# Galois LCD Codes over Finite Fields 

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#### Abstract

In this paper, we study the complementary dual codes in more general setting (which are called Galois LCD codes) by a uniform method. A necessary and sufficient condition for linear codes to be Galois LCD codes is determined, and constacyclic codes to be Galois LCD codes are characterized. Some illustrative examples which constacyclic codes are Galois LCD MDS codes are provided as well. In particular, we study Hermitian LCD constacyclic codes. Finally, we present a construction of a class of Hermitian LCD codes which are also MDS codes.


Key Words: Constacyclic codes, MDS codes, cyclotomic cosets, Galois LCD codes MSC2010: 12E20, 94B60

## 1 Introduction

Linear complementary dual codes (which is abbreviated to LCD codes) are linear codes that meet their dual trivially. These codes were introduced by Massey in [13] and showed that asymptotically good LCD codes exist, and provided an optimum linear coding solution for the two-user binary adder channel. They are also used in counter measure to passive and active side channel analyses on embedded cryto-systems [2]. Guenda, Jitman and Gulliver investigated an application of LCD codes in constructing good entanglement-assisted quantum error correcting codes [11].

Dinh established the algebraic structures in terms of generator polynomial of all repeatedroot constacyclic codes of length $3 p^{s}, 4 p^{s}, 6 p^{s}$ over finite field $\mathbb{F}_{p^{m}}$. Using these structures, constacyclic LCD codes of such lengths were also characterized (see [4, 6, 5]). Yang and Massey in [16] showed that a necessary and sufficient condition for a cyclic code of length $n$ over finite fields to be an LCD code is that the generator polynomial $g(x)$ is self-reciprocal and all the monic irreducible factors of $g(x)$ have the same multiplicity in $g(x)$ as in $x^{n}-1$. In [15], Sendrier indicated that linear codes with complementary-duals meet the asymptotic Gilbert-Varshamov bound. Esmaeiliand Yari in [9] studied complementary-dual quasi-cyclic codes. Necessary and sufficient conditions for certain classes of quasi-cyclic codes to be LCD codes were obtained [9]. Dougherty, Kim, Ozkaya, Sok and Solé developed a linear programming bound on the largest size of an LCD code of given length and minimum distance [8]. In recently, Ding, C. Li and $\mathrm{S} . \mathrm{Li}$ in [3] constructed LCD BCH codes. In addition, Boonniyoma and Jitman gave a study on linear codes with Hermitian complementary dual [1] and we also in 14 studied LCD codes over finite chain rings.

Constacyclic codes over finite fields are important classes of linear codes in theoretical and practical viewpoint. In [10], Fan and Zhang studied Galois self-dual constacyclic codes over finite fields. Motivated by this work, we will investigate Galois complementary dual codes (which is abbreviated to Galois LCD codes) over finite fields. Some of them have better parameters.

In this work, we study the complementary dual constacyclic codes in more general setting by a uniform method. The necessary background materials of Galois dual and the definition of Galois LCD codes are given in Section 2. Moreover, we obtain a criteria of Galois LCD codes. In Section 3, we characterize the generator polynomials of Galois LCD constacyclic codes. Next, we obtain a sufficient and necessary condition for a code $C$ to be an Galois LCD constacyclic code over finite fields and give examples that $C$ is an Galois LCD MDS constacyclic over finite fields. Finally, in Section 4, we address the Hermitian LCD constacyclic codes over finite fields and get a family of Hermitian LCD MDS constacyclic codes.

## 2 Galois LCD codes over $\mathbb{F}_{q}$

Throughout this paper, we denote by $\mathbb{F}_{q}$ the finite field with cardinality $\left|\mathbb{F}_{q}\right|=q=p^{e}$, where $p$ is a prime and $e$ is a positive integer. Let $\lambda \in \mathbb{F}_{q}^{*}$, where $\mathbb{F}_{q}^{*}$ denotes the multiplicative group of units of $\mathbb{F}_{q}$, and let $n$ be a positive integer coprime to $q$. Any ideal $C$ of the quotient ring $R_{n, \lambda}=\mathbb{F}_{q}[X] /\left\langle X^{n}-\lambda\right\rangle$ is said to be a $\lambda$-constacyclic code of length $n$ over $\mathbb{F}_{q}$. Let $\mathbb{F}_{q}^{n}=\left\{\mathbf{x}=\left(x_{1}, \cdots, x_{n}\right) \mid x_{j} \in \mathbb{F}_{q}\right\}$ be $n$ dimensional vector space over $\mathbb{F}_{q}$. A subspace $C$ of $\mathbb{F}_{q}^{n}$ is called a linear code of length $n$ over $\mathbb{F}_{q}$. We assume that all codes are linear. If a linear code $C$ over $\mathbb{F}_{q}$ with parameters $[n, k, d]$ attains the Singleton bound $d=n-k+1$, then it is called a maximum-distance-separable (MDS) code.

Let $\mathbf{x}, \mathbf{y} \in \mathbb{F}_{q}^{n}$. In [10], Fan and Zhang introduce a kind of inner products, called Galois inner product, as follows: for each integer $k$ with $0 \leq k<e$, define:

$$
[\mathbf{x}, \mathbf{y}]_{k}=x_{1} y_{1}^{p^{k}}+\cdots+x_{n} y_{n}^{p^{k}}
$$

It is just the usual Euclidean inner product if $k=0$. And, it is the Hermitian inner product if $e$ is even and $k=\frac{e}{2}$. we call

$$
C^{\perp_{k}}=\left\{\mathbf{x} \in \mathbb{F}_{q}^{n} \mid[\mathbf{c}, \mathbf{x}]_{k}=0, \forall \mathbf{c} \in C\right\}
$$

as the Galois dual code of $C$. It is easy to see that $C^{\perp_{0}}\left(\operatorname{simply}, C^{\perp}\right)$ is just the Eucilidean dual code of $C$ and $C^{\perp_{\frac{e}{2}}}$ (simply, $C^{\perp_{H}}$ ) is just the Hermitian dual code of $C$.

Notice that $C^{\perp_{k}}$ is linear if $C$ is linear or not.
From the fact that Galois inner product is nondegenerate, it follows immediately that $\operatorname{dim}_{\mathbb{F}_{q}} C+\operatorname{dim}_{\mathbb{F}_{q}} C^{\perp_{k}}=n$.

A linear code $C$ is called Galois LCD if $C^{\perp_{k}} \cap C=\{\mathbf{0}\}$, and an Galois LCD code $C$ is called Galois LCD MDS if $C$ attains the Singleton bound.

Given a vector $\mathbf{a}=\left(a_{1}, a_{2}, \ldots, a_{n}\right) \in \mathbb{F}_{q}^{n}$, we define the $p^{e-k}$ th power of $\mathbf{a}$ as

$$
\mathbf{a}^{p^{e-k}}=\left(a_{1}^{p^{e-k}}, a_{2}^{p^{e-k}}, \ldots, a_{n}^{p^{e-k}}\right) .
$$

For a linear code $C$ of length $n$ over $\mathbb{F}_{q}^{n}$, we define $C^{p^{e-k}}$ to be the set $\left\{\mathbf{a}^{p^{e-k}} \mid\right.$ for all $\left.\mathbf{a} \in C\right\}$. Then, it is easy to see that for a linear code $C$ of length $n$ over $\mathbb{F}_{q}^{n}$, the Galois dual $C^{\perp_{k}}$ is equal to the Euclidean $\left(C^{p^{e-k}}\right)^{\perp}$ dual of $C^{p^{e-k}}$.

