# ON THE CLASSIFICATION OF SELF-DUAL [20, 10, 9] CODES OVER GF(7)

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ABSTRACT. It is shown that the extended quadratic residue code of length 20 over GF(7) is a unique self-dual [20, 10, 9] code C such that the lattice obtained from C by Construction A is isomorphic to the 20-dimensional unimodular lattice  $D_{20}^+$ , up to equivalence. This is done by converting the classification of such self-dual codes to that of skew-Hadamard matrices of order 20.

### 1. INTRODUCTION

Let GF(p) be the finite field of order p, where p is prime. As described in [16], self-dual codes are an important class of linear codes for both theoretical and practical reasons. For  $p \equiv 1 \pmod{4}$ , a self-dual code of length n over GF(p) exists if and only if n is even, and for  $p \equiv 3 \pmod{4}$ , a self-dual code of length n over GF(p) exists if and only if  $n \equiv 0 \pmod{4}$ . It is a fundamental problem to classify self-dual codes over GF(p) and determine the largest minimum weight among self-dual codes over GF(p) for a fixed length. Much work has been done towards classifying self-dual codes over GF(p) and determining the largest minimum weight among self-dual codes of a given length over GF(p) for p = 2 and 3 (see [16]).

Self-dual codes over GF(7) have been classified for lengths up to 12 (see [9]), and the largest minimum weight  $d_7(n)$  among self-dual codes of length n over GF(7) has been determined for  $n \leq 28$  (see [7, Table 2]). For example, it is known that  $d_7(20) = 9$  and the extended quadratic residue code  $QR_{20}$  of length 20 over GF(7) is a self-dual [20, 10, 9] code (see [5]).

There are 12 nonisomorphic 20-dimensional unimodular lattices having minimum norm 2 (see [3, Table 16.7]), and one of them is  $D_{20}^+$ . Let  $A_7(C)$  denote the unimodular lattice obtained from a self-dual code Cover GF(7) by Construction A.

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In this paper, we convert the classification of self-dual [20, 10, 9] codes C over GF(7) such that  $A_7(C)$  is isomorphic to  $D_{20}^+$  to that of skew-Hadamard matrices of order 20. The main aim of this paper is to give the following partial classification of self-dual [20, 10, 9] codes over GF(7).

**Theorem 1.** Up to equivalence, the extended quadratic residue code of length 20 over GF(7) is a unique self-dual [20, 10, 9] code C over GF(7) such that  $A_7(C)$  is isomorphic to  $D_{20}^+$ .

All computer calculations in this paper were done with the help of MAGMA [1].

### 2. Preliminaries

In this section, we give definitions and notions on self-dual codes, unimodular lattices and skew-Hadamard matrices. Some basic facts on these subjects are also provided.

2.1. Self-dual codes. An [n, k] code C over GF(p) is a k-dimensional subspace of  $GF(p)^n$ . The value n is called the *length* of C. The *weight* wt(x) of a vector  $x \in GF(p)^n$  is the number of non-zero components of x. A vector of C is called a *codeword* of C. The minimum non-zero weight of all codewords in C is called the *minimum weight* of C and an [n, k] code with minimum weight d is called an [n, k, d] code. The *weight enumerator* W(C) of C is given by  $W(C) = \sum_{i=0}^{n} A_i y^i$ , where  $A_i$  is the number of codewords of weight i in C. The *dual code*  $C^{\perp}$  of C is defined as

$$C^{\perp} = \{ x \in \mathrm{GF}(p)^n \mid x \cdot y = 0 \text{ for all } y \in C \},\$$

under the standard inner product  $x \cdot y$ . A code C is called *self-dual* if  $C = C^{\perp}$ . Two codes C and C' are *equivalent* if there exists a (1, -1, 0)-monomial matrix M with  $C' = \{cM \mid c \in C\}$ .

