BOOLEAN DIMENSION AND LOCAL DIMENSION

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ABSTRACT. Dimension is a standard and well-studied measure of complexity of posets. Recent research has provided many new upper bounds on the dimension for various structurally restricted classes of posets. Bounded dimension gives a succinct representation of the poset, admitting constant response time for queries of the form "is x < y?". This application motivates looking for stronger notions of dimension, possibly leading to succinct representations for more general classes of posets. We focus on two: *boolean dimension*, introduced in the 1980s and revisited in recent research, and *local dimension*, a very new one. We determine precisely which values of dimension/boolean dimension/local dimension imply that the two other parameters are bounded.

1. INTRODUCTION

Dimension. The dimension of a poset $P = (X, \leq)$ is the minimum number of linear extensions of \leq on X the intersection of which gives \leq . More precisely, a *realizer* of a poset $P = (X, \leq)$ is a set $\{\leq_1, \ldots, \leq_d\}$ of linear extensions of \leq on X such that

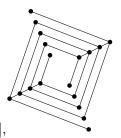
 $x \leq y \iff (x \leq_1 y) \land \dots \land (x \leq_d y), \text{ for any } x, y \in X,$

and the dimension is the minimum size of a realizer. The concept of dimension was introduced by Dushnik and Miller [8] and has been widely studied since. There are posets with arbitrarily large dimension: the standard example $S_k = (\{a_1, \ldots, a_k, b_1, \ldots, b_k\}, \leqslant)$, where a_1, \ldots, a_k are minimal elements, b_1, \ldots, b_k are maximal elements, and $a_i < b_j$ if and only if $i \neq j$, has dimension k when $k \ge 2$ [8]. On the other hand, the dimension of a poset is at most the width [12], and it is at most $\frac{n}{2}$ when $n \ge 4$, where n denotes the number of elements [12].

The cover graph of a poset $P = (X, \leq)$ is the graph on X with edge set $\{xy : x < y \text{ and there is} no z \text{ with } x < z < y\}$. A poset is planar if its cover graph has a non-crossing upward drawing in the plane, which means that every cover graph edge xy with x < y is drawn as a curve that goes monotonically up from x to y. Planar posets that contain a least element and a greatest element are well known to have dimension at most 2 [1]. By contrast, spherical posets (i.e. posets with upward non-crossing drawings on a sphere) with least and greatest elements can have arbitrarily large dimension [24]. Trotter and Moore [25] proved that planar posets that contain a least element have dimension at most 3 (and so do posets whose cover graphs are forests) and asked whether all planar

posets have bounded dimension. The answer is no—Kelly [17] constructed planar posets with arbitrarily large dimension (pictured). Another property of Kelly's posets is that their cover graphs have path-width and treewidth 3. Recent research brought a plethora of new bounds on dimension for structurally restricted posets. In particular, dimension is bounded for

- posets with height 2 and planar cover graphs [9],
- posets with bounded height and planar cover graphs [23],
- posets with bounded height and cover graphs of bounded tree-width [14],
- posets with bounded height and cover graphs excluding a topological minor [27],
- posets with bounded height and cover graphs of bounded expansion [16],



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- posets with cover graphs of path-width 2 [3],
- posets with cover graphs of tree-width 2 [15],
- posets with planar cover graphs excluding two incomparable chains of bounded length [13].

Boolean dimension. The *boolean dimension* of a poset $P = (X, \leq)$ is the minimum number of linear orders on X a boolean combination of which gives \leq . More precisely, a *boolean realizer* of P is a set $\{\leq_1, \ldots, \leq_d\}$ of linear orders on X for which there is a d-ary boolean formula ϕ such that

(1) $x \leqslant y \iff \phi((x \leqslant_1 y), \dots, (x \leqslant_d y))$ for any $x, y \in X$,

and the boolean dimension is the minimum size of a boolean realizer. The boolean dimension is at most the dimension, because a realizer is a boolean realizer for the formula $\phi(\alpha_1, \ldots, \alpha_d) = \alpha_1 \wedge \cdots \wedge \alpha_d$. Beware that the relation \leq defined by (1) from arbitrary linear orders \leq_1, \ldots, \leq_d on X and formula ϕ is not necessarily a partial order.

