

On the average number of elements in a finite field with order or index in a prescribed residue class

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Abstract

For any prime p the density of elements in \mathbb{F}_p^* having order, respectively index, congruent to $a \pmod{d}$ is being considered. These densities on average are determined, where the average is taken over all finite fields of prime order. Some connections between the two densities are established. It is also shown how to compute these densities with high numerical accuracy.

Keywords: Order, residue class, natural density.

Mathematics Subject Classification (2000). 11N37, 11T99.

1 Introduction

Let \mathbb{F}_q^* be the multiplicative group of a finite field \mathbb{F}_q and let $x \in \mathbb{F}_q^*$. The order of x is the smallest positive integer k such that $x^k = 1$, the index is the largest number t such that $x^{(q-1)/t} = 1$. Note that $t = [\mathbb{F}_q^* : \langle x \rangle]$. Let $\pi(x)$ denote the number of primes $p \leq x$. Let $\delta(p; a, d)$ and $\rho(p; a, d)$ denote the density of elements in \mathbb{F}_p^* having order, respectively index, congruent to $a \pmod{d}$. It is not so difficult to show that both $\lim_{x \rightarrow \infty} N(a, d)(x)/\pi(x)$ and $\lim_{x \rightarrow \infty} N'(a, d)(x)/\pi(x)$ exist, where

$$N(a, d)(x) = \sum_{p \leq x} \delta(p; a, d) \text{ and } N'(a, d)(x) = \sum_{p \leq x} \rho(p; a, d)$$

These limits are denoted by $\delta(a, d)$, respectively $\rho(a, d)$. In this note these quantities are investigated. The following result is characteristic of the type of results that will be established. (If a and b are natural numbers, then by (a, b) , respectively $[a, b]$, the greatest common divisor, respectively the lowest common multiple of a and b are denoted. By $\gamma(a) = \prod_{p|a} p$ the square free kernel of a is denoted. Throughout the letter p will be used to indicate primes.)

Theorem 1

1) For every $B > 0$ one has

$$N(a, d)(x) = \delta(a, d)\text{Li}(x) + O_B\left(\frac{x}{\log^B x}\right),$$

where

$$\delta(a, d) = \sum_{\substack{r=1 \\ (1+ra, d)=1}}^{\infty} \sum_{\substack{m=1 \\ (m, d)|a}}^{\infty} \frac{\mu(m)}{mr\varphi(r[m, d])},$$

and the implied constant depends at most on B .

2) One has

$$\delta(a, d) = \frac{\varphi((a, d))}{(a, d)} \sum_{\substack{r=1 \\ (1+ra, d)=1}}^{\infty} \sum_{\substack{m=1 \\ (m, d)=1}}^{\infty} \frac{\mu(m)}{mr\varphi(mrd)}.$$

3) If $d|d_1$ and $\gamma(d_1) = \gamma(d)$, then $\delta(a, d_1) = \frac{d}{d_1}\delta(a, d)$.

4) One has

$$\delta(0, d) = \frac{1}{d} \prod_{p|d} \frac{1}{1 - \frac{1}{p^2}} \text{ and } \delta(d, 2d) = \begin{cases} \frac{1}{2}\delta(0, 2d) & \text{if } d \text{ is odd;} \\ \delta(0, 2d) & \text{if } d \text{ is even.} \end{cases}$$

5) For $s \geq 1$ one has

$$\delta(a, 2^s) = \begin{cases} 2^{2-s}/3 & \text{if } a \text{ is even;} \\ 2^{1-s}/3 & \text{if } a \text{ is odd.} \end{cases}$$

6) Let q be a prime and $q \nmid a$. Then

$$\delta(a, q) = \frac{q^2}{(q-1)(q^2-1)} - \frac{q}{q^2-q-1} \rho\left(-\frac{1}{a}, q\right).$$

7) Put $W_d(a) = \{0 \leq r < d : (1+ra, d) = 1\}$. One has

$$\delta(a, d) = \frac{\varphi((a, d))}{(a, d)\varphi(d) \prod_{p|d} (1 - \frac{1}{p(p-1)})} \sum_{\alpha \in W_d(a)} \rho(\alpha, d) \prod_{p|(\alpha, d)} \frac{p^2 - p - 1}{p^2 - 1}.$$

In the next subsection a characteristic zero version of $\delta(a, d)$ and $\rho(a, d)$ will be discussed. Indeed, these characteristic zero quantities (exhibiting far more complicated behaviour) motivated the author to study $\delta(a, d)$ and $\rho(a, d)$. The behaviour of these characteristic zero quantities turns out to have many resemblances with that of $\delta(a, d)$ and $\rho(a, d)$.

1.1 Connections with characteristic zero

Let $g \in \mathbb{Q} \setminus \{-1, 0, 1\}$ and p be a prime. By $\nu_p(g)$ the exponent of p in the canonical factorisation of g is denoted. If $\nu_p(g) = 0$, then g can be considered as an element of \mathbb{F}_p^* with order $\text{ord}_g(p)$ and index $r_g(p)$. Let $N_g(a, d)$ and $N'_g(a, d)$ denote the set of primes p with $\nu_p(g) = 0$ such that the order, respectively index of $g(\text{mod } p)$ is congruent to $a(\text{mod } d)$.

In case $g = 2$ the set $N'_g(a, d)$ was first considered by Pappalardi [15], who proved that it has a natural density $\rho_g(a, d)$ under the assumption of the Generalized Riemann Hypothesis (GRH). For the general case see [8].

The methods of [8] can be extended (see [10]) to show that under GRH the set $N_g(a, d)$ has a natural density $\delta_g(a, d)$, the evaluation of which seems to be far

less easy than that of $\rho_g(a, d)$. By $\bar{\delta}_g(d)$ the vector $(\delta_g(0, d), \dots, \delta_g(d-1, d))$ (if it exists) is denoted. Up to this century only $\bar{\delta}_g(2)$ had been evaluated. Recently Chinen and Murata [2, 12] computed $\bar{\delta}_g(4)$ (on GRH) under the assumption that g is a positive integer that is not a pure power. In [8], on GRH, $\bar{\delta}_g(3)$ and $\bar{\delta}_g(4)$ are evaluated for each $g \in \mathbb{Q} \setminus \{-1, 0, 1\}$. If $d|2(a, d)$, then $\delta_g(a, d)$ can be evaluated unconditionally, cf. [1, 14, 19, 20].

Let G be the set of rational numbers g that cannot be written as $-g_0^h$ or g_0^h with $h > 1$ an integer and g_0 a rational number. By $D(g)$ the discriminant of the number field $\mathbb{Q}(\sqrt{|g|})$ is denoted. The functions $N(a, d)(x)$ and $N'(a, d)(x)$ can be considered as naive heuristic approximations of $N_g(a, d)(x)$ and $N'_g(a, d)(x)$ (if S is any set of non-negative integers then $S(x)$ denotes the number of elements in S not exceeding x). For more on heuristics and primitive root theory, see [4, 5]. Theorem 2 and Theorem 3 say that as $D(g)$ becomes large, the naive heuristic for $\delta_g(a, d)$ and $\rho_g(a, d)$ become more and more accurate. Some related numerical material is provided in Table 3 and Table 4. The next result is proved in [10].

Theorem 2 (GRH). *If $D(g) \rightarrow \infty$ with $g \in G$, then $\delta_g(a, d)$ tends to $\delta(a, d)$.*

In this paper the following similar (but easier) result will be proved.

Theorem 3 (GRH). *If $D(g) \rightarrow \infty$ with $g \in G$, then $\rho_g(a, d)$ tends to $\rho(a, d)$.*

It turns out that both $\delta(a, d)$ and $\rho(a, d)$ are much more accessible quantities than $\delta_g(a, d)$, respectively $\rho_g(a, d)$. In the light of the latter two theorems it thus seems of some importance to compute $\delta(a, d)$ and $\rho(a, d)$, which is the purpose of this paper. The more complicated nature of $N_g(a, d)$ versus $N'_g(a, d)$ is mirrored in the fact that $\delta(a, d)$ is rather more difficult to compute than $\rho(a, d)$.

As a prelude to proving Theorem 3, its ‘index equals t ’ analog is proved in Section 2. Theorem 3 is then proved in Section 3 (this involves evaluating $\rho(a, d)$). In Section 4 Theorem 1 is considered. In Section 5 an Euler product A_χ involving a Dirichlet character χ is studied and it is shown how $\rho(a, d)$ and $\delta(a, d)$ can be expressed in terms of A_χ ’s. Since A_χ can be evaluated with high numerical accuracy (Sections 6 and 7) this then allows us to evaluate $\rho(a, d)$ and $\delta(a, d)$ with high numerical precision.

