# EXISTENCE OF PRIMITIVE 2-NORMAL ELEMENTS IN FINITE FIELDS 

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#### Abstract

An element $\alpha \in \mathbb{F}_{q^{n}}$ is normal over $\mathbb{F}_{q}$ if $\mathcal{B}=\left\{\alpha, \alpha^{q}, \alpha^{q^{2}}, \cdots, \alpha^{q^{n-1}}\right\}$ forms a basis of $\mathbb{F}_{q^{n}}$ as a vector space over $\mathbb{F}_{q}$. It is well known that $\alpha \in \mathbb{F}_{q^{n}}$ is normal over $\mathbb{F}_{q}$ if and only if $g_{\alpha}(x)=\alpha x^{n-1}+\alpha^{q} x^{n-2}+\cdots+\alpha^{q^{n-2}} x+\alpha^{q^{n-1}}$ and $x^{n}-1$ are relatively prime over $\mathbb{F}_{q^{n}}$, that is, the degree of their greatest common divisor in $\mathbb{F}_{q^{n}}[x]$ is 0 . Using this equivalence, the notion of $k$-normal elements was introduced in Huczynska et al. (2013): an element $\alpha \in \mathbb{F}_{q^{n}}$ is $k$-normal over $\mathbb{F}_{q}$ if the greatest common divisor of the polynomials $g_{\alpha}[x]$ and $x^{n}-1$ in $\mathbb{F}_{q^{n}}[x]$ has degree $k$; so an element which is normal in the usual sense is 0 -normal.

Huczynska et al. made the question about the pairs $(n, k)$ for which there exist primitive $k$-normal elements in $\mathbb{F}_{q^{n}}$ over $\mathbb{F}_{q}$ and they got a partial result for the case $k=1$, and later Reis and Thomson (2018) completed this case. The Primitive Normal Basis Theorem solves the case $k=0$. In this paper, we solve completely the case $k=2$ using estimates for Gauss sum and the use of the computer, we also obtain a new condition for the existence of $k$-normal elements in $\mathbb{F}_{q^{n}}$.


## 1. Introduction

Let $\mathbb{F}_{q^{n}}$ be a finite field with $q^{n}$ elements, where $q$ is a prime power and $n$ is a positive integer. An element $\alpha \in \mathbb{F}_{q^{n}}^{*}$ is primitive if $\alpha$ generates the cyclic multiplicative group $\mathbb{F}_{q^{n}}^{*}\left(\alpha\right.$ has multiplicative order $\left.q^{n}-1\right)$. Also, $\alpha \in \mathbb{F}_{q^{n}}$ is normal over $\mathbb{F}_{q}$ if the set $B_{\alpha}=\left\{\alpha^{q^{i}} \mid 0 \leq i \leq n-1\right\}$ spans $\mathbb{F}_{q^{n}}$ as a $\mathbb{F}_{q^{-}}$-vector space, in this case we say that $B_{\alpha}$ is a normal basis. Normal basis are frequently used in cryptography and computer algebra systems due to the efficiency of exponentiation. Primitive elements are constantly used in cryptographic applications such as discrete logarithm problem and pseudorandom number generators [12]. If we put these two properties together, we obtain a primitive normal element. We can study the
multiplicative structure of $\mathbb{F}_{q^{n}}$ and at the same time see $\mathbb{F}_{q^{n}}$ as a vector space over $\mathbb{F}_{q}$. The Primitive Normal Basis Theorem states that for any extension field $\mathbb{F}_{q^{n}}$ of $\mathbb{F}_{q}$, there exists a basis composed by primitive normal elements; this result was first proved by Lenstra and Schoof [7] using a combination of character sums, sieving results and a computer search.

One can prove that an element $\alpha \in \mathbb{F}_{q^{n}}$ is normal if and only if the polynomial $g_{\alpha}(x)=\alpha x^{n-1}+\alpha^{q} x^{n-2}+\ldots+\alpha^{q^{n-2}} x+\alpha^{q^{n-1}}$ and $x^{n}-1$ are relatively prime over $\mathbb{F}_{q^{n}}$ [8, Theorem 2.39]. With this as motivation, Huczynska et al. [4] introduced the concept of $k$-normal elements, as an extension of the usual normal elements:

Definition 1.1. Let $\alpha \in \mathbb{F}_{q^{n}}$ and let $g_{\alpha}(x)=\sum_{i=0}^{n-1} \alpha^{q^{i}} x^{n-1-i} \in \mathbb{F}_{q^{n}}[x]$. If $\operatorname{gcd}\left(x^{n}-\right.$ $1, g_{\alpha}(x)$ ) over $\mathbb{F}_{q^{n}}$ has degree $k$ (where $0 \leq k \leq n-1$ ), then $\alpha$ is a $k$-normal element of $\mathbb{F}_{q^{n}}$ over $\mathbb{F}_{q}$. ${ }^{1}$

The $k$-normal elements can be used to reduce the multiplication process in finite fields, see [9]. From the above definition, elements which are normal in the usual sense are 0-normal and from the Primitive Normal Basis Theorem, we know that they always exist. There are several criteria in the literature for the existence of $k$-normal elements (see for example [10], [15], [16]). In [4] the authors worked out the case $k=1$, and partially established a Primitive 1-normal element Theorem. Reis and Thompson completed the case $k=1$ in [11].

A question which naturally arises is: for which values of $k$ one has a Primitive $k$ normal element Theorem (see [4, Problem 6.3])? On this line, in [10], Reis obtained a sufficient condition for the existence of primitive $k$-normal elements, and he proved that given $\epsilon>0$, for $q$ sufficiently large, there exist primitive $k$-normal elements for $k \in\left[0,\left(\frac{1}{2}-\epsilon\right) n\right]$, whenever $k$-normal elements actually exist in $\mathbb{F}_{q^{n}}$. Since this is an asymptotic result for $q$, it is not possible to conclude the result for specific values, but from the condition that he obtained it is possible to generalize and study particular cases of $k$.

Since the cases $k=0$ and $k=1$ are completely finished, in this paper we study the case $k=2$ as follows: in Section 2, we provide background material that is used along the paper. In Section 3, we present two general conditions for the existence of primitive $k$-normal elements in $\mathbb{F}_{q^{n}}$ over $\mathbb{F}_{q}$, as well as some weaker conditions

[^0]for some particular cases. In Section 4, we apply the results from previous sections to prove all cases for $n \geq 8$ and also for $q \leq 19$. In Section 5 we study the cases $n=5,6,7$ developing new ideas based on the factorization of certain divisors of $q^{n}-1$. Finally, in Section 6 we study the remaining case $n=4$, where we prove that there exist primitive 2-normal elements in $\mathbb{F}_{q^{4}}$ if and only if $q \equiv 1(\bmod 4)$. For this last case we develop Gauss sum which is different from the ones used in the previous cases.

Our results can be summarized in the following theorem.
Theorem 1.2 (The Primitive 2-Normal Theorem). Let $q$ be a prime power and $n$ be a natural number. There exists a primitive 2-normal element in $\mathbb{F}_{q^{n}}$ if and only if $n \geq 5$ and $\operatorname{gcd}\left(q^{3}-q, n\right) \neq 1$ or $n=4$ and $q \equiv 1(\bmod 4)$.

In Appendix A we show the SageMath procedures that we used in the paper and in Appendix B we present tables with primitive 2-normal elements for specific cases.

## 2. Preliminaries

In this section, we present some definitions and results that will be useful in the rest of this paper. We start with the following definitions.

Definition 2.1. (a) Let $f(x)$ be a monic polynomial with coefficients in $\mathbb{F}_{q}$. The Euler Totient Function for polynomials over $\mathbb{F}_{q}$ is given by

$$
\Phi_{q}(f)=\left|\left(\frac{\mathbb{F}_{q}[x]}{\langle f\rangle}\right)^{*}\right|
$$

where $\langle f\rangle$ is the ideal generated by $f(x)$ in $\mathbb{F}_{q}[x]$.
(b) If $t$ is a positive integer (or a monic polynomial over $\mathbb{F}_{q}$ ), $W(t)$ denotes the number of square-free (monic) divisors of $t$.
(c) If $f(x)$ is a monic polynomial with coefficients in $\mathbb{F}_{q}$, the Polynomial Möbius Function $\mu_{q}$ is given by $\mu_{q}(f)=0$ if $f$ is not square-free and $\mu_{q}(f)=(-1)^{r}$ if $f$ is a product of $r$ distinct irreducible factors over $\mathbb{F}_{q}$.

We have an interesting formula for the number of $k$-normal elements over finite fields:

Theorem 2.2. (4], Theorem 3.5) The number of $k$-normal elements of $\mathbb{F}_{q^{n}}$ over $\mathbb{F}_{q}$ is given by

$$
\sum_{\substack{h \mid x^{n}-1 \\ \operatorname{deg}(h)=n-k}} \Phi_{q}(h)
$$

where the divisors are monic and the polynomial division is over $\mathbb{F}_{q}$.
2.1. Linearized polynomials and the $\mathbb{F}_{q}$-order. Here we present some definitions and basic results on linearized polynomials over finite fields that are frequently used in this paper. A useful feature of these polynomials is the structure of the set of roots that facilitates the determination of the roots, see [8, Section 3.4].

Definition 2.3. Let $f \in \mathbb{F}_{q}[x]$ with $f(x)=\sum_{i=0}^{r} a_{i} x^{i}$.
(a) The polynomial $L_{f}(x):=\sum_{i=0}^{r} a_{i} x^{q^{i}}$ is the linearized $q$-associate of $f$.
(b) For $\alpha \in \mathbb{F}_{q^{n}}$, we set $L_{f}(\alpha)=\sum_{i=0}^{r} a_{i} \alpha^{q^{i}}$.

The polynomial $L_{f}$ induces a linear transformation of $\mathbb{F}_{q^{n}}$ over $\mathbb{F}_{q}$ that also has additional properties:

Lemma 2.4. [8, Lemma 3.59] Let $f, g \in \mathbb{F}_{q}[x]$. The following hold:
(a) $L_{f}(x)+L_{g}(x)=L_{f+g}(x)$;
(b) $L_{f g}(x)=L_{f}\left(L_{g}(x)\right)=L_{g}\left(L_{f}(x)\right)$.

Lemma 2.5. Let $f, g \in \mathbb{F}_{q}[x]$ such that $f g=x^{n}-1$. For every $\alpha \in \mathbb{F}_{q^{n}}$, we have that $L_{g}(\alpha)=0$ if and only if $\alpha=L_{f}(\beta)$ for some $\beta \in \mathbb{F}_{q^{n}}$.

Proof. Observe that $f g=x^{n}-1$ implies $L_{g} \circ L_{f}=L_{f} \circ L_{g}=L_{x^{n}-1}=0$, so $\operatorname{Im} L_{f} \subset \operatorname{Ker} L_{g}$ and $\operatorname{Im} L_{g} \subset \operatorname{Ker} L_{f}$. On the other hand, $L_{g}$ has degree $q^{\operatorname{deg} g}$ and Ker $L_{g}$ has at most dimension $\operatorname{deg} g$. Conversely, we have that $\operatorname{Im} L_{g}$ has at most dimension $\operatorname{deg} f=n-\operatorname{deg} g$. So, we get that $\operatorname{Ker} L_{g}$ has dimension exactly $\operatorname{deg} g$ and $\operatorname{Im} L_{f}=\operatorname{Ker} L_{g}$, since $n=\operatorname{dim}_{\mathbb{F}_{q}} \operatorname{Im} L_{g}+\operatorname{dim}_{\mathbb{F}_{q}} \operatorname{Ker} L_{g} \leq(n-\operatorname{deg} g)+\operatorname{deg} g$.

Let $D \in \mathbb{F}_{q}[x]$ be a monic polynomial. We say that an element $\alpha \in \mathbb{F}_{q^{n}}$ has $\mathbb{F}_{q}$-order $D$ if $D$ is the lowest degree monic polynomial such that $L_{D}(\alpha)=0$. It is known that the $\mathbb{F}_{q^{-}}$-order of an element $\alpha \in \mathbb{F}_{q^{n}}$ divides $x^{n}-1$ and we also have the following equivalences.

Theorem 2.6. ([4], Theorem 3.2) Let $\alpha \in \mathbb{F}_{q^{n}}$. The following three properties are equivalent:
(i) $\alpha$ is $k$-normal over $\mathbb{F}_{q}$.
(ii) Let $V_{\alpha}$ be the $\mathbb{F}_{q}$-vector space generated by $\left\{\alpha, \alpha^{q}, \ldots, \alpha^{q^{n-1}}\right\}$, then $\operatorname{dim} V_{\alpha}$ is $n-k$.
(iii) $\alpha$ has $\mathbb{F}_{q}$-order of degree $n-k$.
2.2. Freeness and Characters. We present the concept of freeness, introduced in Carlitz [1] and Davenport [3], and refined in Lenstra and Schoof (see [7]). This concept is useful in the construction of certain characteristic functions over finite fields.

Definition 2.7. (a) Let $m \mid\left(q^{n}-1\right)$, we say that $\alpha \in \mathbb{F}_{q^{n}}^{*}$ is $m$-free if, for every $d \mid m$ and $\beta \in \mathbb{F}_{q^{n}}, \alpha=\beta^{d}$ implies that $d=1$.
(b) Let $M \mid\left(x^{n}-1\right)$, we say that $\alpha \in \mathbb{F}_{q^{n}}$ is $M$-free if, for every $h \mid M$ and $\beta \in \mathbb{F}_{q^{n}}, \alpha=L_{h}(\beta)$ implies that $h=1$.

It is well known that an element $\alpha \in \mathbb{F}_{q^{n}}^{*}$ is primitive if and only if $\alpha$ is $\left(q^{n}-1\right)$-free and $\alpha \in \mathbb{F}_{q^{n}}$ is normal if and only if $\alpha$ is $\left(x^{n}-1\right)$-free.

Also, from the definition, we have that if $\alpha$ is $m$-free then $\alpha$ is $e$-free, for any $e \mid m$ (analogous result for polynomial).

