SKLAR'S THEOREM IN AN IMPRECISE SETTING

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ABSTRACT. Sklar's theorem is an important tool that connects bidimensional distribution functions with their marginals by means of a copula. When there is imprecision about the marginals, we can model the available information by means of p-boxes, that are pairs of ordered distribution functions. Similarly, we can consider a set of copulas instead of a single one. We study the extension of Sklar's theorem under these conditions, and link the obtained results to stochastic ordering with imprecision.

Keywords. Sklar's theorem, copula, p-boxes, natural extension, independent products, stochastic orders.

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1. INTRODUCTION

In this paper, we deal with the problem of combining two marginal models representing the probabilistic information about two random variables X, Y into a bivariate model of the joint behaviour of (X, Y). In the classical case, this problem has a simple solution, by means of *Sklar's* well-known *theorem* [26], that tells us that any bivariate distribution function can be obtained as the combination of its marginals by means of a copula [19].

Here we investigate to what extent Sklar's theorem can be extended in the context of imprecision, both in the marginal distribution functions and in the copula that links them. The imprecision in marginal distributions shall be modelled by a *probability box* [10] (p-box, for short), that summarizes a set of distribution functions by means of its lower and upper envelopes. Regarding the imprecision about the copula, we shall also consider a set of copulas. This set shall be represented by means of the newly introduced notion of *imprecise copula*, that we study in Section 3.1. This imprecision means that in the bivariate case we end up with a set

of bivariate distribution functions, that we can summarize by means of a coherent bivariate p-box, a notion recently studied in [22].

Interestingly, we shall show in Section 3.1 that Sklar's theorem can be only partly extended to the imprecise case; although the combination of two marginal p-boxes by means of a set of copulas (or its associated imprecise copula) always produces a coherent bivariate p-box, the most important aspect of the theorem does not hold: not every coherent bivariate p-box can be obtained in this manner. In Sections 3.2 and 3.3, we consider two particular cases of interest: that where we have no information about the copula that links the two variables together, and that where we assume that the two variables are independent. In those cases, we use Walley's notions of natural extension [28] and (epistemic) independent products [4, 28] to derive the joint model.

In Section 4, we connect our results to decision making by applying the notion of stochastic dominance in this setting, and we establish a number of cases in which the order existing on the marginals is preserved by their respective joints. We conclude the paper with some additional comments and remarks in Section 5.

2. Preliminary concepts

2.1. Coherent lower previsions.
Let us introduce the basic notions from the theory of coherent lower previsions that we shall use later on in this paper. For a more detailed exposition of the theory and for a behavioural interpretation of the concepts below in terms of betting rates, we refer to [28].

Let Ω be a possibility space. A *gamble* is a bounded real-valued function $f : \Omega \to \mathbb{R}$. We shall denote by $\mathcal{L}(\Omega)$ the set of all gambles on Ω, and by $\mathcal{L}^+(\Omega)$ the set of non-negative gambles. It includes in particular the indicator functions of subsets B of Ω, i.e., the gambles that take value 1 on the elements of B and 0 elsewhere. In this paper, we shall use the same symbol for an event B and for its indicator function.

A *lower prevision* is a functional $\underline{P} : \mathcal{K} \to \mathbb{R}$ defined on some set of gambles $\mathcal{K} \subseteq \mathcal{L}(\Omega)$. Here we are interested in lower previsions satisfying the property of *coherence*:

Definition 1 (**Coherent lower previsions**). A lower prevision $\underline{P} : \mathcal{L}(\Omega) \to \mathbb{R}$ is called *coherent* when it satisfies the following conditions for every pair of gambles $f, g \in \mathcal{L}(\Omega)$ and every $\lambda > 0$:

  (C1) $\underline{P}(f) \geq \inf_{\omega \in \Omega} f(\omega)$.
  (C2) $\underline{P}(\lambda f) = \lambda \underline{P}(f)$.
  (C3) $\underline{P}(f + g) \geq \underline{P}(f) + \underline{P}(g)$.