Let $A=\left(a_{i j}\right)$ be an $s \times s$ matrix with entries in $\mathbb{F}_{q}$, we define $A^{\left(p^{e-k}\right)}=\left(a_{i j}^{p^{e-k}}\right)$. The following lemma is clear.

Lemma 2.1. If $C$ is an $[n, l, d]$ linear code over $\mathbb{F}_{q}$ with a generating matrix $G$, then $C^{p^{e-k}}$ is also an $[n, l, d]$ linear code over $\mathbb{F}_{q}$ with a generating matrix $G^{\left(p^{e-k}\right)}$. Moreover, $C$ is Galois $L C D$ if and only if $C \cap\left(C^{p^{e-k}}\right)^{\perp}=\{\mathbf{0}\}$.

The following theorem gives a criteria of Galois LCD codes and is analogous to the result of Eucilidean LCD codes in [13].

Theorem 2.2. Let $C$ be an $[n, l, d]$ linear code over $\mathbb{F}_{q}$ with generator matrix $G$. Then $C$ is Galois LCD if and only if $G\left(G^{\left(p^{e-k}\right)}\right)^{T}$ is nonsingular.

Proof. Suppose that $G\left(G^{\left(p^{e-k}\right)}\right)^{T}$ is singular. Then there exists a nonzero a $\in \mathbb{F}_{q}^{n}$ such that $\mathbf{a} G\left(G^{\left(p^{e-k}\right)}\right)^{T}=\mathbf{0}$. Taking $\mathbf{c}=\mathbf{a} G$, it is clear that $\mathbf{c} \in C \backslash\{\mathbf{0}\}$ and it satisfies $\mathbf{c}\left(G^{\left(p^{e-k}\right)}\right)^{T}=\mathbf{0}$. It follows that $\mathbf{c} \in\left(C^{p^{e-k}}\right)^{\perp}$, which is a contraction.

Conversely, assume that $G\left(G^{\left(p^{e-k}\right)}\right)^{T}$ is nonsingular. Let $\mathbf{u} \in \mathbb{F}_{q}^{n}$. If $\mathbf{u} \in C$, then there exists $\mathbf{v} \in \mathbb{F}_{q}^{l}$ such that $\mathbf{u}=\mathbf{v} G$. It follows that

$$
\mathbf{u}\left(G^{\left(p^{e-k}\right)}\right)^{T}\left(G\left(G^{\left(p^{e-k}\right)}\right)^{T}\right)^{-1} G=\mathbf{v} G\left(G^{\left(p^{e-k}\right)}\right)^{T}\left(G\left(G^{\left(p^{e-k}\right)}\right)^{T}\right)^{-1} G=\mathbf{v} G=\mathbf{u} .
$$

If $\mathbf{u} \in C^{\perp_{k}}$, then $\mathbf{u}\left(G^{\left(p^{e-k}\right)}\right)^{T}=\mathbf{0}$, and hence

$$
\mathbf{u}\left(G^{\left(p^{e-k}\right)}\right)^{T}\left(G\left(G^{\left(p^{e-k}\right)}\right)^{T}\right)^{-1} G=\mathbf{0}\left(G\left(G^{\left(p^{e-k}\right)}\right)^{T}\right)^{-1} G=\mathbf{0} .
$$

For any $\mathbf{a} \in C \cap C^{\perp_{k}}$, by $\mathbf{a} \in C$, we have $\mathbf{a}=\mathbf{a}\left(G^{\left(p^{e-k}\right)}\right)^{T}\left(G\left(G^{\left(p^{e-k}\right)}\right)^{T}\right)^{-1} G$, and by $\mathbf{a} \in$ $C^{\perp_{k}}$ again, we have also $\mathbf{a}=\mathbf{a}\left(G^{\left(p^{e-k}\right)}\right)^{T}\left(G\left(G^{\left(p^{e-k}\right)}\right)^{T}\right)^{-1} G=\mathbf{0}$. Therefore, $C \cap C^{\perp_{k}}=\{\mathbf{0}\}$, i.e., $C$ is Galois LCD.

It is well known that, for a given $[n, l, d]$ code over $\mathbb{F}_{q}$, there exists an equivalent code with the same parameters such that its generator matrix is of the form $G=\left[I_{l} A\right]$ for some $l \times(n-l)$ matrix $A$ over $\mathbb{F}_{q}$, where $I_{l}$ is a $l \times l$ identity matrix. The generator matrix of a linear code of this form plays an important role in constructing Galois LCD codes.

The following fact is well known.
Lemma 2.3. Let $p$ be a prime. If $p \equiv 1 \bmod 4$, then -1 is a quadratic modulo $p$.
Theorem 2.4. Let $\widetilde{C}$ be an $[n, l, \widetilde{d}]$ linear code over $\mathbb{F}_{q}$ with generator matrix $\widetilde{G}=\left[\begin{array}{ll}I_{l} & A\end{array}\right]$.
(1) If char $\mathbb{F}_{q}=2$, then exists a Galois $L C D$ code $C$ over $\mathbb{F}_{q}$ with parameters $[2 n-l, l, d]$ and $d \geq \widetilde{d}$.
(2) If $\operatorname{char} \mathbb{F}_{q} \equiv 1 \bmod 4$, then there exists $\eta \in \mathbb{F}_{q}$ such that $\eta^{2}=-1$ and a linear code $C$ generated by $G=\left[\begin{array}{lll}I_{l} & A & \eta\end{array}\right]$ is a Galois $L C D$ code over $\mathbb{F}_{q}$ with parameters $[2 n-l, l, d]$ and $d \geq \widetilde{d}$.

Proof. (1) When char $\mathbb{F}_{q}=2$. Let $C$ be a linear code generated by $G=\left[\begin{array}{lll}I_{l} & A & A\end{array}\right]$ over $\mathbb{F}_{q}$. Then

$$
G\left(G^{\left(p^{e-k}\right)}\right)^{T}=I_{l}+A\left(A^{\left(p^{e-k}\right)}\right)^{T}+A\left(A^{\left(p^{e-k}\right)}\right)^{T}=I_{l} .
$$

