Almost Affinely Disjoint Subspaces

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Abstract

In this work, we introduce a natural notion concerning finite vector spaces. A family of k-dimensional subspaces of \mathbb{F}_q^n , which forms a partial spread, is called almost affinely disjoint if any (k+1)-dimensional subspace containing a subspace from the family non-trivially intersects with only a few subspaces from the family. The central question discussed in the paper is the polynomial growth (in q) of the maximal cardinality of these families given the parameters k and n. For the cases k=1 and k=2, optimal families are constructed. For other settings, we find lower and upper bounds on the polynomial growth. Additionally, some connections with problems in coding theory are shown.

Keywords: Partial spread, affinely disjoint subspaces, subspace design

1. Introduction

Let q be a prime power and let \mathbb{F}_q denote the finite field with q elements. By \mathbb{F}_q^n we denote the standard vector space over \mathbb{F}_q , whose elements are n-tuples $\boldsymbol{x}=(x_1,\ldots,x_n)$ with $x_i\in\mathbb{F}_q$. A partial k-spread in \mathbb{F}_q^n is a collection of k-dimensional subspaces with pairwise trivial intersection. For a vector $\boldsymbol{u}\in\mathbb{F}_q^n$ and a subspace $S\subseteq\mathbb{F}_q^n$, we define the affine subspace $\boldsymbol{u}+S:=\{\boldsymbol{u}+\boldsymbol{v}:\boldsymbol{v}\in S\}$. Denote by [m] the set of integers $\{1,\ldots,m\}$.

In this paper, we discuss two families of k-dimensional subspaces in \mathbb{F}_q^n satisfying the following definitions.

Definition 1 (Almost affinely disjoint subspace family). Given positive integers k and n such that 2k < n, let \mathcal{F} be a family of k-dimensional linear subspaces in \mathbb{F}_q^n . This family is said to be L-almost affinely disjoint (or, briefly, $[n, k, L]_q$ -AAD) if the two properties hold:

- 1. The family \mathcal{F} is a partial k-spread of \mathbb{F}_q^n .
- 2. For any $S \in \mathcal{F}$ and $\boldsymbol{u} \in \mathbb{F}_q^n \setminus S$, the affine subspace $\boldsymbol{u} + S$ intersects at most L subspaces from the family \mathcal{F} .

Note that $n \geq 2k$ is required to construct a partial spread of size larger than 1. We also exclude n = 2k, since if the family \mathcal{F} is a partial k-spread of \mathbb{F}_q^n , the affine subspace u + S, $S \in \mathcal{F}$ and $u \in \mathbb{F}_q^n \setminus S$, intersects every subspace from \mathcal{F} except S.

Definition 2 (Almost sparse subspace family). Given positive integers k and n such that 2k < n, let \mathcal{F} be a family of k-dimensional linear subspaces in \mathbb{F}_q^n . This family is said to be L-almost sparse (or, shortly, $[n, k, L]_q$ -AS) if the two properties hold:

- 1. The family \mathcal{F} is a partial k-spread of \mathbb{F}_q^n .
- 2. Any (k+1)-dimensional subspace in \mathbb{F}_q^n intersects non-trivially at most L subspaces from the family \mathcal{F} .

Remark 1. We also remark that the spread property does not affect the asymptotic analysis when $q \to \infty$ (see the proof of Theorem 1).

Given k, n, L and q we denote the maximal size of $[n,k,L]_q$ -AAD and $[n,k,L]_q$ -AS families by $m_q^{AAD}(n,k,L)$ and $m_q^{AS}(n,k,L)$. Define the polynomial growth of the maximal size of AAD and AS families by

$$\begin{split} p^{AAD}(n,k,L) &:= \limsup_{q \to \infty} \log_q(m_q^{AAD}(n,k,L)), \\ p^{AS}(n,k,L) &:= \limsup_{q \to \infty} \log_q(m_q^{AS}(n,k,L)). \end{split}$$

1.1. Related work

The concept of almost sparse families is closely related to so-called subspace designs introduced by Guruswami and Xing in [5] and further developed by Guruswami, Kopparty, Xing and Yuan in [2, 6]. A collection \mathcal{F} of subspaces in \mathbb{F}_q^n is called an $[n,k,L]_q$ -weak subspace design (c.f. [2]) if every k-dimensional subspace in \mathbb{F}_q^n intersects non-trivially at most L subspaces from \mathcal{F} . Despite it is not required by definition, many known constructions of weak subspace designs contain subspaces with a fixed co-dimension at least k. Weak subspace designs, almost sparse and almost affinely disjoint families have the following (trivial) relations:

- an $[n,k,L]_q$ -AS family is also an $[n,k+1,L]_q$ -weak subspace design and an $[n,k,L-1]_q$ -AAD family;
- for $n \ge 2k+1$, a partial k-spread of \mathbb{F}_q^n is an $[n,k+1,L]_q$ -weak subspace family if and only if it is also an $[n,k,L]_q$ -AS family.

By explicit constructions presented in [2], we derive that $p^{AS}(n,k,L) \geq \left\lfloor \frac{n-k}{k+1} \right\rfloor$ for $L \geq \frac{(n-1)(k+1)}{\lfloor (n-k)/(k+1) \rfloor}$. We note that the motivating application for subspace designs has n growing and, thus, a straightforward application of these results does not give an optimal result for our problem except for the case n=2k+1. Constructions of subspace designs have found several applications, such as constructing list-decodable error-correcting codes [5], rank-metric codes [4], dimensional expanders [3]. In particular, using this concept the first deterministic polynomial time construction of list-decodable codes over constant-sized large alphabets and sub-logarithmic list size achieving the optimal rate has been designed [2].

Almost affinely disjoint subspace families with n = 2k+1 were first introduced in [10]. It was proposed to use such families for constructing primitive batch codes defined by Ishai et al. in [8]. A primitive binary [N, K, s]-batch code encodes a binary string xof length K into a binary string y of length N, such that each multiset of k symbols from x has s mutually disjoint recovering sets from y. The basic question is how to minimize the redundancy, N-K, for given parameters K and s. Suppose that an $[n,k,L]_q$ -AAD family \mathcal{F} is given. Let $K:=q^n$. To construct a systematic batch code, we associate K information bits with K points in \mathbb{F}_q^n and set $y_i := x_i$ for $i \in [K]$. For an affine subspace of the form $\mathbf{v} + S$ with $S \in \mathcal{F}$ and $\mathbf{v} \in \mathbb{F}_q^n$, we define a parity-check bit as a sum of information bits lying in this affine subspace. As the number of distinct affine subspaces of such a form is $|\mathcal{F}|q^{n-k}$, the constructed systematic code has length $N=q^n+|\mathcal{F}|q^{n-k}$ and the redundancy is $N-K=|\mathcal{F}|q^{n-k}$. Moreover, it can be shown that this construction is an [N, K, s]-batch code with $s := ||\mathcal{F}|/L|$.

A naive way for constructing AAD families is by exploiting constructions of long linear codes with fixed minimum distance. Suppose that $H(\mathcal{C})$ is a parity-check matrix of a q-ary linear $[N, K, d]_q$ -code \mathcal{C} of length N and dimension K with minimum distance d=3k+1. Let the subspace S_i be the linear span of k consecutive columns, from the (ik+1)-th to the (i+1)k-th column, of $H(\mathcal{C})$. Then $\mathcal{F}:=\{S_1,\ldots,S_{\lfloor N/k\rfloor}\}$ is an $[N-K,k,1]_q$ -AAD family. Thus, for fixed N-K, the longer the code, the larger the constructed family. Yekhanin and Dumer have developed a class of long non-binary codes with a fixed distance [11]. For k = 1 and d = 3k + 1, linear $[N, K, 4]_q$ -codes are known to be equivalent to caps in projective geometries and have been studied extensively under this name [7, 9, 1]. By the results of [11, 1], for fixed k and large enough n, it holds that $p^{ADD}(n, k, 1) > (3k - 1)(n + 1)/(9k^2 - 9k + 1).$

1.2. Our contribution

The main results of our paper are presented in Theorems 1-3 showing that

$$n-2k-\frac{(k+1)(n-k)}{L+1} \le p^{AS}(n,k,L) \le p^{AAD}(n,k,L-1) \le n-2k$$

for all ranges of parameters. Moreover, by providing explicit constructions of AAD and AS subspace families we prove that $p^{AS}(n,1,L)=n-2$ for $L\geq n$ and $p^{AAD}(n,2,L)=n-2$ n-4 for $L \ge 4n^2-18n+21$, respectively. Note that for the special case n=2k+1, by constructions from [4,10] we have $p^{AAD}(2k+1,k,L)=1$ for $L \ge k$ and $p^{AS}(2k+1,k,L)=1$ 1 for $L \ge 2k(k+1)$.