The restriction to events of a coherent lower prevision is called a *coherent lower probability*, and more generally a lower prevision $\underline{P}$ on $\mathcal{K}$ is said to be coherent whenever it can be extended to a coherent lower prevision on $\mathcal{L}(\Omega)$. On the other hand, if $\underline{P}$ is a coherent lower prevision on $\mathcal{L}(\Omega)$ and it satisfies (C3) with equality for every f and g in $\mathcal{L}(\Omega)$, then it is called a *linear* prevision, and its restriction to events is a finitely additive probability. In fact, coherent lower previsions can be given the following sensitivity analysis interpretation: a lower prevision $\underline{P}$ on $\mathcal{K}$ is coherent if and only if it is the lower envelope of its associated *credal set*,

$$(1) \qquad \mathcal{M}(\underline{P}) := \{P : \mathcal{L}(\Omega) \to \mathbb{R} \text{ linear prevision} : P(f) \geq \underline{P}(f) \ \forall f \in \mathcal{K}\},$$

and as a consequence the lower envelope of a set of linear previsions is always a coherent lower prevision [28, Section 3.3.3(b)].

One particular instance of coherent lower probabilities are those associated with p-boxes.

Definition 2. [10] A (univariate) *p-box* is a pair $(\underline{F}, \overline{F})$ where $\underline{F}, \overline{F} : \mathbb{R} \to [0, 1]$ are cumulative distribution functions (i.e., monotone and such that $\underline{F}(-\infty) = \overline{F}(-\infty) = 0, \underline{F}(+\infty) = \overline{F}(+\infty) = 1$) satisfying $\underline{F}(x) \le \overline{F}(x)$ for every $x \in \mathbb{R}$.

Define the set $A_x = [-\infty, x]$ for every $x \in \overline{\mathbb{R}}$, and let

$$\mathcal{E}_0 := \{A_x : x \in \overline{\mathbb{R}}\} \cup \{A_x^c : x \in \overline{\mathbb{R}}\}.$$

Then [27] a p-box $(\underline{F}, \overline{F})$ induces a coherent lower probability $\underline{P}_{(\underline{F}, \overline{F})} : \mathcal{E}_0 \to [0, 1]$ by

$$(2) \qquad \underline{P}_{(\underline{F}, \overline{F})}(A_x) = \underline{F}(x) \text{ and } \underline{P}_{(\underline{F}, \overline{F})}(A_x^c) = 1 - \overline{F}(x) \ \forall x \in \mathbb{R}.$$

2.2. **Bivariate p-boxes.** In [22], the notion of p-box from Definition 2 has been extended to the bivariate case, to describe couples of random variables (X, Y) in presence of imprecision.

Definition 3. [22] A map $F : \overline{\mathbb{R}} \times \overline{\mathbb{R}} \to [0, 1]$ is called *standardized* when it is component-wise increasing, that is, $F(t_1, z) \le F(t_2, z)$ and $F(z, t_1) \le F(z, t_2)$ for all $t_1 \le t_2$ and z, and satisfies

$$F(-\infty, y) = F(x, -\infty) = 0 \ \forall x, y \in \overline{\mathbb{R}}, \ F(+\infty, +\infty) = 1.$$

It is called a *distribution function* for (X, Y) when it is standardized and satisfies

$$F(x_2, y_2) + F(x_1, y_1) - F(x_1, y_2) - F(x_2, y_1) \ge 0$$

for all $x_1, x_2, y_1, y_2 \in \overline{\mathbb{R}}$ such that $x_1 \le x_2, y_1 \le y_2$ (with equality holding whenever $(x_1 \le X < x_2) \wedge (y_1 \le Y < y_2)$ is impossible). Given two standardized functions $\underline{F}, \overline{F} : \overline{\mathbb{R}} \times \overline{\mathbb{R}} \to [0, 1]$ satisfying $\underline{F}(x, y) \le \overline{F}(x, y)$ for every $x, y \in \overline{\mathbb{R}}$, the pair $(\underline{F}, \overline{F})$ is called a *bivariate p-box*.

Bivariate p-boxes are introduced as a model for the imprecise knowledge of a bivariate distribution function. The reason why the lower and upper functions in a bivariate p-box are not required to be distribution functions is that the lower and upper envelopes of a set of bivariate distribution functions need not be distribution functions themselves, as showed in [22].

