality iff the unit ball of the space is a parallelotope, see [6]. K. Swanepoel [8] proved that if the unit ball of a d-dimensional Minkowski space is a parallelotope then a k-distance set has cardinality at most  $(k + 1)^d$ , where the bound is tight. Therefore, he [8] conjectured that the maximal cardinality of kdistance sets in Minkowski spaces is  $(k+1)^d$ . Also, it was proved in [8] that the cardinality of a k-distance set in a d-dimensional Minkowski space is at most min $\{2^{kd}, (k+1)^{(11^d-9^d)/2}\}$ . Moreover, the last bound was recently replaced by  $(k+1)^{5^{d+o(d)}}$ , see [9]. Our second result is the following improvement.

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**Theorem 2.** The cardinality of a k-distance set (k > 1) in a d-dimensional Minkowski space is at most  $k^{O(3^d d)}$ , where the constant in  $O(\cdot)$  does not depend on d and k.

Our proof is based on Theorem 3, which seems to be of independent interest.

**Theorem 3.** Assume that  $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$  are points in a d-dimensional Minkowski space with an o-symmetric convex body K as the unit ball, such that  $\|\mathbf{v}_i - \mathbf{v}_j\|_K = \lambda_i$  for any  $1 \leq i < j \leq n$ , where  $\lambda_i, i \in [n-1]$ , are some positive numbers. Then

$$n \le d\left(1 + \frac{2}{2 - 2^{1/(d-1)}}\right)^{d+1} = O(3^d d).$$

It is important to note that M. Naszódi, J. Pach and K. Swanepoel [4] proved that if the conditions of Theorem 3 hold then  $n = O(6^d (d \log d)^2)$ .

For more links dealing with k-distance sets we refer the interested readers to [8, 9].

One of the main ingredients of the proofs of Theorems 1 and 3 is the following simple lemma which is a generalization of the well-known Danzer-Grünbaum Theorem about the maximal cardinality antipodal sets, i.e. such sets that satisfy conditions of Lemma 1 when  $\lambda = 1$  (see [1] and also Lemma 7 in [4]).

**Lemma 1.** Suppose that  $\lambda \geq 1$  is a real number and  $X = {\mathbf{x}_1, \ldots, \mathbf{x}_n} \subset \mathbb{R}^d$  is a set of points such that for any  $i \neq j \in [n]$  there are two distinct parallel hyperplanes  $k_{i,j}$  and  $k_{j,i}$  with  $X \subset \operatorname{conv}(k_{i,j}, k_{j,i})$  and

$$\frac{\operatorname{dist}(k_{i,j}, k_{j,i})}{\operatorname{dist}(g_{i,j}, g_{j,i})} \le \lambda,\tag{1}$$

where  $g_{i,j}$  and  $g_{j,i}$  are hyperplanes passing through  $\mathbf{x}_i$  and  $\mathbf{x}_j$  respectively and parallel to  $k_{i,j}$  (and  $k_{j,i}$ ). Then  $n \leq (1+\lambda)^d$ .

Another key tool in our proofs is the lifting method developed in [5] (see also [3]), where M. Naszódi showed that the maximal number of pairwise touching positive homothets of a convex body K that is not necessary o-symmetric is at most  $2^{d+1}$ . We develop this method further by new ideas.

The article is organized in the following way. In Section 2.1 we prove Lemma 1. In Section 2.2 we discuss some properties of a set of pairwise intersecting homothets, which we will use in Sections 3.1 and 3.2, where we present the proofs of Theorem 1 and Theorem 3 respectively. In Section 3.3 we prove Theorem 2 using Theorem 3.

## 2. Auxiliary Lemmas

2.1. **Proof of Lemma 1.** We may clearly assume that  $P := \operatorname{conv}(X)$  is a *d*-dimensional polytope in  $\mathbb{R}^d$ , otherwise  $d_1 := \dim(\operatorname{aff}(P)) < d$ , i.e. by induction hypothesis, we have  $n \leq (1+\lambda)^{d_1} < (1+\lambda)^d$ . It is easy to see that  $P_i = \mathbf{x}_i + \frac{1}{1+\lambda}(P - \mathbf{x}_i) \subset P$ . Without loss of generality we assume that  $\mathbf{x}_i$  is closer to  $k_{i,j}$  than  $\mathbf{x}_j$ . We claim that  $P_i$  and  $P_j$  do not share a common interior point. Indeed,  $P_i \subset \operatorname{conv}(k_{i,j} \cup l_{i,j}), P_j \subset \operatorname{conv}(k_{j,i} \cup l_{j,i})$ , where  $l_{i,j} = \mathbf{x}_i + \frac{1}{1+\lambda}(k_{j,i} - \mathbf{x}_i), l_{j,i} = \mathbf{x}_j + \frac{1}{1+\lambda}(k_{i,j} - \mathbf{x}_j)$ . Note that  $\operatorname{conv}(k_{i,j} \cup l_{i,j})$  and  $\operatorname{conv}(k_{j,i} \cup l_{j,i})$  do not have a common interior point because