Following the notation in [2], we can characterize the freeness of an element. For the multiplicative part: a multiplicative character $\eta$ of $\mathbb{F}_{q^{n}}^{*}$ is a group homomorphism of $\mathbb{F}_{q^{n}}^{*}$ to $\mathbb{C}^{*}$, whose order is the least positive integer $d$ such that $\eta(\alpha)^{d}=1$ for any $\alpha \in \mathbb{F}_{q^{n}}^{*}$. Let $m$ be a divisor of $q^{n}-1$ and let $\int_{d \mid m} \eta_{d}$ denote the sum $\sum_{d \mid m} \frac{\mu(d)}{\varphi(d)} \sum_{(d)} \eta_{d}$, where $\eta_{d}$ is a multiplicative character of $\mathbb{F}_{q^{n}}$, and the sum $\sum_{(d)} \eta_{d}$ runs over all multiplicative characters of order $d$. It is known that there exist $\varphi(d)$ of those characters.

For the additive part: if $p$ is the characteristic of $\mathbb{F}_{q}$, for $\alpha \in \mathbb{F}_{q^{n}}$, let $\chi_{\alpha}: \mathbb{F}_{q^{n}} \longrightarrow \mathbb{C}$ be the additive character defined by

$$
\chi_{\alpha}(\beta)=e^{\frac{2 \pi i}{p} \operatorname{Tr}_{q^{n} / p}(\alpha \beta)}, \quad \beta \in \mathbb{F}_{q^{n}}
$$

where $\operatorname{Tr}_{q^{n} / p}$ is the trace function of $\mathbb{F}_{q^{n}}$ over $\mathbb{F}_{p}$. It is well known that any additive character of $\mathbb{F}_{q^{n}}$ is of this form. We say that the additive character $\chi_{\alpha}$ has $\mathbb{F}_{q^{\prime}}$-order $D$ if $\alpha$ has $\mathbb{F}_{q}$-order $D$. We use the notation $\int_{D \mid T} \chi_{\delta_{D}}$ to represent $\sum_{D \mid T} \frac{\mu_{q}(D)}{\Phi_{q}(D)} \sum_{\left(\delta_{D}\right)} \chi_{\delta_{D}}$ where $\chi_{\delta_{D}}$ runs through all characters of $\mathbb{F}_{q}$-order $D$. It is known that there exist $\Phi_{q}(D)$ of those characters.

For each divisor $m$ of $q^{n}-1$ and each monic divisor $T \in \mathbb{F}_{q}[x]$ of $x^{n}-1$, set $\theta(m)=\frac{\varphi(m)}{m}$ and $\Theta(T)=\frac{\Phi_{q}(T)}{q^{\operatorname{deg}(T)}}$.

Proposition 2.8. Let $m$ be a divisor of $q^{n}-1$ and $T \in \mathbb{F}_{q}[x]$ be a monic divisor of $x^{n}-1$. For any $\alpha \in \mathbb{F}_{q^{n}}$ we have
(i)

$$
w_{m}(\alpha)=\theta(m) \int_{d \mid m} \eta_{d}(\alpha)= \begin{cases}1, & \text { if } \alpha \text { is } m-\text { free } \\ 0, & \text { otherwise }\end{cases}
$$

(ii)

$$
\Omega_{T}(\alpha)=\Theta(T) \int_{D \mid M} \chi_{D}(\alpha)= \begin{cases}1, & \text { if } \alpha \text { is } T \text {-free }, \\ 0, & \text { otherwise }\end{cases}
$$

Proof. See [4, section 5.2] or [10, Theorem 2.15]. Extending the multiplicative characters $\eta$ to $0 \in \mathbb{F}_{q^{n}}$ by setting $\eta(0)=0$, we can easily see that $w_{m}(0)=0$.
2.3. Estimates. To finish this section, we present some estimates that are used along the next sections.

We will need the following result, which is modeled after [2, Lemma 3.3] and [5. Lemma 4.1] and, like these results, is proved using the multiplicativity of the function $W(\cdot)$ and the fact that if a positive integer $M$ has $l$ distinct prime divisors then $W(M)=2^{l}$.

Lemma 2.9. Let $M$ be a positive integer and $t$ be a positive real number. Then $W(M) \leq A_{t} \cdot M^{\frac{1}{t}}$, where

$$
A_{t}=\prod_{\substack{\wp^{\alpha_{\odot}} 2^{t} \\ \wp_{i} \text { is prime } \\ \wp_{\wp}^{\alpha_{\wp}} \mid M}} \frac{2}{\sqrt[t]{\wp^{\alpha_{\S}}}},
$$

and for any prime $\wp, \alpha_{\wp}$ is defined as the largest positive integer such that $\wp^{\alpha_{\wp}}<2^{t}$ and $\wp^{\alpha_{\wp}} \mid M$.

Proof. Let $M=p_{1}^{\alpha_{1}} \cdots p_{l}^{\alpha_{l}}$, so that $W(M)=2^{l}$. If $\wp$ is a prime such that $\wp>2^{t}$ then $\frac{2}{\sqrt[t]{\gamma}}<1$. Let $\beta_{i} \leq \alpha_{i}$ be the greatest integer such that $p_{i}^{\beta_{i}} \leq 2^{t}$, thus

$$
\frac{W(M)}{M^{\frac{1}{t}}}=\frac{2^{l}}{\sqrt[t]{p_{1}^{\alpha_{1}}} \cdots \sqrt[t]{p_{l}^{\alpha_{l}}}} \leq \prod_{i=1}^{l} \frac{2}{\sqrt[t]{p_{i}^{\beta_{i}}}} \leq \prod_{\substack{\wp_{\wp_{\varphi}} \\ \wp_{\text {is }} \\ \text { orime } \\ \wp_{\wp}^{\alpha_{\wp}} \mid M}} \frac{2}{\sqrt[t]{\wp^{\alpha_{\wp}}}}=A_{t} .
$$

The result follows immediately.
Now, we present some estimates involving sum of characters:
Lemma 2.10. [8, Theorem 5.41] Let $\eta$ be a multiplicative character of $\mathbb{F}_{q^{n}}$ of order $r>1$ and $f \in \mathbb{F}_{q^{n}}[x]$ be a monic polynomial of positive degree such that $f$ is not of
the form $g(x)^{r}$ for some $g \in \mathbb{F}_{q^{n}}[x]$ with degree at least 1 . Let $e$ be the number of distinct roots of $f$ in its splitting field over $\mathbb{F}_{q^{n}}$. For every $a \in \mathbb{F}_{q^{n}}$,

$$
\left|\sum_{\alpha \in \mathbb{F}_{q^{n}}} \eta(a f(\alpha))\right| \leq(e-1) q^{n / 2}
$$

Lemma 2.11. [14, Theorem 2G] Let $\eta$ be a multiplicative character of $\mathbb{F}_{q^{n}}$ of order $d \neq 1$ and $\chi$ be a non-trivial additive character of $\mathbb{F}_{q^{n}}$. If $F, G \in \mathbb{F}_{q^{n}}[x]$ are such that $F$ has exactly $m_{1}$ roots and $\operatorname{deg}(G)=m_{2}$ with $\operatorname{gcd}(d, \operatorname{deg}(F))=\operatorname{gcd}\left(m_{2}, q\right)=1$, then

$$
\left|\sum_{\alpha \in \mathbb{F}_{q^{n}}} \eta(F(\alpha)) \chi(G(\alpha))\right| \leq\left(m_{1}+m_{2}-1\right) q^{n / 2}
$$

Lemma 2.12 ([6, Theorem 1). Let $F$ be a finite field, let $n \geq 1$ be an integer and let $E$ be an extension field of $F$ of degree $n$. Let $\chi$ be any nontrivial complex-valued multiplicative character of $E^{\times}$(extended by zero to all of $E$ ), and $x$ in $E$ any element that generates $E$ over $F$. Then

$$
\left|\sum_{t \in F} \chi(t-x)\right| \leq(n-1) \sqrt{\#(F)}
$$

## 3. General Results

In [10], Reis gives a method to construct $k$-normal elements: let $\beta \in \mathbb{F}_{q^{n}}$ be a normal element and $f \in \mathbb{F}_{q}[x]$ be a divisor of $x^{n}-1$ of degree $k$, then $\alpha=L_{f}(\beta)$ is $k$-normal (see [10, Lemma 3.1]). From Theorem [2.2, we also know that there exists a $k$-normal element in $\mathbb{F}_{q^{n}}$ if and only if $x^{n}-1$ has a divisor of degree $n-k$ (or, equivalently, a divisor of degree $k$ ). So, if $x^{n}-1$ has a divisor of degree $k$ and

$$
\begin{equation*}
q^{\frac{n}{2}-k} \geq W\left(q^{n}-1\right) W\left(x^{n}-1\right) \tag{1}
\end{equation*}
$$

then there exists a primitive $k$-normal element in $\mathbb{F}_{q^{n}}$ (see [10, Theorem 3.3]).
When $k=2$, it is easy to prove that the existence of primitive 2-normal elements is only possible for $n \geq 4$ (see Theorem 2.6). Note that we cannot use condition 1 for the case $n=4$ because the exponent on the left side is equal to zero, so we need a different approach in that case. We will discuss this case in Section 6. First, we are going to use the ideas of Huczynska [4] and Reis [10] to get a more general result than condition 1 .

From Theorem 2.2 we know that the existence of a factor of degree $k$ of $x^{n}-1$ is a necessary and sufficient condition for the existence of $k$-normal elements. Thus, the following result is very important to know the number of these factors.

Lemma 3.1. Let $q$ be a prime power and let $n$ be a positive integer prime to $q$. Let $I_{n}(r)$ be the number of irreducible monic factors of $x^{n}-1$ with degree $r$ over $\mathbb{F}_{q}$. We have

$$
I_{n}(r)=\frac{1}{r} \sum_{d \mid r} t_{d} \cdot \mu\left(\frac{r}{d}\right),
$$

where $t_{d}=\operatorname{gcd}\left(q^{d}-1, n\right)$.
Proof. Let $\alpha$ be a primitive element in $\mathbb{F}_{q^{r}}^{*}$. For $0 \leq s \leq q^{r}-1$, we have $\left(\alpha^{s}\right)^{n}=1$ if and only if $q^{r}-1$ divides $s n$. Since $\frac{q^{r}-1}{t_{r}}$ and $\frac{n}{t_{r}}$ are coprimes, we get that $\left(\alpha^{s}\right)^{n}=1$ if and only if $\frac{q^{r}-1}{t_{r}}$ divides $s$. Therefore, there exist $t_{r}$ possibilities for $s$, which implies that the number of elements $\alpha$ in $\mathbb{F}_{q^{r}}^{*}$ with $\alpha^{n}=1$ is $t_{r}$.

Observe that for each irreducible polynomial of degree $r$ defined over $\mathbb{F}_{q}$ which divides $x^{n}-1$, there are $r$ elements $\alpha$ in $\mathbb{F}_{q^{r}}$ such that $\alpha \notin \mathbb{F}_{q^{d}}$ for any $d<r$, with $\alpha^{n}=1$. So, from the definition of $I_{n}(r)$, the number of elements $\alpha$ in $\mathbb{F}_{q^{r}}$ which are not in $\mathbb{F}_{q^{d}}$ for $d<r$, with $\alpha^{n}=1$ is $r \cdot I_{n}(r)$. Note that if $\alpha \in \mathbb{F}_{q^{r}} \cap \mathbb{F}_{q^{d}}$ and $d<r$, then $d \mid r$. We conclude by using the inclusion-exclusion principle.

Lemma 3.2. Let $q$ be a prime power and let $n$ be a positive integer. There exists a 2 -normal element in $\mathbb{F}_{q^{n}}$ over $\mathbb{F}_{q}$ if and only if $\operatorname{gcd}\left(q^{3}-q, n\right) \neq 1$.

Proof. The result follows directly from Theorem 2.2 and Lemma 3.1.
For the purpose of proving The Primitive Normal Basis Theorem without computational calculations, in [2], the authors defined, for $m \mid\left(q^{n}-1\right)$ and $g \mid\left(x^{n}-1\right)$, the number $N(m, g)$ of non-zero elements of $\mathbb{F}_{q^{n}}$ that are both $m$-free and $g$-free. So they needed to prove that $N\left(q^{n}-1, x^{n}-1\right)$ is positive. Similarly, we define:

Definition 3.3. Let $f, T \in \mathbb{F}_{q}[x]$ be divisors of $x^{n}-1$ such that $\operatorname{deg} f=k$ and let $m \in \mathbb{N}$ be a divisor of $q^{n}-1$. We denote by $N_{f}(T, m)$ the number of $T$-free elements $\alpha \in \mathbb{F}_{q^{n}}$ such that $L_{f}(\alpha)$ is $m$-free.

The following theorem generalizes [10, Theorem 3.3] using Definition 3.3.
Theorem 3.4. Let $f, T \in \mathbb{F}_{q}[x]$ be divisors of $x^{n}-1$ such that $\operatorname{deg} f=k$ and let $m \in$ $\mathbb{N}$ be a divisor of $q^{n}-1$. We have $N_{f}(T, m)>\theta(m) \Theta(T)\left(q^{n}-q^{n / 2+k} W(m) W(T)\right)$. In particular, if $q^{n / 2-k} \geq W(m) W(T)$ then $N_{f}(T, m)>0$.