2 Index t

Let $\rho_p(t)$ denote the density of elements in \mathbb{F}_p^* having index t . Note that $\rho_p(t) = \varphi((p-1)/t)/(p-1)$ if $p \equiv 1 \pmod{t}$ and $\rho_p(t) = 0$ otherwise. By the method of proof of Theorem 4 (cf. [5, p. 161]) it is easy to show that, for every $B > 0$,

$$\sum_{p \leq x} \rho_p(t) = \sum_{\substack{p \leq x \\ p \equiv 1 \pmod{t}}} \frac{\varphi\left(\frac{p-1}{t}\right)}{p-1} = \text{Li}(x) \sum_{n=1}^{\infty} \frac{\mu(n)}{tn\varphi(tn)} + O\left(\frac{x}{\log^B x}\right), \quad (1)$$

where $\text{Li}(x)$ denotes the logarithmic integral. Now

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{tn\varphi(tn)} = \frac{1}{t\varphi(t)} \sum_{n=1}^{\infty} \frac{\mu(n)\varphi(t)}{n\varphi(tn)}.$$

The latter sum has as argument a multiplicative function in n . On applying Euler's identity, it is then inferred that

$$\sum_{n=1}^{\infty} \frac{\mu(t)}{nt\varphi(nt)} = Ar(t), \quad (2)$$

where

$$A = \prod_p \left(1 - \frac{1}{p(p-1)}\right) = 0.37395581361920228805472805434641641511 \dots$$

denotes Artin's constant and

$$r(t) = \frac{1}{t^2} \prod_{p|t} \frac{p^2 - 1}{p^2 - p - 1} = \frac{1}{t\varphi(t)} \prod_{p|t} \frac{1 - \frac{1}{p^2}}{1 - \frac{1}{p(p-1)}}.$$

Note that

$$\frac{1}{t\varphi(t)} \leq r(t) \leq \frac{6}{A\pi^2 t\varphi(t)}. \quad (3)$$

Combination of (1) and (2) yields the following result.

Proposition 1 *For every $B > 0$, one has*

$$\sum_{p \leq x} \rho_p(t) = Ar(t) \text{Li}(x) + O\left(\frac{x}{\log^B x}\right),$$

Thus, the density of elements in \mathbb{F}_p^ with index t equals $Ar(t)$ on average.*

Let $N_g(t)$ denote the set of primes p with $r_g(p) = t$. If $g \in \mathbb{Q} \setminus \{-1, 0, 1\}$, then it can be shown [3, 18], under GRH, that $N_g(t)$ has density

$$\delta(N_g(t)) = \sum_{n=1}^{\infty} \frac{\mu(n)}{[\mathbb{Q}(\zeta_{nt}, g^{1/nt}) : \mathbb{Q}]}. \quad (4)$$

For an explicit evaluation of this density see [11, 18]. We can now prove the following result.

Proposition 2 (GRH). *Let $g \in G$. If $g > 0$, set $m = [2, D(g)]$. If $g < 0$, set $m = D(g)/2$ if $D(g) \equiv 4 \pmod{8}$ and $m = [4, D(g)]$ otherwise. Put $m_1 = m/(t, m)$. If $g \in G$, then*

$$|\delta(N_g(t)) - Ar(t)| \leq \frac{2.21}{tm_1\varphi(tm_1)}.$$

Corollary 1 (GRH). *The density of $N_g(t)$ exists and if $D(g) \rightarrow \infty$ with $g \in G$, tends to the average density, $Ar(t)$, of elements in \mathbb{F}_p^* having index t .*

The latter corollary is the ‘index equals t ’ analog of Theorem 3.

Proof of Proposition 2. By (4) and the evaluation of the degree $[\mathbb{Q}(\zeta_k, g^{1/k}) : \mathbb{Q}]$ as given in [18], it is deduced that

$$\delta(N_g(t)) = Ar(t) + \sum_{\substack{k=1 \\ m|kt}}^{\infty} \frac{\mu(k)}{kt\varphi(kt)} = Ar(t) + \sum_{\substack{k=1 \\ m_1|k}}^{\infty} \frac{\mu(k)}{kt\varphi(kt)}. \quad (5)$$

On noting that $\varphi(zw) \geq \varphi(w)\varphi(z)$, with w and z arbitrary integers and using that $\sum_k 1/(k\varphi(k)) < 2.21$, the result then follows. \square

Remark 1. The sum $\sum_k 1/(k\varphi(k))$ can be written as an Euler product of the form $\prod_p F_1(p)/F_2(p)$, with $F_j(X) \in \mathbb{Z}[X]$ for $j = 1, 2$ and monic. Using Theorem 2 of [6] such Euler products can be expressed in terms of values at integer points of the (partial) Riemann zeta-function. This enables one to evaluate these constants with hundreds of decimals of precision, see [13]. A similar idea forms the basis of Theorem 6 and Theorem 7.

3 Computation of $\rho(a, d)$

Equation (4) suggests that, under GRH, one should have

Proposition 3 (GRH). *If $g \in \mathbb{Q} \setminus \{-1, 0, 1\}$, then*

$$\rho_g(a, d) = \sum_{t \equiv a \pmod{d}} \delta(N_g(t)) = \sum_{t \equiv a \pmod{d}} \sum_{n=1}^{\infty} \frac{\mu(n)}{[\mathbb{Q}(\zeta_{nt}, g^{1/nt}) : \mathbb{Q}]}. \quad (6)$$

(In this proposition and in the sequel sums over t are assumed to run over positive integers only.) Indeed by [15], cf. [8], Proposition 3 is known to be true. Similarly one would expect that $\rho(a, d)$ satisfies (7) as can indeed be proved.

Theorem 4 *For every $B > 0$ one has*

$$N'(a, d)(x) = \rho(a, d)\text{Li}(x) + O\left(\frac{x}{\log^B x}\right), \quad (6)$$

where

$$\rho(a, d) = A \sum_{t \equiv a \pmod{d}} r(t) = \sum_{t \equiv a \pmod{d}} \frac{A}{t^2} \prod_{p|t} \frac{p^2 - 1}{p^2 - p - 1} \quad (7)$$

and the implied constant depends at most on B .

Proof. One has

$$N'(a, d)(x) = \sum_{p \leq x} \sum_{\substack{t|p-1 \\ t \equiv a \pmod{d}}} \frac{\varphi\left(\frac{p-1}{t}\right)}{p-1}.$$

On using that $\varphi(n)/n = \sum_{m|n} \mu(m)/m$, one obtains

$$N'(a, d)(x) = \sum_{p \leq x} \sum_{\substack{t|p-1 \\ t \equiv a \pmod{d}}} \sum_{m|\frac{p-1}{t}} \frac{\mu(m)}{mt}.$$

Writing $v = mt$ and bringing the summation over p to the inside, one obtains

$$N'(a, d)(x) = \sum_{v \leq x-1} \frac{1}{v} \sum_{\substack{t|v \\ t \equiv a \pmod{d}}} \mu\left(\frac{v}{t}\right) \sum_{\substack{p \leq x \\ p \equiv 1 \pmod{v}}} 1.$$

The summation range is split up into the range $v \leq \log^{B+1} x$ and $\log^{B+1} x < v \leq x$. In the former range the Siegel-Walfisz theorem (see e.g. [17, Satz 4.8.3]) is invoked and for the latter range the trivial estimate $\sum_{p \leq x, p \equiv 1 \pmod{v}} 1 < x/v$ is employed. Let $d(v)$ denote the number of divisors of v . Together with the trivial estimate $|\sum_{t|v, t \equiv a \pmod{d}} \mu(v/t)| \leq d(v) \ll v^\epsilon$, which holds for every $\epsilon > 0$, one concludes (cf. [5, p. 161]) that (6) holds with

$$\rho(a, d) = \sum_{v=1}^{\infty} \frac{\sum_{t|v, t \equiv a \pmod{d}} \mu\left(\frac{v}{t}\right)}{v\varphi(v)}. \quad (8)$$

Interchanging the order of summation and using (2) one infers that

$$\rho(a, d) = \sum_{t \equiv a \pmod{d}} \sum_{v_1=1}^{\infty} \frac{\mu(v_1)}{tv_1\varphi(tv_1)} = A \sum_{t \equiv a \pmod{d}} r(t).$$

This concludes the proof. \square

Let $a > 0$. As d becomes large, the first term in the second summation in (7), $Ar(a)$, tends to be dominant by Corollary 4. In particular, $\lim_{d \rightarrow \infty} \rho(a, d) = Ar(a)$.

Proposition 4 *One has*

$$\rho(0, d) = \frac{1}{d\varphi(d)} \text{ and } \rho(d, 2d) = \begin{cases} \rho(0, 2d) & \text{if } d \text{ is odd;} \\ 3\rho(0, 2d) & \text{if } d \text{ is even.} \end{cases}$$

Proof. Note that

$$\rho(0, d) = r(d) \sum_{m=1}^{\infty} \frac{r(dm)}{r(d)}.$$

Since $r(dm)/r(d)$ is a multiplicative function in m , the identity for $\rho(0, d)$ then follows on applying Euler's identity and noting that $\varphi(d)/d = \prod_{p|d} (1 - 1/p)$. The identity for $\rho(0, d)$ together with the observation that $\rho(0, 2d) + \rho(d, 2d) = \rho(0, d)$, then yields the truth of the remainder of the assertion. \square

By $\omega(m)$ the number of distinct prime divisors of m is denoted.

