Homogeneous orthocomplete effect algebras are covered by MV-algebras

Josef Niederle[a], Jan Paseka[a,*]

[a] *Department of Mathematics and Statistics, Faculty of Science, Masaryk University, Kotlářská 2, CZ-611 37 Brno, Czech Republic*

Abstract

The aim of our paper is twofold. First, we thoroughly study the set of meager elements $\mathrm{M}(E)$ and the set of hypermeager elements $\mathrm{HM}(E)$ in the setting of homogeneous effect algebras E. Second, we study the property (W+) and the maximality property introduced by Tkadlec as common generalizations of orthocomplete and lattice effect algebras. We show that every block of an Archimedean homogeneous effect algebra satisfying the property (W+) is lattice ordered. Hence such effect algebras can be covered by ranges of observables. As a corollary, this yields that every block of a homogeneous orthocomplete effect algebra is lattice ordered. Therefore finite homogeneous effect algebras are covered by MV-algebras.

Keywords: homogeneous effect algebra, orthocomplete effect algebra, lattice effect algebra, center, atom, sharp element, meager element, hypermeager element, ultrameager element

Introduction

The history of quantum structures started at the beginning of the 20th century. Observable events constitute a Boolean algebra in a classical physical system. Because event structures in quantum mechanics cannot be described by Boolean algebras, Birkhoff and von Neumann introduced orthomodular lattices which were considered as the standard quantum logic. Later on, orthoalgebras were introduced as the generalizations of orthomodular posets, which were considered as "sharp" quantum logic.

In the nineties of the twentieth century, two equivalent quantum structures, D-posets and effect algebras were extensively studied, which were considered as "unsharp" generalizations of the structures which arise in quantum mechanics, in particular, of orthomodular lattices and MV-algebras.

*Corresponding author

Email addresses: niederle@math.muni.cz (Josef Niederle), paseka@math.muni.cz (Jan Paseka)

Effect algebras are fundamental in investigations of fuzzy probability theory. In the fuzzy probability frame, the elements of an effect algebra represent fuzzy events which are used to construct fuzzy random variables.

In the present paper, we continue the study of homogeneous effect algebras started in [9]. This class of effect algebras includes orthoalgebras, lattice ordered effect algebras and effect algebras satisfying the Riesz decomposition property.

In [9] it was proved that every homogeneous effect algebra is a union of its blocks, which are defined as maximal sub-effect algebras satisfying the Riesz decomposition property. In [19] Tkadlec introduced the property (W+) as a common generalization of orthocomplete and lattice effect algebras.

Riečanová in [16] proved one of the most important results in the theory of effect algebras that each lattice ordered effect algebra can be covered by MV-subalgebras which form blocks. Dvurečenskij extended in [3] this result for effect algebras with the Riesz interpolation property and with the decomposition-meet property. Pulmannová [15] proved that every homogeneous effect algebra E such that every block B of E satisfies the decomposition-meet property can be covered by MV-algebras.

The aim of our paper is to show that every block of an Archimedean homogeneous effect algebra satisfying the property (W+) is lattice ordered. Hence Archimedean homogeneous effect algebras fulfilling the condition (W+) can be covered by ranges of observables. As a corollary, this yields that every block of a homogeneous orthocomplete effect algebra is lattice ordered. Therefore finite homogeneous effect algebras are covered by MV-algebras which form blocks.

As a by-product of our study we extend the results on sharp and meager elements of [10] into the realm of Archimedean homogeneous effect algebras satisfying the property (W+). We also thoroughly study the set of meager elements $\mathrm{M}(E)$ and the set of hypermeager elements $\mathrm{HM}(E)$ in the setting of homogeneous effect algebras E.

1. Preliminaries and basic facts

Effect algebras were introduced by Foulis and Bennett (see [4]) for modelling unsharp measurements in a Hilbert space. In this case the set $\mathcal{E}(H)$ of effects is the set of all self-adjoint operators A on a Hilbert space H between the null operator 0 and the identity operator 1 and endowed with the partial operation $+$ defined iff $A + B$ is in $\mathcal{E}(H)$, where $+$ is the usual operator sum.

In general form, an effect algebra is in fact a partial algebra with one partial binary operation and two unary operations satisfying the following axioms due to Foulis and Bennett.

The basic reference for the present text is the classic book by Dvurečenskij and Pulmannová [2], where the interested reader can find unexplained terms and notation concerning the subject.

Definition 1.1. [4, 17] A partial algebra $(E; \oplus, 0, 1)$ is called an *effect algebra* if 0, 1 are two distinct elements, called the *zero* and the *unit* element, and $\oplus$ is a partially defined binary operation called the *orthosummation* on E which satisfy the following conditions for any $x, y, z \in E$:

(Ei) $x \oplus y = y \oplus x$ if $x \oplus y$ is defined,

(Eii) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$ if one side is defined,

(Eiii) for every $x \in E$ there exists a unique $y \in E$ such that $x \oplus y = 1$ (we put $x' = y$),

(Eiv) if $1 \oplus x$ is defined then $x = 0$.

$(E; \oplus, 0, 1)$ is called an *orthoalgebra* if $x \oplus x$ exists implies that $x = 0$ (see [5]).

We often denote the effect algebra $(E; \oplus, 0, 1)$ briefly by E. On every effect algebra E a partial order $\leq$ and a partial binary operation $\ominus$ can be introduced as follows:

$$x \leq y \text{ and } y \ominus x = z \text{ iff } x \oplus z \text{ is defined and } x \oplus z = y\,.$$

If E with the defined partial order is a lattice (a complete lattice) then $(E; \oplus, 0, 1)$ is called a *lattice effect algebra* (*a complete lattice effect algebra*).

Mappings from one effect algebra to another one that preserve units and orthosums are called *morphisms of effect algebras*, and bijective morphisms of effect algebras having inverses that are morphisms of effect algebras are called *isomorphisms of effect algebras*.

Definition 1.2. Let E be an effect algebra. Then $Q \subseteq E$ is called a *sub-effect algebra* of E if

(i) $1 \in Q$

(ii) if out of elements $x, y, z \in E$ with $x \oplus y = z$ two are in Q, then $x, y, z \in Q$.

If E is a lattice effect algebra and Q is a sub-lattice and a sub-effect algebra of E, then Q is called a *sub-lattice effect algebra* of E.

Note that a sub-effect algebra Q (sub-lattice effect algebra Q) of an effect algebra E (of a lattice effect algebra E) with inherited operation $\oplus$ is an effect algebra (lattice effect algebra) in its own right.

For an element x of an effect algebra E we write $\mathrm{ord}(x) = \infty$ if $nx = x \oplus x \oplus \cdots \oplus x$ (n-times) exists for every positive integer n and we write $\mathrm{ord}(x) = n_x$ if n_x is the greatest positive integer such that $n_x x$ exists in E. An effect algebra E is *Archimedean* if $\mathrm{ord}(x) < \infty$ for all $x \in E$.

A minimal nonzero element of an effect algebra E is called an *atom* and E is called *atomic* if under every nonzero element of E there is an atom.