The remaining part of this paper is structured as follows. In Section 2, we prove a nonexistence result for AAD families. Random and explicit constructions for AS and AAD families are presented in Section 3. We close our paper with a conjecture in Section 4.

2. Converse bound

Theorem 1. Fix arbitrary positive integers k, n and L such that 2k < n. Let \mathcal{F} be an $[n, k, L]_q$ -AAD family. Then

$$|\mathcal{F}| \le 1 + L \frac{q^{n-k} - 1}{q^k - 1}.\tag{1}$$

For $L = q^{o(1)}$, it follows that $p^{AS}(n, k, L) \leq n - 2k$.

Proof. By m denote the cardinality of \mathcal{F} and let S_i be the ith subspace from \mathcal{F} . For some $i \in [m]$, let $G_i \in \mathbb{F}_q^{n \times k}$ be a matrix whose columns span S_i and let $H_i \in \mathbb{F}_q^{(n-k) \times n}$ be a matrix whose rows span S_i^{\perp} . Notice that $H_mG_j \in \mathbb{F}_q^{(n-k) \times k}$ has full column rank because S_m and S_j have only trivial intersection for any $j \in [m-1]$, since they are from the AAD family which is a partial spread according to Definition 1. Let $\hat{G}_j \in \mathbb{F}_q^{(n-2k) \times (n-k)}$ be a matrix whose rows form a basis of the space orthogonal to the column span of H_mG_j .

Next, we shall prove that for any non-zero vector $\boldsymbol{w} \in \mathbb{F}_q^{n-k}$, $\hat{G}_j \boldsymbol{w}$ is the all-zero vector for at most L different j's. To see this, suppose that for some multiset $\{j_1,\ldots,j_{L+1}\} \subset [m-1]$ of size L+1, we have $\hat{G}_{j_t}\boldsymbol{w}=\boldsymbol{0}$ for every $t\in [L+1]$. This implies that \boldsymbol{w} is in the column span of $H_mG_{j_t}$, say that $\boldsymbol{w}=H_mG_{j_t}\boldsymbol{y}_t$ for some $\boldsymbol{y}_t\in \mathbb{F}_q^k$. Define $\boldsymbol{v}:=G_{j_1}\boldsymbol{y}_1$. So we have

$$\boldsymbol{w} = H_m G_{j_t} \boldsymbol{y}_t = H_m \boldsymbol{v}$$

which means that

$$G_{j_t} \boldsymbol{y}_t = \boldsymbol{v} + G_m \boldsymbol{x}_t$$

for some $x_t \in \mathbb{F}_q^k$ (e.g., $x_1 = \mathbf{0}$). But this implies that $\mathbf{v} + S_m$ and S_{j_t} intersect. By Definition 1, there are at most L different j's so that $\mathbf{v} + S_m$ and S_j intersect. This leads to a contradiction.

Whenever we have a collection of m subspaces with the desired property, we have (m-1) matrices $\hat{G}_j \in \mathbb{F}_q^{(n-2k)\times (n-k)}$ so that the span of any L+1 of them has rank n-k. We claim that we must have (1) for such a collection of matrices $\{\hat{G}_j,\ j\in [m-1]\}$ to exist. Indeed, let x be a random non-zero vector in \mathbb{F}_q^{n-k} . We observe that the expectation is

$$\mathbb{E}\left|\{j \in [m-1]: \ \hat{G}_j \boldsymbol{x} = \boldsymbol{0}\}\right| = (m-1)\Pr\left\{\hat{G}_j \boldsymbol{x} = \boldsymbol{0}\right\} = (m-1)\frac{q^k - 1}{q^{n-k} - 1}.$$

So if $m > L \frac{q^{n-k}-1}{q^k-1} + 1$, there exists some vector $\boldsymbol{w} \in \mathbb{F}_q^{n-k}$ so that $\hat{G}_j \boldsymbol{w} = \boldsymbol{0}$ for at least L+1 different j's. This contradiction completes the proof.