Proposition 5 (GRH). *Let $g \in G$ and m be as in Proposition 2, then*

$$\left| \rho_g(a, d) - \rho(a, d) \right| \leq \frac{2^{\omega(m)+2}}{m\varphi(m)}.$$

Proof. By Theorem 3, Proposition 3 and (5) one infers on putting $kt = v$ that, under GRH,

$$\left| \rho_g(a, d) - \rho(a, d) \right| \leq \sum_{t \equiv a \pmod{d}} \sum_{\substack{k=1 \\ m|kt}}^{\infty} \frac{|\mu(k)|}{kt\varphi(kt)} = \sum_{m|v} \frac{\sum_{t|v, t \equiv a \pmod{d}} |\mu(\frac{v}{t})|}{v\varphi(v)}.$$

On noting that

$$\sum_{m|v} \frac{\sum_{t|v, t \equiv a \pmod{d}} |\mu(\frac{v}{t})|}{v\varphi(v)} \leq \sum_{v=1}^{\infty} \frac{2^{\omega(mv)}}{mv\varphi(mv)} \leq \frac{2^{\omega(m)}}{m\varphi(m)} \sum_{v=1}^{\infty} \frac{2^{\omega(v)}}{v\varphi(v)} \leq \frac{2^{\omega(m)+2}}{m\varphi(m)},$$

the result follows. \square

Since $2^{\omega(m)}/(m\varphi(m))$ tends to zero with increasing m and m tends to infinity as $D(g)$ tends to infinity, Theorem 3 is a consequence of Proposition 5.

The following result is concerned with \mathbb{Q} -linear relations between the $\rho(a, d)$'s with d fixed.

Lemma 1 *Let $\alpha_1, \dots, \alpha_{\varphi(d)}$ be representatives of the reduced residue classes mod d . Then, for every integer a ,*

$$\rho(a, d) \in \mathbb{Q}[\rho(\alpha_1, d), \dots, \rho(\alpha_{\varphi(d)-1}, d)].$$

Corollary 2 *If $d|d_1$ and $\beta_1, \dots, \beta_{\varphi(d_1)}$ are representatives of the reduced residue classes mod d_1 , then*

$$\mathbb{Q}[\rho(\alpha_1, d), \dots, \rho(\alpha_{\varphi(d)-1}, d)] \subseteq \mathbb{Q}[\rho(\beta_1, d_1), \dots, \rho(\beta_{\varphi(d_1)-1}, d_1)].$$

Proof of Lemma 1. It is easy to show that

$$\sum_{j=1}^{\varphi(d)} \rho(\alpha_j, d) = A \sum_{(t,d)=1} r(t) = \prod_{p|d} \left(1 - \frac{1}{p(p-1)} \right) \in \mathbb{Q}. \quad (9)$$

It is thus enough to show that $\rho(a, d) \in V_d := \mathbb{Q}[\rho(\alpha_1, d), \dots, \rho(\alpha_{\varphi(d)}, d)]$.

Let $\alpha = a/(a, d)$ and $\delta = d/(a, d)$. Note that $(\alpha, \delta) = 1$. Let δ_1 be the largest divisor of (a, d) with $(\delta, \delta_1) = 1$ and write $(a, d) = \delta_1 \delta_2$. If α, δ, δ_1 and δ_2 are being used for integers other than a and d , then this will be made explicit in the notation. Thus the meaning of $\delta_1(a_j, d)$, which appears later in the proof, should be obvious. Note that

$$\rho(a, d) = A \sum_{t \equiv \alpha \pmod{\delta}} r(\delta_1 \delta_2 t) = Ar(\delta_2) \sum_{t \equiv \alpha \pmod{\delta}} r(\delta_1 t).$$

The proof proceeds with induction with respect to the number of distinct prime divisors of δ_1 . If $\omega(\delta_1) = 0$, then $\delta_1 = 1$ and one has to show that $\rho(\alpha, \delta) \in V_d$, where $\alpha \pmod{\delta}$ is a reduced residue class mod δ . Since $\alpha \pmod{\delta}$ lifts to d/δ reduced residue classes mod d , this is clear. If $\omega(\delta_1) = 1$, then $\delta_1 = q^e$ with q a prime and $e \geq 1$. Then one has

$$\rho(a, d) = Ar(\delta_2) \sum_{t \equiv \alpha \pmod{\delta}} r(q^e t)$$

$$\begin{aligned}
&= r(q^e)Ar(\delta_2) \sum_{\substack{t \equiv \alpha \pmod{\delta} \\ t \not\equiv 0 \pmod{q}}} r(t) + \frac{A}{q^2}r(\delta_2) \sum_{t \equiv \frac{\alpha}{q} \pmod{\delta}} r(q^e t) \\
&= c_1 + \frac{A}{q^2}r(\delta_2) \sum_{t \equiv \frac{\alpha}{q} \pmod{\delta}} r(q^e t) \\
&\quad \vdots \\
&= c_n + \frac{A}{q^{2n}}r(\delta_2) \sum_{t \equiv \frac{\alpha}{q^n} \pmod{\delta}} r(q^e t),
\end{aligned}$$

where $c_i \in V_d$. On choosing $n \geq 1$ to be such that $q^n \equiv 1 \pmod{\delta}$, one infers that $\rho(a, d) \in V_d$. Suppose the result has been proved for all a and d with $\omega(\delta_1) \leq m$ for some $m \geq 1$. Then consider next a and d with $\omega(\delta_1) = m + 1$. One has

$$\rho(a, d) = \sum_{j=1}^{\delta_1} Ar(\delta_2) \sum_{\substack{t \equiv \alpha \pmod{\delta} \\ t \equiv j \pmod{\delta_1}}} r(\delta_1 t). \quad (10)$$

Note that

$$A \sum_{\substack{t \equiv \alpha \pmod{\delta} \\ t \equiv j \pmod{\delta_1}}} r(\delta_1 t) = A \sum_{\substack{t \equiv \alpha \pmod{\delta} \\ t \equiv \frac{j}{(j, \delta_1)} \pmod{\frac{\delta_1}{(j, \delta_1)}}}} r(\delta_2) r(\delta_1(j, \delta_1)t).$$

The latter sum equals a rational multiple times

$$A \sum_{\substack{t \equiv \alpha \pmod{\delta} \\ t \equiv \frac{j}{(j, \delta_1)} \pmod{\frac{\delta_1}{(j, \delta_1)}}}} r(\delta_2) r((j, \delta_1)t) = \rho(a_j, d),$$

for some integer a_j . Note that $\delta_1(a_j, d) = (j, \delta_1)$. Thus by the induction hypothesis all terms in (10) with $\gamma(\delta_1) \nmid j$ are in V_d (where $\gamma(\delta_1)$ denotes the squarefree kernel of δ_1). One infers that

$$\begin{aligned}
\rho(a, d) &= d_1 + Ar(\delta_2) \sum_{\substack{t \equiv \alpha \pmod{\delta} \\ t \equiv 0 \pmod{\gamma(\delta_1)}}} r(\delta_1 t) \\
&= d_1 + \frac{A}{\gamma(\delta_1)^2}r(\delta_2) \sum_{t \equiv \frac{\alpha}{\gamma(\delta_1)} \pmod{\delta}} r(\delta_1 t) \\
&\quad \vdots \\
&= d_n + \frac{A}{\gamma(\delta_1)^{2n}}r(\delta_2) \sum_{t \equiv \frac{\alpha}{\gamma(\delta_1)^n} \pmod{\delta}} r(\delta_1 t),
\end{aligned}$$

where $\delta_i \in V_d$. On choosing $n \geq 1$ to be such that $\gamma(\delta_1)^n \equiv 1 \pmod{\delta}$ one infers that $\rho(a, d) \in V_d$. \square

Example 1. The result says that $\rho(a, 6) \in \mathbb{Q}[\rho(1, 6)]$. Indeed, $\rho(0, 6) = \rho(3, 6) = 1/12$. Furthermore $\rho(2, 6) = 1/12 + 3\rho(1, 6)/5$, $\rho(4, 6) = 1/3 - 3\rho(1, 6)/5$ and $\rho(5, 6) = 5/12 - \rho(1, 6)$.

4 Proof of Theorem 1

Proof of Theorem 1. 1) Note that

$$N(a, d)(x) = \sum_{p \leq x} \delta(p; a, d) = \sum_{p \leq x} \sum_{\substack{r|p-1 \\ \frac{p-1}{r} \equiv a \pmod{d}}} \frac{\varphi\left(\frac{p-1}{r}\right)}{p-1}.$$

Proceeding as in the proof of Theorem 4, one infers that

$$N(a, d)(x) = \sum_{r \leq x-1} \sum_{m \leq \frac{x-1}{r}} \frac{\mu(m)}{mr} \sum_{\substack{p \leq x, p \equiv 1 \pmod{rm} \\ p \equiv 1+ra \pmod{rd}}} 1. \quad (11)$$

Now for the inner sum to be non-zero the two congruences must be compatible. By the Chinese remainder theorem this is the case if and only if $1 \equiv 1 + ra \pmod{r(m, d)}$, that is if and only if $a \equiv 0 \pmod{(m, d)}$. If the two congruences are compatible, then they form a reduced residue class if and only if $(1 + ra, d) = 1$. If the residue class is not reduced it contains at most one prime and the contribution of these primes to $N(a, d)(x)$ is bounded in absolute value by $\sum_{v \leq x} d(v)/v = O(\log^2 x)$.