Definition 1.3. We say that a finite system $F = (x_k)_{k=1}^n$ of not necessarily different elements of an effect algebra E is *orthogonal* if $x_1 \oplus x_2 \oplus \cdots \oplus x_n$ (written $\bigoplus\limits_{k=1}^{n} x_k$ or $\bigoplus F$) exists in E. Here we define $x_1 \oplus x_2 \oplus \cdots \oplus x_n = (x_1 \oplus x_2 \oplus \cdots \oplus x_{n-1}) \oplus x_n$ supposing that $\bigoplus\limits_{k=1}^{n-1} x_k$ is defined and $(\bigoplus\limits_{k=1}^{n-1} x_k) \oplus x_n$ exists.

We also define $\bigoplus \emptyset = 0$. An arbitrary system $G = (x_\kappa)_{\kappa \in H}$ of not necessarily different elements of E is called *orthogonal* if $\bigoplus K$ exists for every finite $K \subseteq G$. We say that for a orthogonal system $G = (x_\kappa)_{\kappa \in H}$ the element $\bigoplus G$ exists iff $\bigvee\{\bigoplus K \mid K \subseteq G \text{ is finite}\}$ exists in E and then we put $\bigoplus G = \bigvee\{\bigoplus K \mid K \subseteq G$ is finite$\}$. We say that $\bigoplus G$ is the *orthogonal sum* of G and G is *orthosummable*. (Here we write $G_1 \subseteq G$ iff there is $H_1 \subseteq H$ such that $G_1 = (x_\kappa)_{\kappa \in H_1}$). We denote $G^{\oplus} := \{\bigoplus K \mid K \subseteq G \text{ is finite}\}$.

Definition 1.4. E is called *orthocomplete* if every orthogonal system is orthosummable.

Every orthocomplete effect algebra is Archimedean.

Definition 1.5. An element x of an effect algebra E is called

(i) *sharp* if $x \wedge x' = 0$. The set $\mathrm{S}(E) = \{x \in E \mid x \wedge x' = 0\}$ is called a *set of all sharp elements* of E (see [7]).

(ii) *principal*, if $y \oplus z \leq x$ for every $y, z \in E$ such that $y, z \leq x$ and $y \oplus z$ exists.

(iii) *central*, if x and x' are principal and, for every $y \in E$ there are $y_1, y_2 \in E$ such that $y_1 \leq x, y_2 \leq x'$, and $y = y_1 \oplus y_2$ (see [6]). The *center* $\mathrm{C}(E)$ of E is the set of all central elements of E.

If $x \in E$ is a principal element, then x is sharp and the interval $[0, x]$ is an effect algebra with the greatest element x and the partial operation given by restriction of $\oplus$ to $[0, x]$.

Observation 1.6. *Clearly, E is an orthoalgebra if and only if* $\mathrm{S}(E) = E$.

Statement 1.7. [6, Theorem 5.4] *The center $\mathrm{C}(E)$ of an effect algebra E is a sub-effect algebra of E and forms a Boolean algebra. For every central element x of E, $y = (y \wedge x) \oplus (y \wedge x')$ for all $y \in E$. If $x, y \in \mathrm{C}(E)$ are orthogonal, we have $x \vee y = x \oplus y$ and $x \wedge y = 0$.*

Statement 1.8. [11, Lemma 3.1.] *Let E be an effect algebra, $x, y \in E$ and $c, d \in \mathrm{C}(E)$. Then:*

(i) *If $x \oplus y$ exists then $c \wedge (x \oplus y) = (c \wedge x) \oplus (c \wedge y)$.*

(ii) *If $c \oplus d$ exists then $x \wedge (c \oplus d) = (x \wedge c) \oplus (x \wedge d)$.*

Definition 1.9. A subset M of an effect algebra E is called *compatible* (*internally compatible*) if for every finite subset M_F of M there is a finite orthogonal family $(x_1, \ldots, x_n)$ of elements in E (in M) such that for every $m \in M_F$ there is a set $A_F \subseteq \{1, \ldots, n\}$ with $m = \bigoplus_{i \in A_F} x_i$. If $\{x, y\}$ is a compatible set, we write $x \leftrightarrow y$ (see [10, 13]).

Evidently, $x \leftrightarrow y$ iff there are $p, q, r \in E$ such that $x = p \oplus q$, $y = q \oplus r$ and $p \oplus q \oplus r$ exists iff there are $c, d \in E$ such that $d \leq x \leq c$, $d \leq y \leq c$ and $c \ominus x = y \ominus d$. Moreover, if $x \wedge y$ exists then $x \leftrightarrow y$ iff $x \oplus (y \ominus (x \wedge y))$ exists.

Definition 1.10. An effect algebra E satisfies the *Riesz decomposition property* (or RDP) if, for all $u, v_1, v_2 \in E$ such that $u \leq v_1 \oplus v_2$, there are u_1, u_2 such that $u_1 \leq v_1, u_2 \leq v_2$ and $u = u_1 \oplus u_2$.

An effect algebra E is called *homogeneous* if, for all $u, v_1, v_2 \in E$ such that $u \leq v_1 \oplus v_2 \leq u'$, there are u_1, u_2 such that $u_1 \leq v_1$, $u_2 \leq v_2$ and $u = u_1 \oplus u_2$ (see [9]).

An effect algebra E satisfies the *difference-meet property* (or DMP) if, for all $x, y, z \in E$ such that $x \leq y$, $x \wedge z \in E$ and $y \wedge z \in E$, then $(y \ominus x) \wedge z \in E$ (see [3]).

Statement 1.11. [9, Proposition 2.3] *Let E be a homogeneous effect algebra. Let $u, v_1, \dots, v_n \in E$ be such that $v_1 \oplus \dots \oplus v_n$ exists, $u \leq v_1 \oplus \dots \oplus v_n \leq u'$. Then there are $u_1, \dots, u_n$ such that, for all $1 \leq i \leq n$, $u_i \leq v_i$ and $u = u_1 \oplus \dots \oplus u_n$.*

Statement 1.12. [10, Proposition 2]

(i) *Every orthoalgebra is homogeneous.*

(ii) *Every lattice effect algebra is homogeneous.*

(iii) *An effect algebra E has the Riesz decomposition property if and only if E is homogeneous and compatible.*

Let E be a homogeneous effect algebra.

(iv) *A subset B of E is a maximal sub-effect algebra of E with the Riesz decomposition property (such B is called a* block *of E) if and only if B is a maximal internally compatible subset of E containing* 1.

(v) *Every finite compatible subset of E is a subset of some block. This implies that every homogeneous effect algebra is a union of its blocks.*

(vi) $\mathrm{S}(E)$ *is a sub-effect algebra of E.*

(vii) *For every block B,* $\mathrm{C}(B) = \mathrm{S}(E) \cap B$.

(viii) *Let $x \in B$, where B is a block of E. Then $\{y \in E \mid y \leq x$ and $y \leq x'\} \subseteq B$.*

Hence the class of homogeneous effect algebras includes orthoalgebras, effect algebras satisfying the Riesz decomposition property and lattice effect algebras.