2.2. Unimodular lattices. An *n*-dimensional (Euclidean) lattice is a discrete subgroup of rank n in  $\mathbb{R}^n$ . A lattice L is unimodular if  $L = L^*$ , where the dual lattice  $L^*$  is defined as

$$L^* = \{ x \in \mathbb{R}^n \mid (x, y) \in \mathbb{Z} \text{ for all } y \in L \},\$$

under the standard inner product (x, y). The norm  $||x||^2$  of a vector  $x \in \mathbb{R}^n$  is (x, x). The minimum norm of L is the smallest norm among all nonzero vectors of L. Two lattices L and L' are isomorphic, denoted  $L \cong L'$ , if there exists an orthogonal matrix A with  $L' = \{xA \mid x \in L\}$ .

Let C be a code of length n over GF(p) and let  $\varepsilon_1, \ldots, \varepsilon_n$  be an orthogonal basis of  $\mathbb{R}^n$  satisfying  $(\varepsilon_i, \varepsilon_j) = p\delta_{i,j}$ , where  $\delta_{i,j}$  is the Kronecker delta. Then we define the lattice  $A_p(C)$  obtained from C by Construction A as

$$A_p(C) = \{ \frac{1}{p} \sum_{i=1}^n x_i \varepsilon_i \mid x = (x_1, \dots, x_n) \in \mathbb{Z}^n, x \bmod p \in C \}.$$

It is known that  $A_p(C)$  is unimodular if and only if C is self-dual. A set  $\{f_1, \ldots, f_n\}$  of n vectors  $f_1, \ldots, f_n$  in an n-dimensional lattice Lwith  $(f_i, f_j) = k\delta_{i,j}$  is called a k-frame of L. Clearly,  $A_p(C)$  contains a p-frame. Conversely, if a unimodular lattice L contains a p-frame, then there is a self-dual code C over GF(p) with  $A_p(C) \cong L$  (see [8]).

Let C be a self-dual [20, 10, d] code over GF(7) with  $d \in \{8, 9\}$ . Then it is easy to see that  $A_7(C)$  has minimum norm 2. It is known that there are 12 nonisomorphic 20-dimensional unimodular lattices having minimum norm 2 (see [3, Table 16.7]), and one of them is  $D_{20}^+$ . The lattice  $D_{20}^+$  is defined from the root lattice  $D_{20}$  as follows:

$$D_{20} = \{ \sum_{i=1}^{20} \alpha_i e_i \mid (\alpha_1, \dots, \alpha_{20}) \in \mathbb{Z}^{20}, \sum_{i=1}^{20} \alpha_i \equiv 0 \pmod{2} \}, D_{20}^+ = \langle D_{20}, \frac{1}{2} \mathbf{1} \rangle,$$

where  $e_i = (\delta_{1,i}, \ldots, \delta_{20,i})$   $(1 \le i \le 20)$  and **1** denotes the all-one vector. Note that  $D_{20}$  is the even sublattice of  $D_{20}^+$ , that is, the sublattice consisting of vectors of even norm in  $D_{20}^+$ .

2.3. Skew-Hadamard matrices. A Hadamard matrix of order n is an  $n \times n$  (1, -1)-matrix H such that  $HH^{\top} = nI$ , where I is the identity matrix and  $H^{\top}$  denotes the transposed matrix of H. It is well known that the order n is necessarily 1, 2, or a multiple of 4. Two Hadamard matrices H and K are said to be *equivalent* if there are (1, -1, 0)monomial matrices P and Q with K = PHQ. All Hadamard matrices of orders up to 32 have been classified (see [10, Chap. 7] for orders up to 28 and [11] for order 32, see also [18]).

A Hadamard matrix H of order n is called a *skew-Hadamard matrix* if  $H+H^{\top}=2I$ . Skew-Hadamard matrices are a class of Hadamard matrices, which has been widely studied (see e.g., [4], [13]). The numbers of inequivalent skew-Hadamard matrices of orders 4, 8, 12, 16, 20, 24 are 1, 1, 1, 3, 2, 11, respectively [13]. We denote by  $S_1$  the Paley Hadamard matrix of order 20, which is a skew-Hadamard matrix. The other skew-Hadamard matrix of order 20 can be found in [12] and we denote the

matrix by  $S_2$ . Moreover, we have verified with the help of MAGMA that  $S_2$  is equivalent to had.20.toncheviv in [18].

The following lemma can be proved in the same manner as [15, Lemma 3].