$$dist(k_{i,j}, l_{i,j}) + dist(k_{j,i}, l_{j,i}) = \\ = dist(k_{i,j}, g_{i,j}) + \frac{1}{1+\lambda} dist(g_{i,j}, k_{j,i}) + dist(k_{j,i}, g_{j,i}) + \frac{1}{1+\lambda} dist(g_{j,i}, k_{i,j}) = \\ = dist(k_{i,j}, k_{j,i}) - dist(g_{i,j}, g_{j,i}) + \frac{1}{1+\lambda} dist(k_{i,j}, k_{j,i}) + \frac{1}{1+\lambda} dist(g_{i,j}, g_{j,i}) \leq dist(k_{i,j}, k_{j,i}).$$

The last inequality holds because of (1). Therefore,  $\sum_{i=1}^{n} \operatorname{vol}(P_i) \leq \operatorname{vol}(P)$ , i.e.  $\frac{n}{(1+\lambda)^d} \operatorname{vol}(P) \leq \operatorname{vol}(P)$ ,  $n \leq (1+\lambda)^d$ . Lemma 1 is proved.

2.2. Properties of pairwise intersecting homothets. Throughout Section 2.2,  $\ell(\mathbf{x}, \mathbf{y})$  denotes the line passing through points  $\mathbf{x}$  and  $\mathbf{y}$ ,  $\ell(\mathbf{x}, l)$  and  $h(\mathbf{x}, h)$  stand for the line and the k-dimensional flat passing through a point  $\mathbf{x}$  and parallel to a line l and to a k-dimensional flat h respectively, we write  $[\mathbf{x}, \mathbf{y}]$  for the segment with endpoints  $\mathbf{x}$  and  $\mathbf{y}$ ,  $\Delta(\mathbf{x}, \mathbf{y}, \mathbf{z})$  denotes the triangle with vertices  $\mathbf{x}, \mathbf{y}, \mathbf{z}$ . We write  $\Delta(\mathbf{x}_1, \mathbf{y}_1, \mathbf{z}_1) \sim \Delta(\mathbf{x}_2, \mathbf{y}_2, \mathbf{z}_2)$  if the triangles  $\Delta(\mathbf{x}_1, \mathbf{y}_1, \mathbf{z}_1)$  and  $\Delta(\mathbf{x}_2, \mathbf{y}_2, \mathbf{z}_2)$  are similar.  $(\mathbf{x}_1, \mathbf{x}_2; \mathbf{x}_3, \mathbf{x}_4)$  stands for the cross-ratio of points  $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$  on the real line, i.e.

$$(\mathbf{x}_1, \mathbf{x}_2; \mathbf{x}_3, \mathbf{x}_4) = \frac{x_1 - x_3}{x_2 - x_3} : \frac{x_1 - x_4}{x_2 - x_4},$$

where  $x_1, x_2, x_3, x_4$  are coordinates of the points  $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$  respectively. If one of the points is the point at infinity then the two distances involving that point are dropped from the formula. Also, we will use the fact that if  $\mathbf{p} : \mathbb{R}^d \to \mathbb{R}^d$  is a projective transformation and distinct points  $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4 \in \mathbb{R}^d$  are collinear then  $\mathbf{p}(\mathbf{x}_1), \mathbf{p}(\mathbf{x}_2), \mathbf{p}(\mathbf{x}_3), \mathbf{p}(\mathbf{x}_4)$  are also collinear and

$$(\mathbf{p}(\mathbf{x}_1),\mathbf{p}(\mathbf{x}_2);\mathbf{p}(\mathbf{x}_3),\mathbf{p}(\mathbf{x}_4)) = (\mathbf{x}_1,\mathbf{x}_2;\mathbf{x}_3,\mathbf{x}_4).$$

Let us identify  $\mathbb{R}^d$  with the *d*-dimensional flat

$$h := \{(x_1, \dots, x_{d+2}) \in \mathbb{R}^{d+2} : x_{d+1} = 0, x_{d+2} = 1\}$$
 in  $\mathbb{R}^{d+2}$ 

and consider the following hyperplanes

$$h_0 := \{ (x_1, \dots, x_{d+2}) \in \mathbb{R}^{d+2} : x_{d+2} = 1 \} \text{ in } \mathbb{R}^{d+2}, h_1 := \{ (x_1, \dots, x_{d+2}) \in \mathbb{R}^{d+2} : x_{d+1} = 1 \} \text{ in } \mathbb{R}^{d+2}.$$

Note that  $h \subset h_0$ . Let  $\{\mathbf{e}_i : i \in [d+2]\}$  be the standard basis of  $\mathbb{R}^{d+2}$ .

