

# The Geodetic Hull Number is Hard for Chordal Graphs

Stéphane Bessy<sup>1</sup>    Mitre C. Dourado<sup>2</sup>    Lucia D. Penso<sup>3</sup>    Dieter Rautenbach<sup>3</sup>

<sup>1</sup> Laboratoire d'Informatique, de Robotique et de Microélectronique de Montpellier  
Université de Montpellier, Montpellier, France, stephane.bessy@lirmm.fr

<sup>2</sup> Departamento de Ciência da Computação, Instituto de Matemática  
Universidade Federal do Rio de Janeiro, Rio de Janeiro, Brazil, mitre@dcc.ufrj.br

<sup>3</sup> Institute of Optimization and Operations Research  
Ulm University, Ulm, Germany, {lucia.penso,dieter.rautenbach}@uni-ulm.de

## Abstract

We show the hardness of the geodetic hull number for chordal graphs.

**Keywords:** Geodetic convexity; shortest path; hull number; chordal graphs

## 1 Introduction

One of the most well studied convexity notions for graphs is the *shortest path convexity* or *geodetic convexity*, where a set  $X$  of vertices of a graph  $G$  is considered *convex* if no vertex outside of  $S$  lies on a shortest path between two vertices inside of  $S$ . Defining the *convex hull* of a set  $S$  of vertices as the smallest convex set containing  $S$ , a natural parameter of  $G$  is its *hull number*  $h(G)$  [7], which is the minimum order of a set of vertices whose convex hull is the entire vertex set of  $G$ . The hull number is NP-hard for bipartite graphs [2], partial cubes [1], and  $P_9$ -free graphs [5], but it can be computed in polynomial time for cographs [4],  $(q, q - 4)$ -graphs [2], {paw,  $P_5$ }-free graphs [3, 5], and distance-hereditary graphs [9]. Bounds on the hull number are given in [2, 6, 7].

In [9] Kanté and Nourine present a polynomial time algorithm for the computation of the hull number of chordal graphs. Unfortunately, their correctness proof contains a gap described in detail at the end of the present paper. As our main result we show that computing the hull number of a chordal graph is NP-hard, which most likely rules out the existence of a polynomial time algorithm.

Before we proceed to our results, we collect some notation and terminology. We consider finite, simple, and undirected graphs. A graph  $G$  has vertex set  $V(G)$  and edge set  $E(G)$ . A graph  $G$  is *chordal* if it does not contain an induced cycle of order at least 4. A *clique* in  $G$  is the vertex set of a complete subgraph of  $G$ . A vertex of a graph  $G$  is *simplicial* in  $G$  if its neighborhood is a clique. The *distance*  $\text{dist}_G(u, v)$  between two vertices  $u$  and  $v$  in  $G$  is the minimum number of edges of a path in  $G$  between  $u$  and  $v$ . The *diameter*  $\text{diam}(G)$  of  $G$  is the maximum distance between any two vertices of  $G$ . The *eccentricity*  $e_G(u)$  of a vertex  $u$  of  $G$  is the maximum distance between  $u$  and any other vertex of  $G$ . For a positive integer  $k$ , let  $[k]$  be the set of the positive integers at most  $k$ .

Let  $G$  be a graph, and let  $S$  be a set of vertices of  $G$ . The *interval*  $I_G(S)$  of  $S$  in  $G$  is the set of all vertices of  $G$  that lie on shortest paths in  $G$  between vertices from  $S$ . Note that  $S \subseteq I_G(S)$ , and that  $S$  is *convex* in  $G$  if  $I_G(S) = S$ . The set  $S$  is *concave* in  $G$  if  $V(G) \setminus S$  is convex. Note that  $S$  is concave

if and only if  $S \cap I_G(\{v, w\}) = \emptyset$  for every two vertices  $v$  and  $w$  in  $V(G) \setminus S$ . The *hull*  $H_G(S)$  of  $S$  in  $G$ , defined as the smallest convex set in  $G$  that contains  $S$ , equals the intersection of all convex sets that contain  $S$ . The set  $S$  is a *hull set* if  $H_G(S) = V(G)$ , and the *hull number*  $h(G)$  of  $G$  [5, 7] is the smallest order of a hull set of  $G$ .