**Lemma 2.** Let F be a square matrix all of whose entries are integers. If  $FF^{\top} = kI$  and p is a prime divisor of k such that  $p^2 \nmid k$ , then F generates a self-dual code over GF(p).

Hence, the code over GF(7) generated by the row vectors of H + 2I is self-dual, where H is a skew-Hadamard matrix of order 20.

## 3. Proof of Theorem 1

In this section, we give a proof of Theorem 1, which is the main result of this paper.

**Lemma 3.** Let C be a self-dual [20, 10, 9] code over GF(7). If  $\xi \in A_7(C)$  and  $\|\xi\|^2 = 2$ , then

$$|\{i \mid 1 \le i \le 20, \ |(\xi, \varepsilon_i)| \ge 2\}| \le 1.$$

Proof. Write

$$\xi = \frac{1}{7} \sum_{i=1}^{20} x_i \varepsilon_i, \ x = (x_1, \dots, x_n) \in \mathbb{Z}^{20}, \ x \bmod 7 \in C.$$

Since for each  $j \in \{1, \ldots, 20\}$ ,

$$x_{j}^{2} = 7 \|\frac{1}{7} x_{j} \varepsilon_{j}\|^{2}$$
  

$$\leq 7 \|\frac{1}{7} \sum_{i=1}^{20} x_{i} \varepsilon_{i}\|^{2}$$
  

$$= 7 \|\xi\|^{2}$$
  

$$= 14,$$

we have

(1)  $x_j \equiv 0 \pmod{7} \iff x_j = 0 \iff (\xi, \varepsilon_j) = 0.$ Set

$$a_{1} = |\{i \mid 1 \le i \le 20, \ |(\xi, \varepsilon_{i})| = 1\}|, \\ a_{2} = |\{i \mid 1 \le i \le 20, \ |(\xi, \varepsilon_{i})| \ge 2\}|.$$

Then by (1) we have

$$a_1 + a_2 = \operatorname{wt}(x) \ge 9,$$

and we have

$$a_1 + 4a_2 \le \sum_{i=1}^{20} (\xi, \varepsilon_i)^2$$
  
=  $7 \|\xi\|^2$   
= 14.

Thus  $a_2 \leq \frac{5}{3}$ , and hence  $a_2 \leq 1$ .

**Proposition 4.** Let C be a self-dual [20, 10, 9] code over GF(7) with  $A_7(C) \cong D_{20}^+$ . Then there exists a skew-Hadamard matrix H of order 20 such that C is generated by the row vectors of H + 2I over GF(7).

Proof. Let  $\Psi : A_7(C) \to D_{20}^+$  be an isomorphism. Since  $\|\Psi(\varepsilon_j)\|^2 = \|\varepsilon_j\|^2 = 7$  is odd,  $\Psi(\varepsilon_j) \notin D_{20}$ . Thus  $\Psi(\varepsilon_j) \in \frac{1}{2}\mathbf{1} + D_{20} \subset \frac{1}{2}(1+2\mathbb{Z})^{20}$ , and hence there exist odd integers  $f_{i,j}$  such that

$$\Psi(\varepsilon_j) = \frac{1}{2} \sum_{i=1}^{20} f_{i,j} e_i.$$

Let F denote the  $20 \times 20$  matrix whose (i, j) entry is  $f_{i,j}$ . Then  $F^{\top}F = 28I$ . In particular,

$$\sum_{h=1}^{20} f_{h,i}^2 = 28$$

Since  $f_{h,i}$  are odd integers, we see that there exists a unique  $h_i$  such that  $f_{h_i,i} = \pm 3$ . Since  $FF^{\top} = 28I$ , the mapping  $i \mapsto h_i$  is a bijection from  $\{1, \ldots, 20\}$  to itself.

Now we may assume without loss of generality

$$f_{h,i} = \begin{cases} 3 & \text{if } h = i, \\ \pm 1 & \text{otherwise} \end{cases}$$

Set H = F - 2I. Then all the entries of H are  $\pm 1$ , and the diagonal entries are 1.