Boolean dimension was first considered by Gambosi, Nešetřil, and Talamo [10] and by Nešetřil and Pudlák [22]. The definition above follows [22]. That in [10] allows only formulas ϕ of the form $\phi(\alpha_1, \ldots, \alpha_d) = \alpha_i \wedge \psi(\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_d)$ for some *i*. The purpose of this restriction is unclear—it guarantees antisymmetry but not transitivity of the relation \leq defined by (1). Under that modified definition, it is proved in [10] that boolean dimension *d* and dimension *d* are equivalent for $d \in \{1, 2, 3\}$ (we redo that proof in section 2 with no restriction on ϕ). The standard examples S_k with $k \geq 4$ have boolean dimension 4 [10] (see section 2). Easy counting shows that there are posets on *n* elements with boolean dimension $\Theta(\log n)$ [22]. This is optimal—every *n*element poset has boolean dimension $O(\log n)$ witnessed by a formula of length $O(n^2 \log n)$ [22].

Nešetřil and Pudlák [22] asked whether boolean dimension is bounded for planar posets. It was proved already in [10] that posets with height 2 and planar cover graphs have bounded boolean dimension. Spherical posets with a least element also have bounded boolean dimension [6], contrary to ordinary dimension. This and the recent progress on dimension of structurally restricted posets have motivated revisiting boolean dimension in current research.

Local dimension. A partial linear extension of a partial order \leq on X is a linear extension of the restriction of \leq to some subset of X. A local realizer of P of width d is a set $\{\leq_1, \ldots, \leq_t\}$ of partial linear extensions of \leq such that every element of X occurs in at most d of \leq_1, \ldots, \leq_t and

(2) $x \leq y \iff$ there is no $i \in \{1, \dots, t\}$ with $x >_i y$, for any $x, y \in X$.

The *local dimension* of P is the minimum width of a local realizer of P. Thus, instead of the size of a local realizer, we bound the number of times any element of X occurs in it. A set of linear extensions of \leq is a local realizer if and only if it is a realizer. In particular, the local dimension is at most the dimension. For arbitrary partial linear extensions \leq_1, \ldots, \leq_t of \leq on subsets of X, the relation \leq defined by (2) is not necessarily a partial order—it may fail to be antisymmetric or transitive. It is antisymmetric, for example, if one of \leq_1, \ldots, \leq_t is a linear extension of \leq on X.

The concept of local dimension was proposed very recently by Ueckerdt [26] and originates from concepts studied in [4, 19]. Ueckerdt [26] also noticed that the standard examples S_k with $k \ge 3$ have boolean dimension 3.

Results. Extending the results on boolean dimension from [10], for each d, we determine whether posets with dimension/boolean dimension/local dimension d have the other two parameters bounded or unbounded. Here is the full picture:

- **A.** Boolean dimension d and dimension d are equivalent for $d \in \{1, 2, 3\}$ [10].
- **B.** Local dimension d and dimension d are equivalent for $d \in \{1, 2\}$.
- **C.** The standard examples S_k have boolean dimension 4 when $k \ge 4$ [10], local dimension 3 when $k \ge 3$ [26], and dimension k when $k \ge 2$ [8].
- **D.** There are posets with boolean dimension 4 and unbounded local dimension.
- E. Posets with local dimension 3 have bounded boolean dimension.
- **F.** There are posets with local dimension 4 and unbounded boolean dimension.

We present proofs of A–F in the next section.

Other new results concern boolean dimension and local dimension of structurally restricted posets. In particular, posets with cover graphs of bounded path-width have bounded boolean dimension [21] and bounded local dimension [2], while local dimension is unbounded for posets with cover graphs of tree-width 3 [2, 5] and for planar posets [5]. It remains open whether boolean dimension is bounded for posets with cover graphs of bounded tree-width (in particular, tree-width 3) and for planar posets. There are *n*-element posets with local dimension $\Theta(\sqrt{n})$ [2], and conceivably the right bound is $(\frac{1}{2} - o(1))n$.

