

# A characterization of the category $Q\text{-TOP}$

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## 1 Introduction

E.G. Manes in [2] gave a somewhat ‘axiomatic’ characterization (upto isomorphism) of the category **TOP** of topological spaces, among a certain class of categories, satisfying some conditions, in which a Sierpinski space-like object, played a key role. Following that, the category  $[0, 1]\text{-TOP}$  of  $[0, 1]$ -topological spaces (known more commonly as ‘fuzzy’ topological spaces) was analogously characterized by Srivastava [4]. S.A. Solovyov [3] has recently introduced the notion of  $Q$ -topological spaces ( $Q$  being a fixed member of a variety of  $\Omega$ -algebras). In this note, we give a characterization of the category  $Q\text{-TOP}$  of  $Q$ -topological spaces in which the so-called  $Q$ -Sierpinski space, introduced in [3], plays a key role.

## 2 The category $Q\text{-TOP}$

We begin by recalling the (well-known) notions of  $\Omega$ -algebras and their homomorphisms. For details, see [1].

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**Definition 2.1** ([3])

- Let  $\Omega = (n_\lambda)_{\lambda \in I}$  be a class of cardinal numbers. An  $\Omega$ -**algebra** is a pair  $(A, (\omega_\lambda^A)_{\lambda \in I})$  consisting of a set  $A$  and a family of maps  $\omega_\lambda^A : A^{n_\lambda} \rightarrow A$ . Any  $B (\neq \emptyset) \subseteq A$  is called a **subalgebra** of  $(A, (\omega_\lambda^A)_{\lambda \in I})$  if for any  $\lambda \in I$  and any  $(b_i)_{i \in n_\lambda} \in B^{n_\lambda}$ ,  $\omega_\lambda^A((b_i)_{i \in n_\lambda}) \in B$ . Given  $S \subseteq A$ , the intersection of all the subalgebras of  $(A, (\omega_\lambda^A)_{\lambda \in I})$  containing  $S$  is clearly a subalgebra of  $(A, (\omega_\lambda^A)_{\lambda \in I})$ . We shall denote it as  $\langle S \rangle$ .
- Given  $(A, (\omega_\lambda^A)_{\lambda \in I})$  and  $(B, (\omega_\lambda^B)_{\lambda \in I})$ , a map  $f : A \rightarrow B$  is called an  $\Omega$ -**algebra homomorphism** if for every  $\lambda \in I$  the following diagram commutes:

$$\begin{array}{ccc} A^{n_\lambda} & \xrightarrow{f^{n_\lambda}} & B^{n_\lambda} \\ \omega_\lambda^A \downarrow & & \downarrow \omega_\lambda^B \\ A & \xrightarrow{f} & B \end{array}$$

Let  $\mathbf{Alg}(\Omega)$  denote the category of  $\Omega$ -algebras and  $\Omega$ -algebra homomorphisms.

- A **variety** of  $\Omega$ -algebras is a full subcategory of  $\mathbf{Alg}(\Omega)$  which is closed under the formation of products<sup>1</sup>, subalgebras and homomorphic images.

Throughout,  $Q$  denotes a fixed member of a fixed variety of  $\Omega$ -algebras.

- Given a set  $X$ , a subset  $\tau$  of  $Q^X$  is called a  **$Q$ -topology** on  $X$  if  $\tau$  is a subalgebra of  $Q^X$ ; in such a case, the pair  $(X, \tau)$  is called a  **$Q$ -topological space**.
- Given two  $Q$ -topological spaces  $(X, \tau)$  and  $(Y, \eta)$ , a  **$Q$ -continuous function** from  $(X, \tau)$  to  $(Y, \eta)$  is a function  $f : X \rightarrow Y$  such that  $f^\leftarrow(\alpha) \in \tau, \forall \alpha \in \eta$ , where  $f^\leftarrow(\alpha) = \alpha \circ f$ .

It is evident that all  $Q$ -topological spaces, together with  $Q$ -continuous maps, form a category, which we shall denote as  **$Q$ -TOP**.