## 2 Result

We immediately proceed to our main result.

**Theorem 2.1.** *For a given chordal graph  $G$ , and a given integer  $k$ , it is NP-complete to decide whether the hull number  $h(G)$  of  $G$  is at most  $k$ .*

*Proof.* Since the hull of a set of vertices of  $G$  can be computed in polynomial time, the considered decision problem belongs to NP. In order to prove NP-completeness, we describe a polynomial reduction from a restricted version of SATISFIABILITY. Therefore, let  $\mathcal{C}$  be an instance of SATISFIABILITY consisting of  $m$  clauses  $C_1, \dots, C_m$  over  $n$  boolean variables  $x_1, \dots, x_n$  such that every clause in  $\mathcal{C}$  contains at most three literals, and, for every variable  $x_i$ , there are exactly two clauses in  $\mathcal{C}$ , say  $C_{j_i^{(1)}}$  and  $C_{j_i^{(2)}}$ , that contain the literal  $x_i$ , and exactly one clause in  $\mathcal{C}$ , say  $C_{j_i^{(3)}}$ , that contains the literal  $\bar{x}_i$ , and these three clauses are distinct. Using a polynomial reduction from [LO1] [8], it has been shown in [5] that SATISFIABILITY restricted to such instances is still NP-complete.

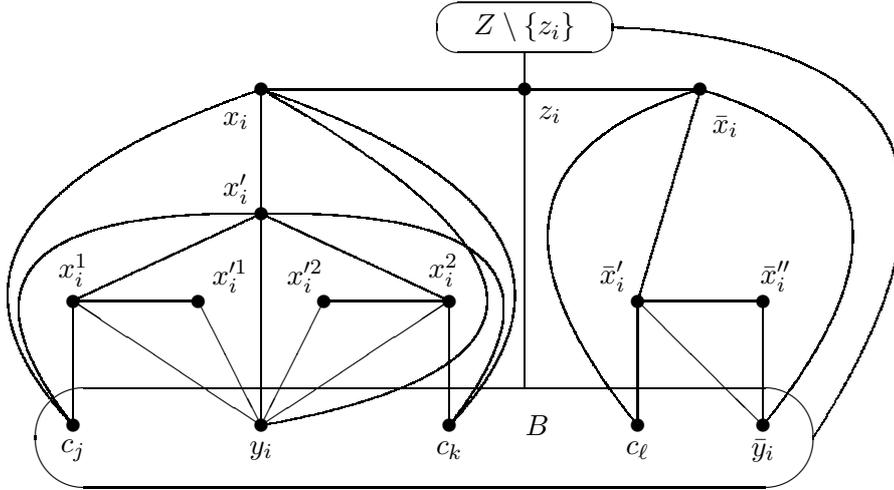


Figure 1: The vertices and edge added for the variable  $x_i$ , where  $j_i^{(1)} = j$ ,  $j_i^{(2)} = k$ , and  $j_i^{(3)} = \ell$ .

Let the graph  $G$  be constructed as follows starting with the empty graph:

- For every  $j \in [m]$ , add a vertex  $c_j$ .
- For every  $i \in [n]$ , add three  $y_i, \bar{y}_i$ , and  $z_i$ .
- Add edges such that  $B \cup Z$  is a clique, where

$$B = \{c_j : j \in [m]\} \cup \{y_i : i \in [n]\} \cup \{\bar{y}_i : i \in [n]\} \text{ and}$$

$$Z = \{z_i : i \in [n]\}, \text{ and}$$

- For every  $i \in [n]$ , add 9 vertices and 25 edges to obtain the subgraph indicated in Figure 2.

Note that  $\text{dist}_G(x_i, \bar{x}'_i) = \text{dist}_G(\bar{x}_i, x_i^1) = 3$  for every  $i$  in  $[n]$ . Since every vertex of  $G$  has a neighbor in the clique  $B \cup Z$ , the diameter of  $G$  is 3. Furthermore, since no vertex is universal, all vertices in  $B \cup Z$  have eccentricity 2.