Let $(\underline{F}, \overline{F})$ be a bivariate p-box on $\overline{\mathbb{R}} \times \overline{\mathbb{R}}$. Define $A_{(x,y)} = [-\infty, x] \times [-\infty, y]$ for every $x, y \in \overline{\mathbb{R}}$, and consider the sets

$$\mathcal{D} := \{A_{(x,y)} : x, y \in \overline{\mathbb{R}}\}, \mathcal{D}_c := \{A_{(x,y)}^c : x, y \in \overline{\mathbb{R}}\}, \ \mathcal{E} := \mathcal{D} \cup \mathcal{D}_c.$$

Note that $A_{(+\infty, +\infty)} = \overline{\mathbb{R}} \times \overline{\mathbb{R}}$, whence both $\overline{\mathbb{R}} \times \overline{\mathbb{R}}$ and $\emptyset$ belong to $\mathcal{E}$. Similarly to Eq. (2), we can define the *lower probability induced by a bivariate p-box* $(\underline{F}, \overline{F})$ on $\overline{\mathbb{R}} \times \overline{\mathbb{R}}$ as the map $\underline{P}_{(\underline{F}, \overline{F})} : \mathcal{E} \to [0, 1]$ given by:

$$(3) \qquad \underline{P}_{(\underline{F}, \overline{F})}(A_{(x,y)}) = \underline{F}(x, y), \quad \underline{P}_{(\underline{F}, \overline{F})}(A_{(x,y)}^c) = 1 - \overline{F}(x, y)$$

for every $x, y \in \overline{\mathbb{R}}$. Conversely, a lower probability $\underline{P} : \mathcal{E} \to [0, 1]$ determines a couple of functions $\underline{F}_{\underline{P}}, \overline{F}_{\underline{P}} : \overline{\mathbb{R}} \times \overline{\mathbb{R}} \to [0, 1]$ defined by

$$(4) \qquad \underline{F}_{\underline{P}}(x, y) = \underline{P}(A_{(x,y)}) \text{ and } \overline{F}_{\underline{P}}(x, y) = 1 - \underline{P}(A_{(x,y)}^c) \ \forall x, y \in \overline{\mathbb{R}}.$$

Then $(\underline{F}_P, \overline{F}_P)$ is a bivariate p-box as soon as the lower probability $\underline{P}$ is 2-coherent [22]. 2-coherence is a weak rationality condition implied by coherence [28, Appendix B], which in the context of this paper, where the domain $\mathcal{E}$ is closed under complementation, is equivalent [22] to $\underline{P}_{(\underline{F},\overline{F})}$ being monotone, normalised, and such that $\underline{P}_{(\underline{F},\overline{F})}(E) + \underline{P}_{(\underline{F},\overline{F})}(E^c) \leq 1$ for every $E \in \mathcal{E}$.

The correspondence between bivariate p-boxes and lower probabilities in terms of precise models is given by the following lemma:[1]

Lemma 1. [22] *Let $(\underline{F}, \overline{F})$ be a p-box and $\underline{P}_{(\underline{F},\overline{F})}$ the lower probability it induces on $\mathcal{E}$ by means of Eq. (3).*

(a) *Let P be (the restriction to $\mathcal{E}$ of) a linear prevision on $\mathcal{L}(\overline{\mathbb{R}} \times \overline{\mathbb{R}})$, and let F_P be its associated distribution function given by $F_P(x,y) = P(A_{(x,y)})$ for every $x, y \in \overline{\mathbb{R}}$. Then*

$$P(A) \geq \underline{P}_{(\underline{F},\overline{F})}(A) \ \forall A \in \mathcal{E} \iff \underline{F} \leq F_P \leq \overline{F}.$$

(b) *Conversely, let F be a distribution function on $\overline{\mathbb{R}} \times \overline{\mathbb{R}}$, and let $P_F : \mathcal{E} \to [0,1]$ be the functional given by $P_F(A_{(x,y)}) = F(x,y), P_F(A_{(x,y)}^c) = 1 - F(x,y)$ for every $x, y \in \overline{\mathbb{R}}$. Then*

$$\underline{F} \leq F \leq \overline{F} \iff P_F(A) \geq \underline{P}_{(\underline{F},\overline{F})}(A) \ \forall A \in \mathcal{E}.$$

Given a bivariate p-box $(\underline{F}, \overline{F})$, Lemma 1 implies that the coherence of its associated lower probability $\underline{P}_{(\underline{F},\overline{F})}$ can be characterised through a set of distribution functions:

Proposition 1. [22] *The lower probability $\underline{P}_{(\underline{F},\overline{F})}$ induced by the bivariate p-box $(\underline{F}, \overline{F})$ by means of Eq. (3) is coherent if and only if $\underline{F}$ (resp., $\overline{F}$) is the lower (resp., upper) envelope of the set*