Therefore, $G\left(G^{\left(p^{e-k}\right)}\right)^{T}$ is nonsingular, which implies code $C$ is Galois LCD.
Next, we show that $d(C) \geq \widetilde{d}$. Let $\mathbf{u} \in C \backslash\{\mathbf{0}\}$. Then there exists $\mathbf{v} \in \mathbb{F}_{q}^{l} \backslash\{\mathbf{0}\}$ such that $\mathbf{u}=\mathbf{v} G=\left[\mathbf{v} I_{l} \mathbf{v} A \mathbf{v} A\right]$. Hence,

$$
W_{H}(\mathbf{u})=W_{H}\left(\left[\mathbf{v} I_{l} \mathbf{v} A \mathbf{v} A\right]\right) \geq W_{H}\left(\left[\mathbf{v} I_{l} \mathbf{v} A\right]\right)=W_{H}(\mathbf{v}(\widetilde{G})) \geq \widetilde{d}
$$

which implies $d \geq \widetilde{d}$.
(2) When char⿷ $\mathbb{F}_{q} \equiv 1 \bmod 4$. For $0 \leq k<e$, we can assume $p^{e-k}=4 t+1$ where $t$ is an integer. Then $\eta^{1+p^{e-k}}=\eta^{2(2 t+1)}=-1$. Therefore,

$$
G\left(G^{\left(p^{e-k}\right)}\right)^{T}=\left[\begin{array}{lll}
I_{l} & A & \eta
\end{array}\right]\left[\begin{array}{ll}
I_{l} & \left.A^{\left(p^{e-k}\right)} \eta^{p^{e-k}} A^{\left(p^{e-k}\right)}\right]^{T}=I_{l}+A\left(A^{\left(p^{e-k}\right)}\right)^{T}+\eta^{p^{-k}+1} A\left(A^{\left(p^{e-k}\right)}\right)^{T}=I_{l} . . .
\end{array}\right.
$$

This means that $C$ is a Galois LCD code over $\mathbb{F}_{q}$.
Similar to (1), we can prove that $C$ is an $[2 n-l, l, d]$ code with $d \geq \widetilde{d}$.
Example 1. Let $C$ be a linear code of length 4 over $\mathbb{F}_{8}=\left\{0,1=\alpha^{7}=\alpha^{0}, \alpha, \alpha^{2}, \alpha^{3}=\right.$ $\left.1+\alpha, \alpha^{4}=\alpha+\alpha^{2}, \alpha^{5}=1+\alpha+\alpha^{2}, \alpha^{6}=1+\alpha^{2}\right\}$ with generator matrix $G=\left(\begin{array}{cccc}1 & 0 & \alpha & \alpha \\ 0 & 1 & 1 & \alpha\end{array}\right)$. Take $k=1$. Then $p^{e-k}=2^{3-1}=4$. Since $\operatorname{det}\left[G\left(G^{(4)}\right)^{T}\right]=\alpha \neq 0$, we have $G\left(G^{(4)}\right)^{T}$ is nonsingular. Hence, $C$ is a Galois $L C D M D S$ code over $\mathbb{F}_{8}$ with parameters $[4,2,3]$.

## 3 Galois LCD constacyclic codes over $\mathbb{F}_{q}$

In this section, we investigate Galois LCD $\lambda$-constacyclic codes over $\mathbb{F}_{q}$. The following proposition in [4] is very usual.

Proposition 3.1. Let $\alpha, \beta$ be distinct nonzero elements of the field $\mathbb{F}_{q}$. Then a linear code $C$ of length $n$ over $\mathbb{F}_{q}$ is both $\alpha$-and $\beta$-constacyclic if and only if $C=\{\mathbf{0}\}$ or $C=\left\{\mathbb{F}_{q}^{n}\right\}$.

The following lemma can be found in [10].
Lemma 3.2. If $C$ is a $\lambda$-constacyclic code of length $n$ over $\mathbb{F}_{q}$, then $C^{\perp_{k}}$ is a $\lambda^{-p^{e-k}}$ constacyclic code of length $n$ over $\mathbb{F}_{q}$.

Corollary 3.3. If $\lambda^{1+p^{e-k}} \neq 1$, then any $\lambda$-constacyclic $C$ of length $n$ over $\mathbb{F}_{q}$ is a Galois LCD code.

Proof. Indeed, by Lemma 3.2, if $C$ is a $\lambda$-constacyclic code then $C^{\perp_{k}}$ is a $\lambda^{-p^{e-k}}$ constacyclic code. Thus, $C \cap C^{\perp_{k}}$ is both $\lambda$-and $\lambda^{-p^{e-k}}$-constacyclic. When $\lambda^{1+p^{e-k}} \neq 1$, as $C \cap C^{\perp_{k}}$ can not be $\mathbb{F}_{q}^{n}$, by Proposition 3.1, $C \cap C^{\perp_{k}}=\{\mathbf{0}\}$, i.e., $C$ is a Galois LCD code.

By Corollary 3.3, when $\lambda^{1+p^{e-k}} \neq 1$, any $\lambda$-constacyclic code $C$ is a Galois LCD code. Thus, in order to obtain all Galois LCD $\lambda$-constacyclic codes, we only need to look at the classes of $\lambda$-constacyclic codes where $\lambda^{1+p^{e-k}}=1$.

We first give the definition of reciprocal polynomial in $\mathbb{F}_{q}[x]$. Then we study the generator polynomials of Galois LCD $\lambda$-constacyclic codes.

For a polynomial $f(x)=\sum_{i=0}^{l} a_{i} x^{i}$ of degree $l\left(a_{0} \neq 0\right)$ over $\mathbb{F}_{q}$, let $\widetilde{f}(x)$ denote the monic reciprocal polynomial of $f(x)$ given by

$$
\widetilde{f}(x)=a_{0}^{-1} x^{l} f\left(\frac{1}{x}\right)=a_{0}^{-1} \sum_{i=0}^{l} a_{i} x^{l-i}
$$

It is well-known that a nonzero $[n, l] \lambda$-constacyclic code $C$ has a unique generator polynomial $g(x)$ of degree $n-l$, where $g(x) \mid x^{n}-\lambda$. The roots of the code $C$ are the roots of $g(x)$. So, if $\xi_{1}, \ldots, \xi_{n-l}$ are the roots of $g(x)$ in some extension field of $\mathbb{F}_{q}$, then $\mathbf{c}=\left(c_{0}, c_{1}, \ldots, c_{n}\right) \in C$ if and only if $c\left(\xi_{1}\right)=\cdots=c\left(\xi_{n-l}\right)=0$, where $c(x)=c_{0}+c_{1} x+\cdots c_{n-1} x^{n-1}$. Let $h(x)=\frac{x^{n}-\lambda}{g(x)}=\sum_{i=0}^{l} h_{i} x^{i}$. Then we have the following lemma.
Lemma 3.4. With notations as above. Let $C$ be an $\lambda$-constacyclic code of length $n$ over $\mathbb{F}_{q}$. Then
(1) $C^{\perp}$ is an $\lambda^{-1}$-constacyclic code generated by $\widetilde{h}(x)=\sum_{i=0}^{l} h_{0}^{-1} h_{i} x^{l-i}$;
(2) $C^{\perp_{k}}$ is an $\lambda^{-p^{e-k}}$-constacyclic code generated by $\widetilde{h}^{p^{e-k}}(x)=\sum_{i=0}^{l} h_{0}^{-p^{e-k}} h_{i}^{p^{e-k}} x^{l-i}$.

Proof. (1) The proof can be found [7.
(2) Set $\widetilde{C}=\left\langle\widetilde{h}^{p^{e-k}}(x)\right\rangle$. Suppose that $\eta_{1}, \ldots, \eta_{l}$ be the zeros of $\widetilde{h}(x)$. Then $\eta_{1}^{p^{e-k}}, \ldots, \eta_{l}^{p^{e-k}}$ are the zeros of $\widetilde{h}^{p^{e-k}}(x)$. This means that if $\left(c_{0}, c_{1}, \ldots, c_{n}\right) \in C^{\perp}$, then $\left(c_{0}^{p^{e-k}}, \ldots, c_{n-1}^{p^{e-k}}\right) \in \widetilde{C}$.