The summation range $mr \leq x-1$ in (11) is split up into the range $r[m, d] \leq \log^C x$ and the range $r[m, d] > \log^C x$, where C is to be chosen later. All error terms arising in this way are easily seen to be of the claimed order of growth, except the error term

$$E(x) = \text{Li}(x) \sum_r \sum_{\substack{(m, d)|a \\ r[m, d] > \log^C x}} \frac{|\mu(m)|}{mr\varphi(r[m, d])},$$

which arises on completing the sum

$$\text{Li}(x) \sum_{\substack{r \leq x-1 \\ (1+ra, d)=1}} \sum_{\substack{m \leq \frac{x-1}{r}, (m, d)|a \\ r[m, d] \leq \log^C x}} \frac{|\mu(m)|}{mr\varphi(r[m, d])},$$

to $\delta(a, d)$. On noting that $r[m, d] > \log^C x$ implies $rm d > \log^C x$ and using that $\varphi(zw) \geq \varphi(z)\varphi(w)$, one obtains, cf. the proof of part 2,

$$E(x) = O \left(\frac{\text{Li}(x)}{\varphi(d)} \sum_{m_1 | (a, d)} \frac{|\mu(m_1)|}{m_1} \sum_r \sum_{\substack{m_2 \\ rm_2 > \log^C x / (dm_1)}} \frac{|\mu(m_2)|}{rm_2\varphi(rm_2)} \right).$$

From this $E(x)$ is easily seen to be $O_B(x/\log^B x)$, when C is chosen to be sufficiently large.

2) By part 1 it is enough to show that

$$I_1 := \sum_{\substack{m=1 \\ (m, d)|a}}^{\infty} \frac{\mu(m)}{mr\varphi(r[m, d])} = \frac{\varphi((a, d))}{(a, d)} \sum_{\substack{m=1 \\ (m, d)=1}}^{\infty} \frac{\mu(m)}{mr\varphi(mrd)}.$$

Note that

$$I_1 = \sum_{m_1|(a,d)} \sum_{\substack{m=1 \\ (m,d)=m_1}}^{\infty} \frac{\mu(m)}{mr\varphi\left(\frac{mrd}{m_1}\right)}.$$

On writing $m = m_1m_2$ one obtains

$$\begin{aligned} I_1 &= \sum_{m_1|(a,d)} \sum_{\substack{m_2=1 \\ (m_2,d/m_1)=1}}^{\infty} \frac{\mu(m_1m_2)}{m_1m_2r\varphi(m_2rd)} \\ &= \sum_{m_1|(a,d)} \frac{\mu(m_1)}{m_1} \sum_{\substack{m_2=1, (m_2,d/m_1)=1 \\ (m_2,m_1)=1}}^{\infty} \frac{\mu(m_2)}{rm_2\varphi(m_2rd)} \\ &= \sum_{m_1|(a,d)} \frac{\mu(m_1)}{m_1} \sum_{\substack{m_2=1 \\ (m_2,d)=1}}^{\infty} \frac{\mu(m_2)}{rm_2\varphi(m_2rd)} \\ &= \frac{\varphi((a,d))}{(a,d)} \sum_{\substack{m=1 \\ (m,d)=1}}^{\infty} \frac{\mu(m)}{rm\varphi(mrd)}. \end{aligned}$$

3) The condition on d and d_1 ensures that $(1+ra, d_1) = 1$ iff $(1+ra, d) = 1$ and $(m, d_1) = 1$ iff $(m, d) = 1$. Furthermore one has $\varphi((a, d_1))/(a, d_1) = \varphi((a, d))/(a, d)$. By part 2 one then finds

$$\delta(a, d_1) = \frac{\varphi((a, d))}{(a, d)} \sum_{\substack{r=1 \\ (1+ra,d)=1}}^{\infty} \sum_{\substack{m=1 \\ (m,d)=1}}^{\infty} \frac{\mu(m)}{mr\varphi(mrd_1)}.$$

On noting that $\varphi(mrd_1) = \varphi(mrd)d_1/d$, the proof of part 2 is completed.

4) By part 2 one has, on writing $mr = v$,

$$\delta(0, d) = \frac{\varphi(d)}{d} \sum_{v=1}^{\infty} \frac{\sum_{m|v, (m,d)=1} \mu(v)}{v\varphi(vd)}.$$

The inner sum equals one if $\gamma(v)|d$ and zero otherwise. Thus

$$\delta(0, d) = \frac{\varphi(d)}{d} \sum_{\gamma(v)|d} \frac{1}{v\varphi(vd)} = \frac{1}{d} \sum_{\gamma(v)|d} \frac{1}{v^2} = \frac{1}{d} \prod_{p|d} \frac{1}{1 - \frac{1}{p^2}}.$$

The formula for $\delta(d, 2d)$ easily follows from that of $\delta(0, d)$ and the observation that $\delta(0, 2d) + \delta(d, 2d) = \delta(0, d)$.

5) An easy consequence of part 3 and part 4.

6) Using part 2 and (2), one infers that

$$\begin{aligned} \delta(a, q) &= \delta(0, q) - \sum_{t \equiv -\frac{1}{a} \pmod{q}} \sum_{\substack{m=1 \\ (m,q)=1}}^{\infty} \frac{\mu(m)}{mt\phi(mtq)} \\ &= \delta(0, q) - \frac{1}{q-1} \sum_{t \equiv -\frac{1}{a} \pmod{q}} \sum_{\substack{m=1 \\ (m,q)=1}}^{\infty} \frac{\mu(m)}{mt\phi(mt)} \end{aligned}$$

$$= \delta(0, q) - \frac{1}{q-1} \sum_{t \equiv -\frac{1}{a} \pmod{q}} \frac{A}{1 - \frac{1}{q(q-1)}} r(t).$$

The proof is then completed on invoking part 3 and (7).

7) By part 2 one can write

$$\delta(a, d) = \frac{\varphi((a, d))}{(a, d)} \sum_{\substack{r=1 \\ (1+ra, d)=1}}^{\infty} \frac{1}{r\varphi(rd)} \sum_{\substack{m=1 \\ (m, d)=1}}^{\infty} \frac{\mu(m)\varphi(rd)}{m\varphi(mrd)}. \quad (12)$$

Denote the inner sum in (12) by I_2 . One has

$$\begin{aligned} I_2 &= \prod_{p \nmid rd} \left(1 - \frac{1}{p(p-1)}\right) \prod_{\substack{p|r \\ p \nmid d}} \left(1 - \frac{1}{p^2}\right) \\ &= \frac{A}{\prod_{p|d} \left(1 - \frac{1}{p(p-1)}\right)} \prod_{p|r} \frac{1 - \frac{1}{p^2}}{1 - \frac{1}{p(p-1)}} \prod_{p|(r, d)} \frac{1 - \frac{1}{p(p-1)}}{1 - \frac{1}{p^2}}. \end{aligned}$$

One thus obtains that

$$\delta(a, d) = \frac{\varphi((a, d))}{(a, d) \prod_{p|d} \left(1 - \frac{1}{p(p-1)}\right)} \sum_{\alpha \in W_d(a)} \sum_{\substack{w=1 \\ w \equiv \alpha \pmod{d}}}^{\infty} \frac{A\varphi(w)}{\varphi(wd)} r(w) \prod_{p|(\alpha, d)} \frac{1 - \frac{1}{p(p-1)}}{1 - \frac{1}{p^2}}.$$

Note that

$$\frac{\varphi(wd)}{\varphi(w)} = d \prod_{\substack{p|d \\ p \nmid w}} \left(1 - \frac{1}{p}\right) = \frac{d \prod_{p|d} \left(1 - \frac{1}{p}\right)}{\prod_{p|(w, d)} \left(1 - \frac{1}{p}\right)} = \frac{\varphi(d)}{\prod_{p|(\alpha, d)} \left(1 - \frac{1}{p}\right)}.$$

This equation together with the latter one derived for $\delta(a, d)$ and (2), then yields the result. \square

5 The densities and A_χ

Given a Dirichlet character $\chi \pmod{d}$, let $h_\chi = \chi \star \mu$, that is h_χ denotes the Dirichlet convolution of χ and the Möbius function μ . Note the following trivial result.

Lemma 2 *The function h_χ is multiplicative and satisfies $h_\chi(1) = 1$ and furthermore $h_\chi(p^r) = \chi(p)^{r-1}[\chi(p) - 1]$, where the convention that $0^0 = 1$ is adopted.*

In particular if χ is the trivial character \pmod{d} , then

$$h_\chi(v) = \begin{cases} \mu(v) & \text{if } v|d; \\ 0 & \text{otherwise.} \end{cases} \quad (13)$$

By using one of the orthogonality relations for Dirichlet characters, the following result is easily inferred.

