# ON THE CLASSIFICATION OF SELF-DUAL [20, 10, 9] CODES OVER GF(7) 

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In memory of Yutaka Hiramine


#### Abstract

It is shown that the extended quadratic residue code of length 20 over $\mathrm{GF}(7)$ is a unique self-dual $[20,10,9]$ code $C$ such that the lattice obtained from $C$ by Construction A is isomorphic to the 20 -dimensional unimodular lattice $D_{20}^{+}$, up to equivalence. This is done by converting the classification of such self-dual codes to that of skew-Hadamard matrices of order 20.


## 1. Introduction

Let $\operatorname{GF}(p)$ be the finite field of order $p$, where $p$ is prime. As described in [16], self-dual codes are an important class of linear codes for both theoretical and practical reasons. For $p \equiv 1(\bmod 4)$, a self-dual code of length $n$ over $\operatorname{GF}(p)$ exists if and only if $n$ is even, and for $p \equiv 3$ $(\bmod 4)$, a self-dual code of length $n$ over $\operatorname{GF}(p)$ exists if and only if $n \equiv 0(\bmod 4)$. It is a fundamental problem to classify self-dual codes over $\operatorname{GF}(p)$ and determine the largest minimum weight among self-dual codes over GF $(p)$ for a fixed length. Much work has been done towards classifying self-dual codes over $\mathrm{GF}(p)$ and determining the largest minimum weight among self-dual codes of a given length over $\operatorname{GF}(p)$ for $p=2$ and 3 (see [16]).

Self-dual codes over GF(7) have been classified for lengths up to 12 (see [9]), and the largest minimum weight $d_{7}(n)$ among self-dual codes of length $n$ over GF(7) has been determined for $n \leq 28$ (see [7, Table 2]). For example, it is known that $d_{7}(20)=9$ and the extended quadratic residue code $Q R_{20}$ of length 20 over $\mathrm{GF}(7)$ is a self-dual [20, 10, 9] code (see [5]).

There are 12 nonisomorphic 20-dimensional unimodular lattices having minimum norm 2 (see [3, Table 16.7]), and one of them is $D_{20}^{+}$. Let $A_{7}(C)$ denote the unimodular lattice obtained from a self-dual code $C$ over GF(7) by Construction A.

[^0]In this paper, we convert the classification of self-dual $[20,10,9]$ codes $C$ over $\mathrm{GF}(7)$ such that $A_{7}(C)$ is isomorphic to $D_{20}^{+}$to that of skewHadamard matrices of order 20. The main aim of this paper is to give the following partial classification of self-dual $[20,10,9]$ codes over GF(7).

Theorem 1. Up to equivalence, the extended quadratic residue code of length 20 over $\mathrm{GF}(7)$ is a unique self-dual $[20,10,9]$ code $C$ over $\operatorname{GF}(7)$ such that $A_{7}(C)$ is isomorphic to $D_{20}^{+}$.

All computer calculations in this paper were done with the help of Magma [1].

## 2. Preliminaries

In this section, we give definitions and notions on self-dual codes, unimodular lattices and skew-Hadamard matrices. Some basic facts on these subjects are also provided.
2.1. Self-dual codes. An $[n, k]$ code $C$ over $\operatorname{GF}(p)$ is a $k$-dimensional subspace of $\operatorname{GF}(p)^{n}$. The value $n$ is called the length of $C$. The weight $\mathrm{wt}(x)$ of a vector $x \in \mathrm{GF}(p)^{n}$ is the number of non-zero components of $x$. A vector of $C$ is called a codeword of $C$. The minimum non-zero weight of all codewords in $C$ is called the minimum weight of $C$ and an $[n, k]$ code with minimum weight $d$ is called an $[n, k, d]$ code. The weight enumerator $W(C)$ of $C$ is given by $W(C)=\sum_{i=0}^{n} A_{i} y^{i}$, where $A_{i}$ is the number of codewords of weight $i$ in $C$. The dual code $C^{\perp}$ of $C$ is defined as