Let  $k = 4n$ .

Note that the order of  $G$  is  $12n + m$ .

It remains to show that  $G$  is chordal, and that  $\mathcal{C}$  is satisfiable if and only if  $h(G) \leq k$ .

In order to show that  $G$  is chordal, we indicate a *perfect elimination ordering*, which is a linear ordering  $v_1, \dots, v_{12n+m}$  of its vertices such that  $v_i$  is simplicial in  $G - \{v_1, \dots, v_{i-1}\}$  for every  $i$  in  $[12n + m]$ . Such an ordering is obtained by

- starting with the vertices  $x_i^1, x_i^2$ , and  $\bar{x}''_i$  for all  $i \in [n]$  (in any order),
- continuing with the vertices  $x_i^1, x_i^2$ , and  $\bar{x}'_i$  for all  $i \in [n]$ ,
- continuing with the vertices  $x'_i$  for all  $i \in [n]$ ,
- continuing with the vertices  $x_i$  and  $\bar{x}_i$  for all  $i \in [n]$ , and
- ending with the vertices in the clique  $B \cup Z$ .

Now, let  $\mathcal{S}$  be a satisfying truth assignment for  $\mathcal{C}$ .

Let

$$S = \bigcup_{i \in [n]} \{x_i^1, x_i^2, \bar{x}''_i\} \cup \bigcup_{i \in [n]: x_i \text{ true in } \mathcal{S}} \{x_i\} \cup \bigcup_{i \in [n]: x_i \text{ false in } \mathcal{S}} \{\bar{x}_i\}.$$

Clearly,  $|S| = k = 4n$ . For every  $i$  in  $[n]$ , we have  $\{z_i, \bar{y}_i\} \subseteq I_G(\{x_i, \bar{x}''_i\})$ ,  $\{z_i, y_i\} \subseteq I_G(\{\bar{x}_i, x_i^1\})$ ,  $y_i \in I_G(\{\bar{y}_i, x_i^1\})$ , and  $\bar{y}_i \in I_G(\{y_i, \bar{x}''_i\})$ , which implies  $\{z_i, y_i, \bar{y}_i\} \subseteq H_G(S)$ . Since  $\mathcal{S}$  is a satisfying truth assignment, for every  $j$  in  $[m]$ , there is a neighbor, say  $v$ , of  $c_j$  in

$$\bigcup_{i \in [n]: x_i \text{ true in } \mathcal{S}} \{x_i\} \cup \bigcup_{i \in [n]: x_i \text{ false in } \mathcal{S}} \{\bar{x}_i\}.$$

If  $v \in \bigcup_{i \in [n]: x_i \text{ true in } \mathcal{S}} \{x_i\}$ , then  $c_j \in I_G(\{v, \bar{x}''_i\})$ , otherwise  $c_j \in I_G(\{v, x_i^1\})$ . Hence,  $B \cup Z \subseteq H_G(S)$ .

Now, for some  $i$  in  $[n]$ , let  $c_j$ ,  $c_k$ , and  $c_\ell$  be the neighbors in  $B \setminus \{y_i, \bar{y}_i\}$  of  $x_i^1$ ,  $x_i^2$ , and  $\bar{x}''_i$ , respectively, similarly as in Figure 2. We have  $x_i^1 \in I_G(\{x_i^1, c_j\})$ ,  $x_i^2 \in I_G(\{x_i^2, c_k\})$ ,  $x'_i \in I_G(\{x_i^1, x_i^2\})$ ,  $\bar{x}'_i \in I_G(\{\bar{x}''_i, c_\ell\})$ ,  $x_i \in I_G(\{x'_i, z_i\})$ , and  $\bar{x}_i \in I_G(\{\bar{x}'_i, z_i\})$ .

Altogether, we obtain that  $S$  is a hull set of  $G$  of order  $4n$ .

Finally, let  $S$  be a hull set of  $G$  of order at most  $4n$ .