Proof. We have that

$$
\begin{aligned}
N_{f}(T, m) & =\sum_{\alpha \in \mathbb{F}_{q^{n}}} \Omega_{T}(\alpha) \cdot w_{m}\left(L_{f}(\alpha)\right) \\
& =\theta(m) \Theta(T) \sum_{\alpha \in \mathbb{F}_{q^{n}}} \int_{d \mid m} \int_{D \mid T} \eta_{d}\left(L_{f}(\alpha)\right) \chi_{\delta_{D}}(\alpha) .
\end{aligned}
$$

If we denote the Gauss sum $S_{f}\left(\eta_{d}, \chi_{\delta_{D}}\right)=\sum_{\alpha \in \mathbb{F}_{q^{n}}} \eta_{d}\left(L_{f}(\alpha)\right) \chi_{\delta_{D}}(\alpha)$, we can write

$$
\frac{N_{f}(T, m)}{\theta(m) \Theta(T)}=S_{0}+S_{1}+S_{2}+S_{3}
$$

where $S_{0}=S_{f}\left(\eta_{1}, \chi_{0}\right), S_{1}=\int_{\substack{D \mid T \\ D \neq 1}} S_{f}\left(\eta_{1}, \chi_{\delta_{D}}\right), S_{2}=\int_{\substack{d \mid m \\ d \neq 1}} S_{f}\left(\eta_{d}, \chi_{0}\right)$ and

$$
S_{3}=\int_{\substack{D \mid T \\ D \neq 1 \\ d \nmid m \neq 1}} \int_{f} S_{f}\left(\eta_{d}, \chi_{\delta_{D}}\right)
$$

We observe that

$$
S_{0}=\sum_{\alpha \in \mathbb{F}_{q^{n}}} \eta_{1}\left(L_{f}(\alpha)\right) \chi_{0}(\alpha)=\sum_{\alpha \in \mathbb{F}_{q^{n}} \backslash \operatorname{Ker} L_{f}} 1=q^{n}-q^{k}
$$

and

$$
S_{1}=\sum_{\alpha \in \mathbb{F}_{q^{n}} \backslash \operatorname{Ker}} \sum_{\substack{D \mid T \\ D \neq 1}} \frac{\mu_{q}(D)}{\Phi_{q}(D)} \sum_{\left(\delta_{D}\right)} \chi_{\delta_{D}}(\alpha)=-\sum_{\alpha \in \operatorname{Ker} L_{f}} \sum_{\substack{D \mid T \\ D \neq 1}} \frac{\mu_{q}(D)}{\Phi_{q}(D)} \sum_{\left(\delta_{D}\right)} \chi_{\delta_{D}}(\alpha),
$$

which implies that $\left|S_{1}\right| \leq q^{k}(W(T)-1)$, since $\left|\chi_{\delta_{D}}(\alpha)\right| \leq 1$.
Now we would like good estimates of the sums $S_{2}$ and $S_{3}$. We have $f(x)=$ $\sum_{i=0}^{k} a_{i} x_{i}$. One can see that the formal derivative of the $q$-associate of $f$ is $a_{0} \neq 0$ (since $f$ divides $x^{n}-1, f$ is not divisible by $x$ ), hence $L_{f}$ does not have repeated roots and is not of the form $G(x)^{r}$ for any $G(x) \in \mathbb{F}_{q^{n}}[x]$ and $r>1$. Therefore, by Lemma 2.10, we have, for each divisor $d \neq 1$ of $q^{n}-1$ :

$$
\left|S_{f}\left(\eta_{d}, \chi_{0}\right)\right|=\left|\sum_{\alpha \in \mathbb{F}_{q^{n}}} \eta_{d}\left(L_{f}(\alpha)\right)\right| \leq\left(q^{k}-1\right) q^{n / 2}<q^{n / 2+k}
$$

From 2.11, we conclude that, for each divisor $D \neq 1$ of $x^{n}-1$ and each divisor $d \neq 1$ of $q^{n}-1$,

$$
\left|S_{f}\left(\eta_{d}, \chi_{\delta_{D}}\right)\right|=\left|\sum_{\alpha \in \mathbb{F}_{q^{n}}} \eta_{d}\left(L_{f}(\alpha)\right) \chi_{\delta_{D}}(\alpha)\right| \leq\left(q^{k}+1-1\right) q^{n / 2}=q^{n / 2+k}
$$

Combining the previous bounds, we have the following inequality:

$$
\begin{aligned}
N_{f}(T, m) & \geq \theta(m) \Theta(T)\left(S_{0}-\left|S_{1}\right|-\left|S_{2}\right|-\left|S_{3}\right|\right) \\
& \geq \theta(m) \Theta(T)\left[q^{n}-q^{k}-q^{k}(W(T)-1)-q^{n / 2+k}(W(m)-1) W(T)\right] \\
& >\theta(m) \Theta(T)\left(q^{n}-q^{n / 2+k} W(m) W(T)\right) .
\end{aligned}
$$

Therefore, if $W(m) W(T) \leq q^{n / 2-k}$ we have $N_{f}(T, m)>0$.
To use Theorem 3.4, we need to have some knowledge about the factorization of $m$ and $T$. Knowing some factors of these values, one can use the next proposition, which helps to decrease the estimates of the function $W$ by adding an offset factor. Before that, we present a result which will be needed in what follows.

For any natural number $m, \operatorname{rad}(m)$ denotes the largest square-free factor of $m$ and for any polynomial $T \in \mathbb{F}_{q}[x], \operatorname{rad}(T)$ denotes the square-free factor of $T$ of largest degree over $\mathbb{F}_{q}$.

The sieving technique from the next two results follows the ideas of [2].
Lemma 3.5. Let $f, T \in \mathbb{F}_{q}[x]$ be divisors of $x^{n}-1$ such that $\operatorname{deg} f=k$ and let $m \in \mathbb{N}$ be a divisor of $q^{n}-1$. Let $Q_{1}, \ldots, Q_{s}$ be irreducible polynomials and let $p_{1}, \ldots, p_{r}$ be prime numbers such that $\operatorname{rad}\left(x^{n}-1\right)=\operatorname{rad}(T) \cdot Q_{1} \cdot Q_{2} \cdots Q_{s}$ and $\operatorname{rad}\left(q^{n}-1\right)=\operatorname{rad}(m) \cdot p_{1} \cdot p_{2} \cdots p_{r}$. We have that
(2) $N_{f}\left(x^{n}-1, q^{n}-1\right) \geq \sum_{i=1}^{r} N_{f}\left(T, m p_{i}\right)+\sum_{j=1}^{s} N_{f}\left(T \cdot Q_{j}, m\right)-(r+s-1) N_{f}(T, m)$.

Proof. The left side of (2) counts every $\alpha \in \mathbb{F}_{q^{n}}$ for which $\alpha$ is normal and $L_{f}(\alpha)$ is primitive. Observe that if $\alpha$ is normal and $L_{f}(\alpha)$ is primitive then $\alpha$ is $T \cdot Q_{j}$-free and $T$-free; and $L_{f}(\alpha)$ is $m p_{i}$-free and $m$-free, so $\alpha$ is counted $r+s-(r+s-1)=1$ times on the right side of (2). For any other $\alpha \in \mathbb{F}_{q^{n}}$, we have that either $\alpha$ is not $T \cdot Q_{j}$-free for some $j \in\{1, \ldots, s\}$ or $L_{f}(\alpha)$ is not $m p_{i}$-free for some $i \in\{1, \ldots, r\}$, so the right side of (2) is at most zero.

Proposition 3.6. Let $f, T \in \mathbb{F}_{q}[x]$ be divisors of $x^{n}-1$ such that $\operatorname{deg} f=k$ and let $m \in \mathbb{N}$ be a divisor of $q^{n}-1$. Let $Q_{1}, \ldots, Q_{s}$ be irreducible polynomials and
let $p_{1}, \ldots, p_{r}$ be prime numbers such that $\operatorname{rad}\left(x^{n}-1\right)=\operatorname{rad}(T) \cdot Q_{1} \cdot Q_{2} \cdots Q_{s}$ and $\operatorname{rad}\left(q^{n}-1\right)=\operatorname{rad}(m) \cdot p_{1} \cdot p_{2} \cdots p_{r}$. Suppose that $\delta=1-\sum_{i=1}^{r} \frac{1}{p_{i}}-\sum_{j=1}^{s} \frac{1}{q^{\operatorname{deg} Q_{j}}}>0$ and let $\Delta=\frac{r+s-1}{\delta}+2$. If $q^{\frac{n}{2}-k} \geq W(m) W(T) \Delta$, then $N_{f}\left(x^{n}-1, q^{n}-1\right)>0$.

Proof. From equation 2, we have that

$$
N_{f}\left(x^{n}-1, q^{n}-1\right) \geq \sum_{i=1}^{r} N_{f}\left(T, m p_{i}\right)+\sum_{j=1}^{s} N_{f}\left(T \cdot Q_{j}, m\right)-(r+s-1) N_{f}(T, m)
$$

We can rewrite the equation in the form

$$
\begin{aligned}
N_{f}\left(x^{n}-1, q^{n}-1\right) & \geq \sum_{i=1}^{r}\left[N_{f}\left(T, m p_{i}\right)-\theta\left(p_{i}\right) N_{f}(T, m)\right] \\
& +\sum_{j=1}^{s}\left[N_{f}\left(T \cdot Q_{j}, m\right)-\Theta\left(Q_{j}\right) N_{f}(T, m)\right]+\delta N_{f}(T, m) .
\end{aligned}
$$

Now we need a good bound for $N_{f}\left(T, m p_{i}\right)-\theta\left(p_{i}\right) N_{f}(T, m)$. Since $\theta$ is a multiplicative function, we have

$$
N_{f}\left(T, m p_{i}\right)=\Theta(T) \theta(m) \theta\left(p_{i}\right) \sum_{\alpha \in \mathbb{F}_{q^{n}}} \int_{d \mid m p_{i}} \int_{D \mid T} \eta_{d}\left(L_{f}(\alpha)\right) \chi_{\delta_{D}}(\alpha) .
$$

We split the set of $d$ 's which divide $m p_{i}$ into two sets: the first one contains those which do not have $p_{i}$ as a factor, while the second one contains those which are a multiple of $p_{i}$. This will split the first summation into two sums, and we get

$$
\begin{aligned}
N_{f}\left(T, m p_{i}\right) & =\Theta(T) \theta(m) \theta\left(p_{i}\right) \sum_{\alpha \in \mathbb{F}_{q^{n}}}\left(\int_{d \mid m} \int_{D \mid T} \eta_{d}\left(L_{f}(\alpha)\right) \chi_{\delta_{D}}(\alpha)\right) \\
& +\Theta(T) \theta(m) \theta\left(p_{i}\right) \sum_{\alpha \in \mathbb{F}_{q^{n}}}\left(\int_{\begin{array}{c}
d, p_{i} \mid d \\
d \mid m p_{i}
\end{array}} \int_{D \mid T} \eta_{d}\left(L_{f}(\alpha)\right) \chi_{\delta_{D}}(\alpha)\right) .
\end{aligned}
$$

Hence, $N_{f}\left(T, m p_{i}\right)-\theta\left(p_{i}\right) N_{f}(T, m)$ is equal to

$$
\Theta(T) \theta(m) \theta\left(p_{i}\right) \sum_{\alpha \in \mathbb{F}_{q^{n}}}\left(\int_{\substack{d, p_{i}|d \\ d| m p_{i}}} \int_{D \mid T} \eta_{d}\left(L_{f}(\alpha)\right) \chi_{\delta_{D}}(\alpha)\right) .
$$

So, from Lemma 2.11 we have the following inequality

$$
\left|N_{f}\left(T, m p_{i}\right)-\theta\left(p_{i}\right) N_{f}(T, m)\right| \leq \Theta(T) \theta(m) \theta\left(p_{i}\right) W(T) W(m) q^{n / 2+k}
$$

Analogously we can prove that

$$
\left|N_{f}\left(T \cdot Q_{j}, m\right)-\Theta\left(Q_{j}\right) N_{f}(T, m)\right| \leq \Theta(T) \theta(m) \Theta\left(Q_{j}\right) W(T) W(m) q^{n / 2+k}
$$

Combining these inequalities, we obtain

$$
\begin{aligned}
& N_{f}\left(x^{n}-1, q^{n}-1\right) \geq \delta N_{f}(T, m)- \\
& \Theta(T) \theta(m) W(T) W(m) q^{n / 2+k}\left(\sum_{i=1}^{r} \theta\left(p_{i}\right)+\sum_{j=1}^{s} \Theta\left(Q_{j}\right)\right) .
\end{aligned}
$$

Therefore, from Theorem 3.4, we have

$$
\begin{aligned}
N_{f}\left(x^{n}-1, q^{n}-1\right)> & \delta \Theta(T) \theta(m)\left(q^{n}-q^{n / 2+k} W(m) W(T)\right) \\
& -\Theta(T) \theta(m) W(T) W(m) q^{n / 2+k}\left(\sum_{i=1}^{r} \theta\left(p_{i}\right)+\sum_{j=1}^{s} \Theta\left(Q_{j}\right)\right) \\
= & \delta \Theta(T) \theta(m)\left(q^{n}-q^{n / 2+k} W(m) W(T) \Delta\right)
\end{aligned}
$$

Thus, we obtain the desired result.
For the case $k \geq 2$ we can rewrite the previous condition as follows, depending on the factorization of $x^{n}-1$.

Proposition 3.7. Let $n \geq 5$ be a natural number and let $q$ be a prime power such that $q \geq n^{2}$. If $x^{n}-1$ has a factor of degree $k \geq 2$ in $\mathbb{F}_{q}[x]$ and $q^{\frac{n}{2}-k} \geq$ $(n+2) W\left(q^{n}-1\right)$, then there exists a primitive $k$-normal element in $\mathbb{F}_{q^{n}}$.

Proof. Let $f \in \mathbb{F}_{q}[x]$ be a factor of $x^{n}-1$ of degree $k$. We may use Proposition 3.6 with $T=1$ and $m=q^{n}-1$.

Let $Q_{1}, \ldots, Q_{s}$ be irreducible polynomials such that $\operatorname{rad}\left(x^{n}-1\right)=Q_{1} \cdot Q_{2} \cdots Q_{s}$. Then $\delta=1-\sum_{j=1}^{s} \frac{1}{q^{\operatorname{deg} Q_{j}}} \geq 1-\frac{n}{q} \geq 1-\frac{1}{n}=\frac{n-1}{n}>0$, since $q \geq n^{2}$ and $s \leq n$. We also have that

$$
\Delta=\frac{s-1}{\delta}+2 \leq \frac{n-1}{\frac{n-1}{n}}+2=n+2 .
$$

This means that $W(m) W(T) \Delta \leq(n+2) W\left(q^{n}-1\right)$ and from Proposition 3.6, we get the desired result.

For small values of $q$ we have the following result which will be used in combination with Theorem 3.4 and Lemma [2.9, Note that those results are different from [7, Lemma 2.11].