$$
C^{\perp}=\left\{x \in \mathrm{GF}(p)^{n} \mid x \cdot y=0 \text { for all } y \in C\right\}
$$

under the standard inner product $x \cdot y$. A code $C$ is called self-dual if $C=C^{\perp}$. Two codes $C$ and $C^{\prime}$ are equivalent if there exists a $(1,-1,0)$ monomial matrix $M$ with $C^{\prime}=\{c M \mid c \in C\}$.
2.2. Unimodular lattices. An $n$-dimensional (Euclidean) lattice is a discrete subgroup of rank $n$ in $\mathbb{R}^{n}$. A lattice $L$ is unimodular if $L=L^{*}$, where the dual lattice $L^{*}$ is defined as

$$
L^{*}=\left\{x \in \mathbb{R}^{n} \mid(x, y) \in \mathbb{Z} \text { for all } y \in L\right\}
$$

under the standard inner product $(x, y)$. The norm $\|x\|^{2}$ of a vector $x \in \mathbb{R}^{n}$ is $(x, x)$. The minimum norm of $L$ is the smallest norm among all nonzero vectors of $L$. Two lattices $L$ and $L^{\prime}$ are isomorphic, denoted $L \cong L^{\prime}$, if there exists an orthogonal matrix $A$ with $L^{\prime}=\{x A \mid x \in L\}$.

Let $C$ be a code of length $n$ over $\operatorname{GF}(p)$ and let $\varepsilon_{1}, \ldots, \varepsilon_{n}$ be an orthogonal basis of $\mathbb{R}^{n}$ satisfying $\left(\varepsilon_{i}, \varepsilon_{j}\right)=p \delta_{i, j}$, where $\delta_{i, j}$ is the Kronecker delta. Then we define the lattice $A_{p}(C)$ obtained from $C$ by Construction $A$ as

$$
A_{p}(C)=\left\{\left.\frac{1}{p} \sum_{i=1}^{n} x_{i} \varepsilon_{i} \right\rvert\, x=\left(x_{1}, \ldots, x_{n}\right) \in \mathbb{Z}^{n}, x \bmod p \in C\right\}
$$

It is known that $A_{p}(C)$ is unimodular if and only if $C$ is self-dual. A set $\left\{f_{1}, \ldots, f_{n}\right\}$ of $n$ vectors $f_{1}, \ldots, f_{n}$ in an $n$-dimensional lattice $L$ with $\left(f_{i}, f_{j}\right)=k \delta_{i, j}$ is called a $k$-frame of $L$. Clearly, $A_{p}(C)$ contains a $p$-frame. Conversely, if a unimodular lattice $L$ contains a $p$-frame, then there is a self-dual code $C$ over $\operatorname{GF}(p)$ with $A_{p}(C) \cong L$ (see [8]).

Let $C$ be a self-dual $[20,10, d]$ code over $\operatorname{GF}(7)$ with $d \in\{8,9\}$. Then it is easy to see that $A_{7}(C)$ has minimum norm 2. It is known that there are 12 nonisomorphic 20-dimensional unimodular lattices having minimum norm 2 (see [3, Table 16.7]), and one of them is $D_{20}^{+}$. The lattice $D_{20}^{+}$is defined from the root lattice $D_{20}$ as follows:

$$
\begin{aligned}
& D_{20}=\left\{\sum_{i=1}^{20} \alpha_{i} e_{i} \mid\left(\alpha_{1}, \ldots, \alpha_{20}\right) \in \mathbb{Z}^{20}, \sum_{i=1}^{20} \alpha_{i} \equiv 0 \quad(\bmod 2)\right\}, \\
& D_{20}^{+}=\left\langle D_{20}, \frac{1}{2} \mathbf{1}\right\rangle
\end{aligned}
$$