The following categorical concepts are from Manes [2]

**Definition 2.2** ([2])

1. A **category  $C$  of sets with structure** is defined through the following descriptions of its objects (' $C$ -structured sets') and morphisms (' $C$ -admissible maps'), satisfying the two axioms given below:  
A class  $C(X)$  of ' $C$ -structures' is assigned with each set  $X$  and a ' $C$ -structured set' is a pair  $(X, s)$ , with  $s \in C(X)$ .

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<sup>1</sup>The category  $\mathbf{Alg}(\Omega)$  is closed under products

A subset  $C(s, t)$  of the set of all functions from  $X$  to  $Y$  is assigned with each pair of  $C$ -structured sets  $(X, s)$  and  $(Y, t)$  and a ‘ $C$ -admissible map’ from  $(X, s)$  to  $(Y, t)$  is any  $f \in C(s, t)$  (in which case we write “ $f : (X, s) \rightarrow (Y, t)$ ”).

The axioms are:

Axiom  $A_1$ : If  $f : (X, s) \rightarrow (Y, t)$  and  $g : (Y, t) \rightarrow (Z, u)$  then also  $g \circ f : (X, s) \rightarrow (Z, u)$ .

Axiom  $A_2$ : Given a bijection  $f : X \rightarrow Y$  and  $t \in C(Y)$ , there exists a unique  $s \in C(X)$  such that  $f : (X, s) \rightarrow (Y, t)$  and  $f^{-1} : (Y, t) \rightarrow (X, s)$ .

2. Given a category  $\mathbf{C}$  of structures (as defined above), a family  $\mathcal{F} = \{f_j : (X, s) \rightarrow (Y_j, t_j) | j \in J\}$  of  $C$ -admissible maps is said to be **optimal** if for each  $C$ -structured set  $(Z, u)$  and a function  $g : Z \rightarrow X$ ,  $g : (Z, u) \rightarrow (X, s)$  iff  $f_j \circ g : (Z, u) \rightarrow (Y_j, t_j), \forall j \in J$ . Further, if for a family  $\mathcal{F} = \{f_j : X \rightarrow (Y_j, t_j) | j \in J\}$  of functions, where  $X$  is a set and each  $(Y_j, t_j)$  is a  $C$ -structured set, there exists  $s \in C(X)$  such that the family  $\{f_j : (X, s) \rightarrow (Y_j, t_j) | j \in J\}$  is optimal, then  $s$  is called an **optimal lift** of the family  $\mathcal{F}$ .

3. An object  $S = (S, u)$  in a category  $\mathbf{C}$  of sets with structure is called a **Sierpinski object** if for every  $C$ -object  $X = (X, s)$ , the family of all  $C$ -admissible maps from  $X$  to  $S$  is optimal.

**Remark 2.1** It is easy to verify that, in  $Q\text{-TOP}$ , the optimal lift of a family  $\mathcal{F} = \{f_j : X \rightarrow (Y_j, t_j) | j \in J\}$  of functions, where  $X$  is a set and each  $(Y_j, t_j)$  is a  $Q\text{-TOP}$ -object, is precisely the smallest  $Q$ -topology on  $X$  making each  $f_i$   $Q$ -continuous.

Both  $\mathbf{TOP}$  and  $[0, 1]\text{-TOP}$  are categories of sets with structures and the usual two-point Sierpinski space and the fuzzy Sierpinski space (of [4]) are Sierpinski objects in these categories.

We verify (on expected lines) that  $Q\text{-TOP}$  is also a category of sets with structures. For each set  $X$ ,  $C(X)$  is the family of all  $Q$ -topologies on  $X$  and the  $C$ -admissible maps are just the  $Q$ -continuous maps. Given a bijection  $f : X \rightarrow Y$  and  $t \in C(Y)$ , there exists a unique  $s \in C(X)$ , namely  $s = \langle \mathcal{A} \rangle$ , with  $\mathcal{A} = \{q \circ f | q \in t\}$ , such that both  $f : (X, s) \rightarrow (Y, t)$  and  $f^{-1} : (Y, t) \rightarrow (X, s)$  are  $Q$ -continuous. Also, every family  $\mathcal{F} = \{f_j : X \rightarrow (Y_j, t_j) | j \in J\}$  of functions, where  $X$  is a set and  $(Y_j, t_j), j \in J$  are the  $Q$ -topological spaces, has an optimal lift  $s$ , namely  $s = \langle \mathcal{S} \rangle$ , with  $\mathcal{S} = \{q \circ f_j | q \in t_j, j \in J\}$ . The product of a family  $\{(X_i, t_i) | i \in I\}$  of  $Q$ -topological spaces is the  $Q$ -topological space  $(\prod X_i, t)$ , where  $t$  is the optimal lift of the family of all the projection maps  $p_i : \prod X_i \rightarrow (X_i, t_i)$ . In fact,  $(\prod X_i, t)$  is the categorical product of the  $Q\text{-TOP}$ -objects  $(X_i, t_i), i \in I$ .