We claim  $H + H^{\top} = 2I$ . To prove this, we need to show  $f_{h,i} + f_{i,h} = 0$ for  $1 \le h < i \le 20$ . Suppose  $f_{h,i} = f_{i,h}$  for some  $1 \le h < i \le 20$ . Set  $\xi = \Psi^{-1}(e_h + f_{i,h}e_i)$ . Then  $\|\xi\|^2 = \|e_h + e_i\|^2 = 2$ , and

$$\begin{aligned} (\xi, \varepsilon_i) &= (\Psi(\xi), \Psi(\varepsilon_i)) \\ &= (e_h + f_{i,h} e_i, \frac{1}{2} \sum_{j=1}^{20} f_{j,i} e_j) \\ &= \frac{1}{2} (f_{h,i} + f_{i,h} f_{i,i}) \end{aligned}$$

 $=2f_{i,h}.$ 

Similarly,

$$\begin{aligned} (\xi, \varepsilon_h) &= (\Psi(\xi), \Psi(\varepsilon_h)) \\ &= (e_h + f_{i,h} e_i, \frac{1}{2} \sum_{j=1}^{20} f_{j,h} e_j) \\ &= \frac{1}{2} (f_{h,h} + f_{i,h}^2) \\ &= 2. \end{aligned}$$

These contradict Lemma 3, and complete the proof of the claim. Since

$$H^{\top}H = (F^{\top} - 2I)(F - 2I)$$
  
= 28I - 2(H^{\top} + H + 4I) + 4I  
= 20I,

H is a Hadamard matrix. Finally, since

$$D_{20}^{+} \ni 2e_{i}$$

$$= \frac{1}{14} \sum_{h=1}^{20} 28\delta_{h,i}e_{h}$$

$$= \frac{1}{14} \sum_{h=1}^{20} \sum_{j=1}^{20} f_{i,j}f_{h,j}e_{h}$$

$$= \frac{1}{7} \sum_{j=1}^{20} f_{i,j}\frac{1}{2} \sum_{h=1}^{20} f_{h,j}e_{h}$$

$$= \frac{1}{7} \sum_{j=1}^{20} f_{i,j}\Psi(\varepsilon_{j})$$

$$= \Psi(\frac{1}{7} \sum_{j=1}^{20} f_{i,j}\varepsilon_{j}),$$

we have

$$\frac{1}{7}\sum_{j=1}^{20} f_{i,j}\varepsilon_j \in A_7(C).$$

Thus the *i*-th row of F = H + 2I belongs to C. The fact that F generates the self-dual code C follows from Lemma 2.

We say that skew-Hadamard matrices H and H' of order n are skew-Hadamard equivalent if there exists a (1, -1, 0)-monomial matrix P with  $PHP^{\top} = H'$ . Let H and H' be skew-Hadamard matrices of order 20. Let C(H) denote the code over GF(7) generated by the row vectors of H+2I. If H and H' are skew-Hadamard equivalent, then C(H) and C(H') are equivalent. By Proposition 4, we can convert the classification of self-dual [20, 10, 9] codes C over GF(7) with  $A_7(C) \cong D_{20}^+$ to that of skew-Hadamard matrices of order 20, up to skew-Hadamard equivalence. The existence of a skew-Hadamard matrix of order n is equivalent to the existence of a doubly regular tournament of order n-1 [17]. It is known that there are two doubly regular tournaments of order 19, up to isomorphism (see [14]). This implies that there are two skew-Hadamard matrices of order 20, up to skew-Hadamard equivalence. Indeed, let H be a skew-Hadamard matrix of order 20 and let D be the diagonal matrix whose diagonal entries are the first row of H. Then

$$DHD = \left(\begin{array}{cc} 1 & \mathbf{1} \\ -\mathbf{1}^\top & M \end{array}\right).$$

Here the  $19 \times 19$  (1, 0)-matrix (M+J)/2 - I is the adjacency matrix of a doubly regular tournament of order 19, where J is the  $19 \times 19$  all-one matrix. Hence, isomorphic doubly regular tournaments of order 19 give skew-Hadamard matrices of order 20, which are skew-Hadamard equivalent. The matrices  $S_1$  and  $S_2$  give the two skew-Hadamard matrices of order 20, up to skew-Hadamard equivalence.