where $e_{i}=\left(\delta_{1, i}, \ldots, \delta_{20, i}\right)(1 \leq i \leq 20)$ and 1 denotes the all-one vector. Note that $D_{20}$ is the even sublattice of $D_{20}^{+}$, that is, the sublattice consisting of vectors of even norm in $D_{20}^{+}$.
2.3. Skew-Hadamard matrices. A Hadamard matrix of order $n$ is an $n \times n(1,-1)$-matrix $H$ such that $H H^{\top}=n I$, where $I$ is the identity matrix and $H^{\top}$ denotes the transposed matrix of $H$. It is well known that the order $n$ is necessarily 1,2 , or a multiple of 4 . Two Hadamard matrices $H$ and $K$ are said to be equivalent if there are $(1,-1,0)$ monomial matrices $P$ and $Q$ with $K=P H Q$. All Hadamard matrices of orders up to 32 have been classified (see [10, Chap. 7] for orders up to 28 and [11] for order 32, see also [18]).

A Hadamard matrix $H$ of order $n$ is called a skew-Hadamard matrix if $H+H^{\top}=2 I$. Skew-Hadamard matrices are a class of Hadamard matrices, which has been widely studied (see e.g., [4], [13]). The numbers of inequivalent skew-Hadamard matrices of orders $4,8,12,16,20,24$ are $1,1,1,3,2,11$, respectively [13]. We denote by $S_{1}$ the Paley Hadamard matrix of order 20 , which is a skew-Hadamard matrix. The other skewHadamard matrix of order 20 can be found in [12] and we denote the
matrix by $S_{2}$. Moreover, we have verified with the help of Magma that $S_{2}$ is equivalent to had.20.toncheviv in [18].

The following lemma can be proved in the same manner as [15, Lemma 3].

Lemma 2. Let $F$ be a square matrix all of whose entries are integers. If $F F^{\top}=k I$ and $p$ is a prime divisor of $k$ such that $p^{2} \nmid k$, then $F$ generates a self-dual code over $\operatorname{GF}(p)$.

Hence, the code over GF(7) generated by the row vectors of $H+2 I$ is self-dual, where $H$ is a skew-Hadamard matrix of order 20.

## 3. Proof of Theorem 1

In this section, we give a proof of Theorem 1, which is the main result of this paper.

Lemma 3. Let $C$ be a self-dual $[20,10,9]$ code over $\operatorname{GF}(7)$. If $\xi \in$ $A_{7}(C)$ and $\|\xi\|^{2}=2$, then

$$
\left|\left\{i\left|1 \leq i \leq 20,\left|\left(\xi, \varepsilon_{i}\right)\right| \geq 2\right\} \mid \leq 1\right.\right.
$$

Proof. Write

$$
\xi=\frac{1}{7} \sum_{i=1}^{20} x_{i} \varepsilon_{i}, x=\left(x_{1}, \ldots, x_{n}\right) \in \mathbb{Z}^{20}, x \bmod 7 \in C
$$

Since for each $j \in\{1, \ldots, 20\}$,

$$
\begin{aligned}
x_{j}^{2} & =7\left\|\frac{1}{7} x_{j} \varepsilon_{j}\right\|^{2} \\
& \leq 7\left\|\frac{1}{7} \sum_{i=1}^{20} x_{i} \varepsilon_{i}\right\|^{2} \\
& =7\|\xi\|^{2} \\
& =14,
\end{aligned}
$$

we have

$$
\begin{equation*}
x_{j} \equiv 0 \quad(\bmod 7) \Longleftrightarrow x_{j}=0 \Longleftrightarrow\left(\xi, \varepsilon_{j}\right)=0 \tag{1}
\end{equation*}
$$

Set

$$
\begin{aligned}
& a_{1}=\left|\left\{i\left|1 \leq i \leq 20,\left|\left(\xi, \varepsilon_{i}\right)\right|=1\right\} \mid,\right.\right. \\
& a_{2}=\left|\left\{i\left|1 \leq i \leq 20,\left|\left(\xi, \varepsilon_{i}\right)\right| \geq 2\right\} \mid .\right.\right.
\end{aligned}
$$