### 3 $Q$ -Sierpinski Space

Let  $S = Q$ . It is obvious that  $u = \langle id \rangle$ , where  $id \in Q^Q$  is the identity function, is a  $Q$ -topology on  $S$ .  $(S, u)$  has been called the  $Q$ -**Sierpinski space** in [3]. Theorem 3.2 establishes the appropriateness of this concept of  $Q$ -Sierpinski space. But first, we state the following easy-to-verify result.

**Theorem 3.1** *For any  $Q$ -topological space  $(X, \tau)$ ,  $p \in \tau$  iff  $p : (X, \tau) \rightarrow (S, u)$  is  $Q$ -continuous.*

**Theorem 3.2**  *$(S, u)$  is a Sierpinski object in  $Q\text{-TOP}$ .*

**Proof:** Let  $(X, \tau)$  be a  $Q\text{-TOP}$ -object and  $\mathcal{F} = \{f : (X, \tau) \rightarrow (S, u) \mid f \text{ is } Q\text{-continuous}\}$  be the family of all  $Q$ -continuous maps from  $(X, \tau)$  to  $(S, u)$ . To show that this family is optimal, let  $\tau'$  be any other  $Q$ -topology on  $X$  making each  $f \in \mathcal{F}$   $Q$ -continuous. If  $p \in \tau$  then, as  $id \in u$ ,  $p : (X, \tau) \rightarrow (S, u)$  must be  $Q$ -continuous and so  $p : (X, \tau') \rightarrow (S, u)$  is also  $Q$ -continuous. But then  $p \in \tau'$ . Thus  $\tau \subseteq \tau'$ , showing that  $\tau$  is the smallest  $Q$ -topology on  $X$  making each  $f \in \mathcal{F}$   $Q$ -continuous.  $\square$

### 4 A characterization of $Q\text{-TOP}$

**Theorem 4.1** *A category  $\mathbf{C}$  of sets with structures is isomorphic to the category  $Q\text{-TOP}$  if and only if  $\mathbf{C}$  contains an object  $(S, u)$ , with  $S = Q$ , satisfying the following conditions:*

1. *every family  $f_i : X \rightarrow (S, u)$  has an optimal lift,*
2. *the maps  $\omega_\lambda : (S, u)^{n_\lambda} \rightarrow (S, u)$  are  $\mathbf{C}$ -admissible for each  $\lambda \in I$ , where  $(S, u)^{n_\lambda} = (S^{n_\lambda}, u_{n_\lambda})$ , with  $u_{n_\lambda}$  being the optimal lift of all the projection maps from  $S^{n_\lambda}$  to  $(S, u)$ ,*
3.  *$(S, u)$  is a Sierpinski object in  $\mathbf{C}$ .*

**Proof:** First, it is clear that  $Q\text{-TOP}$  satisfies (1) and (3). That  $Q\text{-TOP}$  also satisfies (2) is shown as follows. Note that  $u_{n_\lambda}$ , the product  $Q$ -topology on  $S^{n_\lambda}$  has a generating set consisting precisely of all the projection maps from  $S^{n_\lambda}$  to  $S$ . To show that  $\omega_\lambda : (S, u)^{n_\lambda} \rightarrow (S, u)$  is  $Q$ -continuous, it suffices to show that  $\omega_\lambda \in u_{n_\lambda}$ . Now if  $a = (a_i)_{i \in n_\lambda} \in S^{n_\lambda}$ , then  $\omega_\lambda^{S^{n_\lambda}}((p_i)_{i \in n_\lambda})(a) = \omega_\lambda^S((p_i(a))_{i \in n_\lambda}) = \omega_\lambda^S((a_i)_{i \in n_\lambda}) = \omega_\lambda(a)$ , where  $p_i$ 's are the projection maps from  $S^{n_\lambda}$  to  $S$ . Hence  $\omega_\lambda = \omega_\lambda^{S^{n_\lambda}}((p_i)_{i \in n_\lambda}) \in u_{n_\lambda}$ . Thus  $\omega_\lambda$  is  $Q$ -continuous,  $\forall \lambda \in I$ .