Lemma 3.8. For $q$ a prime power, there exist $a, b \in \mathbb{N}$ such that

$$
W\left(x^{n}-1\right) \leq 2^{\frac{n+a}{b}}
$$

For $q \geq 29$, we have $a=0$ and $b=1$, for $7 \leq q \leq 27$ we have $a=q-1$ and $b=2$ and for small values of $q$ we may use the following values of $a$ and $b$.

| $q$ | $a$ | $b$ |
| :---: | :---: | :---: |
| 2 | 14 | 5 |
| 3 | 20 | 4 |
| 4 | 12 | 3 |
| 5 | 18 | 3 |

TABLE 1. Values of $a$ and $b$ for small values of $q$

Proof. Let $s_{n, t}$ be the number of distinct monic irreducible polynomials of degree at most $t$ that divide $x^{n}-1$ and let $T_{n, t}$ be the sum of their degrees. Hence $W\left(x^{n}-1\right)=$ $2^{j}$, where

$$
\begin{equation*}
j \leq \frac{n-T_{n, t}}{t+1}+s_{n, t}=\frac{n+(t+1) s_{n, t}-T_{n, t}}{t+1} . \tag{3}
\end{equation*}
$$

Since each term in the sum $T_{n, t}$ is at most $t$, the right-hand side of the expression above maximizes when $s_{n, t}$ is maximal. On the other hand, it is obvious that zero is not a root of $x^{n}-1$, so the sum of the degrees of polynomials of degree $i$ is less or equal than the number of elements of $\mathbb{F}_{q^{i}}^{*}$, which is not an element of $\mathbb{F}_{q^{j}}^{*}$, for any divisor $j$ of $i$.

Table 1 is obtained from (3) and the reasoning above. For $q=2$, we use $t=4$; for $q=3$, we use $t=3$; for $q=4$ or $q=5$, we use $t=2$; and for $7 \leq q \leq 27$, we use $t=1$ to obtain $a=q-1$ and $b=2$. For $q \geq 29$, it is convenient to use the usual inequality.
4. Results for all cases where $n \geq 8$ and the cases $n \geq 5$ for $q \leq 19$

In this section we begin to apply the results of the previous section for the case $k=2$. Thus, we study the values of $q$ and $n$ for which we can guarantee the existence of primitive 2-normal elements in $\mathbb{F}_{q^{n}}$.

Proposition 4.1. Let $q \leq 19$ be a prime power and $n \geq 5$ be a natural number. There exists a primitive 2-normal element in $\mathbb{F}_{q^{n}}$ if and only if $\operatorname{gcd}\left(q^{3}-q, n\right) \neq 1$.

Proof. From Theorem 3.4, if $q^{\frac{n}{2}-2} \geq W\left(q^{n}-1\right) W\left(x^{n}-1\right)$ then $N_{f}\left(x^{n}-1, q^{n}-1\right)>0$. From Lemma 2.9 and Lemma 3.8, it follows that $A_{t} \cdot q^{\frac{n}{t}} \cdot 2^{\frac{n+a}{b}} \geq W\left(q^{n}-1\right) W\left(x^{n}-1\right)$,
where

$$
A_{t}=\prod_{\substack{\wp<2^{t} \\ \wp \text { is prime } \\ \wp \neq p}} \frac{2}{\sqrt[t]{\wp}} \quad \text { and } \quad \operatorname{char} \mathbb{F}_{q}=p .
$$

So, if for some $t \in \mathbb{N}$ we have $q^{\frac{n}{2}-2} \geq A_{t} \cdot q^{\frac{n}{t}} \cdot 2^{\frac{n+a}{b}}$, then $N_{f}\left(x^{n}-1, q^{n}-1\right)>0$. For $b>\log _{q} 4$ and $t>\frac{2 b}{b-\log _{q} 4}$, this inequality is equivalent to

$$
\begin{equation*}
n \geq \frac{2 \ln q+\ln \left(A_{t} \cdot 2^{\frac{a}{b}}\right)}{\left(\frac{1}{2}-\frac{1}{t}\right) \ln q-\frac{1}{b} \ln 2} \tag{4}
\end{equation*}
$$

| $q$ | $a$ | $b$ | (4) satisfied for | $q$ | $a$ | $b$ | (4) satisfied for |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 2 | 14 | 5 | $n \geq 69$ | 8 | 7 | 2 | $n \geq 28$ |
| 3 | 20 | 4 | $n \geq 46$ | 9 | 8 | 2 | $n \geq 27$ |
| 4 | 12 | 3 | $n \geq 38$ | 11 | 10 | 2 | $n \geq 26$ |
| 5 | 18 | 3 | $n \geq 35$ | 16 | 15 | 2 | $n \geq 24$ |
| 7 | 6 | 2 | $n \geq 31$ | $\{13,17,19\}$ | $q-1$ | 2 | $n \geq 25$ |

Table 2. Values of $n$ depending on $q$ such that (4) is satisfied with $t=6$.

We know that for values of $q$ and $n$ from Table 2, the condition $q^{n / 2-2} \geq W\left(q^{n}-\right.$ 1) $W\left(x^{n}-1\right)$ is satisfied, so it remains only a finite number of cases to test for $q \leq 19$.

| $q$ | $n$ | $q$ | $n$ |
| :---: | :---: | :---: | :---: |
| 2 | $6,8,9,10,12,14,15,18,21$ | 9 | $5,6,8,10$ |
| 3 | $6,8,9,10,12,16$ | 11 | $5,6,8,10,12$ |
| 4 | $5,6,8,9,10,12,15$ | 13 | $6,7,8,12$ |
| 5 | $5,6,8,9,10,12,16$ | 16 | $5,6,9,10,15$ |
| 7 | $6,7,8,9,10,12$ | 17 | 6,8 |
| 8 | $6,7,8,9$ | 19 | $5,6,8,9,10,12$ |

Table 3. Values of $q$ and $n$ such that $q \leq 19, n$ is not in Table 2, $\operatorname{gcd}(q(q-1)(q+1), n) \neq 1$ and $q^{n / 2-2}<W\left(q^{n}-1\right) W\left(x^{n}-1\right)$

Table 3 shows the values of $q$ and $n$ which are not in Table 2 with $q \leq 19$ and $\operatorname{gcd}\left(q^{3}-q, n\right) \neq 1$, where $q^{n / 2-2} \geq W\left(q^{n}-1\right) W\left(x^{n}-1\right)$ is not satisfied.

For pairs $(q, n)$ from Table 3, we test condition $q^{\frac{n}{2}-2} \geq W(m) W(T) \Delta$ (see Proposition (3.6). For this, we use the SageMath procedure Test_Delta(q, n,u) (see Appendix A) where $u$ is a given natural number, $m=\operatorname{gcd}\left(q^{n}-1, u\right)$. If $q>5$ we choose $T=1$; if $q \leq 5$ then $T$ is the product of all monic linear factors of $x^{n}-1$.

For $u=2 \cdot 3 \cdot 5$, procedure Test_Delta $(\mathbf{q}, \mathbf{n}, \mathbf{u})$ gets True for $(q, n)=(2,14)$, $(2,15),(2,18),(2,21),(3,9),(3,16),(4,10),(4,12),(4,15),(5,9),(5,10),(5,16)$, $(7,7),(7,9),(7,10),(8,8),(8,9),(9,10),(11,8),(11,10),(11,12),(13,7),(13,8)$, $(16,9),(16,10),(17,8),(19,8),(19,9),(19,10),(19,12)$.

For $(q, n)=(7,12),(13,12),(16,15)$, we take $m=30,30,3$ and $T=x^{2}-1, x^{4}-1$, $x^{15}-1$ respectively, and we get that condition $q^{\frac{n}{2}-2} \geq W(m) W(T) \Delta$ is satisfied.

For the last remaining cases, Tables 4 , 5and 6 (see Appendix B), show explicitly a primitive 2-normal element $\alpha \in \mathbb{F}_{q^{n}}$ such that $g(\alpha)=0$ for some irreducible polynomial $g \in \mathbb{F}_{p}[x]$, where $p$ is the characteristic of $\mathbb{F}_{q}$. Primitivity and Normality can be tested using the programs in Appendix A.

Proposition 4.2. Let $n \geq 8$ be a natural number. There exists a primitive 2 -normal element in $\mathbb{F}_{q^{n}}$ for all prime powers $q$ satisfying $\operatorname{gcd}\left(q^{3}-q, n\right) \neq 1$.

Proof. From the last result, we have that for $q<23$ and $n \geq 5$, there exists a primitive 2-normal element in $\mathbb{F}_{q^{n}}$. So we will focus on $q \geq 23$. From Theorem 3.4, Lemma 2.9 and considering that $W\left(x^{n}-1\right) \leq 2^{n}$, there exists a primitive 2-normal element in $\mathbb{F}_{q^{n}}$ if $q^{\frac{n}{2}-2} \geq 2^{n} \cdot A_{t} \cdot q^{\frac{n}{t}}$ is satisfied for some real number $t$. This condition is equivalent to the following two inequalities:

$$
n \geq \frac{\ln \left(A_{t}\right)+2 \ln (q)}{\left(\frac{1}{2}-\frac{1}{t}\right) \ln (q)-\ln (2)} \quad \text { and } \quad q \geq\left(2^{n} \cdot A_{t}\right)^{\frac{2 t}{(t-2) n-4 t}}
$$

For a fixed value of $t \geq 4$, the right-hand side of the first inequality is a decreasing function of $q \geq 16$. So, fixing $t=7$ in the first inequality, we get that for $q \geq 23$ and $n \geq 28$, there exists a primitive 2-normal element in $\mathbb{F}_{q^{n}}$. Now, if $14 \leq n \leq 27$, from the second inequality (whose right-hand side is also a decreasing function of $n$ ) and with $t=6.3$, we get that, if $n \geq 14$ and $q \geq 144$, there exists a primitive 2-normal element in $\mathbb{F}_{q^{n}}$. Now, using SageMath, we verify that $q^{\frac{n}{2}-2} \geq W\left(q^{n}-1\right) W\left(x^{n}-1\right)$ is true for all prime powers $23 \leq q<144$ and $14 \leq n<28$. Hence, from Theorem 3.4 and the previous considerations we conclude that there exists a primitive 2-normal element in $\mathbb{F}_{q^{n}}$ for every prime power $q$ and for all $n \geq 14$.

Now, let us suppose that $8 \leq n \leq 13$. From Proposition 3.7 and Lemma 2.9 there exists a primitive 2-normal element in $\mathbb{F}_{q^{n}}$ if $q \geq n^{2}$ and $q^{\frac{n}{2}-2} \geq(n+2) A_{t} \cdot q^{\frac{n}{t}}$.

Define

$$
\begin{equation*}
M_{t}(n)=\max \left\{n^{2},\left\lceil\left((n+2) \cdot A_{t}\right)^{\frac{2 t}{(t-2) n-4 t}}\right\rceil\right\}, \tag{5}
\end{equation*}
$$

where for $x \in \mathbb{R},\lceil x\rceil$ is the smallest integer such that $x \leq\lceil x\rceil$. From the inequalities above, if we have $q \geq M_{t}(n)$, for some real number $t$ suficiently large (for $n \geq 8$ this means $t>4$ ), then there exists a primitive 2-normal element in $\mathbb{F}_{q^{n}}$. For $n$ between 8 and 13, we have

$$
\begin{array}{lll}
M_{6.3}(8)=6426, & M_{6}(10)=100, & M_{6}(12)=144 \\
M_{6}(9)=413, & M_{6}(11)=121, & M_{6}(13)=169
\end{array}
$$

For pairs $(q, n)$ such that $8 \leq n \leq 13$ and $23 \leq q<M_{t}(n)$, where $q$ is a prime power, we test $q^{\frac{n}{2}-2} \geq W(m) W(T) \Delta$, from Proposition 3.6. The procedure Test_Delta, with $u=6$, returns True for all those pairs. Proposition 4.2 is now proved.

## 5. Cases $n=5,6,7$

For $5 \leq n \leq 7$, applying Proposition 3.7 and Lemma 2.9, we get that a sufficient condition to have a primitive 2-normal element in $\mathbb{F}_{q^{n}}$ is $q \geq M_{t}(n)$ for some real number $t$, where $M_{t}(n)$ is defined by equation (5). The problem is that $M_{t}(n)$ is very large.
5.1. Case $\mathbf{n}=7$ : The condition $\operatorname{gcd}\left(q^{3}-q, 7\right) \neq 1$ means that $q \equiv 0, \pm 1(\bmod 7)$.

Proposition 5.1. There exists a primitive 2 -normal element in $\mathbb{F}_{q^{7}}$ for every prime power $q$ such that $q \equiv 0, \pm 1(\bmod 7)$.

Proof. Suppose first that $7 \mid q$. In this case, $q=7^{k}$ for some integer $k \geq 1$. We will use Theorem 3.4 in combination with Lemma 2.9. Since $7 \nmid q^{n}-1$, we may use the following constant

$$
\begin{equation*}
A_{t}=\prod_{\substack{\wp \neq 7, \wp<2^{t} \\ \wp \text { is prime }}} \frac{2}{\sqrt[t]{\wp}} \tag{6}
\end{equation*}
$$

from Lemma 2.9. From Theorem 3.4, and taking into account that $W\left(x^{7}-1\right)=$ $W\left((x-1)^{7}\right)=2$, we have that if, for some real number $t$, the inequality $q^{\frac{7}{2}-2} \geq$ $W\left(x^{7}-1\right) \cdot A_{t} \cdot q^{\frac{7}{t}}=2 A_{t} \cdot q^{\frac{7}{t}}$ holds, then $N_{f}\left(x^{7}-1, q^{7}-1\right)>0$. Setting $t=7$, we get that $N_{f}\left(x^{7}-1, q^{7}-1\right)>0$ for $q \geq 104368$. Since for prime powers $q=7^{2}, 7^{3}, 7^{4}, 7^{5}$ the condition $q^{\frac{7}{2}-2} \geq W\left(q^{7}-1\right) W\left(x^{7}-1\right)$ is satisfied, the result follows from Theorem 3.4 .