Remark 2. Note that if we change the definition of an AAD subspace family by dropping the first property in Definition 1, then the matrices H_mG_j would have full rank for at least m-L-1 different j's. This results in the bound $m \leq L\frac{q^{n-k}-1}{q^k-1} + L + 1$.

3. Constructions

In this section, we provide new random and explicit constructions of AAD and AS families. We first show the existence result on AS families in Section 3.1. Section 3.2 presents a novel construction of AAD subspaces family based on Reed-Solomon codes for k = 1 and k = 2.

3.1. Random construction

Theorem 2. For any fixed integers L, n, k and $q \to \infty$, there exists an $[n, k, L]_q$ -AS family of size $\underline{m}_q^*(n, k, L)$, where

$$\underline{m}_{q}^{*}(n,k,L) := q^{n-2k-\frac{(n-k)(k+1)}{(L+1)}}(1+o(1)).$$

For fixed L, it follows that $p^{AS}(n, k, L) \ge n - 2k - (n - k)(k + 1)/(L + 1)$.

Proof. The number of k-dimensional subspaces in \mathbb{F}_q^n equals

$$s_k := \frac{(q^n - 1)\dots(q^n - q^{k-1})}{(q^k - 1)\dots(q^k - q^{k-1})} = \Theta(q^{k(n-k)}).$$

We form a family of k-dimensional subspaces, written as $\mathcal{F} := \{S_1, \ldots, S_M\}$, of size $M = q^{n-2k-(n-k)(k+1)/(L+1)}$ by choosing each subspace S_i independently and equiprobably with probability $1/s_k$. So, it is possible that $S_i = S_j$ for some $i \neq j$.

Define $\xi = |\{(i,j): i,j \in [M], i < j, |S_i \cap S_j| \neq 1\}|$, the number of pairs (i,j) with i < j such that S_i and S_j have the non-trivial intersection. We will estimate the expectation of ξ . The number of k-dimensional subspaces that do not intersect with a fixed k-dimensional subspace (except the origin point) is equal to

$$g_k := \frac{(q^n - q^k) \dots (q^n - q^{2k-1})}{(q^k - 1) \dots (q^k - q^{k-1})}.$$

Thus, two random k-dimensional subspaces have a trivial intersection with probability g_k/s_k . The mathematical expectation of ξ is then upper bounded as follows

$$\mathbb{E}(\xi) \leq \sum_{1 \leq i < j \leq M} \Pr\{|V_i \cap V_j| \neq 1\} = \sum_{1 \leq i < j \leq M} \left(1 - \frac{g_k}{s_k}\right)$$

$$\leq \binom{M}{2} \left(1 - \frac{(q^n - q^k) \dots (q^n - q^{2k-1})}{(q^n - 1) \dots (q^n - q^{k-1})}\right)$$

$$< M^2 \left(1 - \left(q^{nk} - q^{n(k-1)} \sum_{i=k}^{2k-1} q^i\right) / q^{nk}\right)$$

$$= M^2(q^{2k-1-n} + o(q^{2k-1-n})) < M(q^{-1} + o(q^{-1})).$$

By the Markov inequality

$$\Pr(\xi > q^{0.5}\mathbb{E}(\xi)) < \frac{\mathbb{E}(\xi)}{q^{0.5}\mathbb{E}(\xi)} = o(1).$$

Since $q^{0.5}\mathbb{E}(\xi) < M(q^{-0.5} + o(q^{-0.5})) = o(M)$, we obtain that with probability at least 1-o(1) there exists a family \mathcal{F} of size M, which contains at most o(M) pairs of subspaces with the non-trivial intersection. If we delete one of the intersecting subspaces for each pair, then we obtain a family $\mathcal{F}' \subset \mathcal{F}$ of subspaces of size at least M - o(M) satisfying the first property of Definition 2.

Now we compute the probability that the second property of Definition 2 is violated. The number of k-dimensional subspaces that trivially intersect with a fixed (k + 1)-dimensional subspace, written as V, is equal to

$$u_k := \frac{(q^n - q^{k+1}) \dots (q^n - q^{2k})}{(q^k - 1) \dots (q^k - q^{k-1})}.$$

Thus, the probability that S_i does not intersect V equals u_k/s_k . Let $\mathcal{F}_V \subseteq \mathcal{F}$ be the set of subspaces in \mathcal{F} that non-trivially intersect V, i.e., $\mathcal{F}_V := \{S \in \mathcal{F}, |S \cap V| \neq 1\}$.