**Claim 1.** *For every  $i \in [n]$ , the set  $\{x_i, z_i, \bar{x}_i\}$  is concave.*

*Proof of Claim 1:* For a contradiction, suppose that some vertex in  $S' = \{x_i, z_i, \bar{x}_i\}$  lies on a shortest path  $P$  in  $G$  between two vertices  $v$  and  $w$  in  $V(G) \setminus S'$ . Since the diameter of  $G$  is 3, the path  $P$  contains at most 2 vertices of  $S'$ . Since the neighbors outside of  $S'$  of each vertex in  $S'$  form a clique, the path  $P$  contains exactly 2 adjacent vertices of  $S'$ , that is, either  $P = vx_iz_iw$  or  $P = v\bar{x}_iz_iw$ . In both cases, the vertex  $w$  has eccentricity at least 3. However, every neighbor  $w$  of  $z_i$  outside  $S'$  belongs to  $B \cup Z$ , and thus, has eccentricity 2, a contradiction.  $\square$

**Claim 2.** For every  $j \in [m]$ , the set

$$V_j = \{c_j\} \cup \bigcup_{i \in [n]: j=j_i^{(1)}} \{x_i, x'_i, x_i^1\} \cup \bigcup_{i \in [n]: j=j_i^{(2)}} \{x_i, x'_i, x_i^2\} \cup \bigcup_{i \in [n]: j=j_i^{(3)}} \{\bar{x}_i, \bar{x}'_i\}$$

is concave.

*Proof of Claim 2:* First, suppose that  $C_j$  contains the positive literal  $x_i$ . By symmetry, we may assume that  $j = j_i^{(1)}$  and  $j_i^{(2)} = k$  for some  $k$  in  $[m] \setminus \{j\}$ .

First, suppose that some shortest path  $P$  between two vertices  $v$  and  $w$  in  $\bar{V}_j = V(G) \setminus V_j$  contains  $x_i$ . Choosing  $P$  of minimum length, it follows that  $v$  and  $w$  are the only vertices of  $P$  in  $\bar{V}_j$ . Since the diameter of  $G$  is 3, the length of  $P$  is at most 3, and we may assume that  $v$  is a neighbor of  $x_i$ , which implies  $v \in \{z_i, c_k, y_i\}$ . Since  $\{z_i, c_k, y_i\}$  is a clique, the vertex  $w$  is not a neighbor of  $x_i$ , and  $P$  contains exactly one vertex  $u$  of  $V_j$  different of  $x_i$ , which implies  $P = vx_iuw$  and  $u \in \{x'_i, c_j\}$ . Suppose that  $u = x'_i$ . This implies  $w \in \{x_i^2, c_k, y_i\}$ . Since  $c_k, y_i \in N_G(x_i)$ , we obtain  $w = x_i^2$  and  $v = z_i$ . However,  $\text{dist}_G(z_i, x_i^2) = 2$ , which is a contradiction. Hence,  $u = c_j$  and  $w \in B \cup Z$ . However, every vertex in  $B \cup Z$  has eccentricity 2, which is a contradiction. Hence, no shortest path between two vertices in  $\bar{V}_j$  contains  $x_i$ .

Next, suppose that some shortest path  $P$  between two vertices  $v$  and  $w$  in  $\bar{V}_j$  contains  $x'_i$ . Similarly as above, we may assume that  $v$  and  $w$  are the only vertices of  $P$  in  $\bar{V}_j$ , the length of  $P$  is at most 3, and  $v$  is a neighbor of  $x'_i$ , which implies  $v \in \{x_i^2, y_i, c_k\}$ . Since  $\{x_i^2, y_i, c_k\}$  is a clique, the path  $P$  contains exactly one vertex  $u$  of  $V_j$  different of  $x'_i$ , which implies  $P = vx'_iuw$  and  $u \in \{x_i^1, c_j\}$ , where we use that  $P$  does not contain  $x_i$ . Suppose that  $u = x_i^1$ . This implies  $w \in \{x_i^1, y_i\}$ . Since  $y_i \in N_G(x'_i)$ , we obtain  $w = x_i^1$  and  $v = x_i^2$ . However,  $\text{dist}_G(x_i^2, x_i^1) = 2$ , which is a contradiction. Hence,  $u = c_j$  and  $w \in B \cup Z$ . However, every vertex in  $B \cup Z$  has eccentricity 2, which is a contradiction. Hence, no shortest path between two vertices in  $\bar{V}_j$  contains  $x'_i$ .