(5) $$\mathcal{F} = \{F : \overline{\mathbb{R}} \times \overline{\mathbb{R}} \to [0,1] \ distribution \ function : \underline{F} \leq F \leq \overline{F}\}.$$

If $\underline{P}_{(\underline{F},\overline{F})}$ is coherent, the following conditions hold for every $x_1 \leq x_2 \in \mathbb{R}$ and $y_1 \leq y_2 \in \overline{\mathbb{R}}$:

(I-RI1) $$\underline{F}(x_2, y_2) + \overline{F}(x_1, y_1) - \underline{F}(x_1, y_2) - \underline{F}(x_2, y_1) \geq 0.$$

(I-RI2) $$\overline{F}(x_2, y_2) + \underline{F}(x_1, y_1) - \underline{F}(x_1, y_2) - \underline{F}(x_2, y_1) \geq 0.$$

(I-RI3) $$\overline{F}(x_2, y_2) + \overline{F}(x_1, y_1) - \overline{F}(x_1, y_2) - \underline{F}(x_2, y_1) \geq 0.$$

(I-RI4) $$\overline{F}(x_2, y_2) + \overline{F}(x_1, y_1) - \underline{F}(x_1, y_2) - \overline{F}(x_2, y_1) \geq 0.$$

Definition 4. A bivariate p-box $(\underline{F}, \overline{F})$ is *coherent* whenever the lower probability $\underline{P}_{(\underline{F},\overline{F})}$ it induces on $\mathcal{E}$ by means of Eq. (3) is coherent.

2.3. **Copulas.** In this paper, we are going to study to what extent bivariate p-boxes can be expressed as a function of their marginals. In the precise case (that is, when we have only one bivariate distribution function), this is done through the notion of copula.

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[1]We give a brief sketch of the proof: it suffices to establish the equivalences $P(A_{(x,y)}) \geq \underline{P}_{(\underline{F},\overline{F})}(A_{(x,y)}) \iff F_P(x,y) \geq \underline{F}(x,y)$ and $P(A_{(x,y)}^c) \geq \underline{P}_{(\underline{F},\overline{F})}(A_{(x,y)}^c) \iff F_P(x,y) \leq \overline{F}(x,y)$ for every $x, y \in \overline{\mathbb{R}}$. These follow easily from Eqs. (3) and (4).

Definition 5. [19] A function $C : [0, 1] \times [0, 1] \to [0, 1]$ is called a *copula* when it satisfies the following conditions:

(COP1)
$$C(0, u) = C(u, 0) = 0 \; \forall u \in [0, 1].$$

(COP2)
$$C(1, u) = C(u, 1) = u \; \forall u \in [0, 1].$$

(COP3)
$$C(u_2, v_2) + C(u_1, v_1) - C(u_1, v_2) - C(u_2, v_1) \geq 0 \; \forall u_1 \leq u_2, v_1 \leq v_2 \in [0, 1].$$

It follows from the definition above that a copula is component-wise monotone increasing. One of the main features of copulas lies in Sklar's theorem:

Theorem 1 ([26], **Sklar's Theorem**). *Let* $F_{(X,Y)} : \overline{\mathbb{R}} \times \overline{\mathbb{R}} \to [0, 1]$ *be a bivariate distribution function with marginals* $F_X : \overline{\mathbb{R}} \to [0, 1]$ *and* $F_Y : \overline{\mathbb{R}} \to [0, 1]$, *defined by* $F_X(x) = F_{(X,Y)}(x, +\infty)$ *and* $F_Y(y) = F_{(X,Y)}(+\infty, y)$ *for any* x *and* y *in* $\overline{\mathbb{R}}$. *Then there is a copula* C *such that*

$$F_{(X,Y)}(x, y) = C(F_X(x), F_Y(y)) \text{ for all } (x, y) \in \overline{\mathbb{R}} \times \overline{\mathbb{R}}.$$

Conversely, any transformation of marginal distribution functions by means of a copula produces a bivariate distribution function.

Any copula C must satisfy the *Fréchet-Hoeffding bounds* (see [11, 29]):

$$(6) \qquad C_L(u, v) := \max\{u + v - 1, 0\} \leq C(u, v) \leq \min\{u, v\} := C_M(u, v)$$

for every $u, v \in [0, 1]$. C_L is called the *Łukasiewicz copula* and C_M the *minimum copula*. Eq. (6) applies in particular to one instance of copulas that shall be of interest in this paper: the *product copula* C_P, given by $C_P(u, v) = u \cdot v$ for every $u, v \in [0, 1]$. It holds that two random variables X, Y are stochastically independent if and only if their distribution functions are coupled by the product copula.