If $q \equiv-1(\bmod 7)$, then $7 \nmid q^{7}-1$, and we may also use the constant $A_{t}$ given by (6). From Lemma 3.1, we conclude that $x^{7}-1$ has one factor of degree 1 and three factors of degree 2 . We set $m=q^{7}-1$ and $T=1$, so $\delta=1-\frac{1}{q}-\frac{3}{q^{2}}$ and $\Delta=\frac{4-1}{\delta}+2$. Since $q \geq 23$ and $q \equiv-1(\bmod 7)$ then, $q \geq 27$. This means that $\Delta<5.116$ and from Proposition 3.6 we get $N_{f}\left(x^{7}-1, q^{7}-1\right)>0$ for prime powers $q$ satisfying

$$
q^{\frac{7}{2}-\frac{7}{t}-2} \geq 5.116 \cdot A_{t}>A_{t} \cdot \Delta,
$$

for some real number $t$. Setting $t=6.5$, we get $N_{f}\left(x^{7}-1, q^{7}-1\right)>0$ for $q \geq$ 614236. There are 8377 prime powers $q$ between 23 and 614236 such that $q \equiv-1$ $(\bmod 7)$. For those prime powers, we use Theorem 3.4 and we get that condition $q^{\frac{7}{2}-2} \geq W\left(q^{7}-1\right) W\left(x^{7}-1\right)$ is satisfied except for $q=27$. From $27^{7}-1=2 \cdot 13$. $1093 \cdot 368089$, we set $m=27^{7}-1$ and $T=x-1$ in Proposition 3.6 and we get $27^{\frac{7}{2}-2} \geq W(m) W(T) \Delta$, so the proposition is proved for $q \equiv-1(\bmod 7)$.

Finally, suppose that $q \equiv 1(\bmod 7)$. In this case we may use the following constant

$$
A_{t}=\frac{2}{\sqrt[t]{7^{2}}} \cdot \prod_{\substack{\wp \neq 7, \gamma<2^{t} \\ \wp \\ \wp \text { is prime }}} \frac{2}{\sqrt[t]{p}}
$$

from Lemma [2.9, as $7^{2}$ appears in the factorization of $q^{7}-1$ and $7^{2}<2^{t}$ for any $t \geq 6$. From Lemma 3.1 we know that $x^{7}-1$ has seven factors of degree 1 . We set $m=q^{7}-1$ and $T=1$, so $\delta=1-\frac{7}{q}$ and $\Delta=\frac{6}{\delta}+2=8+\frac{42}{q-7}$. Let us suppose that $q \geq 337$. This means that $\Delta<8.128$ and from Proposition 3.6, we get that if $q^{\frac{7}{2}-2} \geq 8.128 \cdot A_{t} \cdot q^{\frac{7}{t}}>W\left(q^{7}-1\right) W(1) \Delta$, then $N_{f}\left(x^{7}-1, q^{7}-1\right)>0$. Setting $t=6.8$, the inequality $q^{\frac{3}{2}-\frac{7}{t}} \geq 8.128 \cdot A_{t}$ is equivalent to $q \geq 2142829$. For those prime powers, we have $N_{f}\left(x^{7}-1, q^{7}-1\right)>0$. There are 26543 prime powers $q$ between 23 and 2142829 such that $q \equiv 1(\bmod 7)$. For those prime powers, we test $q^{\frac{n}{2}-2} \geq W(m) W(T) \Delta$. The procedure Test_Delta, with $u=2$, returns True in all cases, so the proposition is also proved for $q \equiv 1(\bmod 7)$.
5.2. Case $\mathbf{n}=6$ : The condition $\operatorname{gcd}\left(q^{3}-q, 6\right) \neq 1$ is satisfied for every prime power $q$. From the considerations at the beginning of this section, we have $N_{f}\left(x^{6}-1, q^{6}-1\right)>$ 0 for prime powers $q \geq M_{t}(6)$. For $t=8.1$ we get $M_{t}(6)<1.62 \cdot 10^{18}$. So, we will suppose that $q<1.62 \cdot 10^{18}$.

We have that if $q$ is a prime power then $q$ is of the form $2^{k}, 3^{k}$ or $q \equiv \pm 1(\bmod 6)$. Observe also that $\operatorname{gcd}\left(q^{2}-1, q^{4}+q^{2}+1\right)=\operatorname{gcd}\left(q^{2}-1,3\right)=3$. Let

$$
q^{4}+q^{2}+1=3^{\beta_{0}} \cdot \prod_{i=1}^{v} \wp_{i}^{\beta_{i}}
$$

be the prime factorization of $q^{4}+q^{2}+1$, where $3<\wp_{1}<\cdots<\wp_{v}$ are the prime factors of $q^{4}+q^{2}+1$.

Now, we want to apply Proposition 3.6 with $m=q^{2}-1$, and therefore we need to have some control on the prime factors of $q^{4}+q^{2}+1$.

Lemma 5.2. Let $q \equiv \pm 1(\bmod 6)$ be a prime power such that $q<1.62 \cdot 10^{18}$. If $q^{4}+q^{2}+1=3^{\beta_{0}} \cdot \prod_{i=1}^{r} \wp_{i}^{\beta_{i}}$ is the prime factorization of $q^{4}+q^{2}+1$, then $r \leq 34$ and

$$
S=\sum_{i=1}^{r} \frac{1}{\wp_{i}}<0.539 .
$$

Proof. Let $S_{k}$ and $P_{k}$ be, respectively, the sum of the inverses and the product of the first $k$ primes of the form $6 j+1$. We have that $\operatorname{gcd}\left(q^{4}+q^{2}+1, q^{2}-1\right)=1$ or 3 , so the only primes which divide $q^{4}+q^{2}+1$ are 3 and primes of the form $6 j+1$. Thus $q^{4}+q^{2}+1 \geq 3 \cdot P_{r}$ and then $P_{r} \leq\left(M^{4}+M^{2}+1\right) / 3$, where $M=1.62 \cdot 10^{18}$. We have that $P_{r} \leq\left(M^{4}+M^{2}+1\right) / 3$ for $r \leq 34$, so $S \leq S_{34}<0.539$.

Proposition 5.3. There exists a primitive 2-normal element in $\mathbb{F}_{q^{6}}$ for every prime power $q$.

Proof. Consider $10^{5}<q<1.62 \cdot 10^{18}$ and let us suppose first that $q \equiv \pm 1(\bmod 6)$. Now, we will apply Proposition 3.6 with $m=q^{2}-1$ and $T=1$. If $q \equiv 1(\bmod 6)$, then $x^{6}-1$ has six factors of degree 1 and if $q \equiv-1(\bmod 6)$, then $x^{6}-1$ has two factors of degree 1 and two factors of degree 2 . In any case, we have $\frac{6}{q}<\frac{2}{q}+\frac{2}{q^{2}}$ and $s=4$ or $s=6$, so, in any case, $s \leq 6$. From Lemma 5.2, considering the prime factorization of $q^{4}+q^{2}+1$ given in such lemma and the considerations above, we get $\delta \geq 1-S_{34}-\frac{6}{q}, r \leq 34$ and $s \leq 6$. Since $q \geq 10^{5}$, we have $\delta>0.46094$ and $\Delta=2+\frac{r+s-1}{\delta}<86.61$. From Lemma 2.9 we have $W\left(q^{2}-1\right) \leq A_{t} \cdot q^{\frac{2}{t}}$ for any real number $t$. Now, if $q \geq\left(A_{t} \cdot 86.61\right)^{\frac{t}{t-2}}$, then from Proposition 3.6, there exists a primitive 2-normal element in $\mathbb{F}_{q^{6}}$. For $t=4.9$, this condition becomes $q \geq 94870$. Now, let us assume that $q<94870$ and $q \equiv \pm 1(\bmod 6)$. There are 9221 prime powers $q$ between 23 and 94870 such that $q \equiv \pm 1(\bmod 6)$. For those prime powers, we test $q^{\frac{n}{2}-2} \geq W(m) W(T) \Delta$. The procedure Test_Delta, with $u=6$, returns

True in all cases, except for the following prime powers: $23,25,29,31,37,41,43$, 47, 49, 59, 61, 67, 79.

Finally, for prime powers above, table 7 shows an element $\alpha \in \mathbb{F}_{q^{6}}$, primitive 2-normal over $\mathbb{F}_{q}$, such that $g(\alpha)=0$ for some irreducible polynomial $g \in \mathbb{F}_{p}[x]$, where $p$ is the characteristic of $\mathbb{F}_{q}$.

If $q=2^{k}$, then $W\left(x^{6}-1\right) \leq 8$ and if $q=3^{k}$, then $W\left(x^{6}-1\right)=4$. Hence from Theorem 3.4 and Lemma 2.9, we test the inequality $q \geq 8 \cdot A_{t} \cdot q^{\frac{6}{t}}$ with $t=8$ and we conclude that there exists a primitive 2-normal element in $\mathbb{F}_{q^{6}}$ for $k \geq 58$ (if $q=2^{k}$ ) and $k \geq 37$ (if $q=3^{k}$ ). We also know that there exists a primitive 2-normal element in $\mathbb{F}_{q^{6}}$ for $q \leq 19$. We test the condition $q \geq W(m) W(T) \Delta$ from Proposition 3.6, The procedure Test_Delta, with $u=3$ for $q=2^{k}(5 \leq k \leq 57)$ and $u=2$ for $q=3^{k}(3 \leq k \leq 36)$, returns True in all these cases.
5.3. Case $\mathbf{n}=5$ : From Lemma 3.1, if $q \equiv \pm 2(\bmod 5)$, then $x^{5}-1$ has no irreducible quadratic factor and only one linear factor. If $5 \mid q$, we have $x^{5}-1=(x-1)^{5}$, if $q \equiv 1(\bmod 5)$, then $x^{5}-1$ has five linear factors and if $q \equiv-1(\bmod 5)$, then $x^{5}-1$ has one linear factor and two irreducible factors of degree 2 . In particular, there exist 2-normal elements in $\mathbb{F}_{q^{5}}$ if and only if $q \equiv 0, \pm 1(\bmod 5)$.

Lemma 5.4. Let $q \equiv 0, \pm 1(\bmod 5)$ be a prime power. There exists a primitive 2 -normal element in $\mathbb{F}_{q^{5}}$ for $q \geq 507936$.

Proof. Let $t, u$ be positive real numbers such that $t+u \geq 11$ and let

$$
q^{5}-1=\wp_{1}^{a_{1}} \cdots \wp_{v}^{a_{v}} \cdot \varrho_{1}^{b_{1}} \cdots \varrho_{r}^{b_{r}}
$$

be the prime factorization of $q^{5}-1$ such that $2 \leq \wp_{i} \leq 2^{t}$ or $2^{t+u} \leq \wp_{i}$ for $1 \leq i \leq v$ and $2^{t}<\varrho_{i}<2^{t+u}$ for $1 \leq i \leq r$. We use Proposition 3.6, where we set $T=1$ and $m=\wp_{1}^{a_{1}} \cdots \wp_{v}^{a_{v}}$, so we have

$$
\delta=1-\sum_{i=1}^{r} \frac{1}{\varrho_{i}}-\sum_{j=1}^{s} \frac{1}{q^{\operatorname{deg} Q_{j}}}
$$

where $\operatorname{rad}\left(x^{n}-1\right)=Q_{1} \cdots Q_{s}$. From the considerations above, we have $s \in\{1,3,5\}$ and $1 \leq \operatorname{deg} Q_{i} \leq 2$. Before applying Proposition 3.6, we will bound $\delta, \Delta$ and $W(m)$.

Let $S_{t, u}$ be the sum of the inverse of all prime numbers between $2^{t}$ and $2^{t+u}$ and $r(t, u)$ be the number of those primes. If $S_{t, u}+\frac{5}{q}<1$, then $\delta \geq 1-S_{t, u}-\frac{5}{q}$. If we choose $q>10^{6}$ and $(t, u)=(5.8,9.8)$, we get $S_{t, u} \leq 0.962094, \delta>0.037901$,
$r \leq r(t, u)=5085$ and $\Delta=2+\frac{r+s-1}{\delta}<2+\frac{5085+5-1}{0.037901} \leq 134272.87$. To bound $W(m)$ we will use Lemma 2.9, Let $P_{t}$ be the set of all prime numbers less than $2^{t}$. From this, we obtain that $W(m) \leq A_{t, u} m^{\frac{1}{t+u}} \leq A_{t, u} q^{\frac{5}{t+u}}$, where

$$
A_{t, u}=\prod_{\wp \in P_{t}} \frac{2}{\sqrt[t+u]{\wp}} \leq 3678.26, \quad \text { since }(t, u)=(5.8,9.8)
$$

From Proposition [3.6, we conclude that a suficient condition for the existence of a primitive 2-normal element in $\mathbb{F}_{q^{5}}$ is $q^{\frac{1}{2}} \geq \Delta_{\max } \cdot A_{t, u} \cdot q^{\frac{5}{t+u}}$ (where $\Delta_{\max }=134272.87$ ) or, equivalently,

$$
q \geq\left(\Delta_{\max } \cdot A_{t, u}\right)^{\frac{2(t+u)}{t+u-10}} \cong 2.729 \cdot 10^{48} .
$$

Let's suppose now that $q<2.729 \cdot 10^{48}$. We will apply Proposition 3.6 again, but this time we will set $m=q-1$ and $T=1$. We have that $\operatorname{gcd}\left(q^{4}+q^{3}+q^{2}+\right.$ $q+1, q-1)=1$ or 5 and if a prime different from 5 divides $q^{4}+q^{3}+q^{2}+q+1$, then it is of the form $5 j+1$. We will bound $\delta, \Delta$ and $W(m)$. Obviously, from Lemma [2.9, we have $W(q-1) \leq A_{t} \cdot q^{\frac{1}{t}}$ for any real number $t$. Let $S_{k}$ and $P_{k}$ be, respectively, the sum of the inverses and the product of the first $k$ primes of the form $5 j+1$. Let $r$ be the number of prime factors of $q^{4}+q^{3}+q^{2}+q+1$ different from 5 , so $P_{r} \leq q^{4}+q^{3}+q^{2}+q+1<5.55 \cdot 10^{193}$. Therefore $r \leq 69$ and $S_{r}<0.29717$. As before, if $q>10^{5}$ then $\delta \geq 1-S_{r}-\frac{5}{10^{5}}>0.70278$ and $\Delta=2+\frac{r+s-1}{\delta}<105.874$. So, observing that if $q \geq\left(105.874 \cdot A_{t}\right)^{\frac{2 t}{t-2}}$ for some real number $t$, then $q^{\frac{1}{2}} \geq A_{t} \cdot q^{\frac{1}{t}} \cdot 105.874 \geq W(q-1) \cdot W(1) \cdot \Delta$, and using Proposition 3.6, there exists a primitive 2-normal element in $\mathbb{F}_{q^{5}}$

For $t=4.7$, the condition above becomes $q \geq 1.984 \cdot 10^{10}$. If we suppose now $q<1.984 \cdot 10^{10}$ and if we use again Proposition 3.6 with $m=q-1$ and $T=1$, we get $r \leq 19$ and $S_{r}<0.2441801$. For $q>10^{5}$, we also get $\delta>0.7558149$ and $\Delta<32.43074$. From Proposition 3.6 and taking $t=4.8$, we get that there exists a primitive 2-normal element in $\mathbb{F}_{q^{5}}$ for $q \geq 3.208 \cdot 10^{8}$.

