MEASURES, STATES AND DE FINETTI MAPS ON PSEUDO-BCK ALGEBRAS

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Abstract. In this paper, we extend the notions of states and measures presented in [12] to the case of pseudo-BCK algebras and study similar properties. We prove that, under some conditions, the notion of a state in the sense of [12] coincides with the Bosbach state, and we extend to the case of pseudo-BCK algebras some results proved by J. Kühr only for pseudo-BCK semilattices. We characterize extremal states, and show that the quotient pseudo-BCK algebra over the kernel of a measure can be embedded into the negative cone of an archimedean ℓ-group. Additionally, we introduce a Borel state and using results by J. Kühr and D. Mundici from [28], we prove a relationship between de Finetti maps, Bosbach states and Borel states.

1. Introduction

BCK algebras were introduced originally by K. Isèki in [24] with a binary operation $*$ modeling the set-theoretical difference and with a constant element 0 that is a least element. Another motivation is from classical and non-classical propositional calculi modeling logical implications. Such algebras contain as a special subfamily a family of MV-algebras where some important fuzzy structures can be studied. For more about BCK algebras, see [29].

Pseudo-BCK algebras were originally introduced by G. Georgescu and A. Iorgulescu in [18] as algebras with "two differences", a left- and right-difference, instead of one $*$ and with a constant element 0 as the least element. In [16], a special subclass of pseudo-BCK algebras, called Łukasiewicz pseudo-BCK algebras, was introduced and it was shown that it is always a subalgebra of the positive cone of some ℓ-group (not necessarily abelian). The class of Łukasiewicz pseudo-BCK algebras is a variety whereas the class of pseudo-BCK algebras is not; it is only a quasivariety because it is not closed under homomorphic images. Nowadays pseudo-BCK algebras are used in a dual form, with two implications, $\to$ and $\rightsquigarrow$ and with one constant element 1 that is a greatest element. Thus such pseudo-BCK algebras are in a "negative cone" and are also called "left-ones". For a guide through the pseudo-BCK algebras realm, see the monograph [23].

States or measures give a probabilistic interpretation of randomness of events of given algebraic structures. For MV-algebras, Mundici introduced states (an analogue of probability measures) in 1995, [30], as averaging of the truth-value in Łukasiewicz logic. Measures on BCK algebras were introduced by A. Dvurečenskij in [7, 12]. Based on the notion of a measure it was defined a concept of state on these structures if such

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Key words and phrases. Pseudo-BCK algebra, Bosbach state, extremal state, MV-algebra, Borel state, state-morphism, ℓ-group.

The first author thanks SAIA for the fellowship in Slovakia, Summer 2008, 2009, the second author is thankful for the support by Center of Excellence SAS - Quantum Technologies -, Center of Excellence QUTE, the grant VEGA No. 2/0032/09 SAV and by the Slovak Research and Development Agency under the contract No. APVV-0071-06, Bratislava.

BCK algebra admits a smallest element 0. Today the notion of states has many forms: The notion of a Bosbach state has been studied for other algebras of fuzzy structures such as pseudo-BL algebras, [17], bounded non-commutative $R\ell$-monoids, [14, 15, 13], residuated lattices, [4], pseudo-BCK semilattices and pseudo-BCK algebras, [26].

In the present paper, we study Bosbach states and measures on pseudo BCK algebras. In general, it can happen that on bounded BCK algebras states can fail. We will study states, extremal states, Bosbach states, state-morphisms and we show the relationships among them. Such relations were studied for pseudo MV-algebras, [8], and generalized by Kühr [26] for pseudo-BCK algebras that are $\vee$-semilattices. In our paper we show that the existence of the join in the pseudo-BCK algebra is not substantial for our study. The main results say that the quotient pseudo-BCK algebra that is downwards-directed over the kernel of a measure can be embedded as a subalgebra into the negative cone of an abelian and archimedean ℓ-group. In particular that A is with strong unit, the embedding is even onto. We will apply this result to characterize extremal state-measures on unital ℓ-groups.

Finally, we show how Bosbach states and state-measures appear with respect to de Finetti's coherence principle.

The paper is organized as follows: Section 2 contains preliminary notions on pseudo-BCK algebras. Section 3 is dedicated to states, extremal states, state-morphisms and Bosbach states, kernels on pseudo-BCK algebras. Section 4 deals with a generalization of a notion of measures, Section 5 presents results on state-measures on pseudo-BCK algebras with strong unit. The last section deals with de Finetti's notion of coherence and it gives some integral representations of states and state-measures via Borel states.

2. Preliminaries on Pseudo-BCK Algebras

In the present section, we give elements of theory of pseudo-BCK algebras.

Definition 2.1. ([20]) A *pseudo-BCK algebra* is a structure $\mathcal{A} = (A, \leq, \to, \rightsquigarrow, 1)$ where $\leq$ is a partial binary relation on A, $\to$ and $\rightsquigarrow$ are total binary operations on A and 1 is an element of A satisfying, for all $x, y, z \in A$, the axioms:

(A_1) $x \to y \leq (y \to z) \rightsquigarrow (x \to z)$, $x \rightsquigarrow y \leq (y \rightsquigarrow z) \to (x \rightsquigarrow z)$;
(A_2) $x \leq (x \to y) \rightsquigarrow y$, $\quad x \leq (x \rightsquigarrow y) \to y$;
(A_3) $x \leq x$;
(A_4) $x \leq 1$;
(A_5) if $x \leq y$ and $y \leq x$, then $x = y$;
(A_6) $x \leq y$ iff $x \to y = 1$ iff $x \rightsquigarrow y = 1$.

Without loss of generality, we will denote a pseudo-BCK algebra $(A, \leq, \to, \rightsquigarrow, 1)$ simply by A.

Remarks 2.2. ([20]) (1) A pseudo-BCK algebra A is a BCK algebra if $\to = \rightsquigarrow$.
(2) The partial operation $\leq$ is in fact a partial order on A.
(3) If there is an element 0 in A such that $0 \leq x$ (i.e. $0 \to x = 0 \rightsquigarrow x = 1$), for all $x \in A$, then 0 is called the *zero* of A. A pseudo-BCK algebra with zero is called a *bounded pseudo-BCK algebra* and it is denoted by $\mathcal{A} = (A, \leq, \to, \rightsquigarrow, 0, 1)$, and in a simple way also as A. In such a case, we define two negations, $^-$ and $^\sim$, for any element $x \in A$:

$$x^- := x \to 0, \quad x^\sim := x \rightsquigarrow 0.$$

Example 2.3. Let $(G, +, 0, \wedge, \vee)$ be an ℓ-group (= a lattice ordered group that is not necessarily abelian).
On the negative cone $G^- = \{g \in G \mid g \leq 0\}$ we define:

$$\begin{aligned} g \to h &:= h - (g \vee h) = (h - g) \wedge 0, \\ g \rightsquigarrow h &:= -(g \vee h) + h = (-g + h) \wedge 0. \end{aligned}$$

Then $(G^-, \to, \rightsquigarrow, 0)$ is a pseudo-BCK algebra.

Example 2.4. Let $(G, +, 0, \wedge, \vee)$ be an ℓ-group with a strong unit $u \geq 0$ (i.e. given $g \in G$ there is an integer $n \geq 1$ such that $g \leq nu$). On the interval $[-u, 0]$ we define:

$$x \to y := (y - x) \wedge 0,$$
$$x \rightsquigarrow y := (-x + y) \wedge 0.$$

Then $([-u, 0], \to, \rightsquigarrow, -u, 0)$ is a bounded pseudo-BCK algebra with $x^- = -u + x$ and $x^\sim = -x + u$. In a similar way, $((-u, 0], \to, \rightsquigarrow, 0)$ is a pseudo-BCK algebra that is not bounded.

Example 2.5. Let $(G, +, 0, \wedge, \vee)$ be an ℓ-group with a strong unit $u \geq 0$. On the interval $[0, u]$ we define:

$$x \to y := (u - x + y) \wedge u,$$
$$x \rightsquigarrow y := (y - x + u) \wedge u.$$

Then $([0, u], \to, \rightsquigarrow, 0, u)$ is a bounded pseudo-BCK algebra with $x^- = u - x$ and $x^\sim = -x + u$.

If on $[0, u]$ we set $\to_1 = \rightsquigarrow$ and $\rightsquigarrow_1 = \to$, then $([0, u], \to_1, \rightsquigarrow_1, 0, u)$ is isomorphic with $[-u, 0], \to, \rightsquigarrow, -u, 0)$ under the isomorphism $x \mapsto x - u$, $x \in [0, u]$.

Proposition 2.6. ([21], [22]) *In any pseudo-BCK algebra the following properties hold:*
(c_1) $x \to (y \rightsquigarrow z) = y \rightsquigarrow (x \to z)$ *and* $x \rightsquigarrow (y \to z) = y \to (x \rightsquigarrow z)$.
(c_2) $x \leq y \to x$, $\quad x \leq y \rightsquigarrow x$.
(c_3) $[(y \to x) \rightsquigarrow x] \to x = y \to x$, $\quad [(y \rightsquigarrow x) \to x] \rightsquigarrow x = y \rightsquigarrow x$.
(c_4) $1 \to x = x = 1 \rightsquigarrow x$.
(c_5) *If* $x \leq y$, *then* $z \to x \leq z \to y$ *and* $z \rightsquigarrow x \leq z \rightsquigarrow y$.
(c_6) *If* $x \leq y$, *then* $y \to z \leq x \to z$ *and* $y \rightsquigarrow z \leq x \rightsquigarrow z$.

For all $x, y \in A$, define:

$$x \vee_1 y = (x \to y) \rightsquigarrow y, \quad x \vee_2 y = (x \rightsquigarrow y) \to y.$$

Proposition 2.7. *In any bounded pseudo-BCK algebra A the following hold for all $x, y \in A$:*
(1) $1 \vee_1 x = x \vee_1 1 = 1 = 1 \vee_2 x = x \vee_1 1$.
(2) $x \leq y$ *implies* $x \vee_1 y = y$ *and* $x \vee_2 y = y$.
(3) $x \vee_1 x = x \vee_2 x = x$.
(4) *If* $x_1 \leq x_2$ *and* $y_1 \leq y_2$, *then* $x_1 \vee_1 y_1 \leq x_2 \vee_1 y_2$ *and* $x_1 \vee_1 y_2 \leq x_2 \vee_1 y_2$.

Proof. (1) We have: $1 \vee_1 x = (1 \to x) \rightsquigarrow x = 1$ and $x \vee_1 1 = (x \to 1) \rightsquigarrow 1 = 1$, so $1 \vee_1 x = x \vee_1 1 = 1$, by (c_4) of Proposition 2.6. Similarly, $1 \vee_2 x = x \vee_2 1 = 1$.
(2) $x \vee_1 y = (x \to y) \rightsquigarrow y = 1 \rightsquigarrow y = y$. Similarly, $x \vee_2 y = y$.
(3) By definitions of $\vee_1$ and $\vee_2$.
(4) It is evident. $\square$

Proposition 2.8. *In any bounded pseudo-BCK algebra A the following hold for all $x, y \in A$:*

(1) *$x \vee_1 y^{-\sim} = x^{-\sim} \vee_1 y^{-\sim}$ and $x \vee_2 y^{\sim-} = x^{\sim-} \vee_2 y^{\sim-}$.*
(2) *$x \vee_1 y^{\sim} = x^{-\sim} \vee_1 y^{\sim}$ and $x \vee_2 y^{-} = x^{\sim-} \vee_2 y^{-}$.*
(3) *$(x^{-\sim} \vee_1 y^{-\sim})^{-\sim} = x^{-\sim} \vee_1 y^{-\sim}$ and $(x^{\sim-} \vee_2 y^{\sim-})^{\sim-} = x^{\sim-} \vee_2 y^{\sim-}$.*

Proof. The proof follows by direct computations. □

Proposition 2.9. *In any pseudo-BCK algebra the following hold for all $x, y \in A$:*

$$x \vee_1 y \to y = x \to y \text{ and } x \vee_2 y \rightsquigarrow y = x \rightsquigarrow y. \tag{2.1}$$

Proof. It is a consequence of property (c_3). □

Lemma 2.10. *In any pseudo-BCK algebra A we have*
(1) *$x \vee_1 y$ $(y \vee_1 x)$ is an upper bound of $\{x, y\}$.*
(2) *$x \vee_2 y$ $(y \vee_2 x)$ is an upper bound of $\{x, y\}$.*

Proof. (1) By (A_2) we have $x \leq (x \to y) \rightsquigarrow y$.
Since by (c_2), $y \leq (x \to y) \rightsquigarrow y$, we conclude that $x, y \leq x \vee_1 y$.
Similarly we get $x, y \leq y \vee_1 x$.
(2) Similarly as (1). □

Example 2.11. ([3]) Consider $A = \{o_1, a_1, b_1, c_1, o_2, a_2, b_2, c_2, 1\}$ with $o_1 < a_1, b_1 < c_1 < 1$ and a_1, b_1 incomparable, $o_2 < a_2, b_2 < c_2 < 1$ and a_2, b_2 incomparable. Assume also that any element of the set $\{o_1, a_1, b_1, c_1\}$ is incomparable with any element of the set $\{o_2, a_2, b_2, c_2\}$. Consider the operations $\to, \rightsquigarrow$ given by the following tables:

$\to$	o_1	a_1	b_1	c_1	o_2	a_2	b_2	c_2	1
o_1	1	1	1	1	o_2	a_2	b_2	c_2	1
a_1	o_1	1	b_1	1	o_2	a_2	b_2	c_2	1
b_1	a_1	a_1	1	1	o_2	a_2	b_2	c_2	1
c_1	o_1	a_1	b_1	1	o_2	a_2	b_2	c_2	1
o_2	o_1	a_1	b_1	c_1	1	1	1	1	1
a_2	o_1	a_1	b_1	c_1	o_2	1	b_2	1	1
b_2	o_1	a_1	b_1	c_1	c_2	c_2	1	1	1
c_2	o_1	a_1	b_1	c_1	o_2	c_2	b_2	1	1
1	o_1	a_1	b_1	c_1	o_2	a_2	b_2	c_2	1

$\rightsquigarrow$	o_1	a_1	b_1	c_1	o_2	a_2	b_2	c_2	1
o_1	1	1	1	1	o_2	a_2	b_2	c_2	1
a_1	b_1	1	b_1	1	o_2	a_2	b_2	c_2	1
b_1	o_1	a_1	1	1	o_2	a_2	b_2	c_2	1
c_1	o_1	a_1	b_1	1	o_2	a_2	b_2	c_2	1
o_2	o_1	a_1	b_1	c_1	1	1	1	1	1
a_2	o_1	a_1	b_1	c_1	b_2	1	b_2	1	1
b_2	o_1	a_1	b_1	c_1	b_2	c_2	1	1	1
c_2	o_1	a_1	b_1	c_1	b_2	c_2	b_2	1	1
1	o_1	a_1	b_1	c_1	o_2	a_2	b_2	c_2	1

.

Then $(A, \leq, \to, \rightsquigarrow, 1)$ is a pseudo-BCK algebra which is not a BCK algebra.

Example 2.12. ([3]) Consider $A = \{0, a, b, c, 1\}$ with $0 < a, b < c < 1$ and a, b incomparable. Let the operations $\to, \rightsquigarrow$ be given by the following tables:

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	b	1	1
b	a	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

$\rightsquigarrow$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	b	1	1
b	0	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

.

Then $(A, \leq, \to, \rightsquigarrow, 0, 1)$ is a bounded pseudo-BCK algebra.