Lemma 3 *Let $a(\bmod d)$ be a reduced residue class mod d . One has*

$$\sum_{\substack{t \equiv a(\bmod d) \\ t|v}} \mu\left(\frac{v}{t}\right) = \frac{1}{\varphi(d)} \sum_{\chi(\bmod d)} \overline{\chi(a)} h_{\chi}(v),$$

where χ runs over the Dirichlet characters modulo d .

The reader is referred to Section 2.4 of [8] for some further properties of h_{χ} .

In what follows sums of the form

$$\sum_{(v,d)=1} \frac{h_{\chi}(v)}{v\varphi(v)}$$

will feature. It is easy to see that this sum is absolutely convergent. Since its argument is multiplicative, one then obtains that the latter sum equals

$$A_{\chi} = \prod_{p:\chi(p) \neq 0} \left(1 + \frac{[\chi(p) - 1]p}{[p^2 - \chi(p)](p - 1)}\right).$$

Note that if χ is the principal character, then $A_{\chi} = 1$. For a fixed prime p and $\alpha \in \mathbb{R}$, $0 \leq \alpha < 1$ let

$$f_p(\alpha) = \left(1 + \frac{(e^{2\pi i \alpha} - 1)p}{(p^2 - e^{2\pi i \alpha})(p - 1)}\right).$$

A tedious analysis shows that $|f_p(\alpha)|$ as a function of α is decreasing for $0 < \alpha \leq 1/2$ and increasing for $1/2 \leq \alpha \leq 1$. Thus

$$1 - \frac{2p}{(p^2 + 1)(p - 1)} \leq |f_p(\alpha)| \leq 1,$$

where the lower bound holds true iff $\alpha = 1/2$ and the upper bound iff $\alpha = 0$. It follows that $|A_{\chi}| \leq 1$ with $A_{\chi} = 1$ iff χ is the principal character mod d .

If χ' is the primitive Dirichlet character associated with χ , then the Euler products of A_{χ} and $A_{\chi'}$ differ in at most finitely many primes and hence can be simply related.

It will be shown in Theorem 5 that $\rho(a, d)$ and $\delta(a, d)$ can be expressed in terms of A'_{χ} s, where χ ranges over the Dirichlet characters mod d . The proof makes use of the following proposition.

Proposition 6 *Let $a \geq 1$. One has*

$$\begin{aligned} \rho(a, d) &= \frac{1}{\varphi(\delta)w\varphi(w)} \prod_{p|\delta, p \nmid w} \left(1 - \frac{1}{p(p-1)}\right) \prod_{p|\delta, p|w} \left(1 - \frac{1}{p^2}\right) \\ &\quad \sum_{\chi(\bmod \delta)} \overline{\chi(a)} A_{\chi} \prod_{\substack{p|w \\ p \nmid \delta}} \frac{1 + \frac{\chi(p)-1}{p^2-\chi(p)}}{1 + \frac{(\chi(p)-1)p}{(p^2-\chi(p))(p-1)}}, \end{aligned}$$

where $w = (a, d)$, $\alpha = a/w$ and $\delta = d/w$. In particular, if $(a, d) = 1$ then

$$\rho(a, d) = \frac{1}{\varphi(d)} \prod_{p|d} \left(1 - \frac{1}{p(p-1)}\right) \sum_{\chi(\bmod d)} \overline{\chi(a)} A_{\chi}.$$

Corollary 3 *If $\gamma((a, d))(a, d)|d$, then*

$$\rho(a, d) = \frac{\rho(\alpha, \delta)}{w\varphi(w)} \prod_{p|(\delta, w)} \frac{1 - \frac{1}{p^2}}{1 - \frac{1}{p(p-1)}}.$$

Proof of Proposition 6. From (8) and Lemma 3 one easily infers that

$$\rho(a, d) = \frac{1}{\varphi(\delta)w\varphi(w)} \sum_{\chi \pmod{\delta}} \overline{\chi(\alpha)} \sum_{v=1}^{\infty} \frac{h_{\chi}(v)\varphi(w)}{v\varphi(vw)}.$$

On noting that the argument of the inner sum is multiplicative in v , the result follows on applying (13) and Euler's product identity. \square

Example 3. One has $\rho(0, d) = 1/(d\varphi(d))$ (in agreement with Proposition 4). Let χ_3 and χ_4 denote the non-trivial character mod 3, respectively mod 4. One finds $\rho(\pm 1, 3) = 5(1 \pm A_{\chi_3})/12$ and $\rho(\pm 2, 8) = 5(1 \pm A_{\chi_4})/12$. Let χ be the character mod 5 uniquely determined by $\chi(2) = i$. One has

$$\rho(3, 5) = \frac{19}{80} (1 + 2\operatorname{Re}(iA_{\chi}) - A_{\chi^2}).$$

Using Table 2, these densities can then be numerically approximated.

Example 4. One has

$$\begin{cases} \rho(2, 8) = 3\rho(1, 4)/4 \\ \rho(6, 8) = 3\rho(3, 4)/4 \end{cases} \text{ and } \begin{cases} \rho(3, 9) = 8\rho(1, 3)/45 \\ \rho(6, 9) = 8\rho(2, 3)/45 \end{cases} \text{ and } \begin{cases} \rho(2, 12) = 3\rho(1, 6)/4 \\ \rho(10, 12) = 3\rho(5, 6)/4. \end{cases}$$

One can now infer how the densities can be expressed in terms of A_{χ} 's.

Theorem 5 *Let α and δ be as in Proposition 6. Then*

$$\rho(a, d) \in \mathbb{Q}(\zeta_{\operatorname{ord}_{\alpha}(\delta)})[A_{\chi_1}, \dots, A_{\chi_{\varphi(\delta)}}],$$

where $\chi_1, \dots, \chi_{\varphi(\delta)}$ are the characters mod δ .

Let λ denote Carmichael's function, that is $\lambda(d)$ equals the exponent of the group $(\mathbb{Z}/d\mathbb{Z})^*$, then

$$\delta(a, d) \in \mathbb{Q}(\zeta_{\lambda(d)})[A_{\chi_1}, \dots, A_{\chi_{\varphi(d)}}].$$

Proof. The first part is a straightforward consequence of Proposition 6. The second part follows on applying part 7 of Theorem 1 and Proposition 6 together with the observation that if $\delta|d$, then any character $\chi' \pmod{\delta}$ can be lifted to a character $\chi \pmod{d}$, such that $A_{\chi'} = cA_{\chi}$, where $c \in \mathbb{Q}(\zeta_{\lambda(d)})$. \square

The next result follows on combining Proposition 6 with part 7 of Theorem 1.

Proposition 7 *Suppose that q is a prime and $q \nmid a$. Then*

$$\delta(a, q) = \frac{q^2 - q - 1}{(q-1)^2(q+1)} - \frac{1}{(q-1)^2} \sum_{\chi \neq \chi_0} \chi(-a)A_{\chi}.$$

The Euler product A_χ can also be expressed in terms of $\rho(a, d)$'s. This yields

$$A_\chi = \frac{\sum_{a=1}^d \chi(a) \rho(a, d)}{\prod_{p|d} \left(1 - \frac{1}{p(p-1)}\right)}.$$

Thus, by (9) and $\rho(a, d) \geq 0$, one finds

$$|A_\chi| \leq \frac{\sum_{a=1, (a,d)=1}^d \rho(a, d)}{\prod_{p|d} \left(1 - \frac{1}{p(p-1)}\right)} = 1,$$

with equality iff χ is the principal character mod d .

6 The numerical evaluation of $\delta(a, d)$ and $\rho(a, d)$

Consider the numerical evaluation of the constant A_χ . To this end it turns out to be more convenient to consider

$$B_\chi := \prod_p \left(1 + \frac{[\chi(p) - 1]p}{[p^2 - \chi(p)](p-1)}\right) = A_\chi \prod_{p|d} \left(1 - \frac{1}{p(p-1)}\right).$$

Recall that $L(s, \chi)$, the Dirichlet series for the character χ , is defined, for $\text{Re}(s) > 1$ by $L(s, \chi) = \sum_{n=1}^{\infty} \chi(n)/n^s$.