Proposition 1.13. *Let E be a homogeneous effect algebra and $v \in E$. The following conditions are equivalent.*

(i) $v \in \mathrm{S}(E)$*;*

(ii) *$y \leq z$ whenever $w, y, z \in E$ such that $v = w \oplus z$, $y \leq w'$ and $y \leq w$.*

Proof. (i) $\Longrightarrow$ (ii) Evidently, there is a block, say B, such that it contains the following orthogonal system $\{y, w \ominus y, z, 1 \ominus v\}$. Hence B contains also w, w' and $v \in \mathrm{C}(B)$. Since $1 = w \oplus w'$ we obtain by Statement 1.8, (ii) that $v = v \wedge_B w \oplus v \wedge_B w' = w \oplus v \wedge_B w'$. Subtracting w we obtain $z = v \wedge_B w'$. Hence $y \leq w \leq v$ and $y \leq w'$ yields that $y \leq z$.
(ii) $\Longrightarrow$ (i) Let $y \in [0, v] \cap [0, v']$. Put $w = v$ and $z = 0$. Immediately, $y \leq 0$. $\square$

An important class of effect algebras was introduced by Gudder in [7] and [8]. Fundamental example is the standard Hilbert spaces effect algebra $\mathcal{E}(\mathcal{H})$.

For an element x of an effect algebra E we denote

$$\begin{array}{lll} \widetilde{x} & = \bigvee_E\{s \in \mathrm{S}(E) \mid s \leq x\} & \text{if it exists and belongs to } \mathrm{S}(E) \\ \widehat{x} & = \bigwedge_E\{s \in \mathrm{S}(E) \mid s \geq x\} & \text{if it exists and belongs to } \mathrm{S}(E). \end{array}$$

Definition 1.14. ([7], [8].) An effect algebra $(E, \oplus, 0, 1)$ is called *sharply dominating* if for every $x \in E$ there exists $\widehat{x}$.

Obviously, $\widehat{x}$ is the smallest sharp element such that $x \leq \widehat{x}$. That is $\widehat{x} \in \mathrm{S}(E)$ and if $y \in \mathrm{S}(E)$ satisfies $x \leq y$ then $\widehat{x} \leq y$.

Recall that the following conditions are equivalent in any effect algebra E.

- E is sharply dominating;
- for every $x \in E$ there exists $\widetilde{x} \in \mathrm{S}(E)$ such that $\widetilde{x} \leq x$ and if $u \in \mathrm{S}(E)$ satisfies $u \leq x$ then $u \leq \widetilde{x}$;
- for every $x \in E$ there exist a smallest sharp element $\widehat{x}$ over x and a greatest sharp element $\widetilde{x}$ below x.

As proved in [1], $\mathrm{S}(E)$ is always a sub-effect algebra in a sharply dominating effect algebra E.

Statement 1.15. [10, Proposition 15] *Let E be a sharply dominating effect algebra. Then every $x \in E$ has a unique decomposition $x = x_S \oplus x_M$, where $x_S \in \mathrm{S}(E)$ and $x_M \in \mathrm{M}(E)$, namely $x = \widetilde{x} \oplus (x \ominus \widetilde{x})$.*

Lemma 1.16. *Let E be a sharply dominating effect algebra and let $x \in E$.*

$$\widehat{x \ominus \widetilde{x}} = \widehat{\widehat{x} \ominus x} = \widehat{x} \ominus \widetilde{x}.$$

Proof. Clearly $x \ominus \widetilde{x} \leq \widehat{x} \ominus \widetilde{x} \in \mathrm{S}(E)$ and $\widehat{x} \ominus x \leq \widehat{x} \ominus \widetilde{x} \in \mathrm{S}(E)$. Therefore $\widehat{x \ominus \widetilde{x}} \leq \widehat{x} \ominus \widetilde{x}$ and $\widehat{\widehat{x} \ominus x} \leq \widehat{x} \ominus \widetilde{x}$. Now, by adding $\widetilde{x}$, we obtain

$$x = \widetilde{x} \oplus (x \ominus \widetilde{x}) \leq \widetilde{x} \oplus \widehat{x \ominus \widetilde{x}} \leq \widehat{x}$$

which yields $\widetilde{x} \oplus \widehat{x \ominus \widetilde{x}} = \widehat{x}$, and similarly

$$\widetilde{x} = \widehat{x} \ominus (\widehat{x} \ominus \widetilde{x}) \leq \widehat{x} \ominus \widehat{\widehat{x} \ominus x} \leq \widehat{x} \ominus (\widehat{x} \ominus x) = x$$

which yields $\widetilde{x} = \widehat{x} \ominus \widehat{\widehat{x} \ominus x}$. ∎

Lemma 1.17. *Let E be a sharply dominating effect algebra and let $x \in E$.*

$$\widehat{x} \ominus x = x' \ominus (\widehat{x})' = x' \ominus \widetilde{(x')}$$

and

$$x \ominus \widetilde{x} = (\widetilde{x})' \ominus x' = \widehat{(x')} \ominus x'$$

Proof. Transparent. ∎

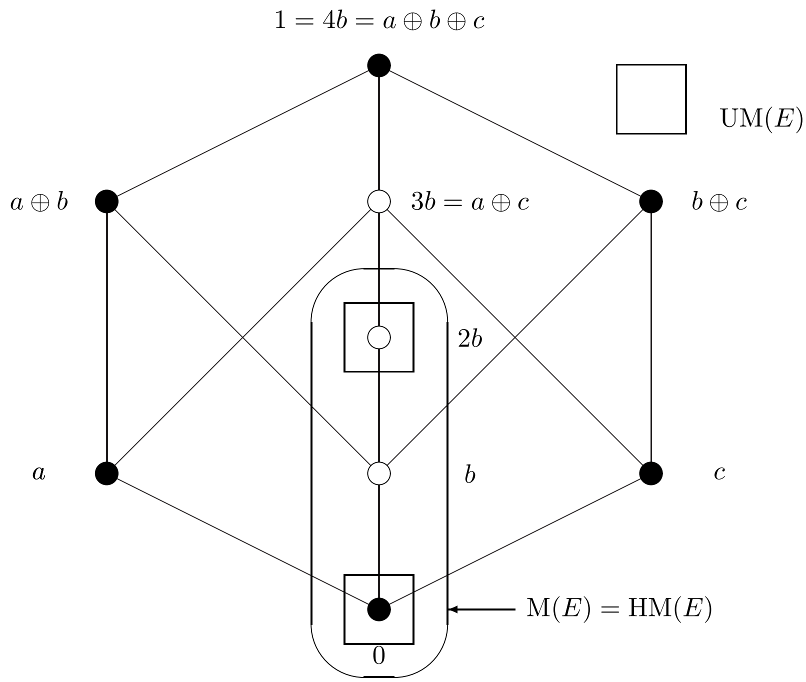


Figure 1: Example 2.2

2. Meager, hypermeager and ultrameager elements

In what follows set (see [10, 18])

$$\mathrm{M}(E) = \{x \in E \mid \text{ if } v \in \mathrm{S}(E) \text{ satisfies } v \leq x \text{ then } v = 0\}.$$

An element $x \in \mathrm{M}(E)$ is called *meager*. Moreover, $x \in \mathrm{M}(E)$ iff $\widetilde{x} = 0$. Recall that $x \in \mathrm{M}(E)$, $y \in E$, $y \leq x$ implies $y \in \mathrm{M}(E)$ and $x \ominus y \in \mathrm{M}(E)$.