Then by (1) we have

$$
a_{1}+a_{2}=\operatorname{wt}(x) \geq 9,
$$

and we have

$$
\begin{aligned}
a_{1}+4 a_{2} & \leq \sum_{i=1}^{20}\left(\xi, \varepsilon_{i}\right)^{2} \\
& =7\|\xi\|^{2} \\
& =14
\end{aligned}
$$

Thus $a_{2} \leq \frac{5}{3}$, and hence $a_{2} \leq 1$.
Proposition 4. Let $C$ be a self-dual $[20,10,9]$ code over $\mathrm{GF}(7)$ with $A_{7}(C) \cong D_{20}^{+}$. Then there exists a skew-Hadamard matrix $H$ of order 20 such that $C$ is generated by the row vectors of $H+2 I$ over $\mathrm{GF}(7)$.

Proof. Let $\Psi: A_{7}(C) \rightarrow D_{20}^{+}$be an isomorphism. Since $\left\|\Psi\left(\varepsilon_{j}\right)\right\|^{2}=$ $\left\|\varepsilon_{j}\right\|^{2}=7$ is odd, $\Psi\left(\varepsilon_{j}\right) \notin D_{20}$. Thus $\Psi\left(\varepsilon_{j}\right) \in \frac{1}{2} \mathbf{1}+D_{20} \subset \frac{1}{2}(1+2 \mathbb{Z})^{20}$, and hence there exist odd integers $f_{i, j}$ such that

$$
\Psi\left(\varepsilon_{j}\right)=\frac{1}{2} \sum_{i=1}^{20} f_{i, j} e_{i} .
$$

Let $F$ denote the $20 \times 20$ matrix whose $(i, j)$ entry is $f_{i, j}$. Then $F^{\top} F=$ 28I. In particular,

$$
\sum_{h=1}^{20} f_{h, i}^{2}=28
$$

Since $f_{h, i}$ are odd integers, we see that there exists a unique $h_{i}$ such that $f_{h_{i}, i}= \pm 3$. Since $F F^{\top}=28 I$, the mapping $i \mapsto h_{i}$ is a bijection from $\{1, \ldots, 20\}$ to itself.

Now we may assume without loss of generality

$$
f_{h, i}= \begin{cases}3 & \text { if } h=i \\ \pm 1 & \text { otherwise }\end{cases}
$$

Set $H=F-2 I$. Then all the entries of $H$ are $\pm 1$, and the diagonal entries are 1.

We claim $H+H^{\top}=2 I$. To prove this, we need to show $f_{h, i}+f_{i, h}=0$ for $1 \leq h<i \leq 20$. Suppose $f_{h, i}=f_{i, h}$ for some $1 \leq h<i \leq 20$. Set $\xi=\Psi^{-1}\left(e_{h}+\bar{f}_{i, h} e_{i}\right)$. Then $\|\xi\|^{2}=\left\|e_{h}+e_{i}\right\|^{2}=2$, and

$$
\begin{aligned}
\left(\xi, \varepsilon_{i}\right) & =\left(\Psi(\xi), \Psi\left(\varepsilon_{i}\right)\right) \\
& =\left(e_{h}+f_{i, h} e_{i}, \frac{1}{2} \sum_{j=1}^{20} f_{j, i} e_{j}\right) \\
& =\frac{1}{2}\left(f_{h, i}+f_{i, h} f_{i, i}\right)
\end{aligned}
$$

$$
\begin{aligned}
& \text { MASAAKI HARADA AND AKIHIRO MUNEMASA } \\
& \qquad=2 f_{i, h}
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\left(\xi, \varepsilon_{h}\right) & =\left(\Psi(\xi), \Psi\left(\varepsilon_{h}\right)\right) \\
& =\left(e_{h}+f_{i, h} e_{i}, \frac{1}{2} \sum_{j=1}^{20} f_{j, h} e_{j}\right) \\
& =\frac{1}{2}\left(f_{h, h}+f_{i, h}^{2}\right) \\
& =2
\end{aligned}
$$