The following we first prove that

$$
\left(C^{\perp}\right)^{p^{e-k}}=\left(C^{p^{e-k}}\right)^{\perp} .
$$

Suppose $G$ is a generator matrix of $C$. It is easy to prove that $G^{p^{e-k}}$ is a generator matrix of $C^{p^{e-k}}$.

Similarly, if $H$ is a parity-check for $C$, then $H^{p^{e-k}}$ is a generator matrix of $\left(C^{\perp}\right)^{p^{e-k}}$.
Suppose that $G=\left(\begin{array}{c}g_{1} \\ g_{2} \\ \vdots \\ g_{k}\end{array}\right)$ and $H=\left(\begin{array}{c}h_{1} \\ h_{2} \\ \vdots \\ h_{n-k}\end{array}\right)$. Then $G^{p^{e-k}}=\left(\begin{array}{c}g_{1}^{p^{e-k}} \\ g_{2}^{p^{e-k}} \\ \vdots \\ g_{k}^{p^{e-k}}\end{array}\right)$ and $H^{p^{e-k}}=$ $\left(\begin{array}{c}h_{1}^{p^{e-k}} \\ h_{2}^{p^{e-k}} \\ \vdots \\ h_{n-k}^{p^{e-k}}\end{array}\right)$.

For $y \in\left(C^{\perp}\right)^{p^{e-k}}$, we can assume that

$$
y=t_{1} h_{1}^{p^{e-k}}+\cdots+t_{n-k} h_{n-k}^{p^{e-k}} .
$$

Then for any $g_{j}^{p^{e-k}} \in G^{p^{e-k}}$, one obtain that

$$
\left[y, g_{j}^{p^{e-k}}\right]=\sum_{i=1}^{n-k} t_{i}\left[h_{i}^{p^{e-k}}, g_{j}^{p^{e-k}}\right]=\sum_{i=1}^{n-k} t_{i}\left[h_{i}, g_{j}\right]^{p^{e-k}}=0
$$

Therefore, $y \in\left(C^{p^{e-k}}\right)^{\perp}$, which implies that $\left(C^{\perp}\right)^{p^{e-k}} \subset\left(C^{p^{e-k}}\right)^{\perp}$.
On the other hand, we verity that if $v_{1}, v_{2}, \ldots, v_{k}$ are linear independent vectors in $\mathbb{F}_{q}$, then $v_{1}^{p^{e-k}}, v_{2}^{p^{e-k}}, \ldots, v_{k}^{p^{e-k}}$ are also linear independent vectors in $\mathbb{F}_{q}$. In fact, assume that

$$
a_{1} v_{1}^{p^{e^{-k}}}+a_{2} v_{2}^{p^{e-k}}+\cdots+a_{k} v_{k}^{p^{e-k}}=0
$$

where $a_{1}, a_{2}, \cdots, a_{k} \in \mathbb{F}_{q}$, then

$$
a_{1}^{p^{k}} v_{1}+a_{2}^{p^{k}} v_{2}+\cdots+a_{k}^{p^{k}} v_{k}=0
$$

Since $v_{1}, v_{2}, \ldots, v_{k}$ are linear independent vectors in $\mathbb{F}_{q}, a_{1}^{p^{k}}=a_{2}^{p^{k}}=\cdots=a_{k}^{p^{k}}=0$. Hence $a_{1}=a_{2}=\cdots=a_{k}=0$. This meant that $v_{1}^{p^{e-k}}, v_{2}^{p^{e-k}}, \ldots, v_{k}^{p^{e-k}}$ are linear independent vectors in $\mathbb{F}_{q}$.

According to the fact above proof, we have

$$
\operatorname{dim}\left(C^{\perp}\right)^{p^{e-k}}=n-k=\operatorname{dim}\left(C^{p^{e-k}}\right)^{\perp}
$$

Summarizing, we have $\left(C^{\perp}\right)^{p^{e-k}}=\left(C^{p^{e-k}}\right)^{\perp}$. Thus

$$
C^{\perp_{k}}=\left(C^{p^{e-k}}\right)^{\perp}=\left\{\left(c_{0}^{p^{e-k}}, \ldots, c_{n-1}^{p^{e-k}}\right) \mid\left(c_{0}, c_{1}, \ldots, c_{n}\right) \in C^{\perp}\right\} .
$$

It follows that $C^{\perp_{k}} \subset \widetilde{C}$. Since $\operatorname{dim}_{\mathbb{F}_{q}} C^{\perp_{k}}=\operatorname{dim}_{\mathbb{F}_{q}} \widetilde{C}$, we get $C^{\perp_{k}}=\widetilde{C}$.
The following gives a criteria of Galois LCD $\lambda$-constacyclic codes. We first take the following notations:

- $\operatorname{ord}_{\mathbb{F}_{q}^{*}}(\lambda)=r$, where $\operatorname{ord}_{\mathbb{F}_{q}^{*}}(\lambda)$ denotes the order of $\lambda$ in multiplicative group;
- $\mathbb{Z}_{r n}$ denotes the residue ring of the integer ring $\mathbb{Z}$ modulo $r n$;
- $\mathbb{Z}_{r n}^{*}$ denotes the multiplicative group consisting of units of $\mathbb{Z}_{r n}$;
- $1+\mathbb{Z}_{r n}=\{1+r t \mid t=0,1, \ldots, n-1\} \subset \mathbb{Z}_{r n}$;
- $\mu_{s}$, where $\operatorname{gcd}(s, r n)=1$, denotes the permutation of the set $\mathbb{Z}_{r n}$ given by $\mu_{s}(x)=s x$ for all $x \in \mathbb{Z}_{r n}$.

Let $m$ be the multiplicative order of $q$ modulo $r n$, i.e., $r n \mid\left(q^{m}-1\right)$ but $r n \nmid\left(q^{m-1}-1\right)$. Then, in $\mathbb{F}_{q^{m}}$, there exists a primitive $r n$th root $\theta$ of unity such that $\theta^{n}=\lambda$. It is easy to check that $\theta^{i}$ for all $i \in\left(1+\mathbb{Z}_{r n}\right)$ are all roots of $x^{n}-\lambda$. In $\mathbb{F}_{q^{m}}[x]$, we have the following decomposition:

$$
x^{n}-\lambda=\prod_{i \in\left(1+\mathbb{Z}_{r n}\right)}\left(x-\theta^{i}\right) .
$$

Since $\operatorname{gcd}(q, n)=1$ and $r \mid(q-1)$, it follows that $q \in \mathbb{Z}_{r n}^{*} \cap\left(1+\mathbb{Z}_{r n}\right)$ and $1+\mathbb{Z}_{r n}$ is $\mu_{q^{-}}$ invariant. Let $\left(1+\mathbb{Z}_{r n}\right) / \mu_{q}$ denote the set of $\mu_{q}$-orbits on $1+\mathbb{Z}_{r n}$, i.e., the set of $q$-cyclotomic cosets on $1+\mathbb{Z}_{r n}$. For any $q$-cyclotomic coset $Q$ on $1+\mathbb{Z}_{r n}$, the polynomial $M_{Q}(x)=$ $\prod_{i \in Q}\left(x-\theta^{i}\right)$ is irreducible in $\mathbb{F}_{q}[x]$. We further get a monic irreducible decomposition as follows:

$$
x^{n}-\lambda=\prod_{Q \in\left(1+\mathbb{Z}_{r n}\right) / \mu_{q}} M_{Q}(x) .
$$

The defining set of the $\lambda$-constacyclic code $C$ is defined as

$$
P=\left\{1+i r \in\left(1+\mathbb{Z}_{r n}\right) \mid \theta^{1+i r} \text { is a root of } C\right\}
$$

It is clearly to see that $P$ is a union of some $q$-cyclotomic cosets modulo $r n$ and $\operatorname{dim} C=$ $n-|P|$.