We also define

Definition 2.1.

$$\mathrm{HM}(E) = \{x \in E \mid \text{ there is } y \in E \text{ such that } x \leq y \text{ and } x \leq y'\}$$

and

$$\mathrm{UM}(E) = \{x \in E \mid \text{ for every } y \in \mathrm{S}(E) \text{ such that } x \leq y \text{ it holds } x \leq y \ominus x\}.$$

An element $x \in \mathrm{HM}(E)$ is called *hypermeager*, an element $x \in \mathrm{UM}(E)$ is called *ultrameager*.

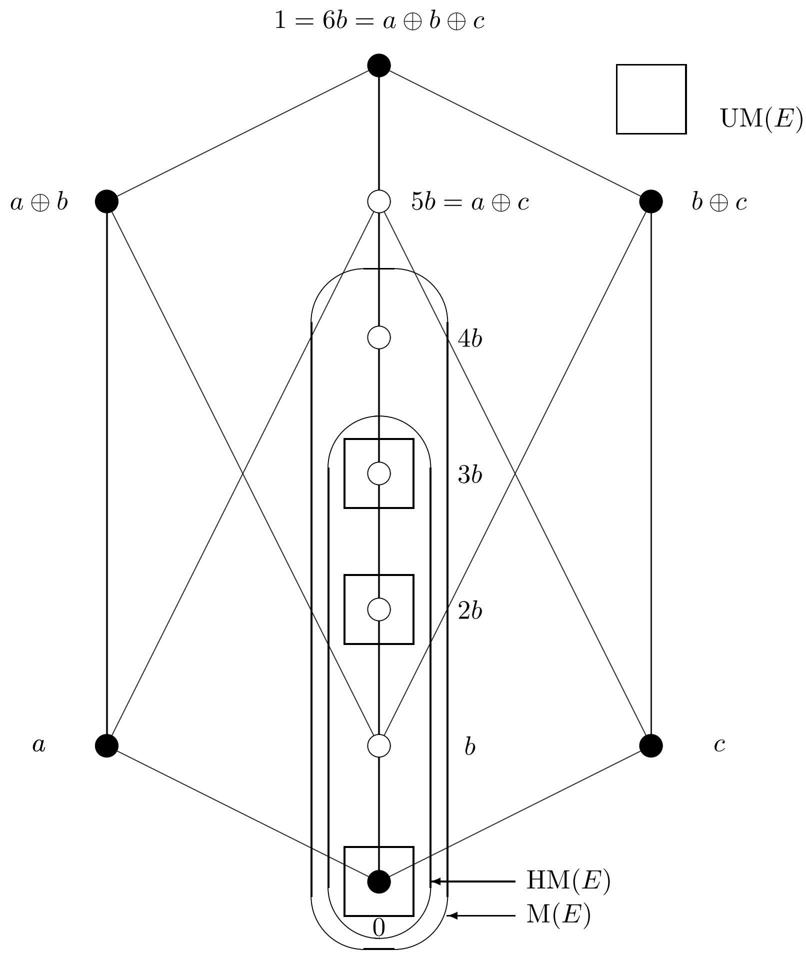


Figure 2: Example 2.3

Example 2.2. In the non-homogeneous non-sharply dominating effect algebra depictured in Figure 1, $\mathrm{M}(E) = \mathrm{HM}(E) \neq \mathrm{UM}(E)$. Sharp elements are denoted in black. One can easily check that E is a sub-effect algebra of the MV-effect algebra $[0,1] \times [0,1]$ such that $a \mapsto (\frac{3}{4}, 0), b \mapsto (\frac{1}{4}, \frac{1}{4}), c \mapsto (0, \frac{3}{4})$. Moreover, since $a \oplus c \notin \mathrm{S}(E)$ we obtain that $\mathrm{S}(E)$ is not a sub-effect algebra of E.

Example 2.3. In the non-homogeneous non-sharply dominating effect algebra depictured in Figure 2, $\mathrm{M}(E) \neq \mathrm{HM}(E) \neq \mathrm{UM}(E)$.

Sharp elements are denoted in black. One can easily check that E is a sub-effect algebra of the MV-effect algebra $[0,1] \times [0,1]$ such that $a \mapsto (\frac{5}{6}, 0), b \mapsto (\frac{1}{6}, \frac{1}{6}), c \mapsto (0, \frac{5}{6})$. Moreover, since $a \oplus c \notin \mathrm{S}(E)$ we obtain that $\mathrm{S}(E)$ is not a sub-effect algebra of E.

Lemma 2.4. *Let E be an effect algebra. Then* $\mathrm{HM}(E) \subseteq \mathrm{M}(E)$*. Moreover, for all $x \in E$, $x \in \mathrm{HM}(E)$ iff $x \oplus x$ exists and, for all $y \in \mathrm{M}(E)$, $y \neq 0$ there is $h \in \mathrm{HM}(E)$, $h \neq 0$ such that $h \leq y$.*

Proof. Let $x \in \mathrm{HM}(E)$. Then there is $y \in E$ such that $x \leq y$ and $x \leq y'$. Therefore also $x \leq y \leq x'$, i.e. $x \oplus x$ exists. Let $v \in \mathrm{S}(E)$, $v \leq x \leq y$. Then $v \leq x \leq y' \leq v'$. Hence $v = v \wedge v' = 0$, i.e., $x \in \mathrm{M}(E)$.

Now, let $x \in E$ such that $x \oplus x$ exists. Then $x \leq x'$ and evidently $x \leq x$. Hence $x \in \mathrm{HM}(E)$.

Assume that $y \in \mathrm{M}(E)$, $y \neq 0$. Since y is meager there is a non-zero element h such that $h \leq y$ and $h \leq y'$ (otherwise we would have $y \in \mathrm{S}(E)$, a contradiction). This invokes that h is hypermeager. □

Lemma 2.5. *Every ultrameager element is hypermeager.*

Proof. Let x be an ultrameager element of an effect algebra E. Because $1 \in \mathrm{S}(E)$, $x \leq 1 \ominus x = x'$, and by Lemma 2.4 x is hypermeager. □

Lemma 2.6. *Let E be a sharply dominating effect algebra and let $y \in E$. Then y is ultrameager if and only if $y \leq \widehat{y} \ominus y$.*

Proof. For every $s \in \mathrm{S}(E)$ for which $y \leq s$, it holds $\widehat{y} \leq s$ and $\widehat{y} \in \mathrm{S}(E)$. □

Lemma 2.7. *In every homogeneous effect algebra E,* $\mathrm{UM}(E) = \mathrm{HM}(E)$*.*