Let K be an o-symmetric d-dimensional convex body such that

$$K \subset h' := \{(x_1, \dots, x_{d+2}) \in \mathbb{R}^{d+2} : x_{d+1} = 0, x_{d+2} = 0\}.$$

Note that the *d*-dimensional flat h' is parallel to the *d*-dimensional flat h. Suppose that  $\{\mathbf{v}_i : i \in [n]\} \subset h$  is a set of n distinct vectors,  $\{\lambda_i : i \in [n]\} \subset \mathbb{R}^+$  is a set of n positive scalars and  $\{\mathbf{v}_i + \lambda_i K : i \in [n]\} \subset h$  is a finite family of pairwise intersecting positive homothets of K.

Section 2.2 is organized in the following way. First, we define the set  $X_0 := \{\mathbf{x}_i := \mathbf{v}_i + \lambda_i \mathbf{e}_{d+1} : i \in [n]\} \subset h_0$  of n points and prove some properties of  $X_0$ . Second, we apply on  $X_0$  the central projection pr  $: h_0 \to h_1$  from the origin of  $\mathbb{R}^{d+2}$  onto the hyperplane  $h_1$ . Finally, we check that the image  $X_1 := \{\mathbf{y}_i := \operatorname{pr}(\mathbf{x}_i) : i \in [n]\} \subset h_1$  of  $X_0$  satisfies some properties.



Choose  $i \neq j \in [n]$ . Write  $r := r_{i,j} := \ell(\mathbf{v}_i, \mathbf{v}_j) \subset h$  and let  $\mathbf{r} := \mathbf{r}_{i,j}$  and  $-\mathbf{r}$  be the points of intersection of  $\partial K$  and  $\ell(\mathbf{o}, r)$ , here we assume that the vectors  $\mathbf{r}$  and  $\mathbf{v}_j - \mathbf{v}_i$  have the same direction (see Figure 1). Denote by  $f := f_{i,j}$  a supporting hyperplane of  $\mathbf{v}_i + \lambda_i K$  in h passing through  $\mathbf{v}_i + \lambda_i \mathbf{r}$ , i.e. f is a (d-1)-dimensional flat.

Let  $\mathbf{v}'_k := \mathbf{v}'_{k,i,j}, \mathbf{x}'_k := \mathbf{x}'_{k,i,j}$  and  $t_k := t_{k,i,j} := [\mathbf{v}'_k - \lambda_k \mathbf{r}, \mathbf{v}'_k + \lambda_k \mathbf{r}] \subset r$  be the projections of  $\mathbf{v}_k$ ,  $\mathbf{x}_k$  and  $\mathbf{v}_k + \lambda_k K$  in the direction of f onto the two-dimensional plane  $\pi$ , where  $\pi := \pi_{i,j}$  passes through  $\mathbf{v}_i, \mathbf{v}_j, \mathbf{x}_i$  and  $\mathbf{x}_j$  (see Figure 1). It follows immediately that  $\mathbf{v}_i = \mathbf{v}'_i, \mathbf{v}_j = \mathbf{v}'_j, \mathbf{x}_i = \mathbf{x}'_i, \mathbf{x}_j = \mathbf{x}'_j, \mathbf{x}'_k = \mathbf{v}'_k + \lambda_k \mathbf{e}_{d+1}, t_i = [\mathbf{v}_i - \lambda_i \mathbf{r}, \mathbf{v}_i + \lambda_i \mathbf{r}]$  and  $t_j = [\mathbf{v}_j - \lambda_j \mathbf{r}, \mathbf{v}_j + \lambda_j \mathbf{r}].$ 

We claim that the segments  $t_k$  share a common point, which we will denote as  $\mathbf{x} := \mathbf{x}_{i,j}$ . Indeed, any two segments  $t_p$  and  $t_q$  share a common point otherwise  $\{\mathbf{v}_p + \lambda_p K\}$  and  $\{\mathbf{v}_q + \lambda_q K\}$  do not intersect each other. Therefore, by Helly's theorem for  $\mathbb{R}$ , we get that  $t_k$  have a common point  $\mathbf{x}$ .