For an in-depth review on copulas we refer to [19].

3. Combining marginal p-boxes into a bivariate one

One particular context where bivariate p-boxes can arise is in the joint extension of two marginal p-boxes. In this section, we explore this case in detail, studying in particular the properties of some bivariate p-boxes with given marginals: the largest one, that shall be obtained by means of the Fréchet bounds and the notion of natural extension, and the one modelling the notion of independence. In both cases, we shall see that the bivariate model can be derived by means of an appropriate extension of the notion of copula.

Related results can be found in [27, Section 7], with one fundamental difference: in [27], the authors use the existence of a total preorder on the product space (in the case of this paper, $\overline{\mathbb{R}} \times \overline{\mathbb{R}}$) that is compatible with the orders in the marginal spaces, and reduce the multivariate p-box to a univariate one. Here we do no such reduction, and we consider only a partial order: the product order, given by

$$(x_1, y_1) \leq (x_2, y_2) \Leftrightarrow x_1 \leq x_2 \text{ and } y_1 \leq y_2.$$

Another related study was made by Yager in [30], considering the case in which the marginal distributions are not precisely described and are defined by means of

Dempster-Shafer belief structures instead. He modelled this situation by considering copulas whose arguments are intervals (the ones determined by the Demspter-Shafer models) instead of crisp numbers, and whose images are also intervals. He showed then that the lower (resp., upper) bound of the interval of images corresponds to the copula evaluated in the lower (resp., upper) bounds of the intervals. This can be seen as a particular case of our subsequent Proposition 4.

3.1. **A generalization of Sklar's theorem.** Let us study to which extent Sklar's theorem can be generalised to a context of imprecision, both in the marginal distribution functions to be combined and in the copula that links them. In order to tackle this problem, we introduce the notion of imprecise copula:

Definition 6. A pair $(\underline{C}, \overline{C})$ of functions $\underline{C}, \overline{C} : [0,1] \times [0,1] \to [0,1]$ is called an *imprecise copula* if:

- $\underline{C}(0,u) = \underline{C}(u,0) = 0, \underline{C}(1,u) = \underline{C}(u,1) = u \ \forall u \in [0,1]$.
- $\overline{C}(0,u) = \overline{C}(u,0) = 0, \overline{C}(1,u) = \overline{C}(u,1) = u \ \forall u \in [0,1]$.
- For any $u_1 \leq u_2, v_1 \leq v_2$:

(CI-1) $$\underline{C}(u_2, v_2) + \overline{C}(u_1, v_1) - \underline{C}(u_1, v_2) - \underline{C}(u_2, v_1) \geq 0.$$

(CI-2) $$\overline{C}(u_2, v_2) + \underline{C}(u_1, v_1) - \underline{C}(u_1, v_2) - \underline{C}(u_2, v_1) \geq 0.$$

(CI-3) $$\overline{C}(u_2, v_2) + \overline{C}(u_1, v_1) - \overline{C}(u_1, v_2) - \underline{C}(u_2, v_1) \geq 0.$$

(CI-4) $$\overline{C}(u_2, v_2) + \overline{C}(u_1, v_1) - \underline{C}(u_1, v_2) - \overline{C}(u_2, v_1) \geq 0.$$

We are using the terminology imprecise copula in the definition above because we intend it as a mathematical model for the imprecise knowledge of a copula; note however that the lower and upper functions $\underline{C}, \overline{C}$ need not be copulas themselves, because they may not satisfy the 2-increasing property (COP3).

(CI-1)÷(CI-4) are useful in establishing the following properties of imprecise copulas.

Proposition 2. *Let $(\underline{C}, \overline{C})$ be an imprecise copula.*

(a) $\underline{C} \leq \overline{C}$.
(b) $\underline{C}$ *and* $\overline{C}$ *are component-wise increasing.*
(c) *The* Lipschitz *condition*

(7) $$|C(u_2, v_2) - C(u_1, v_1)| \leq |u_2 - u_1| + |v_2 - v_1| \ \forall u_1, u_2, v_1, v_2 \in [0,1]$$

is satisfied both by $C = \underline{C}$ and by $C = \overline{C}$.
(d) *The pointwise infimum and supremum of a non-empty set of copulas $\mathcal{C}$ form an imprecise copula.*