Next, suppose that some shortest path  $P$  between two vertices  $v$  and  $w$  in  $\bar{V}_j$  contains  $x_i^1$ . Similarly as above, we may assume that  $v$  and  $w$  are the only vertices of  $P$  in  $\bar{V}_j$ , the length of  $P$  is at most 3, and  $v$  is a neighbor of  $x_i^1$ , which implies  $v \in \{x_i^1, y_i\}$ . Since  $\{x_i^1, y_i\}$  is a clique, the path  $P$  contains exactly one vertex  $u$  of  $V_j$  different of  $x_i^1$ , which implies  $P = vx_i^1c_jw$  and  $w \in B \cup Z$ , where we use that  $P$  does not contain  $x'_i$ . However, every vertex in  $B \cup Z$  has eccentricity 2, which is a contradiction. Hence, no shortest path between two vertices in  $\bar{V}_j$  contains  $x_i^1$ .

Next, suppose that  $C_j$  contains the negative literal  $\bar{x}_i$ , that is,  $j = j_i^{(3)}$ .

First, suppose that some shortest path  $P$  between two vertices  $v$  and  $w$  in  $\bar{V}_j$  contains  $\bar{x}_i$ . Similarly as above, we may assume that  $v$  and  $w$  are the only vertices of  $P$  in  $\bar{V}_j$ , the length of  $P$  is at most 3, and  $v$  is a neighbor of  $\bar{x}_i$ , which implies  $v \in \{z_i, \bar{y}_i\}$ . Since  $\{z_i, \bar{y}_i\}$  is a clique, the vertex  $w$  is not a neighbor of  $\bar{x}_i$ , and  $P$  contains exactly one vertex  $u$  of  $V_j$  different of  $\bar{x}_i$ , which implies  $P = v\bar{x}_iuw$  and  $u \in \{\bar{x}'_i, c_j\}$ . Suppose that  $u = \bar{x}'_i$ . This implies  $w \in \{\bar{x}_i'', \bar{y}_i\}$ . Since  $\bar{y}_i \in N_G(\bar{x}_i)$ , we obtain  $v = z_i$  and  $w = \bar{x}_i''$ . However,  $\text{dist}_G(z_i, \bar{x}_i'') = 2$ , which is a contradiction. Hence,  $u = c_j$  and  $w \in B \cup Z$ . However, every vertex in  $B \cup Z$  has eccentricity 2, which is a contradiction. Hence, no shortest path between two vertices in  $\bar{V}_j$  contains  $\bar{x}_i$ .

Next, suppose that some shortest path  $P$  between two vertices  $v$  and  $w$  in  $\bar{V}_j$  contains  $\bar{x}'_i$ . Similarly as above, we may assume that  $v$  and  $w$  are the only vertices of  $P$  in  $\bar{V}_j$ , the length of  $P$  is at most 3, and  $v$  is a neighbor of  $\bar{x}'_i$ , which implies  $v \in \{\bar{x}_i'', \bar{y}_i\}$ . Since  $\{\bar{x}_i'', \bar{y}_i\}$  is a clique, the path  $P$  contains exactly one vertex  $u$  of  $V_j$  different of  $\bar{x}'_i$ , which implies  $P = v\bar{x}'_ic_jw$  and  $w \in B \cup Z$ , where we use that

$P$  does not contain  $\bar{x}_i$ . However, every vertex in  $B \cup Z$  has eccentricity 2, which is a contradiction. Hence, no shortest path between two vertices in  $\bar{V}_j$  contains  $\bar{x}'_i$ .

Finally, since the neighbors of  $c_j$  outside of  $V_j$  form a clique, no shortest path between two vertices in  $\bar{V}_j$  contains  $c_j$ , which completes the proof of the claim.  $\square$

Note that all  $3n$  simplicial vertices in  $\bigcup_{i \in [n]} \{x_i^1, x_i^2, \bar{x}_i\}$  belong to  $S$ .