Applying the union bound, we can estimate the probability that V intersects at least L+1 subspaces in \mathcal{F} by

$$\Pr[|\mathcal{F}_{V}| \ge L+1] \le {M \choose L+1} \left(1 - \frac{u_{k}}{s_{k}}\right)^{L+1}$$

$$= {M \choose L+1} \left(1 - \frac{(q^{n} - q^{k+1}) \dots (q^{n} - q^{2k})}{(q^{n} - 1) \dots (q^{n} - q^{k-1})}\right)^{L+1}$$

$$< M^{L+1} \left(1 - \left(q^{nk} - q^{n(k-1)} \sum_{i=k+1}^{2k} q^{i}\right) / q^{nk}\right)^{L+1}$$

$$= M^{L+1} (q^{2k-n} + o(q^{2k-n}))^{L+1}$$

$$< q^{-(n-k)(k+1)} (1 + o(1)).$$

Recall that the total number of (k+1)-dimensional subspaces is s_{k+1} , which is $\Theta(q^{(k+1)(n-k-1)})$. Hence, by the union bound,

$$s_{k+1} \cdot \Pr[|\mathcal{F}_V \ge L + 1|] < q^{-1-k} + o(q^{-1-k}),$$

we have that with probability o(1) the second property is violated. This completes the proof of the existence of a [n, k, L]-AS family of size M - o(M).

3.2. Explicit constructions

Construction 1. Let $q \geq nk$, $m = q^{n-2k}$ and γ be a primitive element of \mathbb{F}_q . For $i \in [m]$, define S_i to be the span of vectors $\{v_{i,1}, \ldots, v_{i,k}\}$ with

$$\boldsymbol{v}_{i,j} := \begin{pmatrix} \boldsymbol{e}_j & \Gamma_j(\boldsymbol{c}_i) & h_j(\boldsymbol{c}_i) \end{pmatrix} \text{ for } j \in [k],$$

where e_j is the unit vector of length k having one in the jth position, c_i is a codeword of an $[n-k-1, n-2k, k]_q$ Reed-Solomon code having the following parity-check matrix

$$H_{RS} := \begin{pmatrix} 1 & 1 & 1 & \cdots & 1\\ 1 & \gamma & \gamma^2 & \cdots & \gamma^{n-k-2}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & \gamma^{k-2} & \gamma^{2(k-2)} & \cdots & \gamma^{(n-k-2)(k-2)} \end{pmatrix}$$
(2)

and the map $\Gamma_j: \mathbb{F}_q^{n-k-1} \to \mathbb{F}_q^{n-k-1}$ is defined by

$$\Gamma_i(\boldsymbol{x}) := \begin{pmatrix} \gamma^{j-1} x_1 & \gamma^{2(j-1)} x_2 & \gamma^{3(j-1)} x_3 & \cdots & \gamma^{(n-k-1)(j-1)} x_{n-k-1} \end{pmatrix}$$

and the function $h_j(\boldsymbol{x}) := \sum_{p=1}^{n-k-1} x_p^{(j-1)(n-k-1)+p+1}$. Then we set $\mathcal{F}_{n,k}$ to be the collection of S_i 's.

Theorem 3. The family $\mathcal{F}_{n,k}$ from Construction 1 is a partial spread of k-dimensional subspaces in \mathbb{F}_q^n . Moreover, for k=1 and k=2, $\mathcal{F}_{n,k}$ is $[n,k,L_{n,k}]_q$ -AAD with $L_{n,1}=n-1$ and $L_{n,2}=1+2(n-2)(2n-5)$.

Proof. The linear span of vectors $\{v_{i,1}, \ldots, v_{i,k}\}$ defines a k-dimensional subspace in \mathbb{F}_q^n as the restriction of $v_{i,j}$ to the first k coordinates is e_j . Suppose that S_i and S_j have the non-trivial intersection. Thus, the rank of the system of vectors $\{v_{i_1}, \ldots, v_{i,k}, v_{j,1}, \ldots, v_{j,k}\}$ is at most 2k-1. This yields that the rank of the system of vectors

$$\{\Gamma_1(\boldsymbol{c}_i-\boldsymbol{c}_j),\Gamma_2(\boldsymbol{c}_i-\boldsymbol{c}_j),\ldots,\Gamma_k(\boldsymbol{c}_i-\boldsymbol{c}_j)\}$$