Proof.   (a) This follows from inequality (CI-3), with $u_2 = u_1$.
(b) Use (CI-1) with, alternatively, $v_1 = 0$ and $u_1 = 0$ to obtain, respectively,

$$\underline{C}(u_2, v_2) - \underline{C}(u_1, v_2) \geq 0 \ \forall v_2, u_1, u_2 \in [0,1], s.t. \ u_1 \leq u_2$$
$$\underline{C}(u_2, v_2) - \underline{C}(u_2, v_1) \geq 0 \ \forall u_2, v_1, v_2 \in [0,1], s.t. \ v_1 \leq v_2$$

By these inequalities, $\underline{C}$ is component-wise increasing. Analogously, to prove that $\overline{C}$ is component-wise increasing, apply (CI-4) with $u_1 = 0$ and (CI-3) with $v_1 = 0$.

(c) Applying twice (CI-2) and the boundary conditions in Definition 5, first with $v_2 = 1$ and then with $u_2 = 1$, we obtain, respectively,

$$(8) \qquad \underline{C}(u_2, v_1) - \underline{C}(u_1, v_1) \leq u_2 - u_1$$

$$(9) \qquad \underline{C}(u_1, v_2) - \underline{C}(u_1, v_1) \leq v_2 - v_1.$$

Because they are derived from (CI-2), Eqs. (8) and (9) hold, respectively, for any $v_1, u_1, u_2 \in [0, 1]$ such that $u_1 \leq u_2$, and for any $u_1, v_1, v_2 \in [0, 1]$ such that $v_1 \leq v_2$. In the general case, Eq. (8) is replaced by

$$|\underline{C}(u_2, v_1) - \underline{C}(u_1, v_1)| \leq |u_2 - u_1|$$

and similarly for Eq. (9). Therefore, for arbitrary u_1, u_2, v_1 and v_2 in $[0, 1]$, $|\underline{C}(u_2, v_2) - \underline{C}(u_1, v_1)| \leq |\underline{C}(u_2, v_2) - \underline{C}(u_2, v_1)| + |\underline{C}(u_2, v_1) - \underline{C}(u_1, v_1)| \leq |u_2 - u_1| + |v_2 - v_1|$, which proves the Lipschitz condition for $\underline{C}$. The proof for $\overline{C}$ is similar (use (CI-4) with $v_2 = 1$ and (CI-3) with $u_2 = 1$).

(d) The boundary conditions are trivial, so let us prove (CI-1)÷(CI-4). Define $\underline{C}(x, y) := \inf_{C \in \mathcal{C}} C(x, y)$, $\overline{C}(x, y) := \inf_{C \in \mathcal{C}} C(x, y)$. By applying (COP3) to the copulas in $\mathcal{C}$, we get that, for every $C \in \mathcal{C}$ and every $u_1 \leq u_2, v_1 \leq v_2 \in [0, 1]$,

$$C(u_2, v_2) + C(u_1, v_1) \geq C(u_1, v_2) + C(u_2, v_1) \geq \underline{C}(u_1, v_2) + \underline{C}(u_2, v_1).$$

From this we deduce that, for every $C \in \mathcal{C}$ and every $u_1 \leq u_2, v_1 \leq v_2 \in [0, 1]$,

$$\overline{C}(u_2, v_2) + C(u_1, v_1) \geq \underline{C}(u_1, v_2) + \underline{C}(u_2, v_1),$$
$$C(u_2, v_2) + \overline{C}(u_1, v_1) \geq \underline{C}(u_1, v_2) + \underline{C}(u_2, v_1),$$

whence (CI-2) and (CI-1) hold.

As for (CI-3) and (CI-4), again from (COP3), we get that for every $C \in \mathcal{C}$ and every $u_1 \leq u_2, v_1 \leq v_2 \in [0, 1]$,

$$\overline{C}(u_2, v_2) + \overline{C}(u_1, v_1) \geq C(u_2, v_2) + C(u_1, v_1) \geq C(u_1, v_2) + C(u_2, v_1).$$

This implies that for every $C \in \mathcal{C}$ and every $u_1 \leq u_2, v_1 \leq v_2 \in [0, 1]$,

$$\overline{C}(u_2, v_2) + \overline{C}(u_1, v_1) \geq C(u_1, v_2) + \underline{C}(u_2, v_1),$$
$$\overline{C}(u_2, v_2) + \overline{C}(u_1, v_1) \geq \underline{C}(u_1, v_2) + C(u_2, v_1),$$

whence (CI-3) and (CI-4) hold. $\square$

According to [20, Corollary 2.3], the pointwise infimum and supremum of a set of copulas are also quasi-copulas (see [21] for a study on the lattice structure of copulas). A quasi-copula [19] is a binary operator satisfying conditions (COP1), (COP2) in Definition 5 and the Lipschitz condition given by Eq. (7).