Definition 2.13. ([20]) A pseudo-BCK algebra with the (pP) *condition* (i.e. with the *pseudo-product* condition) or a *pseudo-BCK(pP) algebra* for short, is a pseudo-BCK algebra A satisfying the (pP) condition:
for all $x, y \in A$, there exists $x \odot y = \min\{z \mid x \leq y \to z\} = \min\{z \mid y \leq x \rightsquigarrow z\}$.

Definition 2.14. ([20]) (1) Let $(A, \leq, \rightarrow, \rightsquigarrow, 1)$ be a pseudo-BCK algebra. If the poset $(A, \leq)$ is a lattice, then we say that A is a *pseudo-BCK lattice*.
(2) Let $(A, \leq, \rightarrow, \rightsquigarrow, 1)$ be a pseudo-BCK(pP) algebra. If the poset $(A, \leq)$ is a lattice, then we say that A is a *pseudo-BCK(pP) lattice*.

Definition 2.15. Let A be a bounded pseudo-BCK algebra. Then:
(1) A is called *good* if $x^{-\sim} = x^{\sim -}$ for all $x \in A$.
(2) A is with the (pDN) *condition* (i.e. with the *pseudo-double negation* condition) or a pseudo-BCK(pDN) algebra for short if $x^{-\sim} = x^{\sim -} = x$ for all $x \in A$.

For example, Examples 2.4 and 2.5 are good pseudo-BCK algebras.

Example 2.16. Consider the pseudo-BCK lattice A from Example 2.12. Since $a^{-\sim} = 1$ and $a^{\sim -} = b$, it follows that A is not good. A can be embedded into the good pseudo-BCK lattice $(A_1, \leq, \rightarrow, \rightsquigarrow, 0, 1)$, where $A_1 = \{0, a, b, c, d, 1\}$ with $0 < a < b, c < d < 1$ and b, c are incomparable. The operations $\rightarrow$ and $\rightsquigarrow$ are defined as follows:

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	0	1	1	1	1	1
b	0	a	1	c	1	1
c	0	b	b	1	1	1
d	0	a	b	c	1	1
1	0	a	b	c	d	1

$\rightsquigarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	0	1	1	1	1	1
b	0	c	1	c	1	1
c	0	a	b	1	1	1
d	0	a	b	c	1	1
1	0	a	b	c	d	1

.

One can easily check that A_1 is a good pseudo-BCK algebra. Moreover, we can see that:

$$\min\{z \mid c \leq b \rightarrow z\} = \min\{b, c, d, 1\},$$
$$\min\{z \mid b \leq c \rightsquigarrow z\} = \min\{b, c, d, 1\}$$

do not exist. Thus, $c \odot b$ does not exist, so A_1 is without the (pP) condition.
Since $(A_1, \leq)$ is a lattice, it follows that A_1 is a good pseudo-BCK lattice without the (pP) condition.

Definition 2.17. ([18], [22]) Let A be a pseudo-BCK algebra.
(1) If $x \vee_1 y = y \vee_1 x$ for all $x, y \in A$, then A is called $\vee_1$-*commutative*.
(2) If $x \vee_2 y = y \vee_2 x$ for all $x, y \in A$, then A is called $\vee_2$-*commutative*.

Lemma 2.18. ([18], [22]) *If A is a pseudo-BCK algebra, then:*
(1) A is $\vee_1$ -commutative if and only if it is a join-semilattice (under $\leq$).
(2) A is $\vee_2$ -commutative if and only if it is a join-semilattice (under $\leq$).

Definition 2.19. ([18], [22]) A pseudo-BCK algebra is called *sup-commutative* if it is both $\vee_1$-commutative and $\vee_2$-commutative.

Theorem 2.20. ([18], [22]) *A pseudo-BCK algebra is sup-commutative if and only if it is a semilattice with respect to both $\vee_1$ and $\vee_2$.*

Corollary 2.21. ([18], [22]) *If A is a sup-commutative pseudo-BCK algebra, then $\vee_1 = \vee_2$ for all $x, y \in A$, so A is a* join-semilattice *with $x \vee_1 y = x \vee_2 y$.*

If A is a sup-commutative pseudo-BCK algebra, then $\vee_1 = \vee_2$ for all $x, y \in A$, so A is a semilattice with $x \vee y = x \vee_1 y = x \vee_2 y$.

Proof. It follows applying [18, Cor.1.17]. □

In what follows, we introduce a new class of pseudo-BCK algebras which will be used in the next sections.

Definition 2.22. A bounded pseudo-BCK algebra A is said to be with the *join-negation* (JN for short) if

$$x \vee_1 y = x^{-\sim} \vee_1 y^{-\sim} \quad \text{and} \quad x \vee_2 y = x^{-\sim} \vee_2 y^{-\sim} \quad \text{for all } x, y \in A.$$

Remark 2.23. (1) Every bounded pseudo-BCK(pDN) algebra is with (JN).
(2) Every bounded sup-commutative pseudo-BCK algebra is with (pDN), so it is with (JN).
(3) Every locally finite pseudo-hoop is a bounded pseudo-BCK(pDN) algebra, so it is with (JN) (see [5]).

We recall that a *downwards-directed set* (or a *filtered set*) is a partially ordered set $(A, \leq)$ such that whenever $a, b \in A$, there exists $x \in A$ such that $x \leq a$ and $x \leq b$.

According to [16], we say that a pseudo-BCK algebra A satisfies the *relative cancellation property*, (RCP) for short, if for every $a, b, c \in A$,

$$a, b \leq c \quad \text{and} \quad c \to a = c \to b, \quad c \rightsquigarrow a = c \rightsquigarrow b \quad \text{imply} \quad a = b.$$

We note that a pseudo-BCK algebra A that is sup-commutative and satisfies the (RCP)-condition is said to be a *Łukasiewicz pseudo-BCK algebra,* see [16].

Example 2.24. The pseudo-BCK algebra A from Example 2.12 is downwards-directed with (RCP).

Proposition 2.25. *Any downwards-directed sup-commutative pseudo-BCK algebra has (RCP).*

Proof. Consider $a, b, c \in A$ such that $a, b \leq c$. There exists $x \in A$ such that $x \leq a, b$.
By (c_6), from $a \leq c$ it follows that $c \rightsquigarrow x \leq a \rightsquigarrow x$.
According to Proposition 2.7(2) and (c_1) we have:

$$\begin{aligned} a \rightsquigarrow x &= (c \rightsquigarrow x) \vee_1 (a \rightsquigarrow x) = (a \rightsquigarrow x) \vee_1 (c \rightsquigarrow x) \\ &= [(a \rightsquigarrow x) \to (c \rightsquigarrow x)] \rightsquigarrow (c \rightsquigarrow x) = c \rightsquigarrow [(a \rightsquigarrow x) \to x] \rightsquigarrow (c \rightsquigarrow x) \\ &= [c \rightsquigarrow (a \vee_2 x)] \rightsquigarrow (c \rightsquigarrow x) = (c \rightsquigarrow a) \rightsquigarrow (c \rightsquigarrow x). \end{aligned}$$

Similarly, $b \rightsquigarrow x = (c \rightsquigarrow b) \rightsquigarrow (c \rightsquigarrow x) = a \rightsquigarrow x$.
We have: $a = x \vee_2 a = a \vee_2 x = (a \rightsquigarrow x) \to x = (b \rightsquigarrow x) \to x = b \vee_2 x = x \vee_2 b = b$.
Thus, A has (RCP). ☐

We say that a nonempty subset F of a pseudo-BCK algebra A is a *filter* (or a *deductive system*, [26, 27]) if (i) $1 \in F$, and (ii) if $a \in F$ and $a \to b \in F$, then $b \in F$. It is easy to verify that a set F containing 1 is a filter if and only if (ii)' if $a \in F$ and $a \rightsquigarrow b \in F$, then $b \in F$. A filter F is (i) *maximal* if it is a proper subset of A and not properly contained in another proper filter of A, (ii) *normal* if $a \to b \in F$ if and only if $a \rightsquigarrow b \in F$. Given a normal filter F, the relation Θ_F on A given by

$$(a, b) \in \Theta_F \;\Leftrightarrow a \to b \in F \text{ and } b \to a \in F$$

is a congruence. Then $F = [1]\Theta_F$ and the quotient class A/F defined as A/Θ_F is again a pseudo-BCK algebra, and we write $a/F = [a]_{\Theta_F}$ for every $a \in A$, see [26, 27].

Given an integer $n \geq 1$, we define inductively

$$x \to^0 y = y, \quad x \to^n y = x \to (x \to^{n-1} y), \quad n \geq 1,$$

and

$$x \rightsquigarrow^0 y = y, \quad x \rightsquigarrow^n y = x \rightsquigarrow (x \rightsquigarrow^{n-1} y), \quad n \geq 1.$$

For any nonempty system X of a pseudo-BCK algebra A, there is the least filter of A generated by X, we denote it by $F(X)$; in particular, if $X = \{u\}$ is a singleton, we set $F(u) := F(\{u\})$. By [26, 27], we have

$$F(X) = \{a \in A : \; x_1 \to (\cdots \to (x_n \to a) \cdots) = 1, \; x_1, \ldots, x_n \in X, \; n \geq 1\} \tag{2.2}$$

and

$$F(u) = \{a \in A : \; u \to^n a = 1, \; n \geq 1\} = \{a \in A : \; u \rightsquigarrow^n a = 1, \; n \geq 1\}. \tag{2.3}$$

If F is a filter and $b \in A$, then the filter, F_b, of A generated by $F \cup \{b\}$ is the set

$$F_b = \{a \in A : \; b \to^n a \in F \text{ for some } n \in \mathbb{N}\}. \tag{2.4}$$

3. States on Pseudo-BCK Algebras

We present a notion of states on bounded pseudo-BCK algebras. We characterize extremal states as state-morphisms and we show that the quotient pseudo-BCK algebra through the kernel of a state is always an MV-algebra. We emphasize that our characterizations of states can be studied without the assumption that the pseudo-BCK algebra is a pseudo MV-algebra or a $\vee$-lattice, see [8, 26].

Definition 3.1. A *Bosbach state* on a bounded pseudo-BCK algebra A is a function $s : A \longrightarrow [0, 1]$ such that the following conditions hold for any $x, y \in A$:
(B_1) $s(x) + s(x \to y) = s(y) + s(y \to x)$;
(B_2) $s(x) + s(x \rightsquigarrow y) = s(y) + s(y \rightsquigarrow x)$;
(B_3) $s(0) = 0$ and $s(1) = 1$.

Example 3.2. Consider the bounded pseudo-BCK lattice A from Example 2.16. The function $s : A \longrightarrow [0, 1]$ defined by: $s(0) = 0, s(a) = 1, s(b) = 1, s(c) = 1, s(d) = 1, s(1) = 1$ is a unique Bosbach state on A.

Not every bounded pseudo-BCK algebra has a Bosbach state:

Example 3.3. Consider the bounded pseudo-BCK lattice A from Example 2.12. One can prove that A has no Bosbach state. Indeed, assume that A admits a Bosbach state s such that $s(0) = 0$, $s(a) = \alpha$, $s(b) = \beta$, $s(c) = \gamma$, $s(1) = 1$. From $s(x) + s(x \to y) = s(y) + s(y \to x)$, taking $x = a, y = 0$, $x = b, y = 0$ and respectively $x = c, y = 0$ we get $\alpha = 1$, $\beta = 0$, $\gamma = 1$.
On the other hand, taking $x = b, y = 0$ in $s(x) + s(x \rightsquigarrow y) = s(y) + s(y \rightsquigarrow x)$ we get $\beta + 0 = 0 + 1$, so $0 = 1$ which is a contradiction. Hence, A does not admit a Bosbach state.

Proposition 3.4. *Let A be a bounded pseudo-BCK algebra and s a Bosbach state on A. For all $x, y \in A$, the following properties hold:*
(1) $s(y \to x) = 1 + s(x) - s(y) = s(y \rightsquigarrow x)$ *and* $s(x) \leq s(y)$ *whenever* $x \leq y$.
(2) $s(x \vee_1 y) = s(y \vee_1 x)$ *and* $s(x \vee_2 y) = s(y \vee_2 x)$.
(3) $s(x \vee_1 y^{-\sim}) = s(x^{-\sim} \vee_1 y^{-\sim})$ *and* $s(x \vee_2 y^{\sim-}) = s(x^{\sim-} \vee_2 y^{\sim-})$.
(4) $s(x^{-\sim} \vee_1 y) = s(x \vee_1 y^{-\sim})$ *and* $s(x^{\sim-} \vee_2 y) = s(x \vee_2 y^{\sim-})$.

(5) $s(x^{-\sim}) = s(x) = s(x^{\sim -})$.
(6) $s(x^{-}) = 1 - s(x) = s(x^{\sim})$.

Proof. (1) It is straightforward.

(2) By (2.1) and property (1), we have $s(x \to y) = s(x \vee_1 y \to y) = 1 + s(y) - s(x \vee_1 y)$ and $s(y \to x) = s(y \vee_1 x \to x) = 1 + s(x) - s(y \vee_1 x)$. Then $s(x \to y) = s(y) + s(y \to x) - s(x) = s(y) + (1 + s(x) - s(y \vee_1 x)) - s(x)$ proving $s(x \vee_1 y) = s(y \vee_1 x)$. Similarly, $s(x \vee_2 y) = s(y \vee_2 x)$.

(3) and (4) follow from Proposition 2.8.

(5) Since $x^{-\sim} = x \vee_1 0$, by (2) we have $s(x^{-\sim}) = s(x \vee_1 0) = s(0 \vee_1 x) = s((0 \to x) \rightsquigarrow x) = s(x)$. In a similar way, we have $s(x) = s(x^{\sim -})$.

(6) $s(x^{-}) = s(x \to 0) = s(0) - s(x) + s(0 \to x) = 1 - s(x)$. □

Proposition 3.5. *Let A be a bounded pseudo-BCK algebra and s a Bosbach state on A. For all $x, y \in A$, the following properties hold:*
(1) $s(x^{-\sim} \to y) = s(x \to y^{-\sim})$ and $s(x^{\sim -} \rightsquigarrow y) = s(x \rightsquigarrow y^{\sim -})$.
(2) $s(x \to y^{-\sim}) = s(y^{-} \rightsquigarrow x^{-}) = s(x^{-\sim} \to y^{-\sim}) = s(x^{-\sim} \to y)$ and
$s(x \rightsquigarrow y^{\sim -}) = s(y^{\sim} \to x^{\sim}) = s(x^{\sim -} \rightsquigarrow y^{\sim -}) = s(x^{\sim -} \rightsquigarrow y)$.
(3) $s(x^{\sim} \to y^{-\sim}) = s(x^{\sim} \to y)$ and $s(x^{-} \rightsquigarrow y^{\sim -}) = s(x^{-} \rightsquigarrow y)$.