Proof. Let E be a homogeneous effect algebra. By Lemma 2.5, $\mathrm{UM}(E) \subseteq \mathrm{HM}(E)$. Let conversely $x \in \mathrm{HM}(E)$ and $y \in \mathrm{S}(E)$ such that $x \leq y$. There exists a block B for which $x, y, x \oplus x, y \ominus x \in B$. By Statement 1.12 (vii), y is central in B. Therefore $x \oplus x \leq y$ and consequently $x \leq y \ominus x$. □

Lemma 2.8. *In every sharply dominating homogeneous effect algebra E,*

$$\begin{aligned}\mathrm{UM}(E) = \bigcup_{y \in \mathrm{UM}(E)} {\downarrow}y \cap {\downarrow}(\widehat{y} \ominus y) &= \bigcup_{y \in \mathrm{HM}(E)} {\downarrow}y \cap {\downarrow}(\widehat{y} \ominus y) = \\ &\bigcup_{y \in \mathrm{M}(E)} {\downarrow}y \cap {\downarrow}(\widehat{y} \ominus y) = \bigcup_{y \in E} {\downarrow}y \cap {\downarrow}(\widehat{y} \ominus y) = \mathrm{HM}(E).\end{aligned}$$

Proof. By Lemma 2.7, $\mathrm{UM}(E) = \mathrm{HM}(E)$. Lemma 2.6 yields

$$\mathrm{UM}(E) \subseteq \bigcup_{y \in \mathrm{UM}(E)} {\downarrow}y \subseteq \bigcup_{y \in \mathrm{UM}(E)} {\downarrow}y \cap {\downarrow}(\widehat{y} \ominus y),$$

which implies

$$\begin{aligned}\mathrm{HM}(E) = \mathrm{UM}(E) \subseteq \bigcup_{y \in \mathrm{UM}(E)} {\downarrow}y \cap {\downarrow}(\widehat{y} \ominus y) &\subseteq \bigcup_{y \in \mathrm{HM}(E)} {\downarrow}y \cap {\downarrow}(\widehat{y} \ominus y) \subseteq \\ &\bigcup_{y \in \mathrm{M}(E)} {\downarrow}y \cap {\downarrow}(\widehat{y} \ominus y) \subseteq \bigcup_{y \in E} {\downarrow}y \cap {\downarrow}(\widehat{y} \ominus y) \subseteq \mathrm{HM}(E).\end{aligned}$$

□

Lemma 2.9. *In any homogeneous effect algebra E,*

$$y \wedge_B z = 0 \iff y \wedge z = 0$$

holds for any block B and $y, z \in \mathrm{C}(B)$.

Proof. $\Longrightarrow$ Since $y \wedge_B z = 0$, it holds $y \leq z'$. Therefore $w \in [0, y] \cap [0, z]$ implies $w \in [0, z'] \cap [0, z] = \{0\}$.
$\Longleftarrow$ Trivial. $\square$

Lemma 2.10. *The following conditions are equivalent in any sharply dominating homogeneous effect algebra E.*

(i) *for any block B and $v, y, z \in B$, it holds $\widehat{v} \in B$ and $y \wedge_B z = 0 \Longrightarrow \widehat{y} \wedge_B \widehat{z} = 0$;*

(ii) *for any block B and $v, y, z \in B$, it holds $\widehat{v} \in B$, and $y \wedge_B z = 0 \Longrightarrow \widehat{y} \wedge_B \widehat{z} = 0$ if furthermore $y, z \in B \cap \mathrm{M}(E)$;*

(iii) *for any block B and $v, y, z \in B$, it holds $\widehat{v} \in B$ and $y \wedge_B z = 0 \Longrightarrow \widehat{y} \wedge \widehat{z} = 0$;*

(iv) *for any block B and $v, y, z \in B$, it holds $\widehat{v} \in B$, and $y \wedge_B z = 0 \Longrightarrow \widehat{y} \wedge \widehat{z} = 0$ if furthermore $y, z \in B \cap \mathrm{M}(E)$.*

Proof. Clearly (i) $\iff$ (iii) and (ii) $\iff$ (iv) in virtue of Lemma 2.9. Furthermore, (i) $\Longrightarrow$ (ii) and (iii) $\Longrightarrow$ (iv).
(ii) $\Longrightarrow$ (i) Let B be a block in E and $y, z \in B$. By assumption, $\widehat{y}, \widehat{z} \in B$. Therefore $\widehat{y} \ominus y, \widehat{z} \ominus z \in B \cap \mathrm{M}(E)$. By assumption, from Lemma 1.16 and the fact that $(y \ominus \widetilde{y}) \wedge_B (z \ominus \widetilde{z}) = 0$ we get

$$(\widehat{y} \ominus \widetilde{y}) \wedge_B (\widehat{z} \ominus \widetilde{z}) = \widehat{y \ominus \widetilde{y}} \wedge_B \widehat{z \ominus \widetilde{z}} = 0.$$

Further, $0 \neq w \in [0, \widetilde{y}] \cap [0, \widehat{z}] \cap B$ implies $0 \neq \widehat{w} \in [0, \widetilde{y}] \cap [0, \widehat{z}]$. Clearly $\widehat{w} \wedge_B (z \oplus (\widehat{z} \ominus z)) = \widehat{w} \neq 0$. There exist $w_1 \leq z, w_2 \leq \widehat{z} \ominus z$ for which $w_1 \oplus w_2 = \widehat{w} \leq y$. Because $\widehat{w} \wedge_B z = 0$, $w_1 = 0$ and $\widehat{w} = w_2 \leq \widehat{z} \ominus z \in \mathrm{M}(E)$. Therefore $\widehat{w} = 0$, a contradiction.

This yields $\widehat{y} \wedge_B \widehat{z} = (\widehat{y} \ominus \widetilde{y}) \wedge_B \widehat{z} \oplus (\widetilde{y} \wedge_B \widehat{z}) = (\widehat{y} \ominus \widetilde{y}) \wedge_B \widehat{z} = \widehat{(y \ominus \widetilde{y})} \wedge_B \widehat{z}$.

Applying the above considerations once more we obtain that $\widehat{y} \wedge_B \widehat{z} = \widehat{(y \ominus \widetilde{y})} \wedge_B \widehat{z} = \widehat{(y \ominus \widetilde{y})} \wedge_B (\widehat{z} \ominus \widetilde{z}) = 0$. $\square$

Definition 2.11. A sharply dominating homogeneous effect algebra is *sober* if it satisfies the equivalent conditions in Lemma 2.10.