By Proposition 2 (c), both $\underline{C}$ and $\overline{C}$ in an imprecise copula $(\underline{C}, \overline{C})$ are quasi-copulas. Conversely, given two quasi-copulas C_1 and C_2 such that $C_1 \leq C_2$, (C_1, C_2) may not be an imprecise copula. To see that, it is enough to consider a proper quasi-copula C, i.e. a quasi-copula which is not a copula (see for instance [19, Example 6.3]). Then, the pair (C, C) is not an imprecise copula because it does not satisfy the inequalities in Definition 6: in this case the inequalities all reduce to (COP3). We may then conclude that an imprecise copula is formed by two quasi-copulas $C_1 \leq C_2$, for which the additional inequalities (CI-1)÷(CI-4) hold.

The converse of item (d) in this proposition is still an open problem at this stage; it is formally equivalent to the characterisation of coherent bivariate p-boxes

studied in detail in [22]. So far, we have only established it under some restrictions on the domains of the copulas. If it held, then we could regard imprecise copulas as restrictions of *sets* of bivariate distribution functions of continuous random variables with uniform marginals, similar to the situation for precise copulas.

In the particular case when $\underline{C} = \overline{C} := C$, $(\underline{C}, \overline{C})$ is an imprecise copula if and only if C is a copula. It is also immediate to establish the following:

Proposition 3. *Let C_1 and C_2 be two copulas such that $C_1 \leq C_2$. Then, (C_1, C_2) forms an imprecise copula. In particular, $(C_\mathrm{L}, C_\mathrm{M})$ is the largest imprecise copula, in the sense that, for any imprecise copula $(\underline{C}, \overline{C})$, it holds that $C_\mathrm{L} \leq \underline{C} \leq \overline{C} \leq C_\mathrm{M}$.*

Proof. It is simple to check that (C_1, C_2) satisfies Definition 6. The proof of the remaining part is similar to that of the Fréchet-Hoeffding inequalities. Consider an imprecise copula $(\underline{C}, \overline{C})$. Since $\overline{C}$ is component-wise increasing by Proposition 2 (b), and applying the boundary conditions,

$$\overline{C}(u, v) \leq \min(\overline{C}(u, 1), \overline{C}(1, v)) = \min(u, v).$$

Using (CI-2) we deduce that:

$$1 + \underline{C}(u, v) = \overline{C}(1, 1) + \underline{C}(u, v) \geq \underline{C}(u, 1) + \underline{C}(1, v) = u + v.$$

Then $\underline{C}(u, v) \geq u + v - 1$, and by definition $\underline{C}$ is also non-negative. Finally, the inequality $\underline{C} \leq \overline{C}$ follows from Proposition 2 (a). $\square$

Remark 1. Given a copula $C : [0, 1] \times [0, 1] \to [0, 1]$, it is immediate to see that its extension $C' : \overline{\mathbb{R}} \times \overline{\mathbb{R}} \to [0, 1]$ given by

$$C'(x, y) := \begin{cases} C(x, y) & \text{if } (x, y) \in [0, 1] \times [0, 1] \\ 0 & \text{if } x < 0 \text{ or } y < 0 \\ \min\{x, y\} & \text{if } \min\{x, y\} \in [0, 1] \text{ and } \max\{x, y\} \in [1, +\infty] \\ 1 & \text{otherwise} \end{cases}$$

is a distribution function. Taking this into account, given any non-empty set of copulas $\mathcal{C}$, its infimum $\underline{C}$ and supremum $\overline{C}$ form a coherent bivariate p-box. Moreover, an imprecise copula $(\underline{C}, \overline{C})$ can be extended to $\overline{\mathbb{R}} \times \overline{\mathbb{R}}$ in the manner described above, and then it constitutes a bivariate p-box that satisfies conditions (I-RI1)÷(I-RI4) (although it is still an open problem whether it is coherent). ♦

Let us see to what extent an analogue of Sklar's theorem also holds in an imprecise framework. For this aim, we start by considering marginal imprecise distributions, described by (univariate) p-boxes, and we use imprecise copulas to obtain a bivariate p-box.