Proof. (1) Using Proposition 3.4(4), we have:
$s(x^{-\sim} \to y) = 1 - s(x^{-\sim} \vee_1 y) + s(y) = 1 - s(x \vee_1 y^{-\sim}) + s(y^{-\sim}) = s(x \to y^{-\sim})$.
(2) It follows by (c_4) and (1).
(3) Applying Proposition 3.4(4) we get:
$s(x^{\sim} \to y^{-\sim}) = 1 - s(x^{\sim} \vee_1 y^{-\sim}) + s(y^{-\sim}) = 1 - s(x^{\sim -\sim} \vee_1 y) + s(y) =$
$1 - s(x^{\sim} \vee_1 y) + s(y) = s(x^{\sim} \to y)$.
Similarly, $s(x^{-} \rightsquigarrow y^{\sim -}) = s(x^{-} \rightsquigarrow y)$. □

Proposition 3.6. *Let A be a bounded pseudo-BCK algebra and a function $s : A \longrightarrow [0,1]$ such that $s(0) = 0$, $s(x \vee_1 y) = s(y \vee_1 x)$ and $s(x \vee_2 y) = s(y \vee_2 x)$ for all $x, y \in A$. Then the following are equivalent:*
(a) s is a Bosbach state on A.
(b) for all $x, y \in A$, $y \leq x$ implies $s(x \to y) = s(x \rightsquigarrow y) = 1 - s(x) + s(y)$.
(c) for all $x, y \in A$, $s(x \to y) = 1 - s(x \vee_1 y) + s(y)$ and
$s(x \rightsquigarrow y) = 1 - s(x \vee_2 y) + s(y)$.

Proof. $(a) \Rightarrow (b)$. It follows from Proposition 3.4(1).
$(b) \Rightarrow (c)$. It follows from the proof of Proposition 3.4(2).
$(c) \Rightarrow (a)$. Using Proposition 3.4(2), we get:

$$\begin{aligned} s(x) + s(x \to y) &= s(x) + 1 - s(x \vee_1 y) + s(y) \\ &= 1 - s(y \vee_1 x) + s(x) + s(y) = s(y) + s(y \to x). \end{aligned}$$

Similarly,

$$\begin{aligned} s(x) + s(x \rightsquigarrow y) &= s(x) + 1 - s(x \vee_2 y) + s(y) \\ &= 1 - s(y \vee_2 x) + s(x) + s(y) = s(y) + s(y \rightsquigarrow x). \end{aligned}$$

Moreover, by (c) we have:

$$s(1) = s(x \to x) = 1 - s(x) + s(x) = 1.$$

Thus, s is a Bosbach state on A. □

The following proposition is crucial for our study.

Proposition 3.7. *Let s be a Bosbach state on a bounded pseudo-BCK algebra. Then, for all $x, y \in A$, we have:*
(1) $s(x \vee_1 y) = s(x \vee_2 y)$.
(2) $s(x \to y) = s(x \rightsquigarrow y)$.

Proof. (1) First we prove the equality for $y \leq x$.

Using Proposition 3.4(2), we have $s(x \vee_1 y) = s(y \vee_1 x) = s(x)$ and by Proposition 2.7(2) $s(x \vee_2 y) = s(x)$, i.e., $s(x \vee_1 y) = s(x \vee_2 y)$.

Assume now that x and y are arbitrary elements of A. Using again Proposition 3.4(2) and the first part of the proof, we have

$$\begin{aligned} s(x \vee_1 y) &= s(x \vee_1 (x \vee_1 y)) = s((x \vee_1 y) \vee_1 x) \\ &= s((x \vee_1 y) \vee_2 x) \geq s(y \vee_2 x) \\ &= s(x \vee_2 (y \vee_2 x)) \geq s(x \vee_2 y) \\ &= s(y \vee_2 (x \vee_2 y)) = s((x \vee_2 y) \vee_2 y) \\ &\geq s(x \vee_1 y). \end{aligned}$$

(2) This follows immediately from Proposition 3.6(c) and the first equation. $\square$

Consider the real interval $[0, 1]$ of reals equipped with the Łukasiewicz implication $\to_{\text{Ł}}$ defined by

$$x \to_{\text{Ł}} y = x^- \oplus y = \min\{1 - x + y, 1\}, \quad x, y \in [0, 1].$$

Definition 3.8. Let A be a bounded pseudo-BCK algebra. A *state-morphism* on A is a function $m : A \longrightarrow [0, 1]$ such that:
(SM_1) $m(0) = 0$.
(SM_2) $m(x \to y) = m(x \rightsquigarrow y) = m(x) \to_{\text{Ł}} m(y)$, $x, y \in A$.

Proposition 3.9. *Every state-morphism on a bounded pseudo-BCK algebra A is a Bosbach state on A.*

Proof. It is obvious that $m(1) = m(x \to x) = m(x) \to_{\text{Ł}} m(x) = 1$.
We also have:

$$\begin{aligned} m(x) + m(x \to y) &= m(x) + m(x) \to_{\text{Ł}} m(y) = m(x) + \min\{1 - m(x) + m(y), 1\} \\ &= \min\{1 + m(y), 1 + m(x)\} = m(y) + \min\{1 - m(y) + m(x), 1\} \\ &= m(y) + m(y) \to_{\text{Ł}} m(x) = m(y) + m(y \to x). \end{aligned}$$

Similarly, $m(x) + m(x \rightsquigarrow y) = m(y) + m(y \rightsquigarrow x)$.
Thus, s is a Bosbach state on A. $\square$

Proposition 3.10. *Let A be a bounded pseudo-BCK algebra. A Bosbach state m on A is a state-morphism if and only if:*

$$m(x \vee_1 y) = \max\{m(x), m(y)\}$$

for all $x, y \in A$, or equivalently,

$$m(x \vee_2 y) = \max\{m(x), m(y)\}$$

for all $x, y \in A$.

Proof. In view of Proposition 3.7, the two equations are equivalent. If m is a state-morphism on A, then by Proposition 3.9 m is a Bosbach state. Using the relation:
$m(x \to y) = 1 - m(x \vee_1 y) + m(y)$, we obtain:

$$\begin{aligned} m(x \vee_1 y) &= 1 + m(y) - m(x \to y) = 1 + m(y) - (m(x) \to_{\mathrm{L}} m(y)) \\ &= 1 + m(y) - \min\{1 - m(x) + m(y), 1\} \\ &= 1 + m(y) + \max\{-1 + m(x) - m(y), -1\} = \max\{m(x), m(y)\}. \end{aligned}$$

For the converse, assume that m is a Bosbach state on A such that

$$m(x \vee_1 y) = \max\{m(x), m(y)\} \text{ for all } x, y \in A.$$

Then, using again the relation:
$m(x \to y) = 1 - m(x \vee_1 y) + m(y)$, we have :

$$\begin{aligned} m(x \to y) &= 1 + m(y) - \max\{(m(x), m(y)\} \\ &= 1 + m(y) + \min\{-m(x), -m(y)\} \\ &= \min\{1 - m(x) + m(y), 1\} = m(x) \to_{\mathrm{L}} m(y). \end{aligned}$$

Similarly,
$m(x \rightsquigarrow y) = m(x) \to_{\mathrm{L}} m(y)$.
Thus, m is a state-morphism on A. □

Now we present an example of a linearly ordered pseudo-BCK algebra without the (pP) condition but having a unique Bosbach state (= a unique state-morphism). On the other hand, not every linearly ordered pseudo-BCK algebra admits a Bosbach state, see Example 3.28.

Example 3.11. ([3]) Consider $A = \{0, a, b, c, 1\}$ with $0 < a < b, c < 1$ and b, c incomparable. Consider the operations $\to, \rightsquigarrow$ given by the following tables:

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	1	1	1
b	0	a	1	c	1
c	0	b	b	1	1
1	0	a	b	c	1

$\rightsquigarrow$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	1	1	1
b	0	c	1	c	1
c	0	a	b	1	1
1	0	a	b	c	1

.

Then $(A, \leq, \to, \rightsquigarrow, 0, 1)$ is a bounded pseudo-BCK algebra.
Since $(A, \leq)$ is a lattice, it follows that A is a pseudo-BCK lattice.
Moreover, we can see that $c \odot b = \min\{z \mid c \leq b \to z\} = \min\{b, c, 1\}$ does not exist. Hence, A is a pseudo-BCK lattice without the (pP) condition and the function $m : A \longrightarrow [0, 1]$ defined by:

$$m(0) = 0, m(a) = 1, m(b) = 1, m(c) = 1, m(1) = 1$$

is a unique state-morphism on A.
Moreover, $m(x \vee_1 y) = m(x \vee_2 y) = \max\{m(x), m(y)\}$ for all $x, y \in A$, hence m is also a Bosbach state on A.

The set

$$\mathrm{Ker}(s) := \{a \in A \mid s(a) = 1\}$$

is called the *kernel* of a Bosbach state s on A.

Proposition 3.12. *Let A be a bounded pseudo-BCK algebra and let s be a Bosbach state on A. Then* $\mathrm{Ker}(s)$ *is a proper and normal filter of A.*

Proof. Obviously, $1 \in \mathrm{Ker}(s)$ and $0 \notin \mathrm{Ker}(s)$.
Assume that $a, a \to b \in \mathrm{Ker}(s)$. We have $1 = s(a) \leq s(a \vee_1 b)$, so $s(a \vee_1 b) = 1$.
It follows that $1 = s(a \to b) = 1 - s(a \vee_1 b) + s(b) = s(b)$.
Hence $b \in \mathrm{Ker}(s)$, so $\mathrm{Ker}(s)$ is a proper filter of A.
By Proposition 3.7(2), $s(a \rightsquigarrow b) = s(a \to b)$, and this proves that $\mathrm{Ker}(s)$ is normal. □

Lemma 3.13. *Let s be a Bosbach state on a bounded pseudo-BCK algebra A and $K = \mathrm{Ker}(s)$. In the bounded quotient pseudo-BCK algebra $(A/K, \leq, \to, \rightsquigarrow, 0/K, 1/K)$ we have:*
(1) $a/K \leq b/K$ iff $s(a \to b) = 1$ iff $s(a \vee_1 b) = s(b)$ iff $s(a \vee_2 b) = s(b)$.
(2) $a/K = b/K$ iff $s(a \to b) = s(b \to a) = 1$ iff $s(a) = s(b) = s(a \vee_1 b)$ iff $s(a \rightsquigarrow b) = s(b \rightsquigarrow a) = 1$ iff $s(a) = s(b) = s(a \vee_2 b)$.

Moreover, the mapping $\hat{s} : A/K \to [0, 1]$ defined by $\hat{s}(a/K) := s(a)$ $(a \in A)$ is a Bosbach state on A/K.

Proof. (1) It follows easily: $a/K \leq b/K$ iff $(a \to b)/K = a/K \to b/K = 1/K = K$ iff $a \to b \in K$ iff $s(a \to b) = 1$.
As $s(a \to b) = 1 - s(a \vee_1 b) + s(b)$, we get $a/K \leq b/K$ iff $s(a \vee_1 b) = s(b)$.
Similarly, $a/K \leq b/K$ iff $(a \rightsquigarrow b)/K = a/K \rightsquigarrow b/K = 1/K = K$ iff $a \rightsquigarrow b \in K$ iff $s(a \rightsquigarrow b) = 1$.
As $s(a \rightsquigarrow b) = 1 - s(a \vee_2 b) + s(b)$, we get $a/K \leq b/K$ iff $s(a \vee_2 b) = s(b)$.
(2) It follows easily from (1).

The fact that $\hat{s}$ is a well-defined Bosbach state on A/K is now straightforward. □

Proposition 3.14. *Let s be a Bosbach state on a bounded pseudo-BCK algebra A and let $K = \mathrm{Ker}(s)$. For every element $x \in A$, we have*

$$x^{-\sim}/K = x/K = x^{\sim -}/K,$$

that is, A/K satisfies the (pDN) condition.

Proof. On one side, we have $x \leq x^{-\sim}$. On the other one, by definition of a Bosbach state and Proposition 3.4(5), we have $s(x^{-\sim} \to x) = s(x) + s(x \to x^{-\sim}) - s(x^{-\sim}) = s(x \to x^{-\sim}) = s(1) = 1$. Hence, $x^{-\sim}/K = x/K$. In a similar way, we prove the second identity. □

Remark 3.15. Let s be a Bosbach state on a pseudo-BCK algebra A. According to the proof of Proposition 3.14, we have $s(x^{-\sim} \to x) = 1 = s(x^{\sim -} \to x)$ and $s(x^{-\sim} \rightsquigarrow x) = 1 = s(x^{\sim -} \rightsquigarrow x)$.

Proposition 3.16. *Let s be a Bosbach state on a bounded pseudo-BCK algebra A. Then A/K is $\vee_1$-commutative as well as $\vee_2$-commutative, where $K = \mathrm{Ker}(s)$. In addition, A/K is a $\vee$-semilattice and good.*

Proof. Since s is a Bosbach state, A/K is a BCK algebra. We denote by $\bar{x} := x/K$, $x \in A$ and $\hat{s}(a) := s(a)$ $(a \in A)$ is a Bosbach state on A/K.

(1) We show that if $\bar{x} \leq \bar{y}$, then

$$\bar{x} \vee_1 \bar{y} = \bar{y} = \bar{y} \vee_1 \bar{x}. \qquad (**)$$

By Proposition 2.7, we have $\bar{x} \vee_1 \bar{y} = \bar{y}$. We have to show that $s((y \vee_1 x) \to y) = 1$. Calculate: By Proposition 3.4(1), we have $s(y \vee_1 x) = s((y \to x) \rightsquigarrow x) = \hat{s}((\bar{y} \to \bar{x}) \rightsquigarrow \bar{x}) = 1 + \hat{s}(\bar{x}) - \hat{s}(\bar{y} \to \bar{x}) = 1 + \hat{s}(\bar{x}) - [1 + \hat{s}(\bar{x}) - \hat{s}(\bar{y})] = \hat{s}(\bar{y}) = s(y)$.
Therefore, using Proposition 3.4(2),

$s((y \vee_1 x) \to y) = \hat{s}((\bar{y} \vee_1 \bar{x}) \to \bar{y}) = 1 + \hat{s}(\bar{y}) - \hat{s}(\bar{y} \vee_1 \bar{x}) = 1 + \hat{s}(\bar{y}) - \hat{s}(\bar{x} \vee_1 \bar{y}) = 1 + \hat{s}(\bar{y}) - \hat{s}(\bar{y}) = 1$. Hence, $(**)$ holds for $\bar{x} \leq \bar{y}$.

(2) Now we show that $(**)$ holds for all $x, y \in A$. By (1), we have

$\bar{x} \vee_1 \bar{y} = \bar{x} \vee_1 (\bar{x} \vee_1 \bar{y}) = (\bar{x} \vee_1 \bar{y}) \vee_1 \bar{x} \geq \bar{y} \vee_1 \bar{x} = \bar{y} \vee_1 (\bar{y} \vee_1 \bar{x}) = (\bar{y} \vee_1 \bar{x}) \vee_1 \bar{y} \geq \bar{x} \vee_1 \bar{y}$.

This implies that A/K is $\vee_1$-commutative. In a similar way we prove that A/K is $\vee_2$-commutative. By Lemma 2.10, A/K is a $\vee$-semilattice. □

Proposition 3.17. ([3]) *Let A be a bounded good pseudo-BCK algebra. We define a binary operation $\oplus$ on A by $x \oplus y := x^{\sim} \to y^{\sim -}$. For all $x, y \in A$, the following hold:*
(1) $x \oplus y = y^{-} \rightsquigarrow x^{\sim -}$,
(2) $x, y \leq x \oplus y$,
(3) $x \oplus 0 = 0 \oplus x = x^{\sim -}$,
(4) $x \oplus 1 = 1 \oplus x = 1$,
(5) $(x \oplus y)^{-\sim} = x \oplus y = x^{-\sim} \oplus y^{-\sim}$,
(6) $\oplus$ *is associative.*

An MV-algebra is an algebra $(A, \oplus, \odot, ^{-}, 0, 1)$ of type $\langle 2, 2, 1, 0, 0 \rangle$ such that (i) $\oplus$ is commutative and associative, (ii) $0^{-} = 1$, (iii) $x \oplus 0 = x$, (iv) $x \oplus 1 = 1$, (v) $x^{**} = x$, (vi) $y \oplus (y \oplus x^{-})^{-} = x \oplus (x \oplus y^{-})^{-}$, and (vi) $x \odot y = (x^{-} \oplus y^{-})^{-}$. If we define $x \to y = x \rightsquigarrow y = x^{-} \oplus y$, then $(A, \to, \rightsquigarrow, 0, 1)$ is a bounded pseudo-BCK algebra.