Lemma 2.12. *Let E be a homogeneous effect algebra, and $y \in E$ and $w \in \mathrm{S}(E)$ for which $y \leq w$ and ky exists. It holds $ky \leq w$.*

Proof. The elements y', w, $y, 2y, \ldots, ky$ belong to one block B. For $k = 1$ the statement holds. Suppose $2 \leq k$ and the statement holds for $k - 1$. By Statement 1.8, $w \wedge_B ky = (w \wedge_B (k-1)y) \oplus y = (k-1)y \oplus y = ky$. Therefore $ky \leq w$. $\square$

Lemma 2.13. *Let E be an Archimedean homogeneous effect algebra. For any $a \in E \setminus \{0\}$ for which $a \wedge (n_a a)' = 0$, it holds $\widehat{a}$ exists and $\widehat{a} = n_a a$.*

Proof. Let $a \in E$ such that $a \wedge (n_a a)' = 0$. Clearly $a \not\leq (n_a a)'$. Suppose that there exists an element $b \in E$, $b \leq n_a a$ and $b \leq (n_a a)'$. By Statement 1.11 we have that there are $b_1, \dots, b_{n_a}$ such that $b = b_1 \oplus \dots \oplus b_{n_a}$ and $b_i \leq a$ for all $1 \leq i \leq n_a$. Hence $b_i \leq a \wedge (n_a a)' = 0$ for all $1 \leq i \leq n$, i.e. $b = 0$. Therefore $n_a a \in \mathrm{S}(E)$ and by Lemma 2.12, the statement follows. □

Let us recall the following statement.

Statement 2.14. [14, Theorem 2.10] *Let E be an atomic Archimedean lattice effect algebra and let $x \in \mathrm{M}(E)$. Let us denote $A_x = \{a \mid \ a \text{ an atom of } E, a \leq x\}$ and, for any $a \in A_x$, we shall put $k_a^x = max\{k \in \mathbb{N} \mid ka \leq x\}$. Then*

(i) *For any $a \in A_x$ we have $k_a^x < n_a$.*

(ii) *The set $F_x = \{k_a^x a \mid \ a \in A_x\}$ is orthogonal and*

$$x = \bigoplus\{k_a^x a \mid a \text{ an atom of } E,\ a \leq x\} = \bigvee F_x.$$

Moreover, for all $B \subseteq A_x$ and all natural numbers $l_b < n_b, b \in B$ such that $x = \bigoplus\{l_b b \mid b \in B\}$ we have that $B = A_x$ and $l_a = k_a^x$ for all $a \in A_x$ i.e., F_x is the unique set of multiples of atoms from A_x such that its orthogonal sum is x.

(iii) *If $\widehat{x}$ exists then*

$$\begin{aligned} \widehat{x} = \widehat{\widehat{x} \ominus x} &= \bigoplus\{n_a a \mid a \text{ an atom of } E,\ a \leq x\} \\ &= \bigvee\{n_a a \mid \ a \in A_x\} \end{aligned}$$

and

$$\begin{aligned} \widehat{x} \ominus x &= \bigoplus\{(n_a - k_a^x)a \mid \ a \in A_x\} \\ &= \bigvee\{(n_a - k_a^x)a \mid \ a \in A_x\}. \end{aligned}$$

Statement 2.15. [12, Theorem 2.1] *Let E be a lattice effect algebra. Assume $b \in E$, $A \subseteq E$ are such that $\bigvee A$ exists in E and $b \leftrightarrow a$ for all $a \in A$. Then*

(a) $b \leftrightarrow \bigvee A$.

(b) $\bigvee\{b \wedge a : a \in A\}$ *exists in E and equals $b \wedge (\bigvee A)$.*

Proposition 2.16. *Every atomic Archimedean sharply dominating lattice effect algebra is sober.*

Proof. Let us check the condition (ii) from Lemma 2.10. Let B be a block of E and assume that $v, y, z \in B$. Then $v = \widetilde{v} \oplus x$, $x \in \mathrm{M}(E)$. If $x = 0$ we are finished. Assume that $x \neq 0$. We shall use the same notation as in Statement 2.14. Recall that $\widehat{x} \ominus x = \widehat{v} \ominus v \in \mathrm{M}(E)$. Moreover, let $a \in A_x$. Then $a \leq x \leq v$ and $a \leq \widehat{x} \ominus x = \widehat{v} \ominus v \leq v'$. Hence $a \in B$ by Statement 1.12, (viii). This yields that $(n_a - k_a^x)a \in B$ for all $a \in A_x$. Since $\widehat{x} \ominus x = \bigoplus\{(n_a - k_a^x)a \mid a$ an atom of $E,\ a \leq x\}$ we have by Statement 2.15 that $\widehat{v} \ominus v = \widehat{x} \ominus x \in B$. Hence also $\widehat{v} = (\widehat{v} \ominus v) \oplus v \in B$.

Assume now that $y \in \mathrm{M}(E)$ and $y \wedge z = 0$. Let us put $A_y = \{a \mid\ a$ an atom of E, $a \leq y\}$. Evidently, $a \wedge z = 0$, $n_a a \wedge z \in B$ and $z \leq a'$ for all $a \in A_y$. Therefore by Statement 1.11 $n_a a \wedge z \leq n_a a = a \oplus \cdots \oplus a$ yields that $n_a a \wedge z = b_1 \oplus \dots b_n$, $b_i \leq n_a a \wedge z \wedge a = 0$ for all $a \in A_y$. Then Statements 2.14, (iii) and 2.15, (ii) yield that $\widehat{y} \wedge z = \bigvee\{n_a a \mid\ a \in A_y\} \wedge z = \bigvee\{n_a a \wedge z \mid\ a \in A_y\} = 0$.

Assume now that $y, z \in \mathrm{M}(E)$ and $y \wedge z = 0$. Applying the same considerations as above once more we get that $\widehat{y} \wedge \widehat{z} = 0$. □

3. Meager elements in orthocomplete homogeneous effect algebras

By definition and preceding results, orthocomplete homogeneous effect algebras are always homogeneous, Archimedean, sharply dominating and fulfill the following condition (W+).

Definition 3.1. [19] An effect algebra E *fulfills the condition (W+)* if for each orthogonal subset $A \subseteq E$ and each two upper bounds u, v of $A^{\oplus}$ there exists an upper bound w of $A^{\oplus}$ below u, v.

An effect algebra E has the *maximality property* if $\{u, v\}$ has a maximal lower bound w for every $u, v \in E$.

It is easy to see that an effect algebra E has the maximality property if and only if $\{u, v\}$ has a maximal lower bound w, $w \geq t$ for every $u, v, t \in E$ such that t is a lower bound of $\{u, v\}$. As noted in [19] E has the maximality property if and only if $\{u, v\}$ has a minimal upper bound w for every $u, v \in E$.

Statement 3.2. [19, Theorem 2.2] *Lattice effect algebras and orthocomplete effect algebras fulfill both the condition (W+) and the maximality property.*

Statement 3.3. [19, Theorem 3.1] *Let E be an Archimedean effect algebra fulfilling the condition (W+), and let $y, z \in E$. Every lower bound of y, z is below a maximal one and every upper bound of y, z is above a minimal one. Then E has the maximality property.*

Proposition 3.4. *Let E be an Archimedean effect algebra fulfilling the condition (W+). Then every meager element of E is the orthosum of a system of hypermeager elements.*

Proof. Let $y \in \mathrm{M}(E)$. Consider the set $\mathcal{A}$ of all orthogonal systems, precisely multisets, A of hypermeager elements for which y is an upper bound of $A^{\oplus}$. Since the multiset union of any chain in $\mathcal{A}$ belongs to $\mathcal{A}$, there exists a maximal element Z in $\mathcal{A}$. Since E is Archimedean any element of Z is contained in Z only finitely many times. If y is not the supremum of $Z^{\oplus}$, there exists by the condition (W+) an upper bound z of $Z^{\oplus}$ for which $z < y$. Since y is meager, $y \ominus z \neq 0$ is meager too, and therefore there exists a non-zero hypermeager element h such that $h \leq y \ominus z$. Obviously the multiset sum $Z \uplus \{h\}$ belongs to $\mathcal{A}$, which contradicts the assumption of maximality of Z. □

The following statement generalizes [10, Theorem 13].