2. Proofs

A. Boolean dimension d and dimension d are equivalent for $d \in \{1, 2, 3\}$.

Proof. We basically repeat the argument given in [10] but avoiding the restriction on functions ϕ imposed therein. Let $P = (X, \leq)$ be a poset with boolean dimension d and $\{\leq_1, \ldots, \leq_d\}$ be its boolean realizer for a formula ϕ . Reflexivity of \leq implies $\phi(1, \ldots, 1) = 1$. Without loss of generality, assume $\phi(\alpha) = 0$ when $\phi(\alpha)$ is never used by (1). This and antisymmetry of \leq imply $\phi(\alpha) = 0$ or $\phi(\overline{\alpha}) = 0$ for every $\alpha \in \{0, 1\}^d$. In particular, $\phi(\alpha) = 1$ for at most half of the tuples α . If d = 1, then $\phi(1) = 1$ and $\phi(0) = 0$, so $\{\leq_1\}$ is a realizer of P. This shows that boolean dimension 1 and dimension 1 are equivalent.

Now, let $d \in \{2,3\}$. If $\phi(\alpha) = \phi(\overline{\alpha}) = 0$ for at most one pair $\alpha, \overline{\alpha}$ (as it is for d = 2), then the strict partial order \prec on X defined by $x \prec y \iff ((x <_1 y), \dots, (x <_d y)) = \alpha$ for distinct $x, y \in X$ is a transitive orientation of the incomparability graph of P, so P has dimension at most 2 [8, Theorem 3.61]. This shows that boolean dimension 2 and dimension 2 are equivalent.

For d = 3, to complete the proof that boolean dimension 3 and dimension 3 are equivalent, we consider the three cases (up to symmetry) in which $\phi(\alpha) = 1$ for at most two tuples α .

- 1. If $\phi(\alpha) = 1$ for $\alpha = (1, 1, 1)$ only, then $\{\leq_1, \leq_2, \leq_3\}$ is a realizer of P.
- 2. If $\phi(\alpha) = 1$ for $\alpha \in \{(1, 1, 0), (1, 1, 1)\}$ only, then $\{\leq_1, \leq_2\}$ is a realizer of *P*.
- 3. If $\phi(\alpha) = 1$ for $\alpha \in \{(0,0,1), (1,1,1)\}$ only, then the strict partial order \prec on X defined by $x \prec y \iff ((x <_1 y) \land (x >_2 y))$ for distinct $x, y \in X$ is a transitive orientation of the incomparability graph of P, so P has dimension at most 2 [8, Theorem 3.61].

B. Local dimension d and dimension d are equivalent for $d \in \{1, 2\}$.

Proof. If a poset $P = (X, \leq)$ has local dimension 1, then a local realizer of P of width 1 must consist of a single full linear order on X, because antisymmetry of \leq requires that every pair $x, y \in X$ occurs in at least one partial linear extension.

Now, let $P = (X, \leq)$ be a poset with local dimension 2, and consider a local realizer of P of width 2. If $x, y \in X$ are incomparable in \leq , then both occurrences of x and y are in the same two partial linear extensions, where x < y in one and x > y in the other. Therefore, the subposet of P induced on every connected component C of the incomparability graph of P is witnessed by two partial linear extensions, which restricted to C form a realizer of that subposet. These realizers stacked according to the order \leq form a realizer of P of size 2.

C. The standard examples S_k have boolean dimension 4 when $k \ge 4$, local dimension 3 when $k \ge 3$, and dimension k when $k \ge 2$.

Proof. It was observed in [10] that the standard example S_k has boolean dimension 4 (when $k \ge 4$), witnessed by the formula $\phi(\alpha) = \alpha_1 \land \alpha_2 \land (\alpha_3 \lor \alpha_4)$ and the following four linear orders:

$$a_1 < \dots < a_k < b_1 < \dots < b_k, \qquad b_1 < a_1 < \dots < b_k < a_k, a_k < \dots < a_1 < b_k < \dots < b_1, \qquad b_k < a_k < \dots < b_1 < a_1.$$

Ueckerdt [26] observed that S_k has local dimension 3 (when $k \ge 3$), witnessed by the two linear extensions above on the left and k partial linear extensions each of the form $b_i < a_i$.