For each set  $X$ , let  $Q(X)$  denote the set of all  $Q$ -topologies on  $X$ . It will suffice to prove that, for each set  $X$ , there exists a bijection  $\Phi_X : C(X) \rightarrow Q(X)$  such that  $f : (X, s) \rightarrow (Y, t)$  is  $C$ -admissible iff  $f : (X, \Phi_X(s)) \rightarrow (Y, \Phi_Y(t))$  is  $Q$ -continuous, i.e., iff  $q \circ f \in \Phi_X(s)$  for each  $q \in \Phi_Y(t)$ . For each set  $X$  and

$s \in C(X)$ , put  $\Phi_X(s) = \{p \in Q^X \mid p : (X, s) \rightarrow (S, u) \text{ is } C\text{-admissible}\}$ . We show that  $\Phi_X(s)$  is a  $Q$ -topology on  $X$ , i.e.,  $\Phi_X(s)$  is a subalgebra of  $Q^X$ , or that  $\Phi_X(s) \in Q(X)$ . For a given  $\lambda \in I$ , consider  $(p_i)_{i \in n_\lambda} \in (Q^X)^{n_\lambda}$ , with  $p_i \in \Phi_X(s)$ ,  $\forall i \in n_\lambda$ . We wish to show that  $\omega_\lambda^{Q^X}((p_i)_{i \in n_\lambda}) \in \Phi_X(s)$ , i.e.,  $\omega_\lambda^{Q^X}((p_i)_{i \in n_\lambda})$  is also a  $C$ -admissible map from  $(X, s)$  to  $(S, u)$ . The map  $f : (X, s) \rightarrow (S, u)^{n_\lambda}$ , defined by  $f(x)(j) = p_j(x)$ , for each  $x \in X$  and  $1 \leq j \leq n_\lambda$ , is easily seen to be  $C$ -admissible. Here we note that  $\omega_\lambda^{Q^X}((p_i)_{i \in n_\lambda}) = \omega_\lambda \circ f$ , as  $(\omega_\lambda \circ f)(x) = \omega_\lambda(f(x)) = \omega_\lambda^Q((f(x)(i))_{i \in n_\lambda}) = \omega_\lambda^Q((p_i(x))_{i \in n_\lambda}) = \omega_\lambda^{Q^X}((p_i)_{i \in n_\lambda})(x)$ . So,  $\omega_\lambda^{Q^X}((p_i)_{i \in n_\lambda})$  is also a  $C$ -admissible, being the composition of two  $C$ -admissible maps. Thus  $\Phi_X(s)$  is a  $Q$ -topology on  $X$ .

Let  $\tau \in Q(X)$ ,  $\mathcal{F} = \{f : (X, \tau) \rightarrow (S, u) \mid f \text{ is } Q\text{-continuous}\}$ , and  $s_\tau$  be the optimal lift of the family  $\{f : X \rightarrow (S, u) \mid f \in \mathcal{F}\}$ . Then clearly,  $s_\tau \in C(X)$  and  $f : (X, s_\tau) \rightarrow (S, u)$  is  $C$ -admissible for each  $f \in \mathcal{F}$ . This provides a function from  $Q(X)$  to  $C(X)$ , which, it can be verified, is the inverse of  $\Phi_X : C(X) \rightarrow Q(X)$ . So,  $C(X)$  and  $Q(X)$  are in one-to-one correspondence. Also, condition (3) says that  $f : (X, s) \rightarrow (Y, t)$  is  $C$ -admissible precisely when  $q \circ f \in \Phi_X(s)$  for each  $q \in \Phi_Y(t)$ .

## References

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