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TWO-VALUED σ -MAXITIVE MEASURES AND MESIAR'S HYPOTHESIS

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ABSTRACT. We reformulate Mesiar's hypothesis [Possibility measures, integration and fuzzy possibility measures, Fuzzy Sets and Systems 92 (1997) 191196], which as such was shown to be untrue by Murofushi [Two-valued possibility measures induced by σ -finite σ -additive measures, Fuzzy Sets and Systems 126 (2002) 265268]. We prove that a two-valued σ -maxitive measure can be induced by a σ -additive measure under the additional condition that it is σ -principal.

1. INTRODUCTION

In the sequel, (E, \mathscr{B}) denotes a measurable space. Recall that a σ -maxitive measure on \mathscr{B} is a map $\tau : \mathscr{B} \to \overline{\mathbb{R}}_+$ such that $\tau(\emptyset) = 0$ and

$$\tau\left(\bigcup_{j\in\mathbb{N}}B_j\right) = \sup_{j\in\mathbb{N}}\tau(B_j),$$

for every sequence $(B_j)_{j\in\mathbb{N}}$ of elements of \mathscr{B} . A σ -maxitive measure τ on \mathscr{B} is normed if $\tau(E) = 1$; it is two-valued if $\tau(\mathscr{B}) = \{0, 1\}$, in which case it is normed. A family \mathscr{C} of sets is said to satisfy *CCC* (the *countable chain condition*) if every disjoint subfamily of \mathscr{C} is countable. A measure (either σ -maxitive or σ -additive) μ on \mathscr{B} is then *CCC* if $\{B \in \mathscr{B} : \mu(B) > 0\}$ satisfies CCC.

The following hypothesis was proposed by Mesiar [2].

Hypothesis 1.1. Let τ be a CCC ("countable chain condition") two-valued σ -maxitive measure on \mathscr{B} . Then τ is induced by some σ -finite σ -additive measure m on \mathscr{B} , i.e. $\tau = \delta_m$, where $\delta_m(B) = 1$ if m(B) > 0 and $\delta_m(B) = 0$ otherwise.

Murofushi [3] provided a counterexample and hence showed that this hypothesis as such is wrong. He focused on finding a necessary and sufficient condition for such a τ to be induced by some σ -finite σ -additive measure.

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In this article we propose to give up the constraint " σ -finite" and to replace it by a more adequate condition, namely the σ -principality property.

Definition 1.2. A measure (either σ -maxitive or σ -additive) μ on \mathscr{B} is σ principal if for each σ -ideal \mathscr{I} of \mathscr{B} , there exists some $L \in \mathscr{I}$ such that $\mu(S \setminus L) = 0$ for all $S \in \mathscr{I}$.

Example 1.3. Let *E* be a set endowed with its power set $(\mathscr{B} = 2^E)$. The σ -additive measure $\# : B \mapsto \#B$, where #B is the cardinality of *B*, is σ -principal if and only if *E* is countable.

A σ -principal measure is always CCC. See Sugeno and Murofushi [7] for a proof of the converse statement using Zorn's Lemma. Note also that every finite (or σ -finite) σ -additive measure is σ -principal.

2. MODIFIED MESIAR'S HYPOTHESIS

Mesiar [2] noted that, if his hypothesis were true, then every CCC σ maxitive measure could be represented as an essential supremum with respect to a σ -additive measure. We show first that such a representation holds, then prove a modified version of Mesiar's hypothesis.

Theorem 2.1. Any σ -principal (resp. CCC) σ -maxitive measure can be expressed as an essential supremum with respect to a σ -principal (resp. CCC) σ -additive measure.

Proof. Let τ be a σ -principal σ -maxitive measure on \mathscr{B} and $m = \overline{\tau}$ be the map defined on \mathscr{B} by

$$m(B) = \sup_{\pi} \sum_{B' \in \pi} \tau(B \cap B'),$$

where the supremum is taken over the set of finite \mathscr{B} -partitions π of E. It is not difficult to show that m, called the *disjoint variation* of τ , is the least σ -additive measure greater than τ (see e.g. [4, Theorem 3.2]). Let us show that m is σ -principal. If \mathscr{I} is a σ -ideal of \mathscr{B} , there exists some $L \in \mathscr{I}$ such that $\tau(B \setminus L) = 0$ for all $B \in \mathscr{I}$ (because τ is σ -principal). If $B \in \mathscr{I}$, then $\tau(B \cap B' \setminus L) = 0$ for all $B' \in \mathscr{B}$, since $B \cap B' \in \mathscr{I}$. Hence we have $m(B \setminus L) = 0$. Moreover, m(B) > 0 implies $\tau(B) > 0$, so that mis CCC if τ is CCC. With the Sugeno–Murofushi theorem (see a reminder in the appendix, Theorem A.1) and the fact that τ is absolutely continuous with respect to δ_m , one can write $\tau(B) = \int^{\infty} c d\delta_m = m$ -sup $_{x \in B} c(x)$, where $c: E \to \mathbb{R}_+$ is a \mathscr{B} -measurable map. \Box

Corollary 2.2. A two-valued σ -maxitive measure is σ -principal (resp. CCC) if and only if it is induced by a σ -principal (resp. CCC) σ -additive measure.

Proof. Let τ be a σ -principal σ -maxitive measure on \mathscr{B} . The above construction of m shows that $\tau(B) > 0 \Leftrightarrow m(B) > 0$, which implies that $\tau = \delta_m$ if τ is two-valued.

APPENDIX A.

Sugeno and Murofushi [7] proved a Radon–Nikodym like theorem for the Shilkret integral in the case where the dominating σ -maxitive measure is CCC. Actually their proof remains valid if one just assumes σ -principality (this is straightforward since they showed under Zorn's Lemma that every CCC measure is σ -principal).

Theorem A.1 (Sugeno–Murofushi). Let τ , ν be σ -maxitive measures on \mathcal{B} . Assume that ν is σ -finite and σ -principal. The following are equivalent:

- (1) τ is absolutely continuous with respect to ν , i.e. $\nu(B) = 0$ implies $\tau(B) = 0$, for all $B \in \mathscr{B}$,
- (2) there exists some \mathscr{B} -measurable map $c : E \to \overline{\mathbb{R}}_+$ such that, for all $B \in \mathscr{B}$,

$$\tau(B) = \int_B^\infty c \, d\nu.$$

If (1) or (2) holds, then c is unique ν -almost everywhere.

Here $\int_B^{\infty} c \, d\nu := \sup_{t \in \mathbb{R}_+} t.\nu(B \cap \{c > t\})$ denotes the Shilkret integral, see Shilkret [6]; see also Poncet [5, Chapter I]. The superscript ∞ in the notation of the Shilkret integral finds its justification in Gerritse [1], who stated that the Shilkret integral can be viewed as a limit of Choquet integrals.

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