Theorem 6 *Let $p_1(= 2), p_2, \dots$ be the sequence of consecutive primes. Let χ be any Dirichlet character and $n \geq 31$ (hence $p_n \geq 127$). Then*

$$B_\chi = R_1 A L(2, \chi) L(3, \chi) L(4, \chi) \prod_{k=1}^n \left(1 + \frac{\chi(p_k)}{p_k(p_k^2 - p_k - 1)}\right) \left(1 - \frac{\chi(p_k)}{p_k^3}\right) \left(1 - \frac{\chi(p_k)}{p_k^4}\right),$$

with

$$\frac{1}{1 + p_{n+1}^{-3.85}} \leq |R_1| \leq 1 + \frac{1}{p_{n+1}^{3.85}}.$$

Proof. The first step is to note that

$$B_\chi = A L(2, \chi) L(3, \chi) L(4, \chi) \prod_{k=1}^{\infty} \left(1 + \frac{\chi(p_k)}{p_k(p_k^2 - p_k - 1)}\right) \left(1 - \frac{\chi(p_k)}{p_k^3}\right) \left(1 - \frac{\chi(p_k)}{p_k^4}\right),$$

An upper bound for the k th term in the latter product is given by

$$1 + t^5 \frac{(2 + 2t + t^3 + t^5)}{1 - t - t^2},$$

where $t = 1/p_k$. For $t \geq 127$ some analysis shows that the latter expression is bounded above by $1 + t^{4.85}$. Using this one obtains

$$|R_1| \leq \prod_{p > p_n} \left(1 + \frac{1}{p^{4.85}}\right) < 1 + \sum_{m > p_n} \frac{1}{m^{4.85}} \leq 1 + \frac{1}{p_{n+1}^{4.85}} + \int_{p_{n+1}}^{\infty} \frac{dt}{t^{4.85}} \leq 1 + \frac{1}{p_{n+1}^{3.85}}.$$

A similar argument allows one to obtain the lower bound. \square

Since the Artin constant (see e.g. [13]) and $L(2, \chi)$, $L(3, \chi)$ and $L(4, \chi)$ can be each evaluated with high numerical accuracy, Theorem 6 allows one to compute A_χ with high numerical accuracy. Using Proposition 6 and part 7 of Theorem 1, $\rho(a, d)$, respectively $\delta(a, d)$, can then be evaluated with high numerical precision.

A more straightforward, but numerically much less powerful, approach in computing $\rho(a, d)$ and $\delta(a, d)$, is to invoke part 7 of Theorem 1 and compute $\rho(a, d)$ using the identity $\rho(a, d) = A \sum_{t \equiv a \pmod{d}} r(t)$. One has the following estimates.

Proposition 8 *Let $x \geq 6$. One has*

$$0 < \rho(a, d) - A \sum_{\substack{t \equiv a \pmod{d} \\ t \leq x}} r(t) < \frac{1.28}{x}.$$

Corollary 4 *Let $a > 0$ and $a + d \geq 6$, then*

$$0 < \rho(a, d) - Ar(a) < \frac{1.28}{a + d}.$$

The most important ingredient of the proof will be the following lemma (the idea of which was suggested to the author by Carl Pomerance [16]).

Lemma 4 *For $x \geq 6$ one has*

$$\sum_{n > x} \frac{1}{n\varphi(n)} < \frac{2.1}{x}.$$

Proof. Using that $\varphi(n) \geq \log(2n)/(n \log 2)$ and that $\sum_{k \geq y} 1/k^2 < 1.075/y$ for $y \geq 6$, one finds, for $x \geq 6$,

$$\begin{aligned} \sum_{t > x} \frac{1}{t\varphi(t)} &= \sum_{t > x} \frac{1}{t^2} \sum_{d|t} \frac{|\mu(d)|}{\varphi(d)} \\ &= \sum_{d=1}^{\infty} \frac{|\mu(d)|}{d^2\varphi(d)} \sum_{r > x/d} \frac{1}{r^2} \\ &\leq \zeta(2) \sum_{d > x/6} \frac{|\mu(d)|}{d^2\varphi(d)} + \frac{1.075}{x} \sum_{d \leq x/6} \frac{|\mu(d)|}{d\varphi(d)} \\ &\leq \zeta(2) \sum_{d > x/6} \frac{\log 2d}{d^3 \log 2} + \frac{1.075}{x} \sum_{d \leq x/6} \frac{|\mu(d)|}{d\varphi(d)} \\ &\leq \zeta(2) \int_{[x/6]}^{\infty} \frac{\log 2t}{t^3 \log 2} dt + \frac{1.075}{x} \frac{\zeta(2)\zeta(3)}{\zeta(6)} \\ &\leq \frac{\zeta(2)}{4} \frac{(2 \log(2[x/6]) + 1)}{[x/6]^2 \log 2} + \frac{1.075}{x} \frac{\zeta(2)\zeta(3)}{\zeta(6)}. \end{aligned}$$

For $x \geq 45000$ the latter upper bound is bounded above by $2.1/x$. After calculating $\sum_{n=1}^{\infty} 1/(n\varphi(n))$ with enough precision (see Remark 1) and using that

$$\sum_{t>x} \frac{1}{n\varphi(n)} = \sum_{n=1}^{\infty} \frac{1}{n\varphi(n)} - \sum_{t \leq x} \frac{1}{n\varphi(n)} < 2.20386 - \sum_{t \leq x} \frac{1}{n\varphi(n)},$$

the result follows after verification in the range $(6, 45000)$ (this verification is easily seen to require only a finite amount of computation, cf. [7, Lemma 4]). \square

The above argument can be easily adapted to show that

$$x \sum_{n>x} \frac{1}{n\varphi(n)} \sim \frac{\zeta(2)\zeta(3)}{\zeta(6)} = 1.943 \dots$$

Proof of Proposition 8. From (7) and $r(t) \geq 0$, one infers that

$$A \sum_{\substack{t \equiv a \pmod{d} \\ t \leq x}} r(t) < \rho(a, d) \leq \sum_{\substack{t \equiv a \pmod{d} \\ t \leq x}} r(t) + A \sum_{t>x} r(t)$$

By (3) one has

$$A \sum_{t>x} r(t) \leq \frac{6}{\pi^2} \sum_{t>x} \frac{1}{t\varphi(t)}.$$

On invoking Lemma 4, the result then follows. \square

The terms $AL(2, \chi)L(3, \chi)L(4, \chi)$ in Theorem 6 form the beginning of an expansion of B_χ in terms of special values of L-series.

Theorem 7 Define numbers $G_{j+1}^{(r)}$ by

$$\frac{(-1)^r}{r} \sum_{d|r} \frac{\mu(d)(-1)^{\frac{r}{d}}}{(1 - z^d - z^{2d})^{r/d}} = \sum_{j=0}^{\infty} G_{j+1}^{(r)} z^j.$$

One has

$$B_\chi = A \frac{L(2, \chi)L(3, \chi)}{L(6, \chi^2)} \prod_{r=1}^{\infty} \prod_{k=3r+1}^{\infty} L(k, \chi^r)^{\lambda(k, r)},$$

where $(-1)^{r-1}\lambda(k, r) = G_{k-3r+1}^{(r)} \in \mathbb{Z}_{>0}$.

Note that as formal series $(1 - z - z^2)^{-1} = \sum_{j=0}^{\infty} F_{j+1} z^j$ where F_j denotes the j th Fibonacci number (thus $F_0 = 0$, $F_1 = 1$ etc.). The numbers defined by $(1 - z - z^2)^{-r} = \sum_{j=0}^{\infty} F_{j+1}^{(r)} z^j$ are known as convolved Fibonacci numbers and hence a reasonable term for the integers $G_{j+1}^{(r)}$ might be ‘convoluted convolved Fibonacci numbers’. For the convenience of the reader Table 1 gives a small sample of these numbers.

The positivity of the numbers $(-1)^{r-1}\lambda(k, r)$ is established in [9], where the numbers $G_{j+1}^{(r)}$ are investigated. The argument uses Witt’s dimension formula for free Lie algebras. The remaining part of Theorem 7 follows from the following more general result.

Theorem 8 Suppose that $f(z)$ allows a formal power series in z having only integer coefficients, i.e. $f(z) = \sum_{j \geq 1} a(j)z^j$ with $a(j) \in \mathbb{Z}$. Let $g(z) = \sum_{j \geq 1} |a(j)|z^j$ and let $j_0 \geq 0$ denote the smallest integer such that $a(j) \neq 0$. Let

$$H^{(r)}(z) = \frac{1}{r} \sum_{d|r} \mu(d) f(z^d)^{r/d} = \sum_{j=0}^{\infty} h(j, r) z^j.$$

Then, as formal power series in y and z , one has

$$1 - yf(z) = \prod_{k=1}^{\infty} \prod_{j=kj_0}^{\infty} (1 - z^j y^k)^{h(j,k)}, \quad (14)$$

Moreover, the numbers $h(j, k)$ are integers.