Only one pair a_i, b_i can be ordered as $b_i < a_i$ in a single linear extension, so the dimension of S_k is at least k. A realizer of size exactly k can be constructed easily when $k \ge 2$, see [8]. \Box

D. There are posets with boolean dimension 4 and unbounded local dimension.

Proof. Another well-known construction of posets with arbitrarily large dimension involves incidence posets of complete graphs: $P_n = (V \cup E, \leq)$, where $V = \{v_1, \ldots, v_n\}$ are the minimal elements, $E = \{v_1v_2, v_1v_3, \ldots, v_{n-1}v_n\}$ are the maximal elements, and the only comparable pairs are $v_i < v_iv_j$ and $v_j < v_iv_j$ for $i \neq j$. The dimension of P_n is at least $\log_2 \log_2 n$ [8, Theorem 4.22]. The boolean dimension of P_n is at most 4, witnessed by the formula $\phi(\alpha) = (\alpha_1 \land \alpha_2) \lor (\alpha_3 \land \alpha_4)$ and the following four linear orders:

 $A_1 < \cdots < A_n$, where each A_i has form $v_i < v_i v_{i+1} < \cdots < v_i v_n$, $B_n < \cdots < B_1$, where each B_i has form $v_i < v_i v_n < \cdots < v_i v_{i+1}$, $C_1 < \cdots < C_n$, where each C_i has form $v_i < v_1 v_i < \cdots < v_{i-1} v_i$, $D_n < \cdots < D_1$, where each D_i has form $v_i < v_{i-1} v_i < \cdots < v_1 v_i$.

The local dimension of P_n is unbounded as $n \to \infty$. For suppose P_n has a local realizer of width d. Enumerate the occurrences of each element of $V \cup E$ in the local realizer from 1 to (at most) d. Each triple $v_i v_j v_k$ (i < j < k) can be assigned a color (p,q) so that $v_i v_k < v_j$ in a partial linear extension containing the pth occurrence of v_j and the qth occurrence of $v_i v_k v_k$ (i < j < k) with all four triples of the same color (p,q). It follows that the pth occurrences of v_j and v_k and the qth occurrences of $v_i v_\ell$, $v_i v_k$, and $v_j v_\ell$ are all in the same partial linear extension, which therefore contains a cycle $v_j < v_i v_\ell < v_k < v_i v_k < v_j$, a contradiction.

E. Posets with local dimension 3 have bounded boolean dimension.

Proof. Let $P = (X, \leq)$ be a poset with a local realizer of width 3 consisting of partial linear extensions that we call *gadgets*. We construct a boolean realizer $\{\leq^*, \leq_1, \leq'_1, \ldots, \leq_d, \leq'_d\}$ for a formula of the form $\alpha^* \land (\alpha_1 \lor \alpha'_1) \land \cdots \land (\alpha_d \lor \alpha'_d)$. The order \leq^* is an arbitrary linear extension of \leq on X. Each pair of orders \leq_i, \leq'_i is defined by $X_1 <_i \cdots <_i X_t$ and $X_t <'_i \cdots <'_i X_1$, where $\{X_1, \ldots, X_t\}$ is some partition of X into *blocks* such that every block X_j is completely ordered by some gadget and that order is inherited by \leq_i and \leq'_i . Then, we have x < y for the relation \leq defined by (1) if and only if $x <^* y$ and x < y in every block containing both x and y. It remains to construct a bounded number of partitions of X into blocks so that for any $x, y \in X$, if $x <^* y$ and x > y in some gadget, then x > y in some block in at least one of the partitions.

Without loss of generality, assume that each element $x \in X$ has exactly 3 occurrences in the gadgets—enumerate them as x^1, x^2, x^3 according to a fixed order of the gadgets. For each $p \in \{1, 2, 3\}$, form a partition of X by restricting every gadget to elements of the form x^p . These three partitions witness all comparabilities of the form $x^p > y^p$ within gadgets.

Now, let G be a graph on X where xy (with $x <^* y$) is an edge if and only if $x^p > y^q$ ($p \neq q$) in some gadget. Thus G is a subgraph of the incomparability graph of P. Suppose $\chi(G) > 38$. It follows that G has an edge uv such that $\chi(G[X_{uv}]) \ge 19$, where $X_{uv} = \{x \in X : u <^* x <^* v\}$ [20, Lemma 2.1]. Let $X_u = \{x \in X_{uv} : u \leq x\}$ and $X_v = \{x \in X_{uv} : x \leq v\}$. Thus $X_{uv} = X_u \cup X_v$, as $u \leq v$. Let a color of $x \in X_u$ be a quadruple (p, q, r, s) with p < q and r < s, where either $x^p < u^q$ and $x^r > u^s$ or $x^p > u^q$ and $x^r < u^s$ in some gadgets. There are 9 possible colors (quadruples). The coloring of $G[X_u]$ thus obtained is proper—whenever $x, y \in X_u$ have the same color, x^p and y^p are in the same gadget, as well as x^r and y^r are in the same gadget; this contradicts the fact that the edge xy of G is witnessed by some x^i and y^j with $i \neq j$ occurring in the same gadget (this is where we use the bound 3 on the number of occurrences). Thus $\chi(G[X_u]) \leq 9$, and similarly $\chi(G[X_v]) \leq 9$, which yields $\chi(G[X_{uv}]) \leq 18$. This contradiction shows $\chi(G) \leq 38$.