Similar to cyclic codes, there exists the following BCH bound for constacyclic codes (see[12]).

Theorem 3.5. (The BCH bound for constacyclic codes) Suppose that $\operatorname{gcd}(q, n)=1$. Let $C=\langle g(x)\rangle$ be an $\lambda$-constacyclic code of length $n$ over $\mathbb{F}_{q}$ with the roots $\left\{\theta^{1+r i} \mid i_{1} \leq i \leq\right.$ $\left.i_{1}+d-1\right\}$. Then the minimum distance of $C$ is at least $d$.

Lemma 3.6. If $\lambda^{1+p^{e-k}}=1$, then $-p^{e-k}\left(1+r \mathbb{Z}_{r n}\right)=1+r \mathbb{Z}_{r n}(\bmod r n)$.
Proof. Since $\lambda^{1+p^{e-k}}=1$ and $\operatorname{ord}_{\mathbb{F}_{q}^{*}}(\lambda)=r$, we have $r \mid\left(1+p^{e-k}\right)$. Suppose that $1+p^{e-k}=r t$. Then for $1+i r \in\left(1+\mathbb{Z}_{r n}\right)$ we have

$$
-p^{e-k}(1+i r)=-p^{e-k} i r-\left(p^{e-k}+1\right)+1=\left(-p^{e-k} i-t\right) r+1(\bmod r n) \in\left(1+\mathbb{Z}_{r n}\right)
$$

Thus, we obtain $-p^{e-k}\left(1+r \mathbb{Z}_{r n}\right)=1+r \mathbb{Z}_{r n}(\bmod r n)$.
Theorem 3.7. Let $C_{P}$ be an $\lambda$-constacyclic code of length $n$ over $\mathbb{F}_{q}$ with the defining set $P$, where $\lambda^{1+p^{e-k}}=1$. Let $\bar{P}=\left(1+\mathbb{Z}_{r n}\right) \backslash P$. Then
(1) $-p^{e-k} \bar{P}$ is the defining set of a Galois dual code of $C_{P}$, i.e., $-p^{e-k} \bar{P}$ is the defining set of the $\lambda^{-p^{e-k}}$-constacyclic code $C_{P}^{\perp_{k}}$.
(2) $C_{P}$ is a Galois LCD $\lambda$-constacyclic code if and only if $-p^{k} P=P$.

Proof. (1) According to Lemma 3.6, $-p^{e-k} \bar{P} \subset\left(1+\mathbb{Z}_{r n}\right)$. It is easy to see that $-p^{e-k} \bar{P}$ is a union of some $q$-cyclotomic cosets containing in $\left(1+\mathbb{Z}_{r n}\right)$ and $|P|+\left|-p^{e-k} \bar{P}\right|=n$.

It is clear that

$$
x^{n}-\lambda=\prod_{i \in\left(1+\mathbb{Z}_{r n}\right)}\left(x-\theta^{i}\right)=\prod_{i \in P}\left(x-\theta^{i}\right) \prod_{i \in(-\bar{P})}\left(x-\theta^{i}\right) .
$$

By Lemma 3.4(2), the generator polynomial of $\lambda^{-p^{e-k}}$-constacyclic code $C_{P}^{\perp_{k}}$ is

$$
\widetilde{h}^{p^{e-k}}(x)=h_{0}^{-p^{e-k}} x^{l} \prod_{i \in(-\bar{P})}\left(\frac{1}{x}-\theta^{-i}\right)^{p^{e-k}}=\prod_{i \in(-\bar{P})}\left(x-\theta^{i p^{e-k}}\right)=\prod_{j \in\left(-p^{e-k} \bar{P}\right)}\left(x-\theta^{j}\right) .
$$

Thus, $-p^{e-k} \bar{P}$ is the defining set of the $\lambda^{-p^{e-k}}$-constacyclic code $C_{P}^{\perp_{k}}$.
(2) Let $f_{\bar{P}}(x)=\prod_{i \in \bar{P}}\left(x-\theta^{i}\right)$. Then, according to the definition of $\bar{P}, f_{\bar{P}}(x)$ is check polynomial of $C_{P}$.

Similarly, let $f_{-p^{e-k} P}(x)=\prod_{i \in\left(-p^{e-k} P\right)}\left(x-\theta^{i}\right)$. Then, by $(1), f_{-p^{e-k} P}(x)$ is check polynomial of $C_{P}^{\perp_{k}}$. Therefore, $C_{P} \cap C_{P}^{\perp_{k}}=\{\mathbf{0}\}$ if and only if $\bar{P} \cap\left(-p^{e-k} P\right)=\phi$, i.e., $C_{P} \cap C_{P}^{\perp_{k}}=\{\mathbf{0}\}$ if and only if $-p^{e-k} P=P$. This means that $C_{P}$ is a Galois LCD $\lambda$-constacyclic code if and only if $-p^{k} P=P$.

Example 2. Let $p=11, e=3$, i.e., $q=11^{3}=1331$.Take $k=1, n=5$ and $\lambda=-1$. Then $r=2, r n=10$. Consider

$$
1+2 \mathbb{Z}_{10}=\{1,3,5,7,9\}
$$

The $q$-cyclotomic cosets modulo 10 containing in $1+2 \mathbb{Z}$ are

$$
Q_{1}=\{1\}, Q_{3}=\{3\}, Q_{5}=\{5\}, Q_{7}=\{7\}, Q_{9}=\{9\} .
$$

It is easy to check that $-11 Q_{1}=Q_{9},-11 Q_{9}=Q_{1},-11 Q_{3}=Q_{7},-11 Q_{7}=Q_{3},-11 Q_{5}=Q_{5}$. Take $P=Q_{3} \cup Q_{5} \cup Q_{7}$. Then $-11 P=P$. According to Theorem 3.7(2), $C_{P}$ is a Galois $L C D$ MDS code with parameters $[10,7,4]$.

Theorem 3.8. Let $C_{P}$ is an $\lambda$-constacyclic code of length $n$ over $\mathbb{F}_{q}$ with the defining set $P$, where $\lambda^{1+p^{e-k}}=1$. For any $P \subset\left(1+r \mathbb{Z}_{r n}\right), C_{P}$ is a Galois LCD $\lambda$-constacyclic code if and only if there exists some integer $j$ such that $p^{e j-k} \equiv-1 \bmod r n$.