Let $\epsilon > 0$ be fixed. The identity (14) holds for all complex numbers y and z with $g(|z|)y < 1 - \epsilon$ and $|z| < \rho_c$, where ρ_c is the radius of convergence of the Taylor series of g around $z = 0$. If, moreover, $\rho_c > 1/2$, $g(1/2) < 1$ and $\sum_p g(1/p)$ converges, then

$$\prod_p \left(1 - \chi(p) f\left(\frac{1}{p}\right) \right) = \prod_{k=1}^{\infty} \prod_{j=kj_0}^{\infty} L(j, \chi^k)^{-h(j,k)}. \quad (15)$$

Proof of Theorem 7. Note that

$$(1 - YX^2) \left(\frac{1 + \frac{(Y-1)X^2}{(1-YX^2)(1-X)}}{1 - \frac{X^2}{1-X}} \right) = \left(1 + \frac{YX^3}{1 - X - X^2} \right).$$

By the first part of Theorem 8 one infers that, as formal series,

$$\left(1 + \frac{YX^3}{1 - X - X^2} \right) = (1 - YX^3)^{-1} (1 - Y^2X^6) \prod_{r=1}^{\infty} \prod_{k=3r+1}^{\infty} (1 - X^k Y^r)^{(-1)^r G_{k-3r+1}^{(r)}},$$

on noting that $G_1^{(r)} = 1$ for $r = 2$ and $G_1^{(r)} = 0$ for $r \geq 2$. Apply the second part with $f(z) = -z^3/(1 - z - z^2)$ (and hence $g(z) = z^3/(1 - z - z^2)$). The Taylor series for g has radius of convergence $\rho_c = (\sqrt{5} - 1)/2 > 0.5$. Note that $g(x) \leq 1/2$ for all $0 \leq x \leq 1/2$. Furthermore, $\sum_p g(1/p) < \sum_p \frac{4}{p^3} < \infty$. \square

Remark 2. The Dirichlet character is an example of a completely multiplicative function h , i.e. $h(nm) = h(n)h(m)$ for all natural numbers n and m . If one defines $L(s, h)$ by $L(s, h) = \sum_{n=1}^{\infty} h(n)n^{-s}$, then under the same conditions, one may replace χ in Theorem 8 by any completely multiplicative function h satisfying $|h(n)| \leq 1$.

Remark 3. If $0 < \rho_c \leq 1/2$ and $\sum_{p > 1/\rho_c} g(1/p)$ converges, then an identity of the type (15) still holds, but with Dirichlet L-functions being replaced by partial Dirichlet L-functions. The idea is just to leave out the local factor $1 - \chi(p)f(1/p)$ for sufficiently many small primes p and then proceed as before, cf. the proof of Theorem 1 of [6] (in the formulation of Theorem 1 there, replace $p_{n_0} + 1 > 1/\beta$

(a typo) by $p_{n_0+1} > \beta$).

Remark 4. The conditions in the latter part of the theorem ensure that $f(z) = O(z^2)$ for small z . This ensures on its turn that in the double product in (15) only factors $L(j, \chi^k)$ with $k \geq 1$ and $j \geq 2k \geq 2$ appear.

The proof given here of Theorem 7 rests on the following lemma.

Lemma 5 *Suppose that $f(X, Y) = \sum_{j,k} \alpha(j, k) X^j Y^k$ with $\alpha(j, k)$ integers and $f(0, 0) = 0$. Then there are unique integers $e(j, k)$ such that, as formal series, one has*

$$1 + f(X, Y) = \prod_{j=0}^{\infty} \prod_{\substack{k=0 \\ (j,k) \neq (0,0)}}^{\infty} (1 - X^j Y^k)^{e(j,k)}.$$

Proof. The term $X^{j_1} Y^{k_1}$ is said to be of lower weight than $X^{j_2} Y^{k_2}$ if $k_1 < k_2$ or $k_1 = k_2$ and $j_1 < j_2$. Suppose that $X^j Y^k$ is the term of lowest weight appearing in $f(X, Y)$. Then consider $(1 + f(X, Y))(1 - X^j Y^k)^{-a(j,k)}$. This can be written as $1 + g(X, Y)$ where all the coefficients of $g(X, Y)$ are integers and the term of lowest weight in $g(X, Y)$ has strictly larger weight than the term of lowest weight in $f(X, Y)$. Now iterate.

It is not obvious from this argument that if one starts with a different weight ordering of the terms $X^j Y^k$ we end up with the same integers $e(j, k)$. Suppose that $h(X)$ has integer coefficients, then the coefficients $e(n)$ in $1 + h(X) = \prod_{n=1}^{\infty} (1 - X^n)^{e(n)}$ are unique, cf. [6]. Hence, by setting $X = 0$, respectively $Y = 0$, one obtains that $e(0, k)$, respectively $e(j, 0)$ are uniquely determined. Setting $Y = X^m$ one obtains that $1 + f(X, X^m) = \prod_{n=1}^{\infty} (1 - X^n)^{v(n)}$, where $v(n)$ is uniquely determined and $v(2m) = e(2m, 0) + e(m, 1) + e(0, 2)$. The uniqueness of $e(0, 2)$, $e(2m, 0)$ and $f(2m)$ then implies the uniqueness of $e(m, 1)$. The proof will be completed by using induction. So suppose one has established that $e(j, k)$ with $k \leq r$ for some $r \geq 1$ are uniquely determined. Using that $v((r+2)m) = \sum_{k=0}^{r+2} e((r+2-k)m, k)$, one infers by the induction hypothesis and using that $e(0, r+2)$ and $v((r+2)m)$ are uniquely determined, that $e(m, r+1)$ is uniquely determined. \square

Proof of Theorem 8. By Möbius inversion and the definition of $H^{(r)}(z)$ one infers that

$$f(z)^r = \sum_{d|r} \frac{r}{d} H^{(\frac{r}{d})}(z^d) = \sum_{d|r} \frac{r}{d} h(j, \frac{r}{d}) \sum_{j=0}^{\infty} z^{jd},$$

from which it is inferred that

$$\sum_{r=1}^{\infty} y^r f(z)^r = \sum_{k=1}^{\infty} \sum_{j=0}^{\infty} h(j, k) k \sum_{d=1}^{\infty} z^{jd} y^{kd}.$$

The latter identity with both sides divided out by y can be rewritten as

$$\frac{f(z)}{1 - yf(z)} = \sum_{k=1}^{\infty} \sum_{j=0}^{\infty} \frac{h(j, k) k z^j y^{k-1}}{1 - z^j y^k}.$$

Formal integration of both sides with respect to y gives

$$-\log(1 - yf(z)) = -\sum_{k=1}^{\infty} \sum_{j=0}^{\infty} h(j, k) \log(1 - z^j y^k),$$

whence

$$1 - yf(z) = \prod_{k=1}^{\infty} \prod_{j=0}^{\infty} (1 - z^j y^k)^{h(j, k)}.$$

On writing $f_1(z) = f(z)/z^{j_0}$ and $y_1 = yz^{j_0}$ and expanding $1 - y_1 f_1(z) (= 1 - yf(z))$ in terms of y_1 and z , it is then seen that (14) holds. The integrality of $h(j, k)$ follows by Lemma 5.

The formal argument can be certainly made rigorous in the situation where

$$\sum_{k=1}^{\infty} \sum_{j=j_0}^{\infty} |h(j, k) k z^j y^k| < \infty, \quad (16)$$

where one is in the situation of absolute convergence and interchanges in order of summation are hence allowed. Note that for $x \geq 0$, g is a non-decreasing function of the real variable x . Now note that

$$\sum_{j=j_0}^{\infty} |h(j, r) z^j| \leq \frac{1}{r} \sum_{d|r} g(|z|^d)^{r/d} \leq g(|z|)^r,$$

where the assumption that $|z| < \rho_c$ is being used. The double sum in (16) is thus majorized by $\sum_{r=1}^{\infty} r g(|z|)^r |y|^r$ which in the given (y, z) region converges.

By a similar argument the convergence $\sum_p \sum_{r=1}^{\infty} r g(1/p)^r$, which is a consequence of the convergence of $\sum_p g(1/p)$ (one uses here that g is non-decreasing as a function of the real variable x for $x \geq 0$ and that $g(1/2) < 1$), ensures the convergence of the triple product

$$\prod_p \left(1 - \chi(p) f\left(\frac{1}{p}\right) \right) = \prod_p \prod_{k=1}^{\infty} \prod_{j=kj_0}^{\infty} \left(1 - \frac{\chi(p)^k}{p^j} \right)^{h(j, k)}. \quad (17)$$

(From the theory of infinite products use that a product $\prod (1 + \epsilon_v)$ is called absolutely convergent if $\sum \epsilon_v$ is absolutely convergent and that in an absolutely convergent product the factors can be reordered without changing its value.) On bringing the outer product over the primes p to the inside and using the Euler product for a Dirichlet L-series, the result then follows. \square

7 Tables

Explanation to Table 1. Table 1 gives some values of convoluted convolved Fibonacci numbers $G_j^{(r)}$. These numbers are defined in Theorem 7.

Explanation to Table 2. For every character χ of modulus ≤ 12 , A_χ can be deduced from the table below. In every case the value of χ is given (in at most two arguments) such that χ is uniquely determined by this. If χ itself is not in the table, its complex conjugate $\bar{\chi}$ will be (in which case one has $A_\chi = \overline{A_{\bar{\chi}}}$) or χ is the principal character (in which case $A_\chi = 1$). Although A_χ for χ not a primitive character can be easily related to $A_{\chi'}$ with χ' a primitive character, for the convenience of the reader the numerical approximations to A_χ for the non-primitive characters are listed as well.

Explanation to Table 3. An entry in a column having as header the number a and in a row starting with an integer d , respectively a $-$, gives the first five decimal digits of $\delta(a, d)$, respectively $\delta(a + 6, d)$. If an entry is in a row labelled \approx , let $\delta(a, d)$ be the entry directly above it. Then the number given equals $N_{-19}(a, d)(x)/\pi(x)$ with $x = 2038074743$ (and hence $\pi(x) = 10^8$).