Since  $S$  contains at most  $n$  non-simplicial vertices, Claim 1 implies that, for every  $i$  in  $[n]$ , the set  $S$  contains exactly one of the three vertices in  $\{x_i, z_i, \bar{x}_i\}$ , and that these are the only non-simplicial vertices in  $S$ . Now, Claim 2 implies that, for every  $j$  in  $[m]$ , there is some  $i \in [n]$  such that

- either  $C_j$  contains the literal  $x_i$  and the vertex  $x_i$  belongs to  $S$
- or  $C_j$  contains the literal  $\bar{x}_i$  and the vertex  $\bar{x}_i$  belongs to  $S$ .

Therefore, setting the variable  $x_i$  to true if and only if the vertex  $x_i$  belongs to  $S$  yields a satisfying truth assignment  $S$  for  $\mathcal{C}$ , which completes the proof.  $\square$

As pointed out in the introduction, the correctness proof in [9] contains a gap. In lines 14 and 15 on page 322 of [9] it says

*“At iteration  $i + 1$ , the vertex  $x_{i+1}$  is a simplicial vertex in  $G_{i+1}$ . We first claim that there exists no functional dependency of the form  $zt \rightarrow x_{i+1}$  in  $\Sigma$ .”*

Consider applying the algorithm from [9] to the graph in Figure 2. In iteration 1, it would decide to add  $x_1$  to  $K$ . In iteration 2, it would decide not to add  $x_2$  to  $K$ , because of  $t \rightarrow x_2$ . Furthermore, because of  $t \rightarrow x_2$  and  $z, x_2 \rightarrow x_3$ , it would replace  $z, x_2 \rightarrow x_3$  within  $\Sigma$  with  $z, t \rightarrow x_3$ . Therefore, in iteration 3,  $\Sigma$  would actually contain  $z, t \rightarrow x_3$ , contrary to the claim cited above.

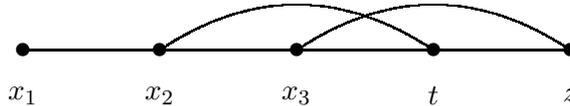


Figure 2: A small chordal graph.

## References

- [1] M. Albenque, K. Knauer, Convexity in partial cubes: The hull number, Lecture Notes in Computer Science 8392 (2014) 421-432.
- [2] J. Araujo, V. Campos, F. Giroire, N. Nisse, L. Sampaio, R. Soares, On the hull number of some graph classes, Theoretical Computer Science 475 (2013) 1-12.
- [3] J. Araujo, G. Morel, L. Sampaio, R. Soares, V. Weber, Hull number:  $P_5$ -free graphs and reduction rules, Discrete Applied Mathematics 210 (2016) 171-175.
- [4] M.C. Dourado, J.G. Gimbel, J. Kratochvíl, F. Protti, J.L. Szwarcfiter, On the computation of the hull number of a graph, Discrete Mathematics 309 (2009) 5668-5674.
- [5] M.C. Dourado, L.D. Penso, D. Rautenbach, On the geodetic hull number of  $P_k$ -free graphs, Theoretical Computer Science 640 (2016) 52-60.

- [6] M.C. Dourado, F. Protti, D. Rautenbach, J.L. Szwarcfiter, On the hull number of triangle-free graphs, *SIAM Journal of Discrete Mathematics* 23 (2010) 2163-2172.
- [7] M.G. Everett, S.B. Seidman, The hull number of a graph, *Discrete Mathematics* 57 (1985) 217-223.
- [8] M.R. Garey, D.S. Johnson, *Computers and Intractability: A Guide to the Theory of NP-Completeness*, W.H. Freeman & Co., New York (1979).
- [9] M.M. Kanté, L. Nourine, Polynomial time algorithms for computing a minimum hull set in distance-hereditary and chordal graphs, *SIAM Journal on Discrete Mathematics* 30 (2016) 311-326.