Let  $u_i := u_{i,j}$  and  $u_j := u_{j,i}$  be the real numbers such that

$$\mathbf{x} - \mathbf{v}_i = u_i \mathbf{r} \text{ and } \mathbf{v}_j - \mathbf{x} = u_j \mathbf{r}.$$
 (2)

Set (see Figure 1)

$$a_{i} := a_{i,j} := \ell(\mathbf{v}_{i} + \lambda_{i}\mathbf{r}, \mathbf{x}_{i}), \ a_{j} := a_{j,i} := \ell(\mathbf{v}_{j} - \lambda_{j}\mathbf{r}, \mathbf{x}_{j}),$$
  

$$b_{i} := b_{i,j} := \ell(\mathbf{x}, a_{i}), \ b_{j} := b_{j,i} := \ell(\mathbf{x}, a_{j}),$$
  

$$f_{0} := f_{0,i,j} := h(\mathbf{x}, f), \ B_{i} := B_{i,j} := \operatorname{aff}(b_{i} \cup f_{0}), \ B_{j} := B_{j,i} := \operatorname{aff}(b_{j} \cup f_{0}).$$

Note that the set  $X_0$  lies in the wedge formed by  $B_i$  and  $B_j$  in  $h_0$  that lies in the halfspace  $\{(x_1, \ldots, x_{d+2}) \in \mathbb{R}^{d+2} : x_{d+1} \ge 0\}$ . Indeed, points  $\mathbf{x}'_k$  lie in the angle formed by  $b_i$  and  $b_j$  that lies in the halfspace  $\{(x_1, \ldots, x_{d+2}) \in \mathbb{R}^{d+2} : x_{d+1} \ge 0\}$  (see Figure 1). Since  $\mathbf{x}'_k$  are the projections of  $\mathbf{x}_k$  in the direction of f onto the plane  $\pi$ , the points  $\mathbf{x}_k$  lie in the corresponding wedge formed by  $B_i$  and  $B_j$ .

Next, we apply the central projection  $\text{pr} : h_0 \to h_1$  from the origin of  $\mathbb{R}^{d+2}$  onto the hyperplane  $h_1$ . The image of h is the "hyperplane at infinity" in  $h_1$ . Therefore, we proved the following lemma.

**Lemma 2.**  $k_{i,j} := \operatorname{pr}(B_i)$  and  $k_{j,i} := \operatorname{pr}(B_j)$  are parallel hyperplanes in  $h_1$  and  $X_1 = \operatorname{pr}(X_0)$  lies in the slab  $\operatorname{conv}(k_{i,j} \cup k_{j,i})$ .



Denote by  $\mathbf{z}_i := \mathbf{z}_{i,j}$  and  $\mathbf{z}_j := \mathbf{z}_{j,i}$  the points of intersection of  $r_0 := r_{0,i,j} = \ell(\mathbf{x}_i, \mathbf{x}_j)$ with  $b_i$  (or  $B_i$ ) and  $b_i$  (or  $B_j$ ) respectively (see Figure 2). Recall that  $\mathbf{y}_k = \operatorname{pr}(\mathbf{x}_k)$ . Let  $\mathbf{s}_i := \mathbf{s}_{i,j} = \operatorname{pr}(\mathbf{z}_i), \mathbf{s}_j := \mathbf{s}_{j,i} = \operatorname{pr}(\mathbf{z}_j)$ . Of course,  $\mathbf{s}_i$  and  $\mathbf{s}_j$  are the points of intersection of  $\ell(\mathbf{y}_i, \mathbf{y}_j)$  with  $k_{i,j}$  and  $k_{j,i}$  respectively because central projections preserve lines. Denote by  $g_{i,j}$  and  $g_{j,i}$  the hyperplanes in  $h_1$  that are parallel to  $k_{i,j}$  and  $k_{j,i}$  and pass through  $\mathbf{y}_i$ and  $\mathbf{y}_j$  respectively.

Lemma 3. We have

$$\frac{\operatorname{dist}(k_{i,j}, k_{j,i})}{\operatorname{dist}(g_{i,j}, g_{j,i})} = \frac{\|\mathbf{s}_i - \mathbf{s}_j\|}{\|\mathbf{y}_i - \mathbf{y}_j\|} = \frac{2\lambda_i \lambda_j}{\lambda_i u_j + \lambda_j u_i}$$

*Proof.* Denote by **c** the point of intersection  $r_0$  with r, where, if  $r_0$  and r are parallel, then we consider **c** as the corresponding point at infinity. Let  $\mathbf{c}' := \operatorname{pr}(\mathbf{c})$ . Since  $\mathbf{c} \in h$ , the point  $\mathbf{c}'$  is a point at infinity. Without loss of generality we assume that points on the line  $r_0$  lie in the following order:  $\mathbf{z}_i, \mathbf{x}_i, \mathbf{x}_j, \mathbf{z}_j$ . Denote by  $\mathbf{w}_i$  and  $\mathbf{w}_j$  the orthogonal projections of  $\mathbf{z}_i$  and  $\mathbf{z}_j$  onto the line r respectively. Note that points on the line r lie in the following order:  $\mathbf{w}_i, \mathbf{v}_j, \mathbf{w}_j$ . Moreover,  $\mathbf{x}$  must lie between  $\mathbf{v}_j - \lambda_j \mathbf{r}$  and  $\mathbf{v}_i + \lambda_i \mathbf{r}$ .