An *MV-state* on an MV-algebra A is a mapping $s : A \to [0, 1]$ such that $s(1) = 1$ and $s(a \oplus b) = s(a) + s(b)$ whenever $a \odot b = 0$. Every MV-algebra admits at least one MV-state, and due to [17], every MV-state on A coincides with a Bosbach state on the BCK algebra A and vice versa.

We note that the *radical*, $\mathrm{Rad}(A)$, of an MV-algebra A is the intersection of all maximal ideals of A, [2].

Proposition 3.18. ([12, Thm 6.1.32]) *In any MV-algebra A the following conditions are equivalent:*
(a) $\mathrm{Rad}(A) = \{0\}$.
(b) $nx \leq x^{-}$ *for all* $n \in \mathbb{N}$ *implies* $x = 0$.
(c) $nx \leq y^{-}$ *for all* $n \in \mathbb{N}$ *implies* $x \wedge y = 0$.
(d) $nx \leq y$ *for all* $n \in \mathbb{N}$ *implies* $x \odot y = x$,
where $nx = x_1 \oplus \cdots \oplus x_n$ *with* $x_1 = \cdots = x_n = x$.

Remark 3.19. An MV-algebra A is archimedean in the sense of [12] if it satisfies the condition (b) of Proposition 3.18 and A is archimedean in Belluce's sense [1] if it satisfies the condition (d) of Proposition 3.18.
By Proposition 3.18 the two definitions of archimedean MV-algebras are equivalent.

Theorem 3.20. *Let s be a Bosbach state on a pseudo-BCK algebra A and let $K = \mathrm{Ker}(s)$. Then $(A/K, \oplus, ^{-}, 0/K)$, where*

$$a/K \oplus b/K = (a^{\sim} \to b)/K \text{ and } (a/K)^{-} = a^{-}/K,$$

is an archimedean MV-algebra and the map $\hat{s}(a/K) := s(a)$ is an MV-state on this MV-algebra.

Proof. By Propositions 3.14 and 3.16, A/K is a good pseudo-BCK algebra that is a $\vee$-semilattice and $\hat{s}$ on A/K is a Bosbach state such that $\mathrm{Ker}(\hat{s}) = \{1/K\}$. Due to [26, Prop 3.4.7], $(A/K)/\mathrm{Ker}(\hat{s})$ is term-equivalent to an MV-algebra that is archimedean and $\hat{s}$ is an MV-state on it. Since $A/K = (A/K)/\mathrm{Ker}(\hat{s})$, the same is true also for A/K, and this proves the theorem. □

We recall that if a pseudo-BCK algebra A is good, in view of Proposition 3.17, we can define a binary operation $\oplus$ via $x \oplus y = x^{\sim} \to y^{\sim -} = y^{-} \rightsquigarrow x^{\sim -}$ that corresponds to an "MV-addition". And for any pseudo MV-algebra A we know, [8], that an MV-state is a state-morphism iff $m(a \oplus b) = m(a) \oplus_{\text{Ł}} m(b)$ for all $a, b \in A$. Inspired by this we can characterize state-morphisms as follows.

Lemma 3.21. *Let m be a Bosbach state on a bounded pseudo-BCK algebra A. The following statements are equivalent:*
(a) m is a state-morphism.
(b) $m(a^{\sim} \to b^{-\sim}) = \min\{m(a) + m(b), 1\}$ for all $a, b \in A$.
(c) $m(b^{-} \rightsquigarrow a^{\sim -}) = \min\{m(a) + m(b), 1\}$ for all $a, b \in A$.

Proof. Due to Theorem 3.20, conditions (b) and (c) are equivalent, and, moreover, we can assume that m is the same as $\hat{m}$.

$(a) \Rightarrow (b)$. Assume that m is a state-morphism on A, so it is a Bosbach state. By Proposition 3.5(3), we have

$$\begin{aligned} m(a^{\sim} \to b^{-\sim}) &= m(a^{\sim} \to b) = m(a^{\sim}) \to_{\text{Ł}} m(b) \\ &= m(a)^{-} \to_{\text{Ł}} m(b) = \min\{m(a) + m(b), 1\}. \end{aligned}$$

$(b) \Rightarrow (a)$. Exchanging m with $\hat{m}$, we have

$$\begin{aligned} m(a \to b) &= m(a^{-\sim} \to b^{-\sim}) = \min\{m(a^{-}) + m(b^{\sim -}), 1\} \\ &= \min\{1 - m(a) + m(b), 1\} = m(a) \to_{\text{Ł}} m(b). \end{aligned}$$

Similarly, $m(a \rightsquigarrow b) = m(a) \to_{\text{Ł}} m(b)$. Hence, m is a state-morphism. $\square$

Proposition 3.22. *Let s be a Bosbach state on a bounded pseudo-BCK algebra A. The following are equivalent:*
(a) s is a state-morphism.
(b) $\mathrm{Ker}(s)$ is a normal and maximal filter of A.

Proof. $(a) \Rightarrow (b)$. It is similar as in [26, Prop.3.4.10].
$(b) \Rightarrow (a)$. Let $K = \mathrm{Ker}(s)$. According to Theorem 3.20, $A/K = (A/K)/\mathrm{Ker}(\hat{s})$ is a BCK algebra that is term equivalent to an MV-algebra. Assume F is a filter of A/K and let $K(F) = \{a \in A \mid a/K \in F\}$. Then $K(F)$ is a filter of A containing K. The maximality of K implies $K = K(F)$ and $F = \{1/K\}$. Due to Theorem 3.20, A/K can be assumed to be an archimedean MV-algebra having only one maximal filter, $\{1/F\}$. Therefore, A/K is an MV-subalgebra of the MV-algebra of the real interval $[0, 1]$. This yields that the mapping $a \mapsto a/K$ $(a \in A)$ is the Bosbach state s that is a state-morphism. $\square$

Lemma 3.23. *Let m be a state-morphism on a bounded pseudo-BCK algebra A and $K = \mathrm{Ker}(m)$. Then*

$$a/K \le b/K \textit{ if and only if } m(a) \le m(b),$$
$$a/K = b/K \textit{ if and only if } m(a) = m(b).$$

Proof. By Proposition 3.9 it follows that m is a Bosbach state on A. Applying Lemma 3.13, $a/K \le b/K$ iff $m(b) = m(a \vee_1 b)$.
But $m(a \vee_1 b) = \max\{m(a), m(b)\}$ and hence $m(a) \le m(b)$.
For the later assertion we apply the first one. $\square$

Proposition 3.24. *Let m be a state-morphism on a bounded pseudo-BCK algebra A. Then $(m(A), \oplus, ^-, 0)$ is a subalgebra of the standard MV-algebra $([0,1], \oplus, ^-, 0)$ and the mapping $a/\mathrm{Ker}(m) \mapsto m(a)$ is an isomorphism of $A/\mathrm{Ker}(m)$ onto $m(A)$.*

Proof. Similarly as [26, Prop. 3.4.12]. □

Proposition 3.25. *Let A be a bounded pseudo-BCK algebra and m_1, m_2 two state-morphisms such that $\mathrm{Ker}(m_1) = \mathrm{Ker}(m_2)$. Then $m_1 = m_2$.*

Proof. By Proposition 3.9, m_1 and m_2 are two Bosbach states. The conditions yield $A/\mathrm{Ker}(m_1) = A/\mathrm{Ker}(m_2)$, and as in the proof of Proposition 3.22, we have that $A/\mathrm{Ker}(m_1)$ is in fact an MV-subalgebra of the MV-algebra of the real interval $[0,1]$. But $\mathrm{Ker}(\hat{m}_1) = \{1/K\} = \mathrm{Ker}(\hat{m}_2)$. Hence, by [8, Prop. 4.5], $\hat{m}_1 = \hat{m}_2$, consequently, $m_1 = m_2$. □

Let A be a bounded pseudo-BCK algebra. We say that a Bosbach state s is *extremal* if for any $0 < \lambda < 1$ and for any two Bosbach states s_1, s_2 on A, $s = \lambda s_1 + (1-\lambda)s_2$ implies $s_1 = s_2$.

Summarizing previous characterizations of state-morphisms, we have the following result.

Theorem 3.26. *Let s be a Bosbach on a bounded pseudo-BCK algebra A. Then the following are equivalent:*
(a) s is an extremal Bosbach state.
(b) $s(x \vee_1 y) = \max\{s(x), s(y)\}$ for all $x, y \in A$.
(c) $s(x \vee_2 y) = \max\{s(x), s(y)\}$ for all $x, y \in A$.
(d) s is a state-morphism.
(e) $\mathrm{Ker}(s)$ is a maximal filter.

Proof. The equivalence of (b)–(e) was proved in Propositions 3.10 and 3.22.

$(d) \Rightarrow (a)$. Let $s = \lambda s_1 + (1-\lambda)s_2$, where s_1, s_2 are Bosbach states and $0 < \lambda < 1$. Then $\mathrm{Ker}(s) = \mathrm{Ker}(s_1) \cap \mathrm{Ker}(s_2)$ and the maximality of $\mathrm{Ker}(s)$ gives that $\mathrm{Ker}(s_1)$ and $\mathrm{Ker}(s_2)$ are maximal and normal filters. (e) yields that s_1 and s_2 are state-morphisms and Proposition 3.25 entails $s_1 = s_2 = s$.

$(a) \Rightarrow (d)$. Let s be an extremal state on A. Define $\hat{s}$ by Proposition 3.13 on $A/\mathrm{Ker}(s)$. We assert that $\hat{s}$ is an extremal MV-state on the MV-algebra $A/\mathrm{Ker}(s)$. Indeed, let $\hat{m} = \lambda\mu_1 + (1-\lambda)\mu_2$, where $0 < \lambda < 1$ and μ_1 and μ_2 are states on $A/\mathrm{Ker}(s)$. There exist two Bosbach states s_1 and s_2 on A such that $s_i(a) := \mu_i(a/\mathrm{Ker}(s))$, $a \in A$ for $i = 1, 2$. Then $s = \lambda s_1 + (1-\lambda)s_2$ which gives $s_1 = s_2 = s$, so that $\mu_1 = \mu_2 = \hat{s}$.

Since $A/\mathrm{Ker}(s)$ is in fact an MV-algebra, we conclude from [12, Thm 6.1.30] that $\hat{s}$ is a state-morphism on $A/\mathrm{Ker}(s)$. Consequently, so is s on A. □

Remark 3.27. In the case of pseudo-BL algebras and bounded non-commutative Rℓ-monoids it was proved that the existence of a state-morphism is equivalent with the existence of a maximal filter which is normal (see [17] and respectively [13]). This result is based on the fact that, if A is one the above mentioned structures and H is a maximal and normal filter of A, then A/H is an MV-algebra.

In the case of pseudo-BCK algebras this result is not true as we can see in the next example.

Example 3.28. Consider $A = \{0, a, b, c, 1\}$ with $0 < a < b < c < 1$ and the operations $\to, \rightsquigarrow$ given by the following tables:

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	1	1	1
b	b	c	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

$\rightsquigarrow$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	1	1	1
b	b	b	1	1	1
c	0	b	b	1	1
1	0	a	b	c	1

Then $(A, \leq, \to, \rightsquigarrow)$ is a good pseudo-BCK lattice. $D = \{1\}$ is a maximal normal filter of A, but there is no Bosbach state on A. Indeed, assume that A admits a Bosbach state s such that $s(0) = 0$, $s(a) = \alpha$, $s(b) = \beta$, $s(c) = \gamma$, $s(1) = 1$. From $s(x) + s(x \to y) = s(y) + s(y \to x)$, taking $x = a, y = 0$, $x = b, y = 0$ and respectively $x = c, y = 0$ we get $\alpha = 1/2$, $\beta = 1/2$, $\gamma = 1$.
On the other hand, we have:

$$s(a) + s(a \rightsquigarrow b) = s(a) + s(1) = 1/2 + 1 = 3/2,$$
$$s(b) + s(b \rightsquigarrow a) = s(b) + s(b) = 1/2 + 1/2 = 1,$$

so condition (B_2) does not hold.
Thus, there is no Bosbach state, in particular, no state-morphism on A.

Remark 3.29. The reason why the above result does not hold is that $A/\{1\} \cong A$ is not an MV-algebra, see Remark 3.27.

Inspired by the latter remark, we have the following characterization of the existence of Bosbach states on a bounded pseudo-BCK algebra.

Theorem 3.30. *Let A be a bounded pseudo-BCK algebra. The following statements are equivalent:*
(a) A admits a Bosbach state.
(b) There exists a normal filter $F \neq A$ of A such that A/F is termwise equivalent to an MV-algebra.
(c) There exists a normal and maximal filter F such that A/F is termwise equivalent to an MV-algebra.

Proof. $(a) \Rightarrow (b)$. Let m be a Bosbach state, then the normal filter $F = \mathrm{Ker}(m)$ according to Theorem 3.20 satisfies (b).

$(b) \Rightarrow (a)$. If A/F is an MV-algebra, then it possesses at least one MV-state, say μ. The function $m(a) := \mu(a/F)$ $(a \in A)$ is a Bosbach state on A.

$(a) \Rightarrow (c)$. If A possesses at least one state, by the Krein-Mil'man theorem (see (3.1) below), $\partial_e \mathcal{BS}(A) = \mathcal{SM}(A) \neq \emptyset$. Then there is a state-morphism m on A and due to Theorem 3.26(d), the filter $F = \mathrm{Ker}(m)$ is maximal and normal, and by Theorem 3.20, F satisfies (c).

$(c) \Rightarrow (a)$. It is the same as that of $(b) \Rightarrow (a)$. □

Remark 3.31. The previous example of a linearly ordered stateless pseudo-BCK algebra shows another difference between pseudo-BCK algebras and pseudo-BL algebras because: *Every linearly ordered pseudo-BL algebra admits a Bosbach states* (see [9, 11]).

We say that a net of Bosbach states $\{s_\alpha\}$ *converges weakly* to a Bosbach state s if $s(a) = \lim_\alpha s_\alpha(a)$ for every $a \in A$. According to the definition of Bosbach states, the set of Bosbach states is a compact Hausdorff topological space (theoretically empty) in the weak topology.

Extremal Bosbach states are very important because they generate all Bosbach states: Due to the Krein–Mil'man theorem, [19, Thm 5.17], every Bosbach state is a weak limit of a net of convex combinations of extremal Bosbach states.

Let $\mathcal{BS}(A)$, $\partial_e\mathcal{BS}(A)$ and $\mathcal{SM}(A)$ denote the set of all Bosbach states, all extremal Bosbach states, and all state-morphisms on $(A, \rightarrow, \rightsquigarrow, 0, 1)$, respectively. Theorem 3.26 says

$$\partial_e\mathcal{BS}(A) = \mathcal{SM}(A) \tag{3.1}$$

and they are compact subsets of $\mathcal{BS}(A)$ in the weak topology.

Definition 3.32. ([3]) Let A be a good bounded pseudo-BCK algebra. The elements $x, y \in A$ are called *orthogonal*, denoted by $x \perp y$ iff $x^{-\sim} \leq y^{\sim}$. If the elements $x, y \in A$ are orthogonal, we define a partial operation $+$ on A by $x + y := x \oplus y$.