Proposition 4. *Let $(\underline{F}_\mathrm{X}, \overline{F}_\mathrm{X})$ and $(\underline{F}_\mathrm{Y}, \overline{F}_\mathrm{Y})$ be two marginal p-boxes on $\overline{\mathbb{R}}$, and let $\mathcal{C}$ be a set of copulas. Consider the imprecise copula $(\underline{C}, \overline{C})$ defined from $\mathcal{C}$ by $\underline{C}(u, v) = \inf_{C \in \mathcal{C}} C(u, v)$ and $\overline{C}(u, v) = \sup_{C \in \mathcal{C}} C(u, v)$ for every $u, v \in [0, 1]$. Define the couple $(\underline{F}, \overline{F})$ by:*

$$(10) \qquad \underline{F}(x, y) = \underline{C}(\underline{F}_\mathrm{X}(x), \underline{F}_\mathrm{Y}(y)) \text{ and } \overline{F}(x, y) = \overline{C}(\overline{F}_\mathrm{X}(x), \overline{F}_\mathrm{Y}(y))$$

for any $(x, y) \in \overline{\mathbb{R}} \times \overline{\mathbb{R}}$. Then, $(\underline{F}, \overline{F})$ is a bivariate p-box and it holds that:

(a) $\underline{P}_{(\underline{F}, \overline{F})}$ *is coherent.*

(b) *The credal set $\mathcal{M}(\underline{P}_{(\underline{F},\overline{F})})$ associated with the lower probability $\underline{P}_{(\underline{F},\overline{F})}$ by means of Eq. (1) is given by*

$$\mathcal{M}(\underline{P}_{(\underline{F},\overline{F})}) = \{P \ linear \ prevision \mid \underline{C}(\underline{F}_X, \underline{F}_Y) \le F_P \le \overline{C}(\overline{F}_X, \overline{F}_Y)\}.$$

Proof. Note that $\underline{F} \le \overline{F}$, since $\underline{F} = \underline{C}(\underline{F}_X, \underline{F}_Y) \le \underline{C}(\overline{F}_X, \overline{F}_Y) \le \overline{C}(\overline{F}_X, \overline{F}_Y) = \overline{F}$. It is easy to check that both $\underline{F}, \overline{F}$ are standardized and as a consequence $(\underline{F}, \overline{F})$ is a bivariate p-box.

(a) Let $\mathcal{F}$ be the set of distribution functions associated with the bivariate p-box $(\underline{F}, \overline{F})$ by means of Eq. (5). Since $\underline{F}_X, \overline{F}_X, \underline{F}_Y, \overline{F}_Y$ are marginal distribution functions, Sklar's theorem implies that $C(\underline{F}_X(x), \underline{F}_Y(y))$ and $C(\overline{F}_X(x), \overline{F}_Y(y))$ are bivariate distribution functions for any $C \in \mathcal{C}$. Moreover, they necessarily belong to $\mathcal{F}$ by Eq. (10). From this we deduce that

$$\underline{F}(x, y) \le \inf_{F \in \mathcal{F}} F(x, y) \le \underline{C}(\underline{F}_X(x), \underline{F}_Y(y)) = \underline{F}(x, y),$$

and therefore $\underline{F}(x, y) = \inf_{F \in \mathcal{F}} F(x, y)$. Similarly, we can prove that $\overline{F}(x, y) = \sup_{F \in \mathcal{F}} F(x, y)$. Applying now Proposition 1, we deduce that $\underline{P}_{(\underline{F},\overline{F})}$ is coherent.

(b) This follows from the first statement and Lemma 1. $\square$

In particular, when the available information about the marginal distributions is precise, and it is given by the distribution functions F_X and F_Y, the bivariate p-box in the proposition above is given by

$$\underline{F}(x, y) = \inf_{C \in \mathcal{C}} C(F_X(x), F_Y(y)) \text{ and } \overline{F}(x, y) = \sup_{C \in \mathcal{C}} C(F_X(x), F_Y(y))$$

for every $(x, y) \in \overline{\mathbb{R}} \times \overline{\mathbb{R}}$. As a consequence, the result above generalizes [20, Theorem 2.4], where the authors only focused on the functions $\underline{F}$ and $\overline{F}$, showing that $\underline{F}(x, y) = \underline{C}(F_X(x), F_Y(y))$ and $\overline{F}(x, y