Proof. Since $-p^{k} \equiv q^{j} \bmod r n$ for some integer $j$, we have $-p^{k} s \equiv q^{j} s \bmod r n$ for any $s \in P$. Thus, $-p^{k} P \subset P$.

On other hand, let $P=\cup_{i=1}^{t} Q_{s_{i}}$ for some positive integer $t$, where $Q_{s_{i}}=\left\{s_{i}, q s_{i}, \ldots, q^{m_{i}-1} s_{i}\right\} \subset$ $\left(1+r \mathbb{Z}_{r n}\right)$ and $m_{i}$ is the smallest integer satisfying $q^{m_{i}} s_{i} \equiv s_{i} \bmod r n$ for $i=1, \ldots, t$. For every $s_{i}$, we have $-s_{i} p^{k} \equiv s_{i} q^{j} \bmod r n$ since $-p^{k} \equiv q^{j} \bmod r n$. Thus, $-s_{i} p^{k} \in Q_{s_{i}}$ for $i=1, \ldots, t$. Furthermore, $s_{i} \equiv s_{i} q^{m^{i}}=s_{i} q^{j} q^{m_{i}-j} \equiv-s_{i} p^{k} q^{m_{i}-j}=-p^{k}\left(s_{i} q^{m_{i}-j}\right) \bmod r n$, which implies $s_{i} \in-p^{k} Q_{s_{i}} \subset-p^{k} P$.

Therefore, $-p^{k} P=P$, i.e., $C_{P}$ is a Galois LCD $\lambda$-constacyclic code.
Conversely, take $P_{1}=\left\{1, q, q^{2}, \ldots, q^{m_{1}-1}\right\}$, where $m_{1}$ is the smallest integer satisfying $q^{m_{1}} \equiv 1 \bmod r n$. By the assumption, $C_{P_{1}}$ is a Galois LCD $\lambda$-constacyclic code. According to Theorem 3.7(2), we have $-p^{k} P_{1}=P_{1}$. This means that there exists some integer $j$ such that $-p^{k}=q^{j} \bmod r n$, i.e., $p^{e j-k} \equiv-1 \bmod r n$.

Remark 3.9. It follows from Theorem 3.8 that if rn divides $1+p^{e j-k}$, then every $\lambda$ constacyclic code of length $n$ over $\mathbb{F}_{q}$ is a Galois LCD code.

In light of the proof of above theorem, the following two corollaries are straightforward.
Corollary 3.10. If $Q_{s} \subset\left(1+r \mathbb{Z}_{r n}\right)$ is an $q$-cyclotomic coset modulo $n r$, then $-p^{k} Q_{s}=Q_{s}$ if and only if $s\left(1+p^{e j-k}\right) \equiv 0 \bmod r n$ for some integer $j$.

Corollary 3.11. Let $P=Q_{s} \cup\left(-p^{k} Q_{s}\right)$ and $-p^{k} Q_{s} \neq Q_{s}$, where $Q_{s} \subset\left(1+r \mathbb{Z}_{r n}\right)$. Then $-p^{k} P=P$ if and only if $p^{2 k} s \equiv q^{j} s \bmod r n$ for some integer $j$.

Theorem 3.12. Let $C_{P}$ is an $\lambda$-constacyclic code of length $n$ over $\mathbb{F}_{q}$ with the defining set $P$, where $\lambda^{1+p^{e-k}}=1$. For any $P \subset\left(1+r \mathbb{Z}_{r n}\right), C_{P}$ is a Galois LCD $\lambda$-constacyclic code if and only if $Q_{1}=Q_{-p^{k}}$, where $Q_{1}=\left\{1, q, q^{2}, \ldots, q^{m_{1}-1}\right\}$, where $m_{1}$ is the smallest integer satisfying $q^{m_{1}} \equiv 1 \bmod r n$.

Proof. If $Q_{1}=Q_{-p^{k}}$, then for any $s \in\left(1+r \mathbb{Z}_{r n}\right), Q_{s}=Q_{-p^{k} s}$. For any $P \subset\left(1+r \mathbb{Z}_{r n}\right)$, we know that $P=\cup_{i=1}^{t} Q_{s_{i}}$ with $Q_{s_{i}} \in\left(1+\mathbb{Z}_{r n}\right) / \mu_{q}$. Thus $-p^{k} P=P$, i.e., $C_{P}$ is a Galois LCD $\lambda$-constacyclic code.

Conversely suppose for any $P \subset\left(1+r \mathbb{Z}_{r n}\right), C_{P}$ is a Galois LCD $\lambda$-constacyclic code. In particular, setting $P=Q_{1}, C_{Q_{1}}$ is a Galois LCD $\lambda$-constacyclic code. Therefore, $-p^{k} Q_{1}=$ $Q_{1}$, which implies that $Q_{1}=Q_{-p^{k}}$.

Lemma 3.13. Let $p$ be an odd prime and $n$ a positive integer such that $\operatorname{ord}_{r n}\left(p^{e-k}\right)=2$. If the group $\mathbb{Z}_{r n}^{*}$ has a unique element of order 2 , i.e., $[-1]_{r n}$ is a unique element of order 2 in $\mathbb{Z}_{r n}^{*}$, where $[-1]_{r n}$ denotes the reside class modulo rn containing -1 , then, for any $s \in\left(1+\mathbb{Z}_{r n}\right)$, we have $Q_{s}=-p^{k} Q_{s}$.

Proof. According to the assumption that $\operatorname{ord}_{r n}\left(p^{e-k}\right)=2$, from the assumption that $[-1]_{r n}$ is a unique element of order 2 in $\mathbb{Z}_{r n}^{*}$, it follows that $p^{e-k} \equiv-1 \bmod r n$, i.e., $p^{e} \equiv$ $-p^{k} \bmod r n$. Therefore, for any $s \in\left(1+\mathbb{Z}_{r n}\right)$, we have $-p^{k} Q_{s}=Q_{s}$.

Example 3. Let $p=5, e=3$, i.e., $q=5^{3}=125$. Take $k=1, n=13$ and $\lambda=-1$. Then $r=2, r n=26$. Consider

$$
1+2 \mathbb{Z}_{26}=\{1,3,5,7,9,11,13,15,17,19,21,23,25\} .
$$

The $q$-cyclotomic cosets modulo 26 containing in $1+2 \mathbb{Z}_{26}$ are

$$
Q_{1}=\{1,5,21,25\}, Q_{3}=\{3,11,15,23\}, Q_{7}=\{7,9,17,19\}, Q_{13}=\{13\}
$$

Since $5^{3-1} \equiv-1 \bmod 26$, according to Theorem 3.8 produce 15 Galois LCD codes, but only 5 types of parameters $[13,12,2],[13,9,4],[13,8,4],[13,4,8],[13,5,7]$, respectively.