We apply now Proposition [3.6, setting $m=\operatorname{gcd}\left(q^{5}-1,2 \cdot 3 \cdot 5\right)$ (so $W(m) \leq 8$ ) and $T=1$, and let $r$ and $s \leq 5$ be the natural numbers defined by Proposition 3.6. Let $S_{k}$ and $P_{k}$ be, respectively, the sum of the inverses and the product of the first $k$ primes greater than 5 . In particular we have $P_{r} \leq M^{5}-1 \leq 3.398 \cdot 10^{42}$, where $M=3.208 \cdot 10^{8}$. This implies that $r \leq 25$, and if we suppose $q \geq 10^{6}$ then $\delta \geq 1-S_{r}-\frac{5}{q}>0.20155$ and $\Delta \leq 145.885$. The condition from Proposition 3.6 is $q^{\frac{1}{2}} \geq 1167.08 \geq 8 \cdot 145.885 \geq W(m) W(T) \Delta$. So, if $q \geq 1.363 \cdot 10^{6}$, then there exists a primitive 2-normal element in $\mathbb{F}_{q^{5}}$.

Finally, we apply one last time Proposition 3.6, setting $m=\operatorname{gcd}\left(q^{5}-1,2 \cdot 3 \cdot 5\right)$ (so $W(m) \leq 8$ ) and $T=1$. This time we get $r \leq 19$ and $S_{r} \leq 0.7359$. If we suppose $q \geq 10^{6}$, we get $\delta \geq 0.264104, \Delta \leq 89.087$ and $q \geq 507936$.

If we try to use procedure Test_Delta, with $u=2 \cdot 3$, for all prime powers such that $q \equiv 0, \pm 1(\bmod 5)$ and $q<507936$, it will produce a list of 127 prime powers for which Test_Delta returns False. For this reason we will try another approach.

Lemma 5.5. Let $q$ be a prime power such that $q \equiv \pm 1(\bmod 5)$. Then $x^{5}-1=$ $(x-1)\left(x^{2}-b x+1\right)\left(x^{2}+(b+1) x+1\right)$, where $b \in \mathbb{F}_{q}$ is a root of $x^{2}+x-1=0$.

Proof. Let $\xi \neq 1$ be a root of $x^{5}-1$ in $\mathbb{F}_{q^{2}}$ and define $b=\xi+\xi^{-1}$. If $q \equiv 1(\bmod 5)$, then obviously $\xi \in \mathbb{F}_{q}$, which implies that $b \in \mathbb{F}_{q}$. If $q \equiv-1(\bmod 5)$, then there exists a primitive element $\alpha$ in $\mathbb{F}_{q^{2}}$ such that $\xi=\alpha^{\frac{q^{2}-1}{5}}$. Observe that

$$
\xi^{q}=\alpha^{\frac{q(q-1)(q+1)}{5}}=\left(\alpha^{\frac{q+1}{5}}\right)^{q^{2}-q}=\left(\alpha^{\frac{q+1}{5}}\right)^{1-q}=\xi^{-1}
$$

This implies that $b^{q}=\xi^{-1}+\xi=b$, so we also have $b \in \mathbb{F}_{q}$. Since $\xi^{4}+\xi^{3}+\xi^{2}+\xi+1=0$, we get that $b^{2}+b=\xi^{-2}\left(\xi^{4}+\xi^{3}+\xi^{2}+\xi+1\right)+1=1$ and $\left(x^{2}-b x+1\right)\left(x^{2}+(b+1) x+1\right)=$ $x^{4}+x^{3}+x^{2}+x+1$.

Lemma 5.6. Let $q$ be a prime power such that $q \equiv \pm 1(\bmod 5), b \in \mathbb{F}_{q}$ be a root of $x^{2}+x-1=0$, $\alpha$ be a normal element in $\mathbb{F}_{q^{5}}$ and $f=x^{2}-b x+1$. Then $L_{f}(\alpha)+a$ is a 2-normal element in $\mathbb{F}_{q^{5}}$ for all $a \in \mathbb{F}_{q}$ except for only one value of $a$.

Proof. If we let $g=(x-1)\left(x^{2}+(b+1) x+1\right)$, we get $f g=x^{5}-1$ and, for every element $\gamma \in \mathbb{F}_{q^{5}}$ we have $0=L_{f g}(\gamma)=L_{g}\left(L_{f}(\gamma)\right)$. Since $x-1$ is a factor of $g$, then $L_{g}(a)=0$ for every $a \in \mathbb{F}_{q}$. In particular, if $\alpha$ is a normal element in $\mathbb{F}_{q^{5}}$ then $L_{g}\left(L_{f}(\alpha)+a\right)=L_{g}\left(L_{f}(\alpha)\right)+L_{g}(a)=0$ for every $a \in \mathbb{F}_{q}$, so $L_{f}(\alpha)+a$ has $F_{q}$-order $\operatorname{deg} h \leq 3$ for some divisor $h$ of $g$. From Theorem 2.6, we get that $L_{f}(\alpha)+a$ is $k$ normal where $k \geq 2$. Let us suppose that $\operatorname{deg} h \leq 2$. If $x-1 \mid h$, then $L_{h}(a)=0$ and $L_{h}\left(L_{f}(\alpha)\right) \neq 0$, since $\alpha$ is normal, so, in this case, $L_{h}\left(L_{f}(\alpha)+a\right) \neq 0$. This means that if $\operatorname{deg} h \leq 2$, then $x-1 \nmid h$ and, in particular, $h \mid x^{2}+(b+1) x+1$. Note that $L_{x^{2}+(b+1) x+1}\left(L_{f}(\alpha)+a\right)=0$ is equivalent to $L_{x^{4}+x^{3}+x^{2}+x+1}(\alpha)+L_{x^{2}+(b+1) x+1}(a)=0$. Since $L_{x^{2}+(b+1) x+1}(a)=(b+3) a$, then $\alpha^{4}+\alpha^{3}+\alpha^{2}+\alpha+1=-(b+3) a$. If $b=-3$, then $(-3)^{2}+(-3)-1=0$, which is not possible because $5 \nmid q$ and hence $b \neq-3$. Therefore there is only one possible value of $a$ such that $T r_{q^{5} / q}(\alpha)=-(b+3) a$. In particular, this means that if $a \neq-(b+3)^{-1} \cdot \operatorname{Tr}_{q^{5} / q}(\alpha)$, then $L_{f}(\alpha)+a$ is 2-normal.

In fact, if $j \in \mathbb{F}_{q}$ is the only value for which $L_{f}(\alpha)+j$ is not 2-normal, then $g(a)=(a-j)\left(L_{f}(\alpha)+a\right)$ is 2-normal for every $a \in \mathbb{F}_{q} \backslash\{j\}$ and for $j$ we have $g(j)=0$. This means that if $g(a)$ is primitive, then $g(a)$ is also 2-normal. We finish the case $n=5$ with a computational approach using the idea from Lemma 5.6.

Proposition 5.7. Let $q$ be a prime power. There exists a primitive element 2normal in $\mathbb{F}_{q^{5}}$ if and only if $q \equiv 0, \pm 1(\bmod 5)$.

Proof. From Lemma5.4, we only need to prove the existence of a primitive 2-normal element in $\mathbb{F}_{q^{5}}$ for $q \equiv 0, \pm 1(\bmod 5)$ such that $q<507936$.

Inspired by Lemma 5.6, we use the SageMath procedure named TestExplicit5 (see Appendix A) to find a primitive 2-normal element in $\mathbb{F}_{q^{5}}$. In this procedure, $a$ generates $\mathbb{F}_{q^{5}}, j \in \mathbb{F}_{p}, b \in \mathbb{F}_{q}$ is a root of $x^{2}+x-1=0$ and $\beta=L_{x^{2}-b x+1}(a)$ for $q \equiv \pm 1(\bmod 5)$. If $5 \mid q$, then $\beta=L_{(x-1)^{2}}(a)$. From Lemma 5.6, we get that if $q \equiv \pm 1(\bmod 5)$, then $\beta+j$ is always 2 -normal except, maybe, for one value of $j$. In any case $(q \equiv 0(\bmod 5)$ or $q \equiv \pm 1(\bmod 5))$, this procedure returns True if $\beta+j$ is primitive 2-normal in $\mathbb{F}_{q^{5}}$ for some $j \in \mathbb{F}_{p}$, where $p=\operatorname{char} \mathbb{F}_{q}$.

For all prime powers $q$ such that $q \equiv 0, \pm 1(\bmod 5)$ for which $q<507936$ and Test_Delta (with $u=2 \cdot 3$ ) returns False, the procedure TestExplicit5(q) returns False only for $q=64$.

For $q=64$, we may use procedures ordmodqn and Normal (see Appendix A) to see that $\alpha$ is a primitive 2-normal element in $\mathbb{F}_{q^{5}}$ where $\alpha$ is a root of

$$
g(x)=x^{30}+x^{27}+x^{26}+x^{23}+x^{22}+x^{21}+x^{16}+x^{14}+x^{12}+x^{9}+x^{6}+x^{5}+x^{3}+x+1 .
$$

This proves the proposition.

## 6. CASE $n=4$

In [10] after Remark 3.5, the author proved that there is no primitive 2-normal element in $\mathbb{F}_{q^{4}}$ if $q \equiv 3(\bmod 4)$. Suppose now that $q$ is a power of 2 . In this case, $x^{4}-1=(x+1)^{4}$ and $f=(x+1)^{2}$, so if $\beta$ is a 2 -normal element, there exists a normal element $\alpha \in \mathbb{F}_{q^{4}}$ such that $\beta=L_{(x+1)^{2}}(\alpha)=\alpha^{q^{2}}+\alpha$. Since $\beta^{q^{2}}=\left(\alpha^{q^{2}}+\alpha\right)^{q^{2}}=$ $\alpha+\alpha^{q^{2}}=\beta$, we have that $\beta$ is not a primitive element in $\mathbb{F}_{q^{4}}$. Therefore, if there exists a primitive 2-normal element in $\mathbb{F}_{q^{4}}$, then $q \equiv 1(\bmod 4)$. In this case, we may factor $x^{4}-1$ into four linear factors, say $x^{4}-1=(x+1)(x-1)(x+b)(x-b)$, where $b \in \mathbb{F}_{q}$ and $b^{2}=-1$.

Throughout this section, we will consider $q \equiv 1(\bmod 4), b \in \mathbb{F}_{q}$ such that $b^{2}=$ $-1, f(x)=(x+1)(x+b) \in \mathbb{F}_{q}[x]$ a factor of $x^{4}-1$ of degree two and $\alpha \in \mathbb{F}_{q^{4}}$ a normal element in $\mathbb{F}_{q^{4}}$. Thus, $L_{f}(\alpha)$ is a 2-normal element in $\mathbb{F}_{q^{4}}$ (see [10], Lemma 3.1). The following result tells us that we can generate more 2-normal elements if they are also primitive, more precisely we have

Lemma 6.1. Let $u, v \in \mathbb{F}_{q}^{*}$ and $f(x)=(x+1)(x+b) \in \mathbb{F}_{q}[x]$, where $b \in \mathbb{F}_{q}$ satisfies $b^{2}=-1$. If $\gamma=u L_{f}(\alpha)+v$ is primitive in $\mathbb{F}_{q^{4}}$, then $\gamma$ is 2 -normal in $\mathbb{F}_{q^{4}}$.

Proof. We know that $L_{f}$ is a linear transformation over $\mathbb{F}_{q}$, so $L_{\left(x^{4}-1\right) / f}\left(u L_{f}(\alpha)+v\right)=$ $u L_{(x-1)(x-b)}\left(L_{f}(\alpha)\right)+L_{(x-1)(x-b)}(v)=u L_{x^{4}-1}(\alpha)+L_{(x-b)}\left(v^{q}-v\right)=0$. Since $\frac{x^{4}-1}{f}$ is a degree two polynomial, we have that the set $\left\{\gamma, \gamma^{q}, \gamma^{q^{2}}\right\}$ is linearly dependent. Suppose that $\gamma$ and $\gamma^{q}$ are linearly dependent, thus $\gamma^{q-1} \in \mathbb{F}_{q}^{*}$ and $\operatorname{ord}_{\mathrm{q}}(\gamma) \leq$ $(q-1)^{2}<q^{4}-1$, which is a contradiction. Thus, $\left\langle\gamma, \gamma^{q}, \gamma^{q^{2}}, \gamma^{q^{3}}\right\rangle=\left\langle\gamma, \gamma^{q}\right\rangle$ and $\gamma$ is 2-normal, by Theorem 2.6.