We have verified with the help of MAGMA that the two self-dual codes  $C(S_1)$  and  $C(S_2)$  have the following weight enumerators:

$$\begin{split} W(C(S_1)) = &1 + 6840y^9 + 47880y^{10} + 200640y^{11} + 957600y^{12} \\ &+ 3625200y^{13} + 10766160y^{14} + 25701984y^{15} \\ &+ 48495600y^{16} + 68276880y^{17} + 68299680y^{18} \\ &+ 43155840y^{19} + 12940944y^{20}, \\ W(C(S_2)) = &1 + 1080y^8 + 5040y^9 + 40320y^{10} + 215760y^{11} \\ &+ 977040y^{12} + 3571200y^{13} + 10751040y^{14} \\ &+ 25814304y^{15} + 48431880y^{16} + 68208840y^{17} \\ &+ 68403000y^{18} + 43106160y^{19} + 12949584y^{20}, \end{split}$$

respectively. In particular,  $C(S_1)$  is a [20, 10, 9] code, while  $C(S_2)$  has minimum weight 8. By Proposition 4,  $C(S_1)$  is a unique selfdual [20, 10, 9] code C over GF(7) with  $A_7(C) \cong D_{20}^+$ . In addition, we have verified with the help of MAGMA that  $A_7(QR_{20}) \cong D_{20}^+$ . This completes the proof of Theorem 1.

Remark 5. The above argument yields that  $C(S_1)$  is equivalent to  $QR_{20}$ . A general case including this fact was described in [2, p. 1041] without proof.

### 4. Some other constructions of self-dual [20, 10, 9] codes

Finally, in this section, we investigate some other constructions of self-dual [20, 10, 9] codes over GF(7). Note that  $D_{20}^+$  is the unique 20-dimensional unimodular lattice with minimum norm 2 and kissing number 760. For a given self-dual [20, 10, 9] code C over GF(7), one can determine whether  $A_7(C)$  is isomorphic to  $D_{20}^+$  or not, by computing the kissing number of  $A_7(C)$  with the help of MAGMA. If  $A_7(C)$  is isomorphic to  $D_{20}^+$ , then by Theorem 1, we have that C is equivalent to  $QR_{20}$ .

- Some self-dual [20, 10, 9] codes over GF(7) were constructed in [6, Table 6] and [7, Table 7] as double circulant codes and quasi-twisted codes, respectively (see [7] for the construction). We have verified that  $A_7(C) \cong D_{20}^+$  for all double circulant selfdual [20, 10, 9] codes C. Also, we have verified that  $A_7(C) \cong$  $D_{20}^+$  for all quasi-twisted self-dual [20, 10, 9] codes C. These imply that all double circulant self-dual [20, 10, 9] codes and all quasi-twisted self-dual [20, 10, 9] codes are equivalent to  $QR_{20}$ .
- Let A and B be  $5 \times 5$  circulant (resp. negacirculant) matrices. A [20, 10] code over GF(7) with the following generator matrix

$$\left(\begin{array}{ccc} I & A & B \\ & -B^{\top} & A^{\top} \end{array}\right)$$

is called a four-circulant (resp. four-negacirculant) code. By exhaustive search, we have verified that  $A_7(C) \cong D_{20}^+$  for all four-circulant self-dual [20, 10, 9] codes C. Also, we have verified that  $A_7(C) \cong D_{20}^+$  for all four-negacirculant self-dual [20, 10, 9] codes C.

• Let C be a self-dual code of length 20 over GF(7). Let x be a vector with  $x \cdot x = 0$ . Then  $C(x) = \langle C \cap \langle x \rangle^{\perp}, x \rangle$  is a selfdual code over GF(7). By exhaustive search, we have verified that  $A_7(QR_{20}(x)) \cong D_{20}^+$  for all vectors x in a set of complete representatives of GF(7)<sup>20</sup>/QR<sub>20</sub> with  $x \cdot x = 0$ .

Moreover, our extensive search failed to discover a self-dual [20, 10, 9] code C over GF(7) with  $A_7(C) \not\cong D_{20}^+$ . We are lead to conjecture that  $QR_{20}$  is a unique self-dual [20, 10, 9] code over GF(7).

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