Corollary 3.5. *Let E be an Archimedean sharply dominating effect algebra fulfilling the condition (W+). Then every element $x \in E$ is the sum of $\widetilde{x}$ and of the orthosum of a system of hypermeager elements.*

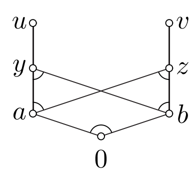
Lemma 3.6. *Let E be an effect algebra having the maximality property, let $u, v \in E$, and let a, b be two maximal lower bounds of u, v. There exist elements y, z for which $y \leq u$, $z \leq v$, a, b are maximal lower bounds of y, z and y, z are minimal upper bounds of a, b.*

Proof. Straightforward. □

Lemma 3.7 (Shifting lemma)**.** *Let E be an effect algebra having the maximality property, let $u, v \in E$, and let a_1, b_1 be two maximal lower bounds of u, v. There exist elements y, z and two maximal lower bounds a, b of y, z for which $y \leq u$, $z \leq v$, $a \leq a_1$, $b \leq b_1$, $a \wedge b = 0$, a, b are maximal lower bounds of y, z and y, z are minimal upper bounds of a, b. Furthermore, $(y \ominus a) \wedge (z \ominus a) = 0$, $(y \ominus b) \wedge (z \ominus b) = 0$, $(y \ominus a) \wedge (y \ominus b) = 0$, $(z \ominus a) \wedge (z \ominus b) = 0$.*

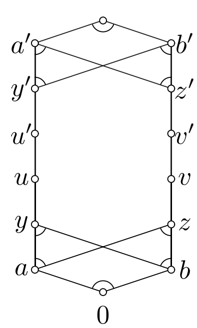
Proof. Let c be a maximal lower bound of a_1, b_1. Let us put $y_1 = u \ominus c$, $z_1 = v \ominus c$, $a = a_1 \ominus c$ and $b = b_1 \ominus c$. Evidently, $a \wedge b = 0$, $y_1 \leq u$, $z_1 \leq v$, $a \leq a_1$, $b \leq b_1$ and a, b are maximal lower bounds of y, z. By Lemma 3.6 there exist elements y, z for which $y \leq y_1$, $z \leq z_1$, a, b are maximal lower bounds of y, z and y, z are minimal upper bounds of a, b. □

The Shifting lemma provides the following *minimax structure*.



Proposition 3.8. *Let E be a homogeneous effect algebra having the maximality property. Every two hypermeager elements u, v possess $u \wedge v$.*

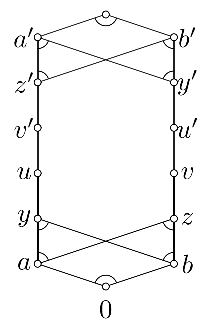
Proof. Consider the minimax structure obtained by the Shifting lemma.



Hence a and b are hypermeager and we have the following implications:
$a \leq (y \ominus b) \oplus b \leq a' \implies (\exists a_1 \leq y \ominus b)(\exists a_2 \leq b)\ a = a_1 \oplus a_2 \overset{a \wedge b = 0}{\Longrightarrow} a_2 = 0, a = a_1 \leq y \ominus b$ and
$a \leq (z \ominus b) \oplus b \leq a' \implies (\exists a_1 \leq z \ominus b)(\exists a_2 \leq b)\ a = a_1 \oplus a_2 \overset{a \wedge b = 0}{\Longrightarrow} a_2 = 0, a = a_1 \leq z \ominus b$.
Since $(y \ominus b) \wedge (z \ominus b) = 0$, it follows $a = 0$. □

Proposition 3.9. *Let E be a homogeneous effect algebra having the maximality property. For every orthogonal elements u, v, $u \wedge v$ and $u \vee_{[0,u \oplus v]} v$ exist and $[0, u \wedge v] \subseteq B$ for every block B containing u or v.*

Proof. Consider the minimax structure obtained by the Shifting lemma.



Then a and b are hypermeager and we have the following implications:
$a \leq (y \ominus b) \oplus b \leq a' \implies (\exists a_1 \leq y \ominus b)(\exists a_2 \leq b)\ a = a_1 \oplus a_2 \overset{a \wedge b = 0}{\Longrightarrow} a_2 = 0, a = a_1 \leq y \ominus b$ and
$a \leq (z \ominus b) \oplus b \leq a' \implies (\exists a_1 \leq z \ominus b)(\exists a_2 \leq b)\ a = a_1 \oplus a_2 \overset{a \wedge b = 0}{\Longrightarrow} a_2 = 0, a = a_1 \leq z \ominus b$.

Since $(y \ominus b) \wedge (z \ominus b) = 0$, it follows $a = 0$. Clearly $(u \oplus v) \ominus (u \wedge v)$ is the supremum of u, v in $[0, u \oplus v]$.

The remaining part of the Proposition follows by Statement 1.12, (viii). □

Corollary 3.10. *Let E be a homogeneous effect algebra having the maximality property. For every element u, $u \wedge u'$ and $u \vee u'$ exist and $[0, u \wedge u'] \subseteq B$ for every block B containing u.*

Corollary 3.11. *Let E be a homogeneous effect algebra having the maximality property. For any block B and every elements $u, v \in B$ for which $u \wedge_B v = 0$, $u \wedge v = 0$.*

Proof. Since $u \wedge_B v = 0$, elements u, v are orthogonal. By Proposition 3.9, $u \wedge v$ exists and $[0, u \wedge v] \subseteq B$. Therefore $u \wedge v = 0$. □

Theorem 3.12. *Let E be a homogeneous effect algebra having the maximality property. Then every block B in E is a lattice, and therefore satisfies the difference-meet property.*

Proof. Let B be a block and $y, z \in B$. There exist elements $a, b, c \in B$ for which $y = a \oplus b$, $z = a \oplus c$ and $a \oplus b \oplus c$ is defined. Hence $b \wedge c$ exists and $b \wedge c \in B$ in virtue of Proposition 3.9. Clearly, $a \oplus (b \wedge c)$ is a maximal lower bound of y, z. Consequently, without loss of generality we may assume $b \wedge c = 0$. Suppose v is a lower bound of y, z in B. Hence $v \leq a \oplus b$ and therefore there exist $a_1 \leq a$ and $b_1 \leq b$ in B for which $a_1 \oplus b_1 = v$. Now $v \ominus a_1 = b_1 \leq b$. Further $v \leq a \oplus c$. Therefore $v \ominus a_1 \leq (a \ominus a_1) \oplus c$. There exist elements $a_2 \leq a \ominus a_1$ and $c_2 \leq c$ in B for which $v \ominus a_1 = a_2 \oplus c_2$. To sum up, $(v \ominus a_1) \ominus a_2 = c_2 \leq c$ and $(v \ominus a_1) \ominus a_2 \leq b \ominus a_2 \leq b$, which together yields $(v \ominus a_1) \ominus a_2 = c_2 \leq b \wedge c = 0$. Consequently, $v = a_1 \oplus a_2 \leq a_1 \oplus (a \ominus a_1) = a$, and a is the infimum of y, z. This yields that B is a lattice. $\square$

The preceding theorem immediately yields the following statements.