Let us define a function $g(x)=x+\beta$, where $\beta$ is a 2-normal element in $\mathbb{F}_{q^{n}}$. We need conditions for the existence of primitive elements of the form $g(a)$, where $a \in \mathbb{F}_{q}^{*}$, because in the case where $n=4$, if we choose $\beta=L_{f}(\alpha)$ then, from Lemma 6.1, primitivity of $g(a)$ implies 2-normality of $g(a)$. Observe also that if $\beta=L_{f}(\alpha) \in \mathbb{F}_{q^{4}}$, then $\beta \notin \mathbb{F}_{q^{2}}$. Indeed, if $\beta \in \mathbb{F}_{q^{2}}$, then $0=L_{\frac{x^{4}-1}{f}}(\beta)=\beta^{q^{2}}-(b+$ 1) $\beta^{q}+b \beta=(b+1)\left(\beta-\beta^{q}\right)$. Since $b^{2}=-1$ and $q \equiv 1(\bmod 4)$, we get $b \neq-1$ and $\beta \in \mathbb{F}_{q}$, which is a contradiction.

Theorem 6.2. Let $q$ be a prime power, let $m \in \mathbb{N}$ be a divisor of $q^{4}-1$, and let $\beta=L_{f}(\alpha)$ be a 2-normal element in $\mathbb{F}_{q^{4}}$, where $\alpha \in \mathbb{F}_{q^{4}}$ is a normal element, $f(x)=(x+1)(x+b) \in \mathbb{F}_{q}[x]$ and $b \in \mathbb{F}_{q}$ satisfies $b^{2}=-1$. Let $N_{\beta}(m)$ be the number of elements $a \in \mathbb{F}_{q}$ such that $g(a)=a+\beta$ is m-free. If $q^{1 / 2} \geq 3 W(m)$, then $N_{\beta}(m)>0$, i.e., there exists an element of the form $g(a)$ in $\mathbb{F}_{q^{4}}^{*}$ which is $m$-free.

Proof. Since $\beta=L_{f}(\alpha) \notin \mathbb{F}_{q^{2}}$, we have that $\mathbb{F}_{q^{4}}=\mathbb{F}_{q}(\beta)$. Therefore, from Lemma 2.12, for any non-trivial multiplicative character $\chi$ over $\mathbb{F}_{q^{4}}$, we have

$$
\begin{equation*}
\left|\sum_{a \in \mathbb{F}_{q}} \chi(g(a))\right| \leq 3 \sqrt{q}, \tag{7}
\end{equation*}
$$

Now, from Proposition 2.8, we have that

$$
N_{\beta}(m)=\sum_{a \in \mathbb{F}_{q}} w_{m}(g(a))=\theta(m)\left(\sum_{a \in \mathbb{F}_{q}} \chi_{1}(g(a))+\int_{d \mid m, d \neq 1} \sum_{a \in \mathbb{F}_{q}} \chi_{d}(g(a))\right) .
$$

Therefore we obtain the following estimative, using inequality (7)

$$
\left|\frac{N_{\beta}(m)}{\theta(m)}-q\right| \leq\left|\sum_{\substack{d \mid m \\ d \neq 1}} \frac{\mu(d)}{\varphi(d)} \sum_{(d)} \sum_{a \in \mathbb{F}_{q}} \chi_{d}(g(a))\right| \leq 3 \sqrt{q} \sum_{\substack{d \mid m \\ d \neq 1}}|\mu(d)|=3(W(m)-1) \sqrt{q},
$$

Then $\frac{N_{\beta}(m)}{\theta(m)} \geq q-3(W(m)-1) \sqrt{q}$, and we obtain the desired result.
The next result's proof is similar to the proof of Proposition 3.6, and hence is omitted.

Proposition 6.3. Let $m \in \mathbb{N}$ be a divisor of $q^{4}-1$ and let $\beta$ be a 2-normal element as in Theorem 6.2. Let $\wp_{1}, \ldots, \wp_{r}$ be prime numbers such that $\operatorname{rad}\left(q^{4}-1\right)=\operatorname{rad}(m) \cdot \wp_{1}$. $\wp_{2} \cdots \wp_{r}$. Suppose that $\delta=1-\sum_{i=1}^{r} \frac{1}{\wp_{i}}>0$ and let $\Delta=\frac{r-1}{\delta}+2$. If $q^{\frac{1}{2}} \geq 3 W(m) \Delta$, then $N_{\beta}\left(q^{4}-1\right)>0$.

Now, we get the following sufficient conditions for the existence of primitive elements in $\mathbb{F}_{q^{4}}$ of the form $g(a)=a+\beta$. If

$$
\begin{equation*}
q^{1 / 2}>3 W\left(q^{4}-1\right) \quad \text { or } \quad q^{1 / 2}>3 W(m) \Delta \tag{8}
\end{equation*}
$$

(for some $m \mid q^{4}-1$ and a specific value of $\Delta$ ), then there exists a primitive element in $\mathbb{F}_{q^{4}}$ of the form $g(a)=a+\beta$ with $a \in \mathbb{F}_{q}$; also this element is 2-normal by Lemma 6.1. Let us use Proposition 6.3 in combination with Lemma 6.1 to find a bound for the values of $q$ such that there exists a primitive 2-normal element in $\mathbb{F}_{q^{4}}$.

Theorem 6.4. Let $q$ be a prime power. There exists a primitive 2-normal element in $\mathbb{F}_{q^{4}}$ if and only if $q \equiv 1(\bmod 4)$.

Proof. We proceed as in Lemma 5.4. Let $t, u$ be positive real numbers such that $t+u>8$ and let $q^{4}-1=\wp_{1}^{a_{1}} \cdots \wp_{v}^{a_{v}} \cdot \varrho_{1}^{b_{1}} \cdots \varrho_{r}^{b_{r}}$ be the prime factorization of $q^{4}-1$ such that $2 \leq \wp_{i} \leq 2^{t}$ or $2^{t+u} \leq \wp_{i}$ for $1 \leq i \leq v$ and $2^{t}<\varrho_{i}<2^{t+u}$ for $1 \leq i \leq r$ and consider $m=\wp_{1}^{a_{1}} \cdots \wp_{v}^{a_{v}}$. Let $S_{t, u}<1$ be the sum of the inverses of all prime numbers between $2^{t}$ and $2^{t+u}$, and $r(t, u)$ be the number of those primes. As in

Lemma $5.4 r \leq r(t, u), \delta \geq 1-S_{t, u}$ and $\Delta \leq 2+\frac{r(t, u)-1}{1-S_{t, u}}$. By Lemma 2.9, considering that $2^{4} \mid q^{4}-1$ and $4<t+u$, we have $W(m)<A_{t, u} \cdot q^{\frac{4}{t+u}}$, where

$$
A_{t, u}=\frac{2}{\sqrt[t+u]{2^{4}}} \cdot \prod_{\substack{2<\wp<2^{t} \\ \wp \text { is prime }}} \frac{2}{\sqrt[t+u]{\wp}}
$$

We know that $\beta=L_{f}(\alpha) \notin \mathbb{F}_{q^{2}}$ and we may apply Proposition 6.3. Therefore, if $q^{\frac{1}{2}} \geq 3 \cdot A_{t, u} \cdot q^{\frac{4}{t+u}} \cdot \Delta$ then $N_{\beta}\left(q^{4}-1\right)>0$. This condition is equivalent to $q \geq\left(3 \cdot A_{t, u} \cdot \Delta\right)^{\frac{2(t+u)}{t+u-8}}$. Taking $t=5$ and $u=8.5$ we get $N_{\beta}\left(q^{4}-1\right)>0$ for $q \geq M=2.12 \cdot 10^{35}$.

Suppose now $q<M=2.12 \cdot 10^{35}$. We will use now Proposition 6.3 with $m=q^{2}-1$. Let $q^{2}+1=2 \cdot \wp_{1}^{a_{1}} \cdots \wp_{r}^{a_{r}}$ be the prime factorization of $q^{2}+1$. For any odd prime number such that $\wp \mid q^{2}+1$, we have $q^{2} \not \equiv 1(\bmod \wp)$ and $q^{4} \equiv 1(\bmod \wp)$. This means that $4 \mid \varphi(\wp)=\wp-1$. Let $S_{k}$ be the sum of the inverses of the first $k$ prime numbers of the form $4 j+1$ and let $P_{k}$ be the product of those $k$ prime numbers. So, from $2 P_{r} \leq q^{2}+1<M^{2}+1$, we get $r \leq 33, S_{r}<0.60520004, \delta>0.39479996$ and $\Delta \leq 83.054$. Let

$$
A_{t}=\frac{2}{\sqrt[t]{2^{3}}} \cdot \prod_{\substack{2<\wp<2^{t} \\ \wp>\text { is prime }}} \frac{2}{\sqrt[t]{p}}
$$

be the constant from Lemma [2.9, considering that $2^{3} \mid q^{2}-1$ and $3<t$. Therefore, if $q^{\frac{1}{2}} \geq 3 \cdot A_{t} \cdot q^{\frac{2}{t}} \cdot \Delta>3 \cdot W\left(q^{2}-1\right) \cdot \Delta$, and applying Proposition 6.3, then $N_{\beta}\left(q^{4}-1\right)>0$. For $t=6.8$, we get $\left(3 \cdot A_{t} \cdot \Delta\right)^{\frac{2 t}{t-4}} \leq 7.321 \cdot 10^{21}$.

Let us suppose now that $M=7.321 \cdot 10^{21}$ and $q<M$. We will use again Proposition 6.3 with $m=\operatorname{gcd}\left(q^{4}-1,2 \cdot 3 \cdot 5 \cdot 7\right)$. Let $S_{k}$ be the sum of the inverses of the first $k$ prime numbers starting with 11 and let $P_{k}$ be the product of those $k$ prime numbers. Observe that if $5 \nmid q^{4}-1$, then $q$ is a prime power of 5 which implies that $3 \mid q^{4}-1$. This means that $2^{4} \cdot 3 \mid q^{4}-1$ or $2^{4} \cdot 5 \mid q^{4}-1$. Let $r$ be the number of prime factors of $q^{4}-1$ greater than 7 . We have $r \leq 44$, since $P_{r}<\frac{M^{4}-1}{48}$. So, $S \leq S_{r}<0.7821, \delta>0.2179$ and therefore $\Delta<2+\frac{44-1}{0.2179}<199.34$. Thus, $q^{\frac{1}{2}} \geq 3 \cdot W(m) \cdot \Delta$ for $q \geq 9.156 \cdot 10^{7}$.

We repeat this last process with $M=9.156 \cdot 10^{7}$ and $m=\operatorname{gcd}\left(q^{4}-1,2 \cdot 3 \cdot 5\right)$. Now $S_{k}$ is the sum of the inverses of the first $k$ prime numbers starting with 7 and $P_{k}$ is the product of those $k$ prime numbers. We have $P_{r}<\frac{M^{4}-1}{48}$ for $r \leq 19$ and $\Delta<70.155$. So, $q^{\frac{1}{2}} \geq 3 \cdot 2^{3} \cdot 70.155 \geq 3 \cdot W(m) \cdot \Delta$ for $q \geq 2834914$. Repeating this process one last time with $M=2834914$ and $m=\operatorname{gcd}\left(q^{4}-1,2 \cdot 3 \cdot 5\right)$, we get $r \leq 16, \Delta<51.253$.

From Proposition 6.3 we get $N_{\beta}\left(q^{4}-1\right)>0$ if $q \geq 1513078>\left(3 \cdot 2^{3} \cdot 51.253\right)^{2}$. From Lemma 6.1, we get that, for $q \geq 1513078$, there exists a primitive 2-normal element in $\mathbb{F}_{q^{4}}$.

There are 57731 prime powers $q \equiv 1(\bmod 4)$ less than 1513078 . We use the test $q^{\frac{1}{2}} \geq 3 \cdot W(m) \cdot \Delta$ using the SageMath procedure Test(q,list) from Appendix A where the variable list is the list of prime numbers which can be factors of $m$. With list $=[2,3,5]$, the procedure $\operatorname{Test}(\mathbf{q}$, list) returns False for 1704 primes powers from all prime powers $q \equiv 1(\bmod 4)$ less than 1513078 . For those prime powers, the procedure $\operatorname{Test}(\mathbf{q}$, list), with list $=[2,3]$, returns False for 934 prime powers. Finally, for those last prime powers, the procedure $\operatorname{Test}(\mathbf{q}$, list), with list $=[2]$, returns False for 918 prime powers.

Now, we use the SageMath procedure named TestExplicit4 (see Appendix A) to find a primitive 2-normal element in $\mathbb{F}_{q^{4}}$. In this procedure, we found first a normal element $\alpha \in \mathbb{F}_{q^{4}}, b \in \mathbb{F}_{q}$ a root of $x^{2}+1=0$ and we define $\beta=L_{(x+1)(x+b)}(a)$. Next, we try to find an element $j \in \mathbb{F}_{p}$ such that $\beta+j$ is primitive. From Lemma 6.1, $\beta+j$ is also 2 -normal. This procedure returns True if $\beta+j$ is primitive 2-normal in $\mathbb{F}_{q^{4}}$ for some $j \in \mathbb{F}_{p}$. For all 918 prime powers for which we didn't conclude with the procedure $\operatorname{Test}(\mathbf{q}, \mathbf{l i s t})$, the procedure TestExplicit4(q) returns False only for $13,17,125$. Table 8 shows for these cases a primitive 2 -normal element $\alpha \in \mathbb{F}_{q^{4}}$, such that $h(\alpha)=0$, for some irreducible polynomial $h \in \mathbb{F}_{p}[x]$, where $p$ is the characteristic of $\mathbb{F}_{q}$. This completes the proof.

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## References

[1] L. Carlitz, Primitive roots in a finite field, Transactions of the American Mathematical Society 73 (1952), 373-382.
[2] S.D. Cohen and S. Huczynska, The primitive normal basis theorem without a computer, Journal of London Mathematical Society, v. 67, n. 1 (2003), 41-56.
[3] H. Davenport, Bases for finite fields, Journal of the London Mathematical Society 43 (1968), 21-39.
[4] S. Huczynska, G.L. Mullen, D. Panario and D. Thomson, Existence and properties of k-normal elements over finite fields, Finite Fields Appl. 24 (2013), 170-183.
[5] G. Kapetanakis and L. Reis, Variations of the Primitive Normal Basis Theorem, Designs, Codes and Cryptography 87 (2019), 1459-1480.
[6] N.M. Katz, An Estimate for Character Sums, Journal of the American Mathematical Society, Vol. 2, No. 2. (1989), 197-200.
[7] H.W. Lenstra and R. Schoof, Primitive normal bases for finite fields, Mathematics of Computation 48 (1987), 217-231.
[8] R. Lidl and H. Niederreiter, Finite Fields, Cambridge university press, 1997.
[9] C. Negre, Finite field arithmetic using quasi-normal bases, Finite Fields Appl. 13 (2007), 635-647.
[10] L. Reis, Existence results on $k$-normal elements over finite fields, Rev. Mat. Iberoam. 35(3) (2019), 805-822.
[11] L. Reis and D. Thompson, Existence of primitive 1-normal elements in finite fields, Finite Fields and Their Applications 51 (2018), 238-269.
[12] G. Meletiou and G. Mullen, A note on discrete logarithms in finite fields, Applicable Algebra in Engineering, Communication and Computing Vol.3(1) (1992), 75-78.
[13] The Sage Developers, SageMath, the Sage Mathematics Software System (Version 8.1), https://www.sagemath.org, 2020.
[14] W.M. Schmidt, Equations over finite fields, an elementary approach, Lecture Notes in Mathematics 536, Springer-Verlag, Berlin-New York, 1976.
[15] J.A. Sozaya-Chan and H. Tapia-Recillas, On k-normal elements over finite fields, Finite Fields and Their Applications 52 (2018), 94-107.
[16] A. Zhang and K. Feng, A New Criterion on k-Normal Elements over Finite Fields, Chinese Annals of Mathematics, Series B 41 (2020), 665-678.

## Appendix A: Procedures in SageMath

```
def Test_Delta(q,n,u):
    A.<a>=GF(q); P.<x>=PolynomialRing(A)
    M1=factor(q^n-1); M2=factor(x^n-1) ; m=1
    count1=0; count2=0; choose=True
    T=1
    for g in M2:
    if g[0].degree()==1 and q<7:
        T=T*g[0]
    r=len(M1); s=len(M2) ; S1=0; S2=0
    for p in M1:
        if gcd(p[0],u)!=1:
            m=m*p [0]
        r=r-1
    else:
```

```
                S1=S1+1/p[0]
    for Q in M2:
    if }\operatorname{gcd}(Q[0],T)!=1
        s=s-1
    else:
        S2=S2+1/q^(Q[0].degree())
delta=1-S1-S2
if delta>0:
    Delta=2+(r+s-1)/delta; A= (q*1.0) ^ (n*0.5-2)
    B=Delta*2^(len(factor(m)))*2^len(factor(T)); Fact=A>=B
    else:
    Fact=False
return Fact
```

\#Given $\mathrm{p}, \mathrm{q}, \mathrm{n}$ and g define:
$\mathrm{B}=\mathrm{GF}(\mathrm{p})$; T. $\langle\mathrm{x}\rangle=$ PolynomialRing (B)
C. $\langle c\rangle=$ B.extension(g); R. $\langle x\rangle=$ PolynomialRing(C)
\#Testing if $c$ is primitive: ordmodqn(c)
\#Testing if c is 2-normal: Normal(c)
\#where:
def ordmodqn(d):
ord=q^n-1
for $m$ in divisors ( $q^{\wedge} n-1$ ):
if $\mathrm{d}^{\wedge} \mathrm{m}==1$ :
if m<ord:
ord=m
return ( $q^{\wedge} n-1$ )/ord
def Normal(e):
pol=0
for $i$ in range $(0, n)$ :
pol=pol+e^(q^i)*x^(n-1-i)
pol_gcd=gcd(pol, $\left.x^{\wedge} n-1\right)$; $k=p o l \_g c d . d e g r e e()$
return k
def TestExplicit5(q):
A. $\langle\mathrm{a}\rangle=G F\left(q^{\wedge} 5\right) ;$ T. $\langle x\rangle=$ PolynomialRing (A)

```
    if mod(q,5)==0:
    beta=a^(q^2)-2*a^q+a
    else:
    Sol=(x^2+x-1).roots(); b=Sol[0] [0]; beta=a^(q^2)-b*a^q+a
j=0; Teste=False; valor=True
while valor:
    c=beta+j; ord=q^5-1
    for m in divisors(q`5-1):
        if c^m==1:
            if m<ord:
                ord=m
    e=(q^n-1)/ord
    if e==1:
        pol=0
        for i in range(0,5):
            pol=pol+c^(q^i)*x^(4-i)
        pol_gcd=gcd(pol,x^n-1); k=pol_gcd.degree()
        if k==2:
            valor=False; Teste=True
    j=j+1
    if beta+j==beta:
        valor=False
return Teste
```

```
def Test(q,list):
    L=factor(q^4-1); m=1; r=len(L); S=0.0
    for p in L:
        if p[0] in list:
            m=m*p [0] ; r=r-1
        else:
            S=S+1/p[0]
    delta=1-S
    if delta>0:
        Delta=2+(r-1)/delta; A=(q*1.0)^(0.5); B=3*Delta*2^(len(factor(m)))
        Fact=A>=B
    else:
        Fact=False
```

return Fact

```
def TestExplicit4(q):
    A.<a>=GF(q^4, modulus="primitive"); T.<x>=PolynomialRing(A)
    z=1
    norm=True
    while norm:
        alpha=a^z
        pol=0
        for i in range(0,n):
            pol=pol+alpha^(q^i)*x^(n-1-i)
        pol_gcd=gcd(pol,x^n-1); k=pol_gcd.degree()
        if k==0:
            norm=False
        z=z+1
        if z==q^4-1 and norm:
            norm=False; Test=False
    Sol=(x^2+1).roots(); b=Sol[0] [0]
    beta=alpha^(q^2)+(b+1)*alpha^q+b*alpha; j=0
    Test=False; valor=True
    while valor:
        c=beta+j; ord=q^n-1
        for m in divisors(q^n-1):
            if c^m==1:
                    if m<ord:
                    ord=m
        e=(q^n-1)/ord
        if e==1:
            pol=0
            for i in range(0,n):
                    pol=pol+c^(q^i)*x^(n-1-i)
            pol_gcd=gcd(pol,x^n-1); k=pol_gcd.degree()
            if k==2:
                valor=False; Test=True
        j=j+1
        if beta+j==beta:
            valor=False
```


## return Test

Appendix B: Tables

| $(q, n)$ | $g(x) \in \mathbb{F}_{p}[x]$ |
| :---: | :---: |
| $(2,6)$ | $x^{6}+x^{5}+x^{3}+x^{2}+1$ |
| $(2,8)$ | $x^{8}+x^{5}+x^{3}+x+1$ |
| $(2,9)$ | $x^{9}+x^{8}+x^{6}+x^{5}+x^{4}+x^{3}+x^{2}+x+1$ |
| $(2,10)$ | $x^{10}+x^{6}+x^{5}+x^{3}+x^{2}+x+1$ |
| $(2,12)$ | $x^{12}+x^{10}+x^{8}+x^{4}+x^{3}+x^{2}+1$ |
| $(3,6)$ | $x^{6}+x^{5}+x^{4}+x^{3}+x+2$ |
| $(3,8)$ | $x^{8}+2 x^{5}+x^{4}+2 x^{2}+2 x+2$ |
| $(3,10)$ | $x^{10}+x^{8}+x^{7}+2 x^{6}+x^{5}+x^{4}+x^{3}+2 x^{2}+x+2$ |
| $(3,12)$ | $x^{12}+x^{10}+2 x^{9}+2 x^{8}+x^{7}+x^{6}+2 x^{4}+2 x^{3}+2$ |

TABLE 4. $\alpha \in \mathbb{F}_{q^{n}}$ is a primitive 2-normal element such that $g(\alpha)=0$

| $(q, n)$ | $g(x) \in \mathbb{F}_{p}[x]$ |
| :---: | :---: |
| $(4,5)$ | $x^{10}+x^{8}+x^{6}+x^{5}+x^{3}+x+1$ |
| $(4,6)$ | $x^{12}+x^{11}+x^{10}+x^{8}+x^{6}+x^{4}+x^{3}+x+1$ |
| $(4,8)$ | $x^{16}+x^{13}+x^{12}+x^{11}+x^{10}+x^{9}+x^{8}+x^{7}+x^{5}+x^{2}+1$ |
| $(4,9)$ | $x^{18}+x^{16}+x^{12}+x^{10}+x^{4}+x+1$ |
| $(5,5)$ | $x^{5}+2 x^{3}+x+2$ |
| $(5,6)$ | $x^{6}+x^{4}+4 x^{3}+x^{2}+2$ |
| $(5,8)$ | $x^{8}+4 x^{7}+x^{6}+3 x^{4}+x^{3}+x+3$ |
| $(5,12)$ | $x^{12}+x^{11}+3 x^{10}+x^{9}+4 x^{7}+3 x^{5}+3 x^{3}+3 x^{2}+4 x+3$ |
| $(7,6)$ | $x^{6}+x^{4}+5 x^{3}+4 x^{2}+6 x+3$ |
| $(7,8)$ | $x^{8}+3 x^{6}+6 x^{5}+x^{4}+6 x^{3}+5 x^{2}+4 x+5$ |
| $(8,6)$ | $x^{18}+x^{16}+x^{15}+x^{14}+x^{13}+x^{6}+x^{2}+x+1$ |
| $(8,7)$ | $x^{21}+x^{16}+x^{14}+x^{11}+x^{7}+x^{6}+x^{5}+x^{3}+1$ |

TABLE 5. $\alpha \in \mathbb{F}_{q^{n}}$ is a primitive 2-normal element such that $g(\alpha)=0$

| $(q, n)$ | $g(x) \in \mathbb{F}_{p}[x]$ |
| :---: | :---: |
| $(9,5)$ | $x^{10}+x^{7}+2 x^{6}+x^{5}+x^{4}+2 x^{3}+2$ |
| $(9,6)$ | $x^{12}+x^{9}+x^{8}+x^{7}+x^{6}+2 x^{4}+2 x^{3}+2 x+2$ |
| $(9,8)$ | $x^{16}+2 x^{14}+2 x^{13}+2 x^{12}+x^{11}+x^{10}+2 x^{9}+x^{8}+x^{5}+x^{4}+2$ |
| $(11,5)$ | $x^{5}+9 x^{3}+4 x^{2}+9 x+3$ |
| $(11,6)$ | $x^{6}+9 x^{5}+x^{4}+3 x^{3}+x^{2}+x+7$ |
| $(13,6)$ | $x^{6}+10 x^{3}+11 x^{2}+11 x+2$ |
| $(16,5)$ | $x^{20}+x^{19}+x^{15}+x^{13}+x^{11}+x^{10}+x^{7}+x^{6}+x^{3}+x+1$ |
| $(16,6)$ | $x^{24}+x^{22}+x^{21}+x^{20}+x^{19}+x^{18}+x^{15}+$ |
| $(17,6)$ | $x^{14}+x^{12}+x^{10}+x^{8}+x^{7}+x^{3}+x^{2}+1$ |
| $(19,5)$ | $x^{6}+9 x^{5}+15 x^{4}+6 x^{3}+x^{2}+4 x+14$ |
| $(19,6)$ | $x^{5}+2 x^{4}+x^{2}+2 x+16$ |

TABLE 6. $\alpha \in \mathbb{F}_{q^{n}}$ is a primitive 2-normal element such that $g(\alpha)=0$

| $q$ | $g(x) \in \mathbb{F}_{p}[x]$ |
| :---: | :---: |
| 23 | $x^{6}+3 x^{5}+20 x^{4}+12 x^{3}+6 x+11$ |
| 25 | $x^{12}+x^{11}+3 x^{10}+x^{9}+4 x^{7}+3 x^{5}+3 x^{3}+3 x^{2}+4 x+3$ |
| 29 | $x^{6}+14 x^{4}+22 x^{3}+6 x^{2}+2 x+15$ |
| 31 | $x^{6}+19 x^{3}+16 x^{2}+8 x+3$ |
| 37 | $x^{6}+35 x^{3}+4 x^{2}+30 x+2$ |
| 41 | $x^{6}+17 x^{4}+19 x^{3}+9 x^{2}+38 x+17$ |
| 43 | $x^{6}+19 x^{3}+28 x^{2}+21 x+3$ |
| 47 | $x^{6}+35 x^{4}+36 x^{3}+36 x^{2}+19 x+31$ |
| 49 | $x^{12}+6 x^{10}+5 x^{9}+6 x^{8}+6 x^{7}+3 x^{6}+x^{5}+4 x^{3}+x^{2}+5 x+3$ |
| 59 | $x^{6}+13 x^{4}+56 x^{3}+15 x^{2}+2 x+11$ |
| 61 | $x^{6}+49 x^{3}+3 x^{2}+29 x+2$ |
| 67 | $x^{6}+32 x^{5}+58 x^{4}+46 x^{3}+22 x^{2}+59 x+61$ |
| 79 | $x^{6}+19 x^{3}+28 x^{2}+68 x+3$ |

TABLE 7. $\alpha \in \mathbb{F}_{q^{6}}$ is a primitive 2-normal element such that $g(\alpha)=0$

| $q$ | $h(x) \in \mathbb{F}_{p}[x]$ |
| :---: | :---: |
| 13 | $x^{4}+11 x^{3}+8 x^{2}+6 x+11$ |
| 17 | $x^{4}+10 x^{2}+5 x+3$ |
| 125 | $x^{12}+x^{10}+3 x^{9}+4 x^{8}+3 x^{6}+2 x^{5}+2 x^{4}+3 x^{3}+x^{2}+4 x+3$ |

TABLE 8. $\alpha \in \mathbb{F}_{q^{4}}$ is a primitive 2-normal element such that $h(\alpha)=0$

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[^0]:    ${ }^{1}$ We use this definition to find primitive 2-normal elements for specific values of $(q, n)$, using Sagemath (cf. [13]) program. Also, throughout this paper, when we talk about a $k$-normal element $\alpha \in \mathbb{F}_{q^{n}}$, it will be over $\mathbb{F}_{q}$.