Using the fact that  $\mathbf{c}'$  is a point at infinity,  $\{\mathbf{s}_i, \mathbf{y}_i, \mathbf{y}_j, \mathbf{s}_j, \mathbf{c}'\} = \operatorname{pr}(\{\mathbf{z}_i, \mathbf{x}_i, \mathbf{x}_j, \mathbf{z}_j, \mathbf{c}\})$  and  $\operatorname{pr}_0(\{\mathbf{z}_i, \mathbf{x}_i, \mathbf{x}_j, \mathbf{z}_j, \mathbf{c}\}) = \{\mathbf{w}_i, \mathbf{v}_i, \mathbf{v}_j, \mathbf{w}_j, \mathbf{c}\}$ , where  $\operatorname{pr}_0 : r_0 \to r$  is the orthogonal projection onto the line r, we easily get

$$\frac{\|\mathbf{s}_{i} - \mathbf{s}_{j}\|}{\|\mathbf{y}_{i} - \mathbf{y}_{j}\|} = (\mathbf{s}_{i}, \mathbf{y}_{i}; \mathbf{s}_{j}, \mathbf{c}') \cdot (\mathbf{s}_{j}, \mathbf{y}_{j}; \mathbf{y}_{i}, \mathbf{c}') = (\mathbf{z}_{i}, \mathbf{x}_{i}; \mathbf{z}_{j}, \mathbf{c}) \cdot (\mathbf{z}_{j}, \mathbf{x}_{j}; \mathbf{x}_{i}, \mathbf{c}) = = (\mathbf{w}_{i}, \mathbf{v}_{i}; \mathbf{w}_{j}, \mathbf{c}) \cdot (\mathbf{w}_{j}, \mathbf{v}_{j}; \mathbf{v}_{i}, \mathbf{c}) = \frac{\|\mathbf{w}_{i} - \mathbf{w}_{j}\|}{\|\mathbf{v}_{i} - \mathbf{v}_{j}\|} \cdot \frac{\|\mathbf{v}_{i} - \mathbf{c}\|}{\|\mathbf{w}_{j} - \mathbf{c}\|} \cdot \frac{\|\mathbf{v}_{j} - \mathbf{c}\|}{\|\mathbf{w}_{i} - \mathbf{c}\|}.$$
(3)

If **c** is not a point at infinity then using  $\Delta(\mathbf{c}, \mathbf{v}_i, \mathbf{x}_i) \sim \Delta(\mathbf{c}, \mathbf{w}_i, \mathbf{z}_i)$  and  $\Delta(\mathbf{c}, \mathbf{v}_j, \mathbf{x}_j) \sim \Delta(\mathbf{c}, \mathbf{w}_j, \mathbf{z}_j)$ , we have

$$\frac{\|\mathbf{v}_i - \mathbf{c}\|}{\|\mathbf{w}_i - \mathbf{c}\|} = \frac{\|\mathbf{v}_i - \mathbf{x}_i\|}{\|\mathbf{w}_i - \mathbf{z}_i\|} = \frac{\lambda_i}{\|\mathbf{w}_i - \mathbf{z}_i\|} \text{ and } \frac{\|\mathbf{v}_j - \mathbf{c}\|}{\|\mathbf{w}_j - \mathbf{c}\|} = \frac{\|\mathbf{v}_j - \mathbf{x}_j\|}{\|\mathbf{w}_j - \mathbf{z}_j\|} = \frac{\lambda_j}{\|\mathbf{w}_j - \mathbf{z}_j\|}.$$

Note that if  $\mathbf{c}$  is a point at infinity then these equalities are obvious. Substituting the last equality into (3), we get

$$\frac{\|\mathbf{s}_i - \mathbf{s}_j\|}{\|\mathbf{y}_i - \mathbf{y}_j\|} = \frac{\|\mathbf{w}_i - \mathbf{w}_j\|}{\|\mathbf{v}_j - \mathbf{v}_i\|} \cdot \frac{\lambda_i}{\|\mathbf{w}_i - \mathbf{z}_i\|} \cdot \frac{\lambda_j}{\|\mathbf{w}_j - \mathbf{z}_j\|}.$$
(4)

Since  $\Delta(\mathbf{w}_i, \mathbf{z}_i, \mathbf{x}) \sim \Delta(\mathbf{v}_i, \mathbf{x}_i, \mathbf{v}_i + \lambda_i \mathbf{r})$  and  $\|\mathbf{v}_i - \mathbf{x}_i\| = \lambda_i$ , we get

$$\frac{\|\mathbf{w}_i - \mathbf{x}\|}{\|\mathbf{w}_i - \mathbf{z}_i\|} = \frac{\|\mathbf{v}_i - \mathbf{v}_i - \lambda_i \mathbf{r}\|}{\|\mathbf{v}_i - \mathbf{x}_i\|} \Leftrightarrow \frac{\|\mathbf{w}_i - \mathbf{x}\|}{\|\mathbf{r}\|} = \|\mathbf{w}_i - \mathbf{z}_i\|.$$
(5)