Example 4. Let $p=13, e=3$, i.e., $q=13^{3}=2197$. Taking $k=2, n=9$ and $\lambda=-1$. Then $r=2, r n=18$. Consider

$$
1+2 \mathbb{Z}_{18}=\{1,3,5,7,9,11,13,15,17\}
$$

The $q$-cyclotomic cosets modulo 18 containing in $1+2 \mathbb{Z}_{18}$ are $Q_{1}=\{1\}, Q_{3}=\{3\}, Q_{5}=$ $\{5\}, Q_{7}=\{7\}, Q_{9}=\{9\}, Q_{11}=\{11\}, Q_{13}=\{13\}, Q_{15}=\{15\}, Q_{17}=\{17\}$. It is easy to check that $-13^{2} Q_{1}=Q_{11},-13^{2} Q_{11}=Q_{13},-13^{2} Q_{13}=Q_{17},-13^{2} Q_{17}=Q_{7},-13^{2} Q_{7}=$ $Q_{5},-13^{2} Q_{5}=Q_{1},-13^{2} Q_{3}=Q_{15},-13^{2} Q_{15}=Q_{3},-13^{2} Q_{9}=Q_{9}$. Taking $P_{1}=\{1,5,7,11,13,17\}$ $P_{2}=\{1,5,7,9,11,13,17\}, P_{3}=\{1,3,5,7,11,13,15,17\}, P_{4}=\{3,15\}, P_{5}=\{3,9,15\}$, and $P_{6}=\{9\}$. Then $-13^{2} P_{i}=P_{i}$ for $1 \leq i \leq 6$. According to Theorem 3.7(2), $C_{P_{1}}$ is a Galois $L C D$ code with parameters $[9,3,3], C_{P_{2}}$ is a Galois LCD code with parameters [9, 2, 6], $C_{P_{3}}$ is a Galois LCD MDS code with parameters $[9,1,9], C_{P_{4}}$ is a Galois LCD code with parameters $[9,7,2], C_{P_{5}}$ is a Galois LCD code with parameters $[9,6,2], C_{P_{6}}$ is a Galois LCD MDS code with parameters $[9,8,2]$.

## 4 Hermitian LCD constacyclic codes over $\mathbb{F}_{q}$

In this section we study constacyclic Hermitian LCD codes over $\mathbb{F}_{q}$, where $q=p^{e}$ and $e=2 a$. By Theorem 3.7, 3.8, 3.12, we have the following corollary.

Corollary 4.1. Let $C_{P}$ is an $\lambda$-constacyclic code of length $n$ over $\mathbb{F}_{q}$ with the defining set $P$, where $\lambda^{1+p^{a}}=1$. For any $P \subset\left(1+r \mathbb{Z}_{r n}\right), C_{P}$ is a Hermitian LCD $\lambda$-constacyclic code if and only if one of the following statements holds
(1) $-p^{a} P=P$.
(2) $Q_{1}=Q_{-p^{a}}$, where $Q_{1}=\left\{1, q, q^{2}, \ldots, q^{m_{1}-1}\right\}$, where $m_{1}$ is the smallest integer satisfying $q^{m_{1}} \equiv 1 \bmod r n$.
(3) There exists some integer $j$ such that $\left(p^{a}\right)^{2 j-1} \equiv-1 \bmod r n$.

By Corollary 3.3, if $\lambda^{1+p^{a}} \neq 1$, then any $\lambda$-constacyclic code of length $n$ over $\mathbb{F}_{p^{2 a}}$ is Hermitian LCD. Then we only need to look at the classes of $\lambda$-constacyclic codes where $\lambda^{1+p^{a}}=1$. In this case, we gives a necessary condition for any $\lambda$-constacyclic code of length $n$ over $\mathbb{F}_{p^{2 a}}$ to be Hermitian LCD.

Theorem 4.2. Let $\lambda$ be a primite $r$ th root of unity over $\mathbb{F}_{p^{2 a}}$ and $r=2^{b_{1}} r^{\prime}\left(b_{1}>0, r^{\prime}\right.$ odd $)$. Let $n=2^{b_{2}} n^{\prime}\left(b_{2}>0, n^{\prime}\right.$ odd $)$, and let $q$ be an odd prime power such that $(n, q)=1$. If for any $\lambda$-constacyclic code of length $n$ over $\mathbb{F}_{p^{2 a}}$ is Hermitian $L C D$, then $r \mid p^{a}+1$ and $p^{a}+1 \equiv 0\left(\bmod 2^{b_{1}+b_{2}}\right)$.

Proof. By Corollary 4.1 (3), there exists some integer $j$ such that $\left(p^{a}\right)^{2 j-1} \equiv-1 \bmod r n$. Therefore,

$$
r n\left|\left(p^{a}\right)^{2 j-1}+1 \Rightarrow 2^{b_{1}+b_{2}} r^{\prime} n^{\prime}\right| \frac{\left(p^{a}\right)^{2 j-1}+1}{p^{a}+1}\left(p^{a}+1\right)
$$

Obviously, $\frac{\left(p^{a}\right)^{2 j-1}+1}{p^{a}+1}$ is odd, we have $p^{a}+1 \equiv 0\left(\bmod 2^{b_{1}+b_{2}}\right)$.
We can check the following lemma.
Lemma 4.3. For some positive integer t, let $A=\cup_{i=1}^{t} Q_{s_{i}}$, where $Q_{s_{i}}=\left\{s_{i}, p^{e} s_{i}, \ldots,\left(p^{e}\right)^{m_{i}-1} s_{i}\right\} \subset$ $\left(1+r \mathbb{Z}_{r n}\right)$ and $m_{i}$ is the smallest integer satisfying $\left(p^{a}\right)^{m_{i}} s_{i} \equiv s_{i} \bmod$ rn for $i=1, \ldots, t$. Assume that $-p^{a} s_{i} \equiv v_{1 i} \bmod r n,-p^{a} s_{i}\left(p^{e}\right) \equiv v_{2 i} \bmod r n, \ldots,-p^{a} s_{i}\left(p^{e}\right)^{m_{i}-1} \equiv v_{m_{i}, i} \bmod r n$, for $i=$ $1, \ldots, t$, and $P=\left(\cup_{i=1}^{t} Q_{s_{i}}\right) \cup\left(\cup_{i=1}^{t}\left\{v_{1 i}, \ldots, v_{m_{i}, i}\right\}\right)$. Then $-p^{a} P=P$.

Theorem 4.4. Let $Q_{s_{1}}, \ldots, Q_{s_{t}}, Q_{w_{1}}, \ldots, Q_{w_{l}}$ be all the $p^{2 a}$-cyclotomic cosets containing in $\left(1+r \mathbb{Z}_{r n}\right)$ which satisfy $-p^{a} Q s_{i}=Q s_{i}, i=1, \ldots, t$, and $-p^{a} Q_{w_{j}} \neq Q_{w_{j}}, j=1, \ldots, l$. Then the total number of Hermitian LCD $\lambda$-constacyclic codes of length $n$ over $\mathbb{F}_{p^{e}}$ is equal to $2^{t+h}-1$, where $l=2 h$ and $h$ is a positive integer.

Proof. Let $-p^{a} Q_{w_{j}}=Q_{w_{\bar{j}}}$ and $\tau$ be a permutation of $\{1,2, \ldots, l\}$ which satisfies $\tau(j)=\bar{j}$ for $j=1,2, \ldots, l$. Then we have $Q_{w_{\tau(j)}}=Q_{w_{\tau^{2}(j)}}=Q_{w_{j}}$ since $p^{2 a} Q_{w_{j}}=Q_{w_{j}}$.