is not full. Denote by $c_{i,j}$ the jth entry of c_i . Since c_i-c_j is a non-zero codeword of the Reed-Solomon code with minimum distance k, there exist k coordinates $p_1,\ldots,p_k\in[n-k-1]$ such that $u_t:=c_{i,p_t}-c_{j,p_t}\neq 0$ for $t\in[k]$. Thus, after restricting each vector of the system onto coordinates p_1,\ldots,p_k , the non-full rank property is equivalent to that the determinant

$$\det \begin{pmatrix} u_1 & u_2 & \cdots & u_k \\ \gamma^{p_1} u_1 & \gamma^{p_2} u_2 & \cdots & \gamma^{p_k} u_k \\ \vdots & \vdots & \ddots & \vdots \\ \gamma^{p_1(k-1)} u_1 & \gamma^{p_2(k-1)} u_2 & \cdots & \gamma^{p_k(k-1)} u_k \end{pmatrix} = \prod_{t=1}^k u_t \prod_{1 \le s < r \le k} (\gamma^{p_r} - \gamma^{p_s})$$

is zero. However, since γ is a primitive element of the field \mathbb{F}_q with $q \geq nk$ and all u_t 's are non-zero, the determinant cannot be zero, which contradicts the assumption that S_i and S_j intersect non-trivially. Thus, the family $\mathcal{F}_{n,k}$ is indeed a partial spread.

Suppose that for some $\mathbf{v} \in \mathbb{F}_q^n \not\in S_i$, the linear span of \mathbf{v} and S_i , written as V, intersects more than $L_{n,k}$ subspaces from the family $\mathcal{F}_{n,k}$. Note that we can replace the vector \mathbf{v} with a non-zero vector from one of the $L_{n,k}+1$ subspaces intersecting V. So, we can assume that the vector \mathbf{v} can be represented by $\mathbf{v} = \mathbf{v}_j(\alpha) := \sum_{t=1}^k \alpha_t \mathbf{v}_{j,t}$ for some $\alpha \in \mathbb{F}_q^k \setminus \{\mathbf{0}\}, j \in [m] \setminus \{i\}$. In what follows, we estimate the number of $\ell \in [m] \setminus \{i,j\}$ such that there exists a $\beta \in \mathbb{F}_q^k$ and $\mathbf{v}_{\ell}(\beta)$ belongs to V. This is equivalent to the property that the system of vectors $\{\mathbf{v}_{i,1}, \ldots, \mathbf{v}_{i,k}, \mathbf{v}_j(\alpha), \mathbf{v}_{\ell}(\beta)\}$ is of rank at most k+1. By the structure of the vectors $\mathbf{v}_{i,t}$'s, this implies that the rank of

$$R_{\boldsymbol{\beta},\ell} := \begin{pmatrix} \sum_{t=1}^{k} \alpha_t \Gamma_t(\boldsymbol{c}_j - \boldsymbol{c}_i) & \sum_{t=1}^{k} \alpha_t (h_t(\boldsymbol{c}_j) - h_t(\boldsymbol{c}_i)) \\ \sum_{t=1}^{k} \beta_t \Gamma_t(\boldsymbol{c}_\ell - \boldsymbol{c}_i) & \sum_{t=1}^{k} \beta_t (h_t(\boldsymbol{c}_\ell) - h_t(\boldsymbol{c}_i)) \end{pmatrix}$$

is one. Observe that the pth element, $p \in [n-k-1]$, of the first row of $R_{\beta,\ell}$ has the form $(c_{i,p}-c_{j,p})\sum_{t=1}^k \alpha_t \gamma^{(t-1)p}$. We can think about $\sum_{t=1}^k \alpha_t \gamma^{(t-1)p}$ as of a polynomial of degree k-1 in a variable $x=\gamma^p$. It has at most k-1 roots in \mathbb{F}_q . For each root x_0 , there exists at most one $p \in [n-k-1]$ such that $\gamma^p=x_0$, since γ is a primitive element in \mathbb{F}_q with $q \geq nk$. Therefore, for any nonzero α , there are at most k-1 different $p \in [n-k-1]$ so that $\sum_{t=1}^k \alpha_t \gamma^{(t-1)p}=0$. Since c_i and c_j are codewords of a linear code with minimum distance k, there are at least k positions $p \in [n-k-1]$ such that $c_{i,p}-c_{j,p} \neq 0$. Hence, there is at least one position $p_0 \in [n-k-1]$ such that the p_0 th entry of the vector $\sum_{t=1}^k \alpha_t \Gamma_t(c_j-c_i)$ is non-zero.