These contradict Lemma 3, and complete the proof of the claim.
Since

$$
\begin{aligned}
H^{\top} H & =\left(F^{\top}-2 I\right)(F-2 I) \\
& =28 I-2\left(H^{\top}+H+4 I\right)+4 I \\
& =20 I
\end{aligned}
$$

$H$ is a Hadamard matrix.
Finally, since

$$
\begin{aligned}
D_{20}^{+} & \ni 2 e_{i} \\
& =\frac{1}{14} \sum_{h=1}^{20} 28 \delta_{h, i} e_{h} \\
& =\frac{1}{14} \sum_{h=1}^{20} \sum_{j=1}^{20} f_{i, j} f_{h, j} e_{h} \\
& =\frac{1}{7} \sum_{j=1}^{20} f_{i, j} \frac{1}{2} \sum_{h=1}^{20} f_{h, j} e_{h} \\
& =\frac{1}{7} \sum_{j=1}^{20} f_{i, j} \Psi\left(\varepsilon_{j}\right) \\
& =\Psi\left(\frac{1}{7} \sum_{j=1}^{20} f_{i, j} \varepsilon_{j}\right),
\end{aligned}
$$

we have

$$
\frac{1}{7} \sum_{j=1}^{20} f_{i, j} \varepsilon_{j} \in A_{7}(C)
$$

Thus the $i$-th row of $F=H+2 I$ belongs to $C$. The fact that $F$ generates the self-dual code $C$ follows from Lemma 2.

We say that skew-Hadamard matrices $H$ and $H^{\prime}$ of order $n$ are skewHadamard equivalent if there exists a $(1,-1,0)$-monomial matrix $P$ with $P H P^{\top}=H^{\prime}$. Let $H$ and $H^{\prime}$ be skew-Hadamard matrices of order 20. Let $C(H)$ denote the code over $\mathrm{GF}(7)$ generated by the row vectors of $H+2 I$. If $H$ and $H^{\prime}$ are skew-Hadamard equivalent, then $C(H)$ and $C\left(H^{\prime}\right)$ are equivalent. By Proposition 4, we can convert the classification of self-dual $[20,10,9]$ codes $C$ over $\mathrm{GF}(7)$ with $A_{7}(C) \cong D_{20}^{+}$ to that of skew-Hadamard matrices of order 20, up to skew-Hadamard equivalence. The existence of a skew-Hadamard matrix of order $n$ is equivalent to the existence of a doubly regular tournament of order $n-1$ [17]. It is known that there are two doubly regular tournaments of order 19, up to isomorphism (see [14]). This implies that there are two skew-Hadamard matrices of order 20 , up to skew-Hadamard equivalence. Indeed, let $H$ be a skew-Hadamard matrix of order 20 and let $D$ be the diagonal matrix whose diagonal entries are the first row of $H$. Then

$$
D H D=\left(\begin{array}{cc}
1 & \mathbf{1} \\
-\mathbf{1}^{\top} & M
\end{array}\right) .
$$

Here the $19 \times 19(1,0)$-matrix $(M+J) / 2-I$ is the adjacency matrix of a doubly regular tournament of order 19 , where $J$ is the $19 \times 19$ all-one matrix. Hence, isomorphic doubly regular tournaments of order 19 give skew-Hadamard matrices of order 20, which are skew-Hadamard equivalent. The matrices $S_{1}$ and $S_{2}$ give the two skew-Hadamard matrices of order 20, up to skew-Hadamard equivalence.

We have verified with the help of Magma that the two self-dual codes $C\left(S_{1}\right)$ and $C\left(S_{2}\right)$ have the following weight enumerators:

$$
\begin{aligned}
W\left(C\left(S_{1}\right)\right)= & 1+6840 y^{9}+47880 y^{10}+200640 y^{11}+957600 y^{12} \\
& +3625200 y^{13}+10766160 y^{14}+25701984 y^{15} \\
& +48495600 y^{16}+68276880 y^{17}+68299680 y^{18} \\
& +43155840 y^{19}+12940944 y^{20}, \\
W\left(C\left(S_{2}\right)\right)= & 1+1080 y^{8}+5040 y^{9}+40320 y^{10}+215760 y^{11} \\
& +977040 y^{12}+3571200 y^{13}+10751040 y^{14} \\
& +25814304 y^{15}+48431880 y^{16}+68208840 y^{17} \\
& +68403000 y^{18}+43106160 y^{19}+12949584 y^{20}
\end{aligned}
$$