Hence, $\tau^{2}(j)=j$ for $j=1,2, \ldots, l$, i.e., $\tau^{2}=I$, where $I$ denote an identity permutation of $\{1,2, \ldots, l\}$

By assumption, we know that $\tau(j) \neq j$ for $j=1,2, \ldots, l$. Since $\tau^{2}=I, \tau$ must be a product of mutually disjoint transpositive like $\left(x_{1} y_{1}\right) \ldots\left(x_{h} y_{h}\right)$. Therefore, $l=2 h$ for some integer $h$. Without loss of generality, for $j=1,2, \ldots, h$, we assume that $\tau(j)=h+j$, then $\tau(h+j)=j$. Hence, by Lemma 4.3, the total number of Hermitian LCD $\lambda$-constacyclic codes of length $n$ over $\mathbb{F}_{p^{e}}$ is equal to $2^{t+h}-1$.

Remark 4.5. It follows from the proof of Theorem 4.4 that if $-p^{a} Q_{w_{j}}=Q_{w_{\bar{j}}}$ then $-p^{a} Q_{w_{\bar{j}}}=$ $Q_{w_{j}}$.

Example 5. Let $p=11, e=2$, i.e., $q=11^{2}=121$. Taking $n=10$ and $\lambda=1$. Then $r=1, r n=10$. Thus, $q$-cyclotomic cosets modulo 10 are $Q_{0}=\{0\}, Q_{1}=\{1\}, Q_{2}=$ $\{2\}, Q_{3}=\{3\}, Q_{4}=\{4\}, Q_{5}=\{5\}, Q_{6}=\{6\}, Q_{7}=\{7\}, Q_{8}=\{8\}, Q_{9}=\{9\}$. It is easy to check that $-11 Q_{1}=Q_{1},-11 Q_{2}=Q_{8},-11 Q_{3}=Q_{7},-11 Q_{4}=Q_{6},-11 Q_{5}=Q_{5}$. By Theorem 4.4, the total number of Hermitian LCD cyclic codes of length 10 over $\mathbb{F}_{11^{2}}$ is equal to 63. Taking $P_{1}=\{4,5,6\}, P_{2}=\{3,4,5,6,7\}$, and $P_{3}=\{2,3,4,5,6,7,8\}$. Then $C_{P_{1}}, C_{P_{2}}$ and $C_{P_{3}}$ are Hermitian LCD MDS codes with parameters $[10,7,4],[10,5,6]$ and $[10,3,7]$, respectively.

Lemma 4.6. Let $p$ be an odd prime and $n$ a positive integer such that $\operatorname{ord}_{r n}\left(p^{a}\right)=2(1+2 j)$ for some integer $j$. If the group $\mathbb{Z}_{r n}^{*}$ has a unique element of order 2 , i.e., $[-1]_{r n}$ is a unique element of order 2 in $\mathbb{Z}_{r n}^{*}$, where $[-1]_{r n}$ denotes the residue class modulo rn containing -1 , then, for $0 \leq i \leq n-1,-p^{a} Q_{1+r i}=Q_{1+r i}$, where $Q_{1+r i}$ is an $p^{2 a}$-cyclotomic coset modulo rn containing $1+r i$ in $1+r \mathbb{Z}_{r n}$, and $\left|Q_{1}\right|=1+2 j$.

Proof. Since $\left(p^{a}\right)^{2(1+2 j)} \equiv 1 \bmod r n$, from the assumption that $[-1]_{r n}$ is a unique element of order 2 in $\mathbb{Z}_{r n}^{*}$, it follows that $\left(p^{a}\right)^{1+2 j} \equiv-1 \bmod r n$. Therefore,

$$
\left(p^{2 a}\right)^{1+j} \equiv-p^{a} \bmod r n
$$

This implies that $-p^{a}(1+r i) \equiv(1+r i)\left(p^{2 a}\right)^{1+j} \bmod r n$, i.e., $-p^{a} Q_{1+r i}=Q_{1+r i}$, for $0 \leq i \leq$ $n-1$.

As $\operatorname{ord}_{r n}\left(p^{a}\right)=2(1+2 j)$, obviously, $\left|Q_{1}\right|=1+2 j$.
Using the aforementioned lemma, some optimal Hermitian LCD $\lambda$-constacyclic codes of length $n$ over $\mathbb{F}_{q}$ can be constructed.

Theorem 4.7. Let $p$ be an odd prime and $n$ a positive integer such that $\operatorname{ord}_{r n}\left(p^{a}\right)=2$. If the group $\mathbb{Z}_{r n}^{*}$ has a unique element of order 2 , i.e., $[-1]_{r n}$ is a unique element of order 2 in $\mathbb{Z}_{r n}^{*}$, where $[-1]_{r n}$ denotes the residue class modulo rn containing -1 , then, for $2 \leq d \leq n$, there exists a Hermitian LCD MDS $\lambda$-constacyclic code with parameters $[n, n+1-d, d]$.

Proof. By Lemma 4.6, we have $Q_{1}=-p^{a} Q_{1}=\{1\}$ and $-p^{a} \equiv 1 \bmod r n$. Therefore, for each $i, 1 \leq i \leq n-1$,

$$
\begin{equation*}
-p^{a}(1+r i) \equiv 1+r i \bmod r n . \tag{4.1}
\end{equation*}
$$

According to the assumption that $\operatorname{ord}_{r n}\left(p^{a}\right)=2$, we obtain

$$
\begin{equation*}
p^{2 a}(1+r i) \equiv 1+r i \bmod r n . \tag{4.2}
\end{equation*}
$$

Combing Equation $(4,1)$ and $(4,2)$, for each $i, 1 \leq i \leq n-1$, we show that $-p^{a} Q_{1+r i}=$ $Q_{1+r i}=\{1+r i\}$.

Let the defining set of an $\lambda$-constacyclic code $C=\langle g(x)\rangle$ of length $n$ over $\mathbb{F}_{q}$ be the set $P=\cup_{i=0}^{d-2} Q_{1+r i}$, where $2 \leq d \leq n$. Then $-p^{a} P=P$. By Corollary 4.1, the code $C$ is a Hermitian LCD $\lambda$-constacyclic code and $\operatorname{dim} C=n+1-d$. Obviously, the defining set $P$ consists of $d-1$ consecutive integers $\{1,1+r, 1+2 r, \ldots, 1+(d-2) r\}$. Using Theorem 3.5, the minimum distance of $C$ is at least $d$. Thus, we conclude that $C$ is an $\lambda$-constacyclic Hermitian LCD code with parameters $[n, n+1-d, \geq d]$. Applying the classical code Singleton bound to $C$ yields a Hermitian LCD MDS $\lambda$-constacyclic code with parameters $[n, n+1-d, d]$.

Example 6. Let $p=3, a=2, r=2$, and $n=5$. Then $p^{2 a} \equiv 1 \bmod 10$ and $\mathbb{Z}_{10}^{*}=\{1,3,7,9\}$ has a unique element 9 of order 2. Applying Theorem 4.7 produce 9 Hermitian LCD MDS negacyclic codes with parameters $[5,1,5],[5,2,4],[5,3,3],[5,4,2]$, respectively.

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