Corollary 3.13. *Let E be a homogeneous effect algebra having the maximality property. Then E can be covered by MV-algebras which form blocks.*

Corollary 3.14. *Let E be an Archimedean homogeneous effect algebra fulfilling the condition (W+). Then E can be covered by Archimedean MV-algebras which form blocks.*

Note that as in [15] we obtain that Archimedean homogeneous effect algebras fulfilling the condition (W+) (in particular orthocomplete homogeneous effect algebras) can be covered by ranges of observables.

Proposition 3.15. *Let E be an orthocomplete homogeneous effect algebra. Then every block in E is a lattice.*

Corollary 3.16. *Finite homogeneous effect algebras are covered by MV-algebras.*

Corollary 3.17. *Let E be a sharply dominating homogeneous effect algebra having the maximality property. For any $y \in \mathrm{M}(E)$, $y \wedge (\widehat{y} \ominus y)$ exists and $y \wedge (\widehat{y} \ominus y) = y \wedge y'$.*

Proof. The meets exist in virtue of Proposition 3.9. Since $y \leq \widehat{y}$, there is a block B for which $y, \widehat{y} \in B$. Now, $\widehat{y} = (y \oplus y') \wedge_B \widehat{y} = (y \wedge_B \widehat{y}) \oplus (y' \wedge_B \widehat{y}) = y \oplus (y' \wedge_B \widehat{y})$, which implies $\widehat{y} \ominus y = y' \wedge_B \widehat{y}$. Consequently, $y \wedge y' = (\widehat{y} \wedge y) \wedge y' = (\widehat{y} \wedge_B y') \wedge_B y) = (\widehat{y} \ominus y) \wedge_B y = (\widehat{y} \ominus y) \wedge y$ because $\widehat{y} \ominus y, y$ are orthogonal. $\square$

Example 3.18. In the finite orthoalgebra $E = \{0, a, b, c, d, e, f, a', b', c', d', e', f', 1\}$ depictured in Figure 3 which is such that $E = \mathrm{S}(E)$ (hence E is homogeneous, Archimedean and orthocomplete), finite joins in blocks do not coincide with finite joins in E. One can easily check that E is a sub-effect algebra of the

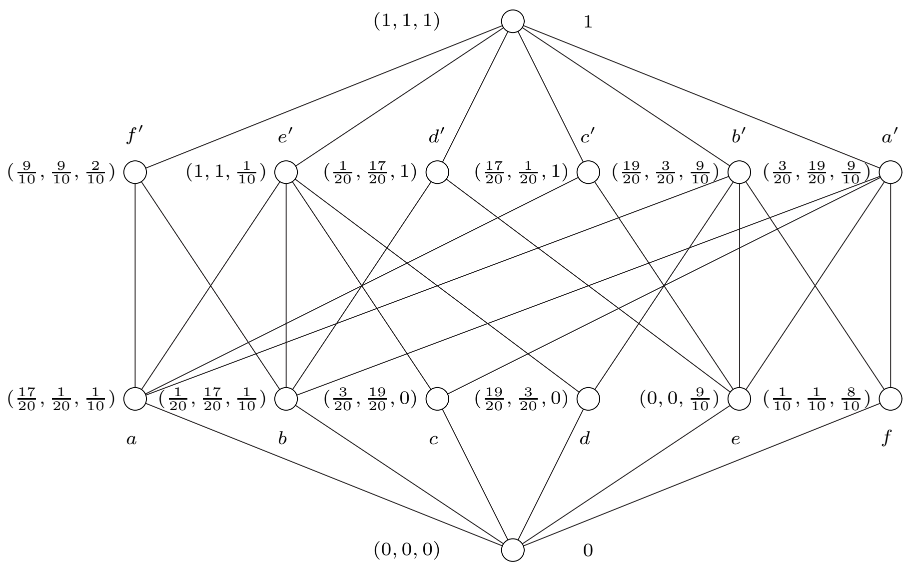


Figure 3: Example 3.18

MV-effect algebra $[0,1]\times[0,1]\times[0,1]$ such that

$$
\begin{aligned}
&0 \mapsto (0,0,0), && 1 \mapsto (1,1,1), && \\
&a \mapsto (\tfrac{17}{20},\tfrac{1}{20},\tfrac{1}{10}), && b \mapsto (\tfrac{1}{20},\tfrac{17}{20},\tfrac{1}{10}), && c \mapsto (\tfrac{3}{20},\tfrac{19}{20},0), \\
&d \mapsto (\tfrac{19}{20},\tfrac{3}{20},0), && e \mapsto (0,0,\tfrac{9}{10}), && f \mapsto (\tfrac{1}{10},\tfrac{1}{10},\tfrac{8}{10}), \\
&a' \mapsto (\tfrac{3}{20},\tfrac{19}{20},\tfrac{9}{10}), && b' \mapsto (\tfrac{19}{20},\tfrac{3}{20},\tfrac{9}{10}), && c' \mapsto (\tfrac{17}{20},\tfrac{1}{20},1), \\
&d' \mapsto (\tfrac{1}{20},\tfrac{17}{20},1), && e' \mapsto (1,1,\tfrac{1}{10}), && f' \mapsto (\tfrac{9}{10},\tfrac{9}{10},\tfrac{2}{10}).
\end{aligned}
$$

Moreover $1=(1,1,1)=a\oplus b\oplus f=a\oplus c\oplus e=b\oplus d\oplus e$. This yields that E has only the following blocks: $B_1=\{0,a,b,f,a\oplus b=f',a\oplus f=b',b\oplus f=a',1\}$, $B_2=\{0,a,c,e,a\oplus c=e',a\oplus e=c',c\oplus e=a',1\}$ and $B_3=\{0,b,d,e,b\oplus d=e',b\oplus e=d',d\oplus e=b',1\}$ which are lattice ordered. Hence $a\not\leftrightarrow d$, $b\not\leftrightarrow c$, $c\not\leftrightarrow d$, $c\not\leftrightarrow f$, $d\not\leftrightarrow f$, $e\not\leftrightarrow f$. In particular, a and b have two different minimal upper bounds, $a\oplus b$ and $a\oplus c$.

Open problem 3.19. *One question still unanswered is whether, if A,B are two blocks of E with $x,y\in A\cap B$, then $x\vee_A y=x\vee_B y$; here E is an Archimedean sharply dominating homogeneous effect algebra fulfilling the condition (W+).*

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