By a similar argument, we obtain

$$\frac{\|\mathbf{x} - \mathbf{w}_j\|}{\|\mathbf{r}\|} = \|\mathbf{w}_j - \mathbf{z}_j\|.$$
(6)

From (5), (6) and (2) we conclude that

$$\frac{\|\mathbf{w}_i - \mathbf{w}_j\|}{\|\mathbf{v}_i - \mathbf{v}_j\|} = \frac{\|\mathbf{w}_i - \mathbf{z}_i\| + \|\mathbf{w}_j - \mathbf{z}_j\|}{u_i + u_j}.$$

Substituting the last equality into (4), we have

$$\frac{\|\mathbf{s}_i - \mathbf{s}_j\|}{|\mathbf{y}_i - \mathbf{y}_j\|} = \frac{\|\mathbf{w}_i - \mathbf{z}_i\| + \|\mathbf{w}_j - \mathbf{z}_j\|}{u_i + u_j} \cdot \frac{\lambda_i}{\|\mathbf{w}_i - \mathbf{z}_i\|} \cdot \frac{\lambda_j}{\|\mathbf{w}_j - \mathbf{z}_j\|}.$$
(7)

Now we are ready to apply twice the following simple fact.

**Lemma 4.** Suppose that  $\mathbf{a}_i$  and  $\mathbf{b}_i$  for  $1 \leq i \leq 3$  are points in  $\mathbb{R}^d$  such that  $\theta_1(\mathbf{a}_1 - \mathbf{a}_2) = \theta_2(\mathbf{a}_2 - \mathbf{a}_3)$  and  $\theta_1(\mathbf{b}_1 - \mathbf{b}_2) = \theta_2(\mathbf{b}_2 - \mathbf{b}_3)$ , where  $\theta_1$  and  $\theta_2$  are real numbers. Then

$$\mathbf{b}_2 - \mathbf{a}_2 = \frac{\theta_1}{\theta_1 + \theta_2} (\mathbf{b}_1 - \mathbf{a}_1) + \frac{\theta_2}{\theta_1 + \theta_2} (\mathbf{b}_3 - \mathbf{a}_3).$$

*Proof.* A simple exercise.

Denote by  $\mathbf{x}'$  the point of intersection of  $\ell(\mathbf{x}, \ell(\mathbf{v}_i, \mathbf{x}_i))$  with  $r_0$  (see Figure 2). Using Lemma 4 for  $\mathbf{w}_i, \mathbf{z}_i, \mathbf{x}, \mathbf{x}', \mathbf{w}_j$  and  $\mathbf{z}_j$ , we obtain

$$\|\mathbf{x} - \mathbf{x}'\| = \frac{2\|\mathbf{w}_i - \mathbf{z}_i\| \|\mathbf{w}_j - \mathbf{z}_j\|}{\|\mathbf{w}_i - \mathbf{z}_i\| + \|\mathbf{w}_i - \mathbf{z}_j\|}.$$
(8)

Using Lemma 4 for  $\mathbf{v}_i, \mathbf{x}_i, \mathbf{x}, \mathbf{x}', \mathbf{v}_j$  and  $\mathbf{x}_j$ , we have

$$\|\mathbf{x} - \mathbf{x}'\| = \frac{u_j}{u_i + u_j} \lambda_i + \frac{u_i}{u_i + u_j} \lambda_j = \frac{\lambda_i u_j + \lambda_j u_i}{u_i + u_j}.$$
(9)

The comparison of (8) and (9) shows that

$$\frac{\|\mathbf{w}_i - \mathbf{z}_i\| + \|\mathbf{w}_j - \mathbf{z}_j\|}{\|\mathbf{w}_i - \mathbf{z}_i\| \|\mathbf{w}_j - \mathbf{z}_j\|} = \frac{2}{\|\mathbf{x} - \mathbf{x}'\|} = 2\frac{u_i + u_j}{\lambda_i u_j + \lambda_j u_i}$$

Substituting the last equality into (7), we get

$$\frac{\|\mathbf{s}_i - \mathbf{s}_j\|}{\|\mathbf{y}_i - \mathbf{y}_j\|} = \frac{2\lambda_i\lambda_j}{\lambda_i u_j + \lambda_j u_i}.$$

Lemma 3 is proved.