Lemma 4. Given a non-zero vector $\boldsymbol{\beta} \in \mathbb{F}_q^k$, there exist at most k(n-k) different \boldsymbol{c}_ℓ 's from the Reed-Solomon code defined by (2) such that the matrix $R_{\boldsymbol{\beta},\ell}$ has rank 1.

Proof. If $R_{\beta,\ell}$ has rank 1, then every column in the left part of $R_{\beta,\ell}$ (consisting of the first n-k-1 columns) is linearly dependent of the p_0 th column, which is known to be

non-zero. Moreover, the dependence can be found by the first row in $R_{\beta,\ell}$. Thus for any $p \in [n-k-1]$, there exists some element $\phi_p \in \mathbb{F}_q$ so that

$$(c_{\ell,p} - c_{i,p}) \sum_{t=1}^{k} \beta_t \gamma^{(t-1)p} = \phi_p(c_{\ell,p_0} - c_{i,p_0}) \sum_{t=1}^{k} \beta_t \gamma^{(t-1)p_0}.$$
 (3)

Recall that $\sum_{t=1}^{k} \beta_t \gamma^{(t-1)p} = 0$ for at most k-1 values of $p \in [n-k-1]$. We have additionally restrictions imposed by the parity-check matrix (2), namely, $\forall t \in [k-1]$

$$\sum_{p=1}^{n-k-1} \gamma^{(p-1)(t-1)}(c_{\ell,p} - c_{i,p}) = 0.$$
(4)

Thus, the system of equations (3)-(4) with at least n-k-2 linear independent equations for variables $\{c_{\ell,p}, p \in [n-k-1] \setminus \{p_0\}\}$ has at most one solution. In other words, $c_{\ell,p} = a_p c_{\ell,p_0} + b_p$ with some $a_p, b_p \in \mathbb{F}_q$ and $p \in [n-k-1]$. This implies that the determinant

$$\det \begin{pmatrix} \sum_{t=1}^{k} \alpha_t \gamma^{(t-1)p_0} (c_{j,p_0} - c_{i,p_0}) & \sum_{t=1}^{k} \alpha_t (h(\boldsymbol{c}_j) - h(\boldsymbol{c}_i)) \\ \sum_{t=1}^{k} \beta_t \gamma^{(t-1)p_0} (c_{\ell,p_0} - c_{i,p_0}) & \sum_{t=1}^{k} \beta_t (h(\boldsymbol{c}_\ell) - h(\boldsymbol{c}_i)) \end{pmatrix}$$

is zero. Note that the entry

$$\sum_{t=1}^{k} \beta_t (h(\mathbf{c}_{\ell}) - h(\mathbf{c}_i))$$

$$= \sum_{t=1}^{k} \beta_t \sum_{p=1}^{n-k-1} \left((a_p c_{\ell,p_0} + b_p)^{(t-1)(n-k-1)+p+1} - c_{i,p}^{(t-1)(n-k-1)+p+1} \right)$$

is a polynomial in c_{ℓ,p_0} of degree at least p_0+1 and at most $k(n-k-1)+1 \le k(n-k)$. Therefore, the determinant represents a non-trivial univariate polynomial in c_{ℓ,p_0} of degree at least p_0+1 and at most k(n-k). Since $q \ge nk$, there are at most k(n-k) solutions for c_{ℓ,p_0} vanishing the determinant.

Let us proceed with proving the remaining statement of this theorem. For this purpose, for k = 1, 2, we estimate the number of possible β 's such that the first n - k - 1 columns of $R_{\beta,\ell}$, treated as vectors, are parallel to the p_0 th column and apply Lemma 4.

For the case k=1, we first note that we can scale vector $\boldsymbol{\beta}$ and, thus, have $\boldsymbol{\beta}=(\beta_1)=(1)$. By Lemma 4, the number of distinct appropriate ℓ 's is at most n-1. Thus, $\mathcal{F}_{n,1}$ is an $[n,1,L]_q$ -AAD family.