Let c be a proper 38-coloring of G. For $1 \leq p < q \leq 3$ and any distinct colors a, b, form a partition of X by restricting every gadget to elements of the form x^p with c(x) = a and y^q with

c(y) = b (adding singletons if necessary to obtain a full partition of X). The 4218 partitions thus obtained have the desired property. The resulting boolean realizer of P has size 8443. \Box

F. There are posets with local dimension 4 and unbounded boolean dimension.

Proof. When (V, E) is an acyclic digraph, $v \in V$, and $X, Y \subseteq V$, let $E(X, v) = \{xv \in E : x \in X\}$, $E(v,Y) = \{vy \in E : y \in Y\}$, and $E(X,Y) = \{xy \in E : x \in X \text{ and } y \in Y\}$ (xy denotes a directed edge from x to y). For every $k \ge 1$, we construct an acyclic digraph G = (V, E) with $\chi(G) > k$, a poset $P = (E, \leq)$, and a local realizer $\{\leq_A, \leq_B\} \cup \{\leq_v : v \in V\}$ of P of width 4, where

- (i) \leq_A is a linear extension of \leq on E such that $E(V, v) <_A E(v, V)$ for every $v \in V$,
- (ii) \leq_B is a linear extension of \leq on E such that $E(V, v) <_B E(v, V)$ and E(v, V) occurs as a contiguous block in \leq_B for every $v \in V$,
- (iii) \leq_v is a gadget—a partial linear extension of the form $E(v, V) <_v E(V, v)$ for every $v \in V$.

The construction is an adaptation of the well-known construction of triangle-free graphs with arbitrarily large chromatic number from [7, 18]. For k = 1, let $V = \{u, v\}, E = \{uv\}$, and $\leq_A, \leq_B, \leq_u, \leq_v$ be trivial orders on E. Now, suppose that $k \geq 2$ and the construction can be performed for k-1. Let r be the number of vertices in that construction, s = k(r-1) + 1, and $n = \binom{s}{r}$. For $1 \leq i \leq n$, let $G^i = (V^i, E^i)$, $P^i = (E^i, \leq^i)$, and $\{\leq^i_A, \leq^i_B\} \cup \{\leq^i_v : v \in V^i\}$ be separate instances of the construction for k-1. Let $X = \{x_1, \ldots, x_s\}$ be yet a separate set of s vertices. Let X^1, \ldots, X^n be the r-element subsets of X. Let $V = X \cup V^1 \cup \cdots \cup V^n$ and $E = \bigcup_{i=1}^{n} (\{x_1^i v_1^i, \dots, x_r^i v_r^i\} \cup E^i),$ where

- x_1^i, \ldots, x_r^i are the vertices in X^i in the same order as in the sequence x_1, \ldots, x_s ,
- v_1^i, \ldots, v_r^i are the vertices in V^i ordered so that $E^i(v_1^i, V^i) <_B^i \cdots <_B^i E^i(v_r^i, V^i)$.