Definition 3.33. ([3]) Let A be a good bounded pseudo-BCK algebra. A *Riečan state* on A is a function $s : A \longrightarrow [0, 1]$ such that the following conditions hold for all $x, y \in A$:
(R_1) If $x \perp y$, then $s(x + y) = s(x) + s(y)$;
(R_2) $s(1) = 1$.

Proposition 3.34. ([3]) *If s is a Riečan state on a good bounded pseudo-BCK algebra A, then the following properties hold for all $x, y \in A$:*
(1) $s(x^-) = s(x^\sim) = 1 - s(x)$.
(2) $s(0) = 0$.
(3) $s(x^{-\sim}) = s(x^{\sim-}) = s(x^{--}) = s(x^{\sim\sim}) = s(x)$.
(4) *if* $x \leq y$, *then* $s(x) \leq s(y)$ *and* $s(y \rightarrow x^{-\sim}) = s(y \rightsquigarrow x^{\sim-}) = 1 + s(x) - s(y)$.
(5) $s((x \vee_1 y) \rightarrow x^{-\sim}) = s((x \vee_1 y) \rightsquigarrow x^{-\sim}) = 1 - s(x \vee_1 y) + s(x)$ *and*
$s((x \vee_2 y) \rightarrow x^{-\sim}) = s((x \vee_2 y) \rightsquigarrow x^{-\sim}) = 1 - s(x \vee_2 y) + s(x)$.
(6) $s((x \vee_1 y) \rightarrow y^{-\sim}) = s((x \vee_1 y) \rightsquigarrow y^{-\sim}) = 1 - s(x \vee_1 y) + s(y)$ *and*
$s((x \vee_2 y) \rightarrow y^{-\sim}) = s((x \vee_2 y) \rightsquigarrow y^{-\sim}) = 1 - s(x \vee_2 y) + s(y)$.

Remark 3.35. According to [3, Thm 3.17] every Bosbach state on a good bounded pseudo-BCK algebra is a Riečan state. The converse is not true in general, as it was proved in [3, Ex. 3.18], see also the next Example 3.36.

In the following example, we show that there exists a Riečan state on a bounded pseudo-BCK algebra A, that is not a Bosbach state. Moreover, this A has no Bosbach state.

Example 3.36. Consider $A = \{0, a, b, c, 1\}$ with $0 < a < b < c < 1$ from Example 3.28. Then $(A, \leq, \rightarrow, \rightsquigarrow)$ is a good pseudo-BCK algebra. The function $s : A \longrightarrow [0, 1]$ defined by

$$s(0) = 0, s(a) = 1/2, s(b) = 1/2, s(c) = 1, s(1) = 1$$

is a unique Riečan state. Indeed, the orthogonal elements of A are the pairs (x, y) in the following table:

x	y	$x^{-\sim}$	$y^{\sim}$	$x+y$
0	0	0	1	0
0	a	0	b	b
0	b	0	b	b
0	c	0	0	1
0	1	0	0	1
a	0	b	1	b
a	a	b	b	1
a	b	b	b	1
b	0	b	1	b
b	a	b	b	1
b	b	b	b	1
c	0	1	1	1
1	0	1	1	1

We prove now that s is not a Bosbach state. Look at $a < b$. We have: $s(b \to a) = s(c) = 1$, but $s(b \rightsquigarrow a) = s(b) = 1/2$. So, s is not a Bosbach state. We recall that the kernel of s is the set $\{c, 1\}$ that is not a filter.

On the other hand, as it was shown in Examle 3.28, A has no Bosbach state.

4. Measures on Pseudo-BCK Algebras

In this section we generalize measures on pseudo-BCK algebras introduced by A. Dvurečenskij in [7] and [12] to pseudo-BCK algebras that are not necessarily bounded. In particular, we show that if A is a downwards-directed pseudo-BCK algebra and m a measure on it, then the quotient over the kernel of m can be embedded into the negative cone of an abelian, archimedean ℓ-group as its subalgebra. This result will enable us to characterize nonzero measure-morphisms as measures whose kernel is a maximal filter.

Consider the bounded $\vee_1$-commutative BCK(P) algebra (i.e. an MV-algebra) $\mathcal{A}_{\text{Ł}} = ([0,1], \leq, \to_{\text{Ł}}, 0, 1)$, where $\to_{\text{Ł}}$ is the Łukasiewicz implication: $x \to_{\text{Ł}} y = \min\{1 - x + y, 1\}$.

Definition 4.1. Let $(A, \leq, \to, \rightsquigarrow, 1)$ be a pseudo-BCK algebra. A mapping $m : A \longrightarrow [0, \infty)$ such that for all $x, y \in A$,
(1) $m(x \to y) = m(x \rightsquigarrow y) = m(y) - m(x)$ whenever $y \leq x$ is said to be a *measure*;
(2) if $0 \in A$ and m is a measure with $m(0) = 1$, then m is said to be a *state-measure*;
(3) if $m(x \to y) = m(x \rightsquigarrow y) = \max\{0, m(y) - m(x)\}$ is said to be a *measure-morphism*;
(4) if $0 \in A$, $m(0) = 1$ and m is a measure-morphism, m is said to be a *state-measure-morphism*.

Of course, the function vanishing on A is always a (trivial) measure.

We note that our definition of a measure (a state-measure) is a definition of a map that maps pseudo-BCK algebra that is in the “negative cone” to the positive cone of the reals $\mathbb{R}$. For a relationship with the previous type of Bosbach state see the second part of the present section and Remark 4.16.

For example, let $(G, +, 0, \wedge, \vee)$ be an ℓ-group, with the negative cone G^-, see Example 2.3. Assume that m is a positive-valued function on G^- that preserves addition in G^-. Then m is a measure on the pseudo-BCK algebra G^-, and conversely if m is a measure on G^-, then m is additive on G^- and positive-valued.

We recall that not every negative cone even of abelian ℓ-group admits a nontrivial measure. To see that look [19, Ex. 9.6].

Proposition 4.2. *Let m be a measure on a pseudo-BCK algebra A. For all $x, y \in A$, we have*
(1) $m(1) = 0$.
(2) $m(x) \geq m(y)$ *whenever* $x \leq y$.
(3) $m(x \vee_1 y) = m(y \vee_1 x)$ *and* $m(x \vee_2 y) = m(y \vee_2 x)$.
(4) $m(x \vee_1 y) = m(x \vee_2 y)$.
(5) $m(x \to y) = m(x \rightsquigarrow y)$.

Proof. (1) Since $1 \leq 1$ we get $m(1) = m(1 \to 1) = m(1) - m(1) = 0$.

(2) Since $x \leq y$ it follows that $m(y \to x) = m(x) - m(y)$, so $m(x) - m(y) \geq 0$.

(3) First, let $x \leq y$. Then by Proposition 2.7(2), we have $m(x \vee_1 y) = m(y)$. Using the property of measures, we have $m((y \vee_1 x) \to x) = m(x) - m(y \vee_1 x) = m(x) - m((y \to x) \rightsquigarrow x) = m(x) - m(x) + m(y \to x) = m(x) - m(y)$ giving $m(y \vee_1 x) = m(y)$.

Let now $x, y \in A$ be arbitrary. Using the first part of the present proof and (2), we have $m(x \vee_1 y) = m(x \vee_1 (x \vee_1 y)) = m((x \vee_1 y) \vee_1 x) \leq m(y \vee_1 x) = m(y \vee_1 (y \vee_1 x)) = m((y \vee_1 x) \vee_1 y) \leq m(x \vee_1 y)$.

In a similar way we prove the second equation in (3).

(4) First again, let $x \leq y$. Then $m(x \vee_1 y) = m(y)$. And $m(x \vee_2 y) = m(y \vee_2 x) = m(y)$.

Let now $x, y \in A$ be arbitrary. Using (3), we have: $m(x \vee_1 y) = m(x \vee_1 (x \vee_1 y)) = m((x \vee_1 y) \vee_1 x) = m((x \vee_1 y) \vee_2 x) \leq m(y \vee_2 x) = m(x \vee_2 y) = m(x \vee_2 (x \vee_2 y)) = m((x \vee_2 y) \vee_2 x) = m((x \vee_2 y) \vee_1 y) \leq m(x \vee_1 y)$.

(5) According to (2.1) and (4), $m(x \to y) = m((x \vee_1 y) \to y) = m(y) - m(x \vee_1 y) = m(y) - m(x \vee_2 y) = m((x \vee_2 y) \rightsquigarrow y) = m(x \rightsquigarrow y)$. □

Proposition 4.3. *Let A be a pseudo-BCK algebra. Then*
(1) $y \leq x$ *implies* $m((x \to y) \rightsquigarrow y) = m((x \rightsquigarrow y) \to y) = m(x)$ *whenever m is a measure on A.*
(2) *If m is a measure on A, then* $\mathrm{Ker}_0(m) = \{x \in A \mid m(x) = 0\}$ *is a normal filter of A.*
(3) *Any measure-morphism on A is a measure on A.*

Proof. (1) From $y \leq x \to y$ we get $m((x \to y) \rightsquigarrow y) = m(y) - m(x \to y) = m(y) - (m(y) - m(x)) = m(x)$.
Similarly, $m((y \rightsquigarrow x) \to x) = m(x)$.
(2) According to Proposition 4.2(1), $1 \in \mathrm{Ker}_0(m)$. Assume that $x, x \to y \in \mathrm{Ker}_0(m)$. Since $x \leq x \vee_1 y$, due to Proposition 4.2(2), we have $0 = m(x) \geq m(x \vee_1 y)$, so $m(x \vee_1 y) = 0$.
In addition,
$0 = m(x \to y) = m(x \vee_1 y \to y) = m(y) - m(x \vee_1 y) = m(y)$, so $y \in \mathrm{Ker}_0(m)$.
(we applied the fact that $y \leq x \vee_1 y$ and Proposition 2.9).
Thus, $\mathrm{Ker}_0(m)$ is a filter of A. The normality of $\mathrm{Ker}_0(m)$ follows from Proposition 4.2(5).
(3) We have $m(1) = m(1 \to 1) = \max\{0, m(1) - m(1)\} = 0$ so that if $y \leq x$ then $0 = m(1) = m(y \to x) = \max\{0, m(x) - m(y)\}$ and $m(x) \leq m(y)$ so that $m(x \to y) = \max\{0, m(y) - m(x)\} = m(y) - m(x)$. Similarly, $m(x \rightsquigarrow y) = m(y) - m(x)$. □

Example 4.4. Consider the bounded pseudo-BCK lattice A from Example 2.16. The function $m : A \longrightarrow [0, \infty)$ defined by: $m(0) = 1, m(a) = m(b) = m(c) = m(d) = m(1) = 0$ is a unique measure on A. Moreover, m is even a state-measure on A.

Proposition 4.5. *Let A be a bounded pseudo-BCK algebra. If M is a Bosbach state, then $m = 1 - M$ is a state-measure.*

Proof. Let $y \leq x$, that is $y \to x = y \rightsquigarrow x = 1$.
Replacing in the axioms $(B1)$ and $(B2)$ of a Bosbach state, we obtain:
$M(x) + M(x \to y) = M(y) + 1$ and $M(x) + M(x \rightsquigarrow y) = M(y) + 1$,
which implies $M(x \to y) = M(x \rightsquigarrow y) = 1 - M(x) + M(y)$.
We get $m(x \to y) = m(x \rightsquigarrow y) = 1 - M(x \to y) = M(x) - M(y) = 1 - M(y) - (1 - M(x)) = m(y) - m(x)$.
Also, $m(0) = 1 - M(0) = 1$ by $(B3)$, and thus, m is a state-measure. □

Proposition 4.6. *Let A be a bounded pseudo-BCK algebra. If m is a state-measure on A, then $M = 1 - m$ is a Bosbach state on A.*

Proof. We have: $y \leq x \vee_1 y$ and using the definition of the measure, we get
$m(x \vee_1 y \to y) = m(x \vee_1 y \rightsquigarrow y) = m(y) - m(x \vee_1 y)$. Using Proposition 2.9, we have: $x \vee_1 y \to y = x \to y$, so we get $m(x \to y) = m(y) - m(x \vee_1 y)$.
Similarly, $m(y \to x) = m(x) - m(y \vee_1 x)$ and also, $m(x \rightsquigarrow y) = m(y) - m(x \vee_2 y)$, $m(y \rightsquigarrow x) = m(x) - m(y \vee_2 x)$. According to Proposition 4.2(3), $m(x \vee_1 y) = m(y \vee_1 x)$, so that $m(x) + m(x \to y) = m(y) + m(y \to x)$.
Similarly, $m(x) + m(x \rightsquigarrow y) = m(y) + m(y \rightsquigarrow x)$.
Therefore, $M(x) + M(x \to y) = M(y) + M(y \to y)$ and $M(x) + M(x \rightsquigarrow y) = M(y) + M(y \rightsquigarrow y)$. Also, $M(0) = 0$ by the hypothesis, and $M(1) = 1$ by Proposition 4.2(1).
Thus, M is a Bosbach state. □

If A is a bounded pseudo-BCK algebra, in a similar way as for Bosbach states, we can define extremal state-measures, as well as the weak-topology. Let us denote the set of state-measures, $\mathcal{SM}_1(A)$, the set of state-measure-morphisms, $\mathcal{SMM}_1(A)$, and the set of extremal state-measures, $\partial_e\mathcal{SM}_1(A)$, respectively.

Theorem 4.7. *Let A be a bounded pseudo-BCK algebra. Define a map $\Psi : \mathcal{SM}_1(A) \to \mathcal{BS}(A)$ via $\Psi(m) = 1 - m$, $m \in \mathcal{SM}_1(A)$. Then Ψ is a affine-homeomorhism such that m is a state-measure-morphism if and only if $\Psi(m)$ is a state-morphism. In particular, m is an extremal state-measure if and only if m is a state-measure-morphism.*

Proof. Propositions 4.5-4.6 show that Ψ is a bijection preserving convex combinations and weak topologies.

If, say, s is a state-morphism on A, i.e. $s(x \to y) = \min\{1 - m(x) + m(y)\}$, then it is straightforward to show that $m = 1 - s$ is a state-measure-morphism on A, i.e. $m(x \to y) = \max\{m(y) - m(x), 0\}$ (as well as for the second arrow $\rightsquigarrow$).

In view of Theorem 3.26, we see that a state-measure is extremal iff it is state-measure-morphism. □

As a corollary of Theorem 4.7 and (3.1), we have that if A is a bounded pseudo-BCK algebra, then
$$\partial_e\mathcal{SM}_1(A) = \mathcal{SMM}_1(A). \tag{4.1}$$

Theorem 4.8. *Let m be a measure on a pseudo-BCK algebra A. Then $A/\mathrm{Ker}_0(m)$ is a pseudo-BCK algebra and the mapping $\hat{m} : A/\mathrm{Ker}_0(m) \longrightarrow [0, +\infty)$ defined by $\hat{m}(\bar{x}) :=$*

$m(x)$, $\bar{x} := x/\mathrm{Ker}_0(m) \in A/\mathrm{Ker}_0(m)$, *is a measure on* $A/\mathrm{Ker}_0(m)$, *and* $A/\mathrm{Ker}_0(m)$ *is* $\vee$*-commutative.*

Proof. By (2.1) and (4) of Proposition 4.2, we have $m(x \to y) = m(x \vee_1 y \to y) = m(y) - m(x \vee_1 y) = m(y) - m(x \vee_2 y) = m(x \vee_2 y \rightsquigarrow y) = m(x \rightsquigarrow y)$. According to (2) of Proposition 4.3, $\mathrm{Ker}_0(m)$ is a normal filter. Since if $x \to y, y \to x \in \mathrm{Ker}_0(m)$, we have $m(x \to y) = m(x \vee_1 y \to y) = 0 = m(y) - m(x \vee_1 y)$. Similarly, $m(x) = m(y \vee_1 x)$. But $m(y) = m(x \vee_1 y) = m(y \vee_1 x) = m(x)$. Hence, $\hat{m}$ is a well-defined function on $A/\mathrm{Ker}_0(m)$.