Explanation to Table 4. Similar to that of Table 2 (and with the same value of x). In case $d = \infty$ one has $\delta(a, d) = Ar(a)$ for $a \geq 1$ and $N'_g(a, \infty)(x)$ denotes the number of primes $p \leq x$ with $v_p(g) = 0$ such that g has index equal to a . Here $x = 1299709$ (and hence $\pi(1299709) = 10^5$).

Acknowledgement. The author thanks Yves Gallot for writing a C^{++} program that was used to create Tables 3 and 4. Furthermore, he thanks the referee for some helpful comments.

Table 1: Convoluted convolved Fibonacci numbers $G_j^{(r)}$

$r \backslash j$	1	2	3	4	5	6	7	8	9	10	11
1	1	1	2	3	5	8	13	21	34	55	89
2	1	1	3	5	11	19	37	65	120	210	376
3	0	1	3	7	17	37	77	158	314	611	1174
4	0	1	3	10	25	64	146	331	710	1505	3091
5	0	1	4	13	38	102	259	626	1457	3287	7224

Table 2: Numerical evaluation of A_χ

d	χ	χ	A_χ
3	$\chi(2) = -1$	-	+ 0.173977122429634...
4	$\chi(3) = -1$	-	+ 0.643650679662525...
5	$\chi(2) = i$	-	+ 0.364689626478581...
-	-	-	+i0.224041094424738...
5	$\chi(2) = -1$	-	+ 0.129307938528080...
6	$\chi(5) = -1$	-	+ 0.869885612148171...
7	$\chi(3) = e^{\pi i/3}$	-	+ 0.218769298429369...
-	-	-	+i0.235418433356679...
7	$\chi(3) = e^{4\pi i/3}$	-	+ 0.212612780475062...
-	-	-	-i0.145188986908610...
7	$\chi(3) = -1$	-	+ 0.611324432919373...
8	$\chi(3) = 1$	$\chi(5) = -1$	+ 0.837998503129360...
8	$\chi(3) = -1$	$\chi(5) = 1$	+ 0.643650679662525...
8	$\chi(3) = -1$	$\chi(5) = -1$	+ 0.603907856267167...
9	$\chi(2) = e^{\pi i/3}$	-	+ 0.578815911632924...
-	-	-	+i0.334468140016295...
9	$\chi(2) = e^{4\pi i/3}$	-	+ 0.250710892521489...
-	-	-	-i0.207858981269346...
9	$\chi(2) = -1$	-	+ 0.173977122429634...
10	$\chi(3) = i$	-	+ 0.779414790379699...
-	-	-	+i0.123970019579663...
10	$\chi(3) = -1$	-	+ 0.646539692640401...
11	$\chi(2) = e^{\pi i/5}$	-	+ 0.657644343795360...
-	-	-	+i0.151998116640767...
11	$\chi(2) = e^{2\pi i/5}$	-	+ 0.373259555803500...
-	-	-	+i0.208638808901506...
11	$\chi(2) = e^{3\pi i/5}$	-	+ 0.187051722258759...
-	-	-	+i0.232381723173172...
11	$\chi(2) = -1$	-	+ 0.184204262987186...
12	$\chi(5) = 1$	$\chi(7) = -1$	+ 0.919500970946465...
12	$\chi(5) = -1$	$\chi(7) = 1$	+ 0.869885612148171...
12	$\chi(5) = -1$	$\chi(7) = -1$	+ 0.841259078358102...

Table 3: $\delta(a, d)$ and approximation to $\delta_{-19}(a, d)$

a	0	1	2	3	4	5
$d = 2$	0.66666	0.33333	-	-	-	-
\approx	0.66667	0.33333	-	-	-	-
3	0.37500	0.35599	0.26900	-	-	-
\approx	0.37502	0.35602	0.26897	-	-	-
4	0.33333	0.16666	0.33333	0.16666	-	-
\approx	0.33334	0.16664	0.33333	0.16669	-	-
5	0.20833	0.23542	0.17799	0.23400	0.14424	-
\approx	0.20831	0.23572	0.17829	0.23373	0.14395	-
6	0.25000	0.06067	0.12134	0.12500	0.29532	0.14766
\approx	0.25001	0.06067	0.12132	0.12501	0.29534	0.14765
7	0.14583	0.15968	0.15483	0.11905	0.16351	0.15567
\approx	0.14584	0.15965	0.15467	0.11915	0.16367	0.15573
-	0.10141	-	-	-	-	-
\approx	0.10129	-	-	-	-	-
8	0.16666	0.08333	0.16666	0.08333	0.16666	0.08333
\approx	0.16667	0.08332	0.16664	0.08335	0.16667	0.08332
-	0.16666	0.08333	-	-	-	-
\approx	0.16669	0.08334	-	-	-	-
9	0.12500	0.11866	0.08966	0.12500	0.11866	0.08966
\approx	0.12501	0.11866	0.08966	0.12501	0.11868	0.08965
-	0.12500	0.11866	0.08966	-	-	-
\approx	0.12500	0.11867	0.08965	-	-	-
10	0.13888	0.07196	0.14393	0.08172	0.06810	0.06944
\approx	0.13888	0.07197	0.14408	0.08159	0.06783	0.06944
-	0.16345	0.03405	0.15227	0.07613	-	-
\approx	0.16374	0.03421	0.15214	0.07612	-	-
11	0.09166	0.09890	0.09811	0.09904	0.09848	0.07170
\approx	0.09166	0.09889	0.09805	0.09904	0.09859	0.07180
-	0.09940	0.09303	0.09297	0.09523	0.06143	-
\approx	0.09939	0.09303	0.09297	0.09526	0.06133	-
12	0.12500	0.03033	0.06067	0.06250	0.14766	0.07383
\approx	0.12500	0.03033	0.06065	0.06251	0.14767	0.07382
-	0.12500	0.03033	0.06067	0.06250	0.14766	0.07383
\approx	0.12501	0.03035	0.06067	0.06249	0.14767	0.07383

Table 4: $\rho(a, d)$ and approximation to $\rho_{65537}(a, d)$

a	0	1	2	3	4	5
$d = 2$	0.50000	0.50000	-	-	-	-
\approx	0.49994	0.50006	-	-	-	-
3	0.16666	0.48915	0.34417	-	-	-
\approx	0.16662	0.48924	0.34414	-	-	-
4	0.12500	0.41091	0.37500	0.08908	-	-
\approx	0.12497	0.41097	0.37497	0.08909	-	-
5	0.05000	0.44143	0.31320	0.10036	0.09498	-
\approx	0.05000	0.44150	0.31322	0.10035	0.09494	-
6	0.08333	0.38955	0.31706	0.08333	0.09959	0.02710
\approx	0.08330	0.38966	0.31705	0.08331	0.09958	0.02709
7	0.02380	0.40253	0.29923	0.08966	0.08471	0.03881
\approx	0.02380	0.40263	0.29923	0.08962	0.08470	0.03881
-	0.06123	-	-	-	-	-
\approx	0.06122	-	-	-	-	-
8	0.03125	0.38569	0.30818	0.07380	0.09375	0.02521
\approx	0.03124	0.38577	0.30818	0.07380	0.09372	0.02521
-	0.06681	0.01528	-	-	-	-
\approx	0.06679	0.01529	-	-	-	-
9	0.01851	0.39347	0.29075	0.08696	0.07829	0.02983
\approx	0.01851	0.39356	0.29075	0.08694	0.07829	0.02981
-	0.06118	0.01738	0.02358	-	-	-
\approx	0.06117	0.01740	0.02358	-	-	-
10	0.02500	0.38063	0.30067	0.07141	0.08456	0.02500
\approx	0.02500	0.38071	0.30068	0.07140	0.08452	0.02500
-	0.06080	0.01253	0.02895	0.01041	-	-
\approx	0.06079	0.01254	0.02895	0.01041	-	-
11	0.00909	0.39040	0.28866	0.07722	0.07791	0.02698
\approx	0.00910	0.39047	0.28865	0.07721	0.07791	0.02698
-	0.05543	0.01768	0.02331	0.01418	0.01909	-
\approx	0.05541	0.01769	0.02331	0.01419	0.01908	-
12	0.02083	0.37819	0.29216	0.07231	0.07926	0.02170
\approx	0.02080	0.37827	0.29215	0.07230	0.07926	0.02169
-	0.06250	0.01136	0.02489	0.01101	0.02033	0.00540
\approx	0.06250	0.01139	0.02491	0.01101	0.02033	0.00539
\dots	\dots	\dots	\dots	\dots	\dots	\dots
∞	0.00000	0.37395	0.28046	0.06648	0.07011	0.01889
\approx	0.00000	0.37367	0.28124	0.06646	0.06913	0.01885
-	0.04986	0.00893	0.01752	0.00738	0.01417	0.00340
\approx	0.04962	0.00915	0.01796	0.00745	0.01449	0.00359

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