**Lemma 5.** If  $t_i \cap t_j \subset [\mathbf{v}_i, \mathbf{v}_j]$  then

$$\frac{2\lambda_i\lambda_j}{\lambda_i u_j + \lambda_j u_i} \le 2$$

*Proof.* Without loss of generality we assume that  $\lambda_i \geq \lambda_j$ . Note that by definition  $u_i, u_j$  are such numbers that  $\mathbf{x} - \mathbf{v}_i = u_i \mathbf{r}$  and  $\mathbf{v}_j - \mathbf{x} = u_j \mathbf{r}$ . Thus if  $x \in t_i \cap t_j \subset [\mathbf{v}_i, \mathbf{v}_j]$  then  $u_i, u_j \geq 0$  and  $u_i + u_j \geq \lambda_i \geq \lambda_j$ , i.e.  $\lambda_i \lambda_j \leq (u_i + u_j) \lambda_j \leq (\lambda_j u_i + \lambda_i u_j)$ . The last inequality proves the statement of Lemma 5.

#### 3. Proofs of theorems

3.1. **Proof of Theorem 1.** Using the notations of Section 2.2, we consider  $X_1 \subset h_1$ , where  $h_1$  is a (d+1)-dimensional plane. Moreover, by Lemmas 2 and 3 for any  $i \neq j \in [n]$  there exist two parallel *d*-dimensional planes  $k_{i,j}$  and  $k_{j,i}$  such that  $\mathbf{y}_k \in \text{conv}(k_{i,j} \cup k_{j,i})$  for any  $k \in [n]$  and

$$\frac{\operatorname{dist}(k_{i,j}, k_{j,i})}{\operatorname{dist}(g_{i,j}, g_{j,i})} = \frac{2\lambda_i \lambda_j}{\lambda_i u_j + \lambda_j u_i}.$$
(10)

Since these homothets form a Minkowski arrangement, we have  $t_i \cap t_j \subset [\mathbf{v}_i, \mathbf{v}_j]$ , i.e. by Lemma 5 we have that (10) is less than or equal to 2. Therefore,  $X_1$  satisfies conditions of Lemma 1 with  $\lambda = 2$ . Thus  $n \leq 3^{d+1}$ .

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3.2. **Proof of Theorem 3.** Consider the following family of pairwise intersecting homothets  $\{\mathbf{v}_i + \lambda_i K : i \in [n]\}$ , where  $\lambda_n := \lambda_{n-1}$ . Without loss of generality assume that  $\max_{i \in [n]} \lambda_i = 1$ . Let us divide the set [n] into d subsets. For any  $l \in [d]$  we consider

$$J_{l} = \{i \in [n] : \lambda_{i} \in \mu^{l-1}I\}, \text{ where } \mu = 2^{-1/(d-1)} < 1, \text{ i.e. } \mu^{d} = \mu/2, \text{ and}$$
$$I = I_{1} \cup I_{2} \cup I_{3} \cup \dots := (\mu, 1] \cup (\mu^{d+1}, \mu^{d}] \cup (\mu^{2d+1}, \mu^{2d}] \cup \dots$$

Obviously, the  $J_l$ s are not pairwise intersecting sets and their union is [n]. We claim that

$$|J_l| \le \left(1 + \frac{2}{2 - \mu^{-1}}\right)^{d+1} \tag{11}$$

Clearly, (11) implies the statement of Theorem 3:

$$n \le d\left(1 + \frac{2}{2 - \mu^{-1}}\right)^{d+1}.$$

It is enough to prove (11) for l = 1. Consider the set of homothets  $\{\mathbf{v}_k + \lambda_k K : k \in J_1\}$ . Using the notations of Section 2.2, we have that for any  $i \neq j$  there exist two parallel d-dimensional planes  $k_{i,j}$  and  $k_{j,j}$  in the (d + 1)-dimensional plane  $h_1$  such that  $\mathbf{y}_k \in \text{conv}(k_{i,j} \cup k_{j,i})$  for any  $k \in J_1$  and

$$\frac{\operatorname{dist}(k_{i,j}, k_{j,i})}{\operatorname{dist}(g_{i,j}, g_{j,i})} = \frac{2\lambda_i \lambda_j}{\lambda_i u_j + \lambda_j u_i}.$$
(12)

By Lemma 1, it is enough to prove that the right hand side of (12) is at most  $\frac{2}{2-\mu^{-1}} > 2$ . Consider two cases:

1)  $i, j \in I_k$  for some k. Assume that i < j thus  $\mathbf{v}_j - \mathbf{v}_i = \lambda_i \mathbf{r}$ . If  $\lambda_i \ge \lambda_j$  then we have  $t_i \cap t_j \subset [\mathbf{v}_i, \mathbf{v}_j]$ , i.e. by Lemma 5, we have that (12) is at most 2. Assume that  $\lambda_j > \lambda_i$ . Since  $\mathbf{x} \in [\mathbf{v}_j - \lambda_j \mathbf{r}, \mathbf{v}_i + \lambda_i \mathbf{r}] = [\mathbf{v}_j - \lambda_j \mathbf{r}, \mathbf{v}_j]$ , we have  $u_i + u_j = \lambda_i$ ,  $0 \le u_j \le \lambda_j$ . Therefore, using  $\lambda_j / \lambda_i < \mu^{-1}$  (because  $i, j \in I_k$ ) we have

$$\frac{\lambda_i u_j + \lambda_j u_i}{\lambda_i \lambda_j} = u_j \left(\frac{1}{\lambda_j} - \frac{1}{\lambda_i}\right) + \frac{u_i + u_j}{\lambda_i} \ge \lambda_j \left(\frac{1}{\lambda_j} - \frac{1}{\lambda_i}\right) + 1 > 2 - \mu^{-1},$$

i.e. the right hand side of (12) is at most  $\frac{2}{2-\mu^{-1}}$ .

2)  $i \in I_k, j \in I_l$  for some k < l. Note that  $\lambda_i > 2\lambda_j$  (see the definition of  $I_m$ ) thus it is impossible that  $\mathbf{v}_j - \mathbf{v}_i = \lambda_j \mathbf{r}$ . Indeed, in such case  $\mathbf{v}_i + \lambda_i \partial K$  and  $\mathbf{v}_j + \lambda_j \partial K$  do not intersect each other because of the triangle inequality, a contradiction. Therefore,  $\mathbf{v}_j - \mathbf{v}_i = \lambda_i \mathbf{r}$ , i.e.  $t_i \cap t_j \subset [\mathbf{v}_i, \mathbf{v}_j]$ , thus (12) is at most 2.

Theorem 3 is proved.

3.3. **Proof of Theorem 2.** Assume that there exists a k-distance set  $\{\mathbf{x}_i : i \in [n]\}$  in the d-dimensional Minkowski space with an o-symmetric convex body K as the unit ball, where

$$n = k^{f(d)}, \ f(d) = \left[ d \left( 1 + \frac{2}{2 - 2^{1/(d-1)}} \right)^{d+1} \right] = O(3^d d).$$

We will construct a set  $Y = {\mathbf{y}_i : i \in [f(d) + 1]}$  in the same *d*-dimensional Minkowski space such that  $\|\mathbf{y}_i - \mathbf{y}_j\|_K = \lambda_i$  for any  $1 \le i < j \le f(d) + 1$ , where  $\lambda_i$  are some positive real numbers, using the following algorithm.

0. Set  $A := [n], Y := \{\mathbf{y}_1 := \mathbf{x}_1\}, l := 1.$ 

1. Let  $\lambda_l$  be a positive real number such that the cardinality of the set

$$A' := \{j : \|\mathbf{y}_l - \mathbf{x}_j\}\|_K = \lambda_l, j \in A\}$$

is at least  $k^{f(d)-l}$  (such  $i_l$  exists because  $|A| \ge k^{f(d)-l+1}$  and there are k distances occurring between points of  $\{\mathbf{x}_i : i \in A \subseteq [n]\}$ ). Put A := A'.

- 2. Choose any  $j \in A$  and put  $\mathbf{y}_{l+1} := \mathbf{x}_j$ . Add  $\mathbf{y}_{l+1}$  to the set Y.
- 3. If l < f(d) then l := l + 1 and return to Step 1, else, output Y, and finish.

Obviously, the existence of the set Y contradicts Theorem 3, therefore, we get a contradiction with our assumption that there exists a k-distance set consisting of  $k^{f(d)}$  points in  $\mathbb{R}^d$ .

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## References

- L. Danzer, B. Grünbaum, Über zwei Probleme bezüglich konvexer Körper von P. Erdős und von V. L. Klee, Math. Z. 79 (1962), 95–99.
- [2] Z. Füredi, P. A. Loeb, On the best constant for the Besicovitch covering theorem, Proc. Amer. Math. Soc. 121(4) (1994), 1063–1073.
- [3] Zs. Lángi, M. Naszódi, On the Bezdek-Pach conjecture for centrally symmetric convex bodies, Canad. Math. Bull. 52(3) (2009), 407–415.
- [4] M. Naszódi, J. Pach, K. Swanepoel, Arrangements of homothets of a convex body, arXiv:1608.04639, submitted.
- [5] M. Naszódi, On a conjecture of Károly Bezdek and János Pach, Period. Math. Hungar. 53(1-2) (2006), 227–230.
- [6] M. Petty, Equilateral sets in Minkowski spaces, Proc. Amer. Math. Soc. 29 (1971), 369–374.
- [7] L.F. Tóth, *Research problem*, Period. Math. Hungar. **31**(2) (1995), 165–166.
- [8] K. Swanepoel, Cardinalities of k-distance sets in Minkowski spaces, Discrete Mathematics 197/198 (1999), 759–767.
- [9] K. Swanepoel, Combinatorial distance geometry in normed spaces, arXiv:1702.00066, submitted.

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