respectively. In particular, $C\left(S_{1}\right)$ is a $[20,10,9]$ code, while $C\left(S_{2}\right)$ has minimum weight 8. By Proposition 4. $C\left(S_{1}\right)$ is a unique selfdual $[20,10,9]$ code $C$ over $\operatorname{GF}(7)$ with $A_{7}(C) \cong D_{20}^{+}$. In addition, we
have verified with the help of Magma that $A_{7}\left(Q R_{20}\right) \cong D_{20}^{+}$. This completes the proof of Theorem [1.

Remark 5. The above argument yields that $C\left(S_{1}\right)$ is equivalent to $Q R_{20}$. A general case including this fact was described in [2, p. 1041] without proof.

## 4. Some other constructions of self-dual [20, 10, 9] CODES

Finally, in this section, we investigate some other constructions of self-dual $[20,10,9]$ codes over $G F(7)$. Note that $D_{20}^{+}$is the unique 20-dimensional unimodular lattice with minimum norm 2 and kissing number 760. For a given self-dual $[20,10,9]$ code $C$ over GF(7), one can determine whether $A_{7}(C)$ is isomorphic to $D_{20}^{+}$or not, by computing the kissing number of $A_{7}(C)$ with the help of Magma. If $A_{7}(C)$ is isomorphic to $D_{20}^{+}$, then by Theorem [1, we have that $C$ is equivalent to $Q R_{20}$.

- Some self-dual $[20,10,9]$ codes over $\mathrm{GF}(7)$ were constructed in [6, Table 6] and [7, Table 7] as double circulant codes and quasi-twisted codes, respectively (see [7] for the construction). We have verified that $A_{7}(C) \cong D_{20}^{+}$for all double circulant selfdual $[20,10,9]$ codes $C$. Also, we have verified that $A_{7}(C) \cong$ $D_{20}^{+}$for all quasi-twisted self-dual $[20,10,9]$ codes $C$. These imply that all double circulant self-dual $[20,10,9]$ codes and all quasi-twisted self-dual $[20,10,9]$ codes are equivalent to $Q R_{20}$.
- Let $A$ and $B$ be $5 \times 5$ circulant (resp. negacirculant) matrices. A $[20,10]$ code over $\mathrm{GF}(7)$ with the following generator matrix

$$
\left(\begin{array}{ccc} 
& A & B \\
I & -B^{\top} & A^{\top}
\end{array}\right)
$$

is called a four-circulant (resp. four-negacirculant) code. By exhaustive search, we have verified that $A_{7}(C) \cong D_{20}^{+}$for all four-circulant self-dual $[20,10,9]$ codes $C$. Also, we have verified that $A_{7}(C) \cong D_{20}^{+}$for all four-negacirculant self-dual [20, 10, 9] codes $C$.

- Let $C$ be a self-dual code of length 20 over GF(7). Let $x$ be a vector with $x \cdot x=0$. Then $C(x)=\left\langle C \cap\langle x\rangle^{\perp}, x\right\rangle$ is a selfdual code over $\mathrm{GF}(7)$. By exhaustive search, we have verified that $A_{7}\left(Q R_{20}(x)\right) \cong D_{20}^{+}$for all vectors $x$ in a set of complete representatives of $\operatorname{GF}(7)^{20} / Q R_{20}$ with $x \cdot x=0$.
Moreover, our extensive search failed to discover a self-dual [20, 10, 9] code $C$ over $\operatorname{GF}(7)$ with $A_{7}(C) \not \not 二 D_{20}^{+}$. We are lead to conjecture that $Q R_{20}$ is a unique self-dual $[20,10,9]$ code over $\mathrm{GF}(7)$.

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