We recall that $\bar{x} = \bar{y}$ iff $m(x) = m(y) = m(x \vee_1 y)$, and $\bar{x} \leq \bar{y}$ iff $m(x \to y) = 0$. Therefore, if $x/\mathrm{Ker}_0(m) = y/\mathrm{Ker}_0(m)$, then $x/\mathrm{Ker}_0(m) = (x \vee_1 y)/\mathrm{Ker}_0(m) = (x \vee_2 y)/\mathrm{Ker}_0(m) = (y \vee_1 x)/\mathrm{Ker}_0(m) = (y \vee_2 x)/\mathrm{Ker}_0(m)$.

To show that $\hat{m}$ is a measure, assume $\bar{y} \leq \bar{x}$. By (2) of Proposition 2.7, $\bar{y} \vee_1 \bar{x} = \bar{x}$. Then $\hat{m}(\bar{x} \to \bar{y}) = m(x \to y) = m(x \vee_1 y \to y) = m(y) - m(x \vee_1 y)$. But $m(x \vee_1 y) = m(y \vee_1 x) = \hat{m}(\bar{y} \vee_1 \bar{x}) = \hat{m}(\bar{x}) = m(x)$. Therefore, $\hat{m}(\bar{x} \to \bar{y}) = \hat{m}(\bar{y}) - \hat{m}(\bar{x})$. Similarly, $\hat{m}(\bar{x} \rightsquigarrow \bar{y}) = \hat{m}(\bar{y}) - \hat{m}(\bar{x})$.

In the same way as in the proof of Proposition 3.16 we can show that $A/\mathrm{Ker}_0(M)$ is $\vee$-commutative. $\square$

In view of Theorem 3.20 and Theorem 4.7 we know that if m is a state-measure on a bounded pseudo-BCK algebra A, then $A/\mathrm{Ker}(m)$ is in fact an MV-algebra, so that according to the famous representation theorem of Mundici, [2], $A/\mathrm{Ker}(m)$ is an interval in an ℓ-group with strong unit. In the following result we generalize this ℓ-group representation of the quotient for measures on unbounded pseudo-BCK algebra that are downwards-directed.

Theorem 4.9. *Let m be a measure on an unbounded pseudo-BCK algebra A that is a downwards-directed set. Then the arrows $\to /\mathrm{Ker}_0(m)$ and $\rightsquigarrow /\mathrm{Ker}_0(m)$ on $A/\mathrm{Ker}_0(m)$ coincide. Moreover, there is a unique (up to isomorphism)) archimedean ℓ-group G such that $A/\mathrm{Ker}_0(m)$ is a subalgebra of the pseudo-BCK algebra G^- and $A/\mathrm{Ker}_0(m)$ generates the ℓ-group G.*

Proof. We note that if a is an arbitrary element of A, then $([a,1], \leq, \to, \rightsquigarrow, a, 1)$ is a pseudo-BCK algebra.

We denote by $K_0 := \mathrm{Ker}_0(m)$. Given $x, y \in A$, choose an element $a \in A$ such that $a \leq x, y$.

If $m(a) = 0$, then $a/K_0 = x \to y/K_0 = x \rightsquigarrow y/K_0 = 1/K_0$.

Assume $m(a) > 0$ and define $m_a(z) := m(z)/m(a)$ for any $z \in [a,1]$. Then m_a is a state-measure on $[a,1]$ and in view of Theorem 4.7, $s_a := 1 - m_a$ is a Bosbach state on $[a,1]$. Theorem 3.20 entails that $[a,1]/\mathrm{Ker}(s_a)$ can be converted into an archimedean MV-algebra. In particular, $(x \to y)/\mathrm{Ker}(s_a) = (x \rightsquigarrow y)/\mathrm{Ker}_0(s_a)$. This yields $s_a((x \to y) \to (x \rightsquigarrow y)) = 1$ and $m((x \to y) \to (x \rightsquigarrow y)) = 0$. In a similar way, $m((x \rightsquigarrow y) \to (x \to y)) = 0$. This proves $\to /\mathrm{Ker}_0(m) = \rightsquigarrow /\mathrm{Ker}_0(m)$.

In addition, we can prove that, for all $x, y \in A$,

$$((x \to y) \vee (y \to x))/\mathrm{Ker}_0(m) = 1/\mathrm{Ker}_0(m) = ((x \rightsquigarrow y) \vee (y \rightsquigarrow x))/\mathrm{Ker}_0(m).$$

It is clear that if $m = 0$, then $\mathrm{Ker}_0(m) = A$ and $A/\mathrm{Ker}_0(m) = \{1/\mathrm{Ker}_0(m)\}$ so that the trivial ℓ-group $G = \{0_G\}$, where 0_G is a neutral element of G, satisfies our conditions.

Therefore, let $m \neq 0$. By [26, Lem 4.1.8], $A/\mathrm{Ker}_0(m)$ is a distributive lattice. As in Proposition 2.25 we can show that $A/\mathrm{Ker}_0(m)$ satisfies the (RCP) condition, and therefore, $A/\mathrm{Ker}_0(m)$ is a Łukasiewicz BCK algebra, see [16]. Therefore, [16, 10], there is a unique (up to isomorphism of ℓ-groups) ℓ-group G such that $A/\mathrm{Ker}_0(m)$ can be embedded into the pseudo-BCK algebra of the negative cone G^-, moreover, $A/\mathrm{Ker}_0(m)$ generates G. Since the arrows in $A/\mathrm{Ker}_0(m)$ coincide, we see that G is abelian, and every interval $[a/K_0, 1/K_0]$ is an archimedean MV-algebra, so is G. □

We note that if m is a measure-morphism on A, then

$$m(u \to^n x) = \max\{0, m(x) - nm(u)\} \tag{4.2}$$

for any $n \geq 0$, and

$$m(x_1 \to (\cdots \to (x_n \to a)\cdots)) = \max\{0, m(a) - m(x_1) - \cdots - m(x_n)\}. \tag{4.3}$$

Proposition 4.10. *Let m be a measure-morphism on a pseudo-BCK algebra A such that $m \neq 0$. Then $\mathrm{Ker}_0(m)$ is a normal and maximal filter of A.*

Proof. Since m is a measure-morphism, by Proposition 4.3, $\mathrm{Ker}_0(m)$ is a normal filter.

Choose $a \in A$ such that $m(a) \neq 0$. Let F be the filter generated by $\mathrm{Ker}_0(m)$ and by the element a. Let $z \in A$ be an arbitrary element of A. There is an integer $n \geq 1$ such that $(n-1)m(a) \leq m(z) < nm(a)$. Due to (4.3), we have that $m(a \to^n z) = \max\{0, m(z) - nm(a)\} = 0$ so that $z \in F$ and $A \subseteq F$ proving that $\mathrm{Ker}_0(m)$ is a maximal filter. □

If $m \neq 0$ is a measure on a bounded pseudo BCK-algebra A, then passing to a state-measure $s_m(a) := m(a)/m(1)$, $a \in A$, and using Theorem 4.7, we see that m is a measure-morphism iff $\mathrm{Ker}_0(m)$ is a maximal filter. The same result is true for unbounded pseudo-BCK algebra that is downwards-directed:

Theorem 4.11. *Let $m \neq 0$ be a measure on an unbounded pseudo-BCK algebra A that is downwards directed. Then m is a measure-morphism if and only if $\mathrm{Ker}_0(m)$ is a maximal filter.*

Proof. By Proposition 4.10, $\mathrm{Ker}_0(m)$ is a maximal filter of A.

Suppose now $\mathrm{Ker}_0(m)$ is a maximal filter of A. In view of Theorem 4.7, $A/\mathrm{Ker}_0(m)$ can be embedded as a subalgebra into the pseudo-BCK algebra G^-, where G^- is the negative cone of an abelian and archimedean ℓ-group G that is generated by $A/\mathrm{Ker}_0(m)$. Let $\hat{m}(a/\mathrm{Ker}_0(m)) := m(a)$ $(a \in A)$. Then $\mathrm{Ker}_0(\hat{m}) = \{1/\mathrm{Ker}_0(m)\}$ and $0_G := 1/\mathrm{Ker}_0(m)$ is the neutral element of G.

Fix an element $a \in A$ with $m(a) > 0$. Since $\mathrm{Ker}_0(m)$ is maximal in A, $\mathrm{Ker}_0(\hat{m}) = \{1/\mathrm{Ker}_0(m)\}$ is maximal in $A/\mathrm{Ker}_0(m)$ and consequently, $\{1/\mathrm{Ker}_0(m)\}$ is a maximal filter of the pseudo-BCK algebra G^- because $A/\mathrm{Ker}_0(m)$ generates G. Therefore, the ℓ-ideal $L := \{0_g\} = \{1/\mathrm{Ker}_0(m)\}$ is a maximal ℓ-ideal of G. We recall that every maximal ℓ-ideal, L, of an ℓ-group is prime ($a, b \in G^+$ with $a \wedge b = 0$ implies $a \in L$ or $b \in L$), whence G/L is a linearly ordered ℓ-group (see e.g. [6, Prop. 9.9]). Since $G = G/L$, G is archimedean and linearly ordered, due to the Hölder theorem, [6, Thm 24.16], G is an ℓ-subgroup of the ℓ-group of real numbers, $\mathbb{R}$. Let s be the unique extension of $\hat{m}$ onto G, then s is additive on G and $s(g) \geq 0$ for any $g \in G^-$. Since G is an ℓ-subgroup of $\mathbb{R}$, s is a unique additive function on G that is positive on the negative cone (see example just after Definition 4.1) with the property $s(a/\mathrm{Ker}_0(m)) = m(a) > 0$ for our fixed element $a \in A$. Because $A/\mathrm{Ker}_0(m)$ can be embedded into $\mathbb{R}^-$, we see that s is a measure-morphism on G^-. Consequently, m is a measure-morphism on A. $\square$

Proposition 4.12. *Let m_1 and m_2 be two measure-morphisms on a downwards-directed pseudo-BCK algebra A such that there is an element $a \in A$ with $m_1(a) = m_2(a) > 0$. If $\mathrm{Ker}_0(m_1) = \mathrm{Ker}_0(m_2)$, then $m_1 = m_2$.*

In addition, let $a \in A$ be fixed. If m is a measure-morphism on A such that $m(a) > 0$, then m cannot be expressed as a convex combination of two measure m_1 and m_2 such that $m_1(a) = m_2(a) = m(a)$.

Proof. (1) Due to Theorem 4.11, $A/\mathrm{Ker}_0(m_1) = A/\mathrm{Ker}_0(m_2)$ is a pseudo-BCK subalgebra of $\mathbb{R}^-$. The condition $m_1(a) = m_2(a) > 0$ entails $\hat{m}_1 = \hat{m}_2$ so that $m_1 = m_2$.

(2) Let $m = \lambda m_1 + (1-\lambda)m_2$ where m_1 and m_2 are measures on A such that $m_1(a) = m_2(a) = m(a)$ and $0 < \lambda < 1$. Then $\mathrm{Ker}_0(m) \subseteq \mathrm{Ker}_0(m_1) \cap \mathrm{Ker}_0(m_2)$. The maximality of $\mathrm{Ker}_0(m)$ entails that both $\mathrm{Ker}_0(m_1)$ and $\mathrm{Ker}_2(m)$ are maximal ideals and by Theorem 4.11, we see that m_1 and m_2 are measure-morphisms on A. The condition $m_1(a) = m_2(a) = m(a)$ yields by (1) that $m = m_1 = m_2$. $\square$

Proposition 4.13. *Let m be a state-measure on a good bounded pseudo-BCK algebra. Then $M = 1 - m$ is a Riečan state.*

Proof. Let x, y be a pair of orthogonal elements, that is $y^{-\sim} \leq x^-$ and using the fact that m is a measure, we obtain: $m(x^- \to y^{-\sim}) = m(x^- \rightsquigarrow y^{-\sim}) = m(y^{-\sim}) - m(x^-)$. Now, because A is good we get: $m(x^- \to y^{\sim -}) = m(y) - 1 + m(x)$, which implies $M(x \oplus y) = M(x) + M(y)$. Thus, M is a Riečan state. $\square$

Proposition 4.14. *Let A be a bounded pseudo-BCK(pDN) algebra and s a Riečan state on A. Then $S = 1 - s$ is a state-measure.*

Proof. Let s be a Riečan state on A.
Consider $y \leq x$. According to [3] we have
$$s(x \to y^{-\sim}) = s(x \rightsquigarrow y^{\sim -}) = 1 - s(x) + s(y).$$
Taking into consideration the (pDN) condition we get
$$s(x \to y) = s(x \rightsquigarrow y) = 1 - s(x) + s(y).$$
It follows that $S(x \to y) = S(x \rightsquigarrow y) = S(y) - S(x)$.
Moreover, we have $S(0) = 1$, so S is a state-measure on A. $\square$

Remark 4.15. We can also define a measure as a map $m : A \longrightarrow (-\infty, 0]$ such that
$$m(x \to y) = m(x \rightsquigarrow y) = m(x) - m(y) \text{ whenever } y \leq x.$$
Properties (2) of Proposition 4.2 and (1) in Proposition 4.3 become:
$(2')$ $m(x) \leq m(y)$ whenever $x \leq y$ and m is a measure on A;
$(1')$ $y \leq x$ implies $m((x \to y) \rightsquigarrow y) = m((x \rightsquigarrow y) \to y) = -m(x)$ whenever m is a measure on A;
If $m(0) = 0$ then m is a state on A.
Proposition 4.5 will be modified such that $m = 1 + M$.
Consider again the bounded pseudo-BCK lattice A from Example 2.16. The function $m : A \longrightarrow (-\infty, 0]$ defined by: $m(0) = -1, m(a) = m(b) = m(c) = m(d) = m(1) = 0$ is the unique measure on A.

Remark 4.16. If a pseudo-BCK algebra is defined on the negative cone, like in Examples 2.3 and 2.4, we map through the negative cone to the positive cone in $\mathbb{R}$. According to the second kind of definition, we map the negative cone to negative numbers.

5. Pseudo-BCK Algebras with Strong Unit

In the present section, we will study state-measures on pseudo-BCK algebras with strong unit. We apply the results of the previous section to show how to characterize state-measure-morphisms as extremal state-measures or as those with the maximal filter. In particular, we show that for unital pseudo-BCK algebras that are downwards directed, the quotient over the kernel can be embedded into the negative cone of an abelian, archimedean ℓ-group with strong unit.

According to [12], we are saying that an element u of a pseudo-BCK algebra A is a *strong unit* if, for the filter $(F9u)$ of A that is generated by u, we have $F(u) = A$. For example, if $(A, \to, \rightsquigarrow, 0, 1)$ is a bounded pseudo-BCK algebra, then $u = 0$ is a strong element. If G is an ℓ-group with strong unit $u \geq 0$, then the negative cone G^- is an unbounded pseudo-BCK algebra with strong unit $-u$.

Remark 5.1. We note that a filter F of a pseudo-BCK algebra with a strong unit u is a proper subset of A if and only if $u \notin F$.