Let G = (V, E). Clearly, G is an acyclic digraph. The assumption that $\chi(G^i) > k - 1$ for all i implies $\chi(G) > k$ [7, 18]. Indeed, in any proper k-coloring of G, at least one of the sets X^i would be monochromatic, which would yield $\chi(G^i) \leq k-1$, a contradiction. For $1 \leq j \leq s$, let $N_j = \{v \in V : x_j v \in E\}$. Let \leq_A and \leq_B be linear orders on V such that

- $E(X^1, V^1) <_A E^1 <_A \dots <_A E(X^n, V^n) <_A E^n$ and the restriction of \leq_A to each E^i is \leq_A^i , • $E(x_1, V) <_B E(N_1, V) <_B \cdots <_B E(x_s, V) <_B E(N_s, V).$

The latter property implies that the restriction of \leq_B to each E^i is \leq_B^i . Finally, for every $x \in X$, let \leq_x be a new gadget on E(x, V), and for $v \in V^i$ and $1 \leq i \leq n$, let \leq_v be \leq_v^i with E(X, v)(which is just one edge) added on top. This guarantees properties (i)–(iii). Let \leq be the relation on E defined from $\{\leq_A, \leq_B\} \cup \{\leq_v : v \in V\}$ by (2). It follows that the restriction of \leq to each E^i is \leq^i . It remains to show that the relation \leq is a partial order, so that $P = (E, \leq)$ is a poset and $\{\leq_A, \leq_B\} \cup \{\leq_v : v \in V\}$ is its local realizer.

Reflexivity and antisymmetry of \leq are clear. For transitivity, suppose $e, f, g \in E, e < f$, and f < g, but $e \not\leq g$. The assumption that e < f and f < g implies $e <_A f <_A g$ and $e <_B f <_B g$. Since $e \not\leq g$, the edges e and g must occur as e > g in some gadget. We consider four cases.

- 1. If $e, g \in E^i$ for some *i*, then the definition of \leq_A implies $f \in E^i$, so $e <^i f <^i g$. This and the assumption that $e \notin g$ contradict the fact that \leq^i is the restriction of \leq to E^i .
- 2. If $e, g \in E(x, V)$ for some $x \in X$, then the definition of \leq_B implies $f \in E(x, V)$. This yields $e <_x f <_x g$, and the only gadget containing both e and g fails to witness $e \not\leq g$.
- 3. If $e = x_j v$ and g = uv for some $x_j \in X$ and $u, v \in V \setminus X$, then $u, v \in V^i$ for some *i*. The definition of E implies that there is an edge $x_{j'}u \in E$, where j' < j, and therefore $g \in E(N_{j'}, V) <_B E(x_j, V) \ni e$, a contradiction.
- 4. If $e = x_j v$ and g = v w for some $x_j \in X$ and $v, w \in V \setminus X$, then $v, w \in V^i$ for some *i*. The definitions of \leq_A and \leq_B imply

$$f \in (E(X^{i}, V^{i}) \cup E^{i}) \cap (E(x_{j}, V) \cup E(N_{j}, V)) = \{x_{j}v\} \cup E(v, V^{i}).$$

This yields $e <_v f <_v g$, and the only gadget containing both e and g fails to witness $e \not\leq g$.

This shows that \leq is transitive, thus completing the proof of correctness of the construction.

Let $k = 2^{2^{2^d}}$. We show that the poset P resulting from the construction above has boolean dimension greater than d. For suppose $\{\leq_1, \ldots, \leq_d\}$ is a boolean realizer of P for a formula ϕ . Let G' = (E, A) be the arc digraph of G, and let G'' = (A, B) be the arc digraph of G'. That is, $A = \{uvw: uv, vw \in E\}$ and $B = \{uvwx: uvw, vwx \in A\} = \{uvwx: uv, vw, wx \in E\}$. It follows that $\chi(G') \ge \log_2 \chi(G)$ and $\chi(G'') \ge \log_2 \chi(G')$ [11, Theorem 9], and thus $\chi(G'') > 2^d$. For $uvw \in A$, let $\alpha(uvw) = ((uv <_1 vw), \ldots, (uv <_d vw)) \in \{0,1\}^d$; the fact that $uv >_v vw$ implies $uv \not\le vw$ and thus $\phi(\alpha(uvw)) = 0$. Let $uvwx \in B$. We have $uv <_A vw <_A wx$ and $uv <_B vw <_B wx$, which implies uv < wx, because no gadget contains both uv and wx. If $\alpha(uvw) = \alpha(vwx) = \alpha$, then transitivity of \leq_1, \ldots, \leq_d implies $((uv <_1 wx), \ldots, (uv <_d wx)) =$ α . This, $\phi(\alpha) = 0$, and uv < wx result in a contradiction. Therefore, $\alpha: A \to \{0,1\}^d$ is a proper 2^d -coloring of G''. This contradicts the fact that $\chi(G'') > 2^d$.

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