Now we discuss the case k=2. Suppose that $\boldsymbol{\beta}$ is not parallel to the vector (1,-1). Since $\boldsymbol{\beta}$ can be appropriately scaled, we shall estimate the number of suspicious $\boldsymbol{\beta}=(\beta_1,1-\beta_1)$ which means that there might exist some \boldsymbol{c}_ℓ from the Reed-Solomon code for such $\boldsymbol{\beta}$ so that $R_{\boldsymbol{\beta},\ell}$ is degenerate. Define the set B that includes $\boldsymbol{\beta}=(1,0)$ and $\boldsymbol{\beta}$'s with the property $\beta_1+(1-\beta_1)\gamma^p=0$ for some $p\in[n-3]$. Consider a $\boldsymbol{\beta}\not\in B$. If $R_{\boldsymbol{\beta},\ell}$ has rank 1, then two rows of $R_{\boldsymbol{\beta},\ell}$, treated as vectors, are parallel and there exists some non-zero $\lambda\in\mathbb{F}_q$ such that

$$c_{\ell,p} - c_{i,p} = \lambda \frac{(\alpha_1 + \alpha_2 \gamma^p)(c_{j,p} - c_{i,p})}{\beta_1 + (1 - \beta_1)\gamma^p} \quad \text{for } p \in [n-3].$$
 (5)

Let w_p denote the numerator of the above fraction. Recall that there exists p_0 such that $w_{p_0} \neq 0$. From the parity-check property $\sum_{p=1}^{n-3} (c_{\ell,p} - c_{i,p}) = 0$ imposed by (2), we thus derive

$$\sum_{p=1}^{n-3} \frac{w_p}{\beta_1 + (1-\beta_1)\gamma^p} = 0 \quad \Leftrightarrow \quad \sum_{p=1}^{n-3} w_p \prod_{t \neq p} (\beta_1 + (1-\beta_1)\gamma^t) = 0.$$

We can think of the left-hand side of the above equation as a non-trivial univariate polynomial in β_1 of degree at most n-4. Indeed, the polynomial is non-trivial as its evaluation at β_1^* satisfying $\beta_1^* + (1-\beta_1^*)\gamma^{p_0} = 0$ is $w_{p_0} \prod_{i=1}^{n-3} (\beta_1^* + (1-\beta_1^*)\gamma^t) \neq 0$.

its evaluation at β_1^* satisfying $\beta_1^* + (1 - \beta_1^*)\gamma^{p_0} = 0$ is $w_{p_0} \prod_{\substack{t=1 \ t \neq p_0}}^{n-3} (\beta_1^* + (1 - \beta_1^*)\gamma^t) \neq 0$. Therefore, there are at most n-4 suspicious β 's not included to B and not parallel to (1,-1) such that the vector $\sum_{t=1}^2 \beta_t \Gamma_t(\mathbf{c}_j - \mathbf{c}_i)$ is parallel to $\sum_{t=1}^2 \beta_t \Gamma_t(\mathbf{c}_\ell - \mathbf{c}_i)$. Define D to be the union of suspicious β 's, the set B and the vector (1,-1). As $|D| \leq 2n-5$, it can be easily verified that by Lemma 4, $\mathcal{F}_{n,2}$ is an $[n,2,L]_q$ -AAD family with L=1+2(n-2)(2n-5).

Remark 3. Note that any 2-dimension subspace, i.e., a plane, that intersects with a 1-dimensional subspace, i.e., a line, from an $[n, k = 1, L]_q$ -ADD family, must contain the line. Therefore, any $[n, 1, L]_q$ -AAD family is also an $[n, 1, L + 1]_q$ -AS family. Therefore, $p^{AS}(n, 1, n) = p^{AAD}(n, 1, n - 1) = n - 2$.

4. Conclusion

In this paper, we have introduced new notions concerning vector finite spaces which we have called almost affinely disjoint and almost sparse families of subspaces. We have presented lower and upper bounds on the polynomial growth of the maximal sizes of these families. For the cases k = 1 and k = 2, our explicit constructions asymptotically achieve the converse bound. We conjecture that for any k, n and a large enough L = L(n, k), the quantities $p^{ADD}(n, k, L)$ and $p^{AS}(n, k, L)$ are equal to n - 2k.

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