By a *unital pseudo-BCK algebra* we mean a couple (A, u) where A is a pseudo-BCK algebra with a fixed strong unit u. We say that a measure m on (A, u) is a *state-measure* if $m(u) = 1$. If, in addition, m is a measure-morphism such that $m(u) = 1$, we call it also a *state-measure-morphism*. We denote by $\mathcal{SM}(A, u)$ and $\mathcal{SMM}(A, u)$ the set of all state-measures and state-measure-morphisms on (A, u), respectively. The set of $\mathcal{SM}(A, u)$ is convex, i.e. if $m_1, m_2 \in \mathcal{SM}(A, u)$ and $\lambda \in [0, 1]$, then $m = \lambda m_1 + (1 - \lambda)m_2 \in \mathcal{SM}(A, u)$; it could be empty. A state-measure m is *extremal* if $m = \lambda m_1 + (1 - \lambda)m_2$ for $\lambda \in (0, 1)$ yields $m = m_1 = m_2$. We denote by $\partial_e \mathcal{SM}(A, u)$ the set of all extremal state-measures on (A, u).

Example 5.2. Let G be an ℓ-group with strong unit $u \geq 0$, i.e., given $g \in G$ there is an integer $n \geq 1$ such that $g \leq nu$. Then a mapping m on G^- is a state-measure on $(G^-, -u)$ if and only if (i) $m: \ G^- \to [0, \infty)$, (ii) $m(g + h) = m(g) + m(h)$ for $g, h \in G^-$, and (iii) $m(-u) = 1$. A state-measure m is extremal if and only if $m(g \wedge h) = \max\{m(g), m(h)\}$, $g, h \in G^-$, see [8, Prop. 4.7]. In addition, $(-u) \to^n g = (g + nu) \wedge 0$ for any $n \geq 1$.

Example 5.3. Let $\Omega \neq \emptyset$ be a compact Hausdorff topological space and let $\mathrm{C}(\Omega)$ be the set of all continuous functions on Ω. Then $\mathrm{C}(\Omega)$ is an ℓ-group with respect to the pointwise ordering and usual addition of functions and the element $u = 1$, the constant function equals 1, is a strong unit. According to Riesz Representation Theorem, see e.g. [19, p. 87], a mapping $m : A \to [0, \infty)$ is a state-measure on $(\mathrm{C}(\Omega)^-, -1)$ if and only if there is a Borel probability measure μ on $\mathcal{B}(\Omega)$ such that

$$m(f) = -\int_\Omega f(x)\, \mathrm{d}\mu(x), \quad f \in \mathrm{C}(\Omega)^-, \tag{5.1}$$

and vice-versa, given a Borel probability measure μ, the integral (5.1) defines always a state-measure. A state-measure is extremal if and only if it is a state-measure-morphism if and only if $\mu = \delta_x$ for some point $x \in \Omega$, where $\delta_x(M) = 1$ iff $x \in M$ otherwise $\delta_x(M) = 0$, then $m(f) = f(x)$.

We say that a net of state-measures $\{m_\alpha\}$ *converges weakly* to a state-measure m if $m(a) = \lim_\alpha m_\alpha(a)$ for every $a \in A$.

Proposition 5.4. *The state spaces $\mathcal{SM}(A,u)$ and $\mathcal{SMM}(A,u)$ are compact Hausdorff topological spaces.*

Proof. If $\mathcal{SM}(A,u)$ is void, the statement is evident. Thus suppose that (A,u) admits at least one state-measure. For any state-measure m and any $x \in A$ we have by (2.1): $m(u \to x) = m(u \vee_1 x \to x) = m(x) - m(u \vee_1 x)$. But $u \le u \vee_1 x$, hence $m(u \vee_1 x) \le m(u) = 1$ so that $m(u \to x) \ge m(x) - 1$ and

$$m(x) \le m(u \to x) + 1.$$

Therefore,

$$m(x) \le m(u \to x) + 1 \le m(u \to^2 x) + 2 \le \cdots \le m(u \to^{n-1} x) + n - 1.$$

Since u is strong, given $x \in A$, let n_x denote an integer $n_x \ge 1$ such that $u \to^{n_x} x = 1$. Then $u \le (u \to^{n_x - 1} x)$ and $m(u \to^{n_x-1} x) \le m(u) = 1$. Consequently, $m(x) \le m(u \to^{n_x-1} x) + n_x - 1 \le n_x$. Hence, $\mathcal{SM}(A,u) \subseteq \prod_{x\in A}[0,n_x]$. By Tychonoff's Theorem, the product of closed intervals is compact. The set of state-measures $\mathcal{SMM}(A,u)$ can be expressed as an intersection of closed subsets of $[0,\infty)^A$, namely of the following sets (for $x, y \in A$)

$$M_{x,y} = \{m \in [0,\infty)^A :\ m(x \to y) = m(x \rightsquigarrow y) = m(y) - m(x)\},\ x \le y,$$
$$M_x = \{m \in [0,\infty)^A : m(x) \ge 0\}, \quad \{m \in [0,\infty)^A : m(u) = 1\}.$$

Therefore, $\mathcal{SM}(A,u)$ is a closed subset of the given product of intervals, and hence, it is compact.

Similarly, the set of state-measure-morphisms $\mathcal{SMM}(A,u)$ is a subset of $\prod_{x\in A}[0,n_x]$ and it can be expressed as an intersection of closed subsets of $[0,\infty)^A$, namely of the following sets (for $x, y \in A$)

$$M_{x,y} = \{m \in [0,\infty)^A :\ m(x \to y) = m(x \rightsquigarrow y) = \max\{0, m(y) - m(x)\}\},$$
$$M_x = \{m \in [0,\infty)^A : m(x) \ge 0\}, \quad \{m \in [0,\infty)^A : m(u) = 1\}.$$

Therefore, $\mathcal{SMM}(A,u)$ is a closed subset of the given product of intervals, and hence, it is compact. $\square$

Proposition 5.5. *Let u be a strong unit of a pseudo-BCK algebra A and m let be a measure on A. Then m vanishes on A if and only if $m(u) = 0$.*

Proof. Assume $m(u) = 0$. Then $m(u \to x) = m(u \vee_1 x \to x) = m(x) - m(u \vee_1 x)$. But $u \le u \vee_1 x$, hence $0 \le m(u \vee_1 x) \le m(u) = 0$ so that $m(x) = m(u \to x)$ and

$$m(x) = m(u \to x) = m(u \to^2 x) = \cdots = (m \to^n x) = m(1) = 0$$

when $u \to^n x = 1$ for some integer $n \ge 1$.

If now $m(u) > 0$, then m does not vanish trivially on A.

$\square$

Lemma 5.6. *Let m_1, m_2 be state-measure-morphisms on a unital pseudo-BCK algebra (A,u). If $\mathrm{Ker}_0(m_1) = \mathrm{Ker}_0(m_2)$, then $m_1 = m_2$.*

In addition, any state-measure-morphism cannot be expressed as a convex combination of other state-measure-morphisms.

Proof. The set $m_1(A) = \{m_1(a) \mid a \in A\}$ and $m_2(A) = \{m_2(a) \mid a \in A\}$ of real numbers can be endowed with a total operation $*_{\mathbb{R}}$ such $(m_1(A), *_{\mathbb{R}}, 0)$ and $(m_2(A), *_{\mathbb{R}}, 0)$ is a subalgebra of the BCK algebra $([0,\infty), *_{\mathbb{R}}, 0)$ in the sense of [12, Chap 5], where $s *_{\mathbb{R}} t = \max\{0, s-t\}$, $s,t \in [0,\infty)$. And the number 1 is a strong unit in all such algebras (for definition see [12]).

If we set $\hat{m}_1$ and $\hat{m}_2$ the state-measure-morphisms on the quotient pseudo-BCK algebras $A/\mathrm{Ker}_0(m_1)$ and $A/\mathrm{Ker}_0(m_2)$ defined by $\hat{m}_i(a/\mathrm{Ker}_0(m_i)) = m_i(a)$, we have again $\hat{m}_i(A/\mathrm{Ker}_0(m_i)) = m_i(A)$ for $i = 1, 2$.

Define a mapping $\phi:\ m_1(A) \to m_2(A)$ by $\phi(m_1(a)) = m_2(a)$ $(a \in A)$. It is possible to show that this is a BCK algebra injective homomorphism. Due to [12, Lem 6.1.22], this means that $m_1(A) = m_2(A)$ and $m_1(a) = m_2(a)$ for all $a \in A$.

Suppose now that $m = \lambda m_1 + (1-\lambda) m_2$, where m, m_1, m_2 are state-measure-morphisms and $\lambda \in (0,1)$. Then $\mathrm{Ker}_0(m) \subseteq \mathrm{Ker}_0(m_1) \cap \mathrm{Ker}_0(m_2)$. Due to Proposition 4.10, all kernels $\mathrm{Ker}_0(m), \mathrm{Ker}_0(m_1), \mathrm{Ker}_0(m_2)$ are maximal filters so that $\mathrm{Ker}_0(m) = \mathrm{Ker}_0(m_1) = \mathrm{Ker}_0(m_2)$ and by the first part of the present proof, $m = m_1 = m_2$. $\square$

Proposition 5.7. *Let u be a strong unit of a pseudo-BCK algebra A and let J be a filter of A, $J_0 := J \cap [u,1]$. Then J_0 is a filter of the pseudo BCK-algebra $([u,1], \leq, \to, \rightsquigarrow, u, 1)$. If $F(J_0)$ is the filter of A generated by J_0, then*

$$F(J_0) = F. \tag{5.2}$$

Moreover, J_0 is maximal in $[u,1]$ if and only if so is J in A.

Proof. Suppose that J is a filter of A. Then $J_0 := J \cap [u,1]$ is evidently a filter of $[u,1]$.

It is clear that $F(J_0) \subseteq F$.

On the other hand, take $x \in J$. Since u is a strong unit, by (2.3), there is an integer $n \geq 1$ such that $u \to^n x = 1 = u \to (\cdots \to (u \to x)\cdots)$. Set $x_n = u \vee_1 x$ and $x_{n-i} = u \vee_1 (u \to^i x)$ for $i = 1, \ldots, n-1$. An easy calculus shows that $x_i \in J_0$ for any $i = 1, \ldots, n$. Moreover, $u \to (u \to (\cdots \to (u \to x)\cdots)) = x_1 \to (x_2 \to (\cdots \to (x_n \to x)\cdots)) = 1$ which by (2.2) proves $x \in F(J_0)$.

Let now J be a maximal filter of A. Assume that F is a filter of $[u,1]$ containing J_0 with $F \neq [u,1]$, and let $\hat{F}(F)$ be the filter of A generated by F. Then $F \subseteq \hat{F}(F) \cap [u,1]$. If now $x \in \hat{F}(F) \cap [u,1]$, there are $f_1, \ldots, f_n \in F$ such that $f_1 \to (\cdots \to (f_n \to x)\cdots) = 1$ giving $x \in F$. Hence, $F = \hat{F}(F) \cap [u,1]$.

We assert that $\hat{F}(F)$ is a filter of A containing J, and $\hat{F}(F) \neq A$. If not, then $u \in \hat{F}(F)$ and therefore by (2.2), there are $x_1, \ldots, x_n \in F$ such that $x_1 \to (\cdots \to (x_n \to u)\cdots) = 1$. If we set $z_n = x_n \vee_1 u$ and $z_{n-i} = x_{n-i} \vee_1 (x_i \to (\cdots \to (x_n \to u)\cdots))$, for $i = 1, \ldots, n-1$, then each z_i belongs to F and $z_1 \to (\cdots \to (z_n \to u)\cdots) = 1$ which implies $u \in F$ that is a contradiction.

The maximality of J entails $J = \hat{F}(F)$. Since $J_0 \subseteq F = \hat{F}(F) \cap [u,1] = J \cap [0,1] = J_0$. That is, J_0 a maximal filter of $[u,1]$ as it was claimed.

Assume now that J_0 is a maximal filter of $[u,1]$ and let $G \neq A$ be a filter of A containing J. Then $G_0 := G \cap [u,1]$ is a filter of $[u,1]$ containing J_0, and by (5.2), $G = F(G_0)$. We assert $u \notin G_0$. Suppose the converse. Then $x \in G$ and for any $x \in A$, there is an integer $n \geq 1$ such that $u \to^n x = u \to (\cdots (u \to x)\cdots) = 1$ proving $x \in G$, so that $A \subseteq G$ that is absurd.

The maximality of J_0 entails $J_0 = G_0$ and in view of (5.2), we have $J = F(J_0) = F(G_0) = G$, thus J is a maximal filter of A. $\square$

Proposition 5.8. *Let m be a state-measure on a unital pseudo-BCK algebra (A,u), and let m_u be the restriction of m onto the interval $[u,1]$. Then m_u is a state-measure-morphism on $([u,1],\le,\to,\rightsquigarrow)$.*

Let us have the following conditions:
(a) m is a state-measure-morphism on (A,u).
(b) m_u is a state-morphism on $[u,1]$.
(c) $\mathrm{Ker}_0(m)$ is a maximal filter of (A,u).
(d) $\mathrm{Ker}_0(m_u)$ is a maximal filter of $[u,1]$.

Then $(b),(c),(d)$ are mutually equivalent and (a) implies each of the conditions $(b),(c)$, and (d).

Proof. Let m_u be the restriction of m onto $[u,1]$. Then m_u is a state-measure on $[u,1]$ and we define $\mathrm{Ker}_0(m_u)$ and $\mathrm{Ker}_0(m)$. Then $\mathrm{Ker}_0(m_u)=\mathrm{Ker}_0(m)\cap[u,1]$ and due to (5.2) we have

$$F(\mathrm{Ker}_0(m_u))=\mathrm{Ker}_0(m). \tag{5.3}$$

$(a)\Rightarrow(b)$. It is evident.

$(b)\Leftrightarrow(d)$: It follows from Theorem 3.26(d)-(e).

$(b)\Leftrightarrow(c)$: We have $\mathrm{Ker}_0(m_u)=\mathrm{Ker}_0(m)\cap[u,1]$. Then by Proposition 5.7, we have the equivalence in question. ☐

Theorem 5.9. *Let (A,u) be an unbounded pseudo-BCK algebra that is downwards-directed and let m be a state-measure on (A,u). Then there is a unique (up to isomorphism) abelian and archimedean ℓ-group G with strong unit $u_G>0$ such that the unbounded unital pseudo-BCK algebra $(A/\mathrm{Ker}_0(m),u/\mathrm{Ker}_0(m))$ is isomorphic with the unbounded unital pseudo-BCK algebra $(G^-,-u_G)$.*

Proof. We define m_u, the restriction of m onto the interval $[u,1]$. Due to Theorem 4.7 and Theorem 3.20, the quotient $[u,1]/\mathrm{Ker}_0(m_u)$ can be converted into an MV-algebra, and in view of $\mathrm{Ker}_0(m_u)=\mathrm{Ker}_0(m)\cap[u,1]$ we have that $[u,1]/\mathrm{Ker}_0(m_u)$ is isomorphic with $[u/\mathrm{Ker}_0(m),1/\mathrm{Ker}_0(m)]=[u,1]/\mathrm{Ker}_0(m)$, so that both can be viewed as isomorphic MV-algebras. Let G be an ℓ-group guaranteed by Theorem 4.9 that is generated by $A/\mathrm{Ker}_0(m)$. Therefore, $u_G:=-(u/\mathrm{Ker}_0(m))$ is a strong unit for G and $0_G:=1/\mathrm{Ker}_0(m)$ is the neutral element of G. By the Mundici famous theorem [2], the unital ℓ-group (G,u_G) is the same for $[u,1]/\mathrm{Ker}_0(m_u)$ and $[u,1]/\mathrm{Ker}_0(m)$. If now $g\in G^-$, then $g=g_1+\cdots+g_n$, where $g_1,\dots,g_n\in[u,1]/\mathrm{Ker}_0(m)$. The set of elements $g\in G^-$ such that $g\in A/\mathrm{Ker}_0(m)$ is a pseudo-BCK algebra containing $A/\mathrm{Ker}_0(m)$. And because $A/\mathrm{Ker}_0(m)$ generates G, this implies that the pseudo-BCK algebra $(G^-,-u_G)$ is isomorphic with the unital pseudo-BCK algebra $(A/\mathrm{Ker}_0(m),u/\mathrm{Ker}_0(m))$. ☐

Theorem 5.10. *Let m be a state-measure on a unital pseudo-BCK algebra (A,u) that is downwards-directed and let m_u be the restriction of m onto the pseudo-BCK algebra $[u,1]$. The following statements are equivalent:*
(a) m is a state-measure-morphism on (A,u).
(b) m_u is a state-morphism on $[u,1]$.
(c) $\mathrm{Ker}_0(m)$ is a maximal filter of (A,u).
(d) $\mathrm{Ker}_0(m_u)$ is a maximal filter of $[u,1]$.
(e) m is an extremal state-measure on (A,u).
(f) m_u is an extremal state-measure on $[u,1]$.

Proof. By Theorem 5.8, $(b),(c),(d)$ are mutually equivalent and (a) implies each of the conditions $(b),(c),(d)$. Theorem 4.11 entails that (c) implies (a). From Theorem 3.26 we see that (b) and (f) are equivalent. Proposition 4.12 gives (a) implies (e).

$(e) \Rightarrow (a)$. Let m an extremal state-measure on (A,u). Define $\mathrm{Ker}_0(m)$, $A/\mathrm{Ker}_0(m)$, and $\hat{m}(a/\mathrm{Ker}_0(m)) := m(a)$ $(a \in A)$. We assert that $\hat{m}$ is extremal on the unital pseudo-BCK algebra $(A/\mathrm{Ker}_0(m), u/\mathrm{Ker}_0(m))$. Indeed, if $\hat{m} = \lambda\mu_1 + (1-\lambda)\mu_2$, $0<\lambda<1$, where μ_1 and μ_2 are two state-measures on $(A/\mathrm{Ker}_0(m), u/\mathrm{Ker}_0(m))$, then there are two state-measures m_1, m_2 on (A,u) such that $\hat{m}_1 = \mu_1$ and $\hat{m}_2 = \mu_2$. Hence, $m = \lambda m_1 + (1-\lambda)m_2$ yielding $m_1 = m_2$ and $\mu_1 = \mu_2$.

Due to Theorem 5.9, $A/\mathrm{Ker}_0(m)$ is isomorphic with the pseudo-BCK algebra (G^-, u_G), where G^- is the negative cone of an abelian and archimedean ℓ-group G that is generated by $A/\mathrm{Ker}_0(m)$ and the element $u_G := -(u/\mathrm{Ker}_0(m))$ is a strong unit for G.

Similarly as in the proof of Theorem 4.11, $\hat{m}$ can be extended to a state-measure on $(G^-, u/\mathrm{Ker}_0(m))$ so that s can be extended to an additive function s on the whole unital ℓ-group $(G, -(u/\mathrm{Ker}_0(m)))$ that is positive on G^- and $s(u/\mathrm{Ker}_0(m)) = 1$. Moreover, s is extremal on $(G, -(u/\mathrm{Ker}_0(m)))$ which by [19, Thm 12.18] is possible if and only if $\mathrm{Ker}_0(\hat{m}) = \{1/\mathrm{Ker}_0(m)\}$ is a maximal filter of the unital pseudo-BCK algebra $(A/\mathrm{Ker}_0(m), u/\mathrm{Ker}_0(m))$. Since the mapping $a \mapsto a/\mathrm{Ker}_0(m)$ is surjective, we have that this implies that $\mathrm{Ker}_0(m)$ is a maximal filter of (A,u). By the equivalence of (c) and (a) we have that m is a measure-morphism. □

As a direct consequence of Theorem 5.10 and the Krein–Mil'man theorem we have:

Corollary 5.11. *Let (A,u) be a unital pseudo-BCK algebra that is downwards-directed. Then*

$$\partial_e \mathcal{SM}(A,u) = \mathcal{SMM}(A,u) \tag{5.4}$$

and every state-measure on (A,u) is a weak limit of a net of convex combinations of states-measure-morphisms.

6. Coherence, de Finetti Maps and Borel States

In this section, we will generalize to pseudo-BCK algebras the identity between de Finetti maps and Bosbach states, following the results proved by Kühr and Mundici in [28] who showed that de Finetti's coherence principle that has an origin in Dutch book making, has a strong relationship with MV-states on MV-algebras. Then we generalize this also for state-measures on unital pseudo-BCK algebras that are downwards-directed.

Finally we present some open questions.

We recall the following definition and notations used in [28]. Let A be a nonempty set, $[0,1]^A$ the set of all functions $V : A \to [0,1]$ endowed with the product topology. If $\mathcal{X} \subseteq [0,1]^A$, by $\mathrm{conv}\mathcal{X}$, $\mathrm{cl}\mathcal{X}$ we denote the convex hull and respectively the closure of $\mathcal{X}$. Also, if $\mathcal{X}$ is convex, $\partial_e\mathcal{X}$ will denote the set of all extremal points of $\mathcal{X}$. We note that the weak topology of Bosbach states is in fact the relativized product topology on $[0,1]^A$.

Definition 6.1. ([28]) Let $A' = \{a_1, a_2, \ldots, a_n\}$ be a finite subset of A. Then a map $\beta : A' \longrightarrow [0,1]$ is said to be *coherent* over A' if,

$$\text{for all } \sigma_1, \sigma_2, \ldots, \sigma_n \in \mathbb{R}, \text{ there is } V \in \mathcal{W} \text{ s.t. } \sum_{i=1}^{n} \sigma_i(\beta(a_i) - V(a_i)) \geq 0. \tag{6.1}$$

By a *de Finetti map* on A we mean a function $\beta : A \longrightarrow [0,1]$ which is coherent over every finite subset of A. We denote by $\mathcal{F}_{\mathcal{W}}$ the set of all de Finetti maps on A.

An interpretation of (6.1) is as follows, [28]: Two players, the bookmaker and the bettor, wager money on the possible occurrence of elementary events $a_1, \dots, a_n \in M$. The bookmaker sets a betting odd $\beta(a_i) \in [0,1]$, and the bettor chooses stakes $\sigma_i \in \mathbb{R}$. The bettor pays the bookmaker $\sigma_i \beta(a_i)$, and will receive $\sigma_i V(a_i)$ from the bookmaker's possible world V. As scholars, we can assume that σ_i could be also positive as well as negative. If the orientation of money transfer is given via bettor-to-bookmaker, then (6.1) means that bookmaker's book should be *coherent* in the sense that the bettor cannot choose stakes $\sigma_1, \dots, \sigma_n$ ensuring him to win money in every $V \in \mathcal{W}$.

Now let A be a bounded pseudo-BCK algebra and denote by $\mathcal{BS}(A)$ the set of Bosbach states on A and by $\mathcal{W}$ the set of state-morphisms on A. Note that, according to Theorem 3.26, $\mathcal{W}$ coincides with the set of extremal Bosbach states, and due to the Krein-Milman'theorem,

$$\mathcal{BS}(A) = \text{cl conv}\partial_e \mathcal{BS}(A) = \text{cl conv}\mathcal{SM}(A). \tag{6.2}$$

Theorem 6.2. *Let A be a bounded pseudo-BCK algebra and let $\mathcal{W} = \mathcal{SM}(A) \neq \emptyset$. Then*

$$\mathcal{F}_{\mathcal{W}} = \mathcal{BS}(A).$$

Proof. According to Theorem 3.26 and (3.1), $\mathcal{W}$ is closed. Now we can apply [28, Prop 3.1] since we have $\mathcal{W} \subseteq \mathcal{S}, \partial_e \mathcal{S} = \mathcal{W} (\subseteq \mathcal{W})$, $\mathcal{W}$ closed. So we get $\mathcal{S} = \mathcal{F}_{\mathcal{W}}$. $\square$

Theorem 6.2 has an important consequence, namely that every Bosbach state (if it exists) on a bounded pseudo-BCK algebra is a de Finetti map coming from the set of $[0,1]$-valued functions on A generated by the set of state-morphisms, and applying (6.2) we have that this de Finetti maps is exactly the weak limit of a net of convex combinations of state-morphisms.

There is also another relationship concerning the representability of Bosbach states via integrals. We introduce the following notions, see e.g. [19, Sec 5]. Let Ω be a nonempty compact Hausdorff topological space. Let $\mathcal{B}(\Omega)$ be the Borel σ-algebra of Ω generated by all open subsets of Ω and any elements of $\mathcal{B}(\Omega)$ is said to be a *Borel set*, and any σ-additive (signed) measure is said to be a *Borel measure*.

Let $\mathcal{P}(\Omega)$ denote all probability measures, that is, all positive regular Borel measures $\mu \in \mathcal{M}(\Omega)$ such that $\mu(\Omega) = 1$. We recall that a Borel measure μ is called regular if

$$\inf\{\mu(O) :\ Y \subseteq O,\ O \text{ open}\} = \mu(Y) = \sup\{\mu(C) :\ C \subseteq Y,\ C \text{ closed}\}$$

for any $Y \in \mathcal{B}(\Omega)$.

Let now A be a bounded pseudo-BCK algebra and let $\mathcal{W} = \mathcal{SM}(A)$. Every element $a \in A$ determines a (continuous) function $f_a :\ \mathcal{W} \to [0,1]$ via

$$f_a(V) = V(a), \quad V \in \mathcal{W}.$$

We say that a mapping $s :\ A \to [0,1]$ is a *Borel state* (of $\mathcal{W}$) if there is a (σ-additive) probability measure μ defined on the Borel σ-algebra of the topological space $\mathcal{W}$ generated by all open subsets of $\mathcal{W}$ such that

$$s(a) = \int_{\mathcal{W}} f_a(V) \, \mathrm{d}\mu(V).$$

Let $\mathcal{B}_{\mathcal{W}}$ be the set of all Borel states of $\mathcal{W}$.

Theorem 6.3. *Let A be a bounded pseudo-BCK algebra. For any Bosbach state s on A there is a Borel probability measure μ on $\mathcal{B}(\mathcal{W})$ such that*

$$s(a)=\int_{\mathcal{W}} f_a(V)\,\mathrm{d}\mu(V). \tag{6.3}$$

Proof. Since $\mathcal{W}=\mathcal{SM}(A)$ is closed, see (3.1), by [28, Thm 4.2] we have $\mathcal{W}\subseteq\mathcal{B}_{\mathcal{W}}$, $\partial_e\mathcal{B}_{\mathcal{W}}\subseteq\mathcal{W}$ and $\mathcal{F}_{\mathcal{W}}=\mathcal{B}_{\mathcal{W}}$. Therefore, by Theorem 6.2, $\mathcal{BS}(A)=\mathcal{B}_{\mathcal{W}}$, i.e. every Bosbach state is a Borel state on A. $\square$

We note that if we set $\Omega=\mathcal{BS}(A)$, then for any $a\in A$, the function $\tilde a:\ \mathcal{BS}(A)\to[0,1]$ defined by $\tilde a(s)=s(a)$, $s\in\mathcal{SB}(A)$, is continuous. Therefore, we can strength Theorem 6.3 as follows.

Theorem 6.4. *Let A be a bounded pseudo-BCK algebra. For any Bosbach state s on A there is a unique Borel probability measure μ on $\mathcal{B}(\mathcal{BS}(A))$ such that*

$$s(a)=\int_{\mathcal{SM}(A)}\tilde a(x)\,\mathrm{d}\mu(x). \tag{6.4}$$

Proof. Suppose that the set of all Bosbach states on A is non empty. Due to the Krein-Mil'man theorem, (6.2), the set of extremal Bosbach state is also nonempty and it coincides with the set of state-morphisms. Denote by $F_0:=\bigcap\{\mathrm{Ker}(s):\ s\in\mathcal{SM}(A)\}$. In view of Propositions 3.20–3.22, F_0 is a normal ideal, and similarly as in Theorem 3.20, we can show that A/F_0 is an archimedean MV-algebra, and for any Bosbach state s on A, the mapping $\hat s(a/F_0)=s(a)$ $(a\in A)$ is an MV-state (= Bosbach state) on A/F_0; we set $\bar a:=a/F_0$ $(a\in A)$. Moreover, the state spaces $\mathcal{BS}(A)$ and $\mathcal{BS}(A/F_0)$ are affinely homeomorphic compact nonempty Hausdorff topological spaces under the mapping $s\in\mathcal{BS}(A)\mapsto\hat s\in\mathcal{SB}(A/F_0)$ (i.e. they are homeomorphic in the weak topologies of states preserving convex combinations of states). In addition, the compact subsets of extremal Bosbach space are also homeomorphic under this mapping. Due to [25], on the Borel σ-algebra $\mathcal{B}(\mathcal{BS}(A))$, there is a unique Borel probability measure μ such that

$$s(a)=\hat s(a/F_0)=\int_{\mathcal{SM}(A/F_0)}\tilde{\bar a}\,\mathrm{d}\mu.$$

This integral can be rewritten identifying the compact spaces and Borel σ-algebras into the form

$$s(a)=\int_{\mathcal{SM}(A)}\tilde a(x)\,\mathrm{d}\mu(x).$$

$\square$

It is interesting to note that de Finetti was a great propagator only of probabilities as finitely additive measures. The result of [25] and formula (6.4) say that whenever s is a Bosbach state, it generates a σ-additive probability such that s is in fact an integral over this Borel probability measure. Thus formula (6.4) joins de Finetti's "finitely additive probabilities" with σ-additive measures on an appropriate Borel σ-algebra.

We now generalize Theorem 6.2 and Theorem 6.3 also for unbounded pseudo-BCK algebras that are downwards-directed.

Theorem 6.5. *Let (A,u) be a pseudo-BCK algebra that is downwards-directed and let $\mathcal{W}=\mathcal{SMM}(A,u)\neq\emptyset$. Then*

$$\mathcal{F}_{\mathcal{W}}=\mathcal{SM}(A,u).$$

Proof. It follows from Theorem 5.10 and using the same steps as those in Theorem 6.2. □

Theorem 6.6. *Let (A, u) be a pseudo-BCK algebra that is downwards-directed. For any state-measure m on (A, u), where $\mathcal{W} = \mathcal{SMM}(A, u) \neq \emptyset$, there is a Borel probability measure μ on $\mathcal{B}(\mathcal{W})$ such that*

$$m(a) = \int_{\mathcal{W}} f_a(V)\,\mathrm{d}\mu(V).$$

Proof. It follows from Theorem 5.10 and it follows the analogous steps as those in Theorem 6.3. □

During our study we have found that the following questions remained open:

(1) Do arrows $\rightarrow /\mathrm{Ker}_0(m)$ and $\rightsquigarrow /\mathrm{Ker}_0(m)$ on $A/\mathrm{Ker}_0(m)$ coincide always in Theorem 4.8 ?

(2) Is Theorem 4.9 true without the assumption on downwards-directness ?

(3) If (A, u) is a unital pseudo-BCK algebra is then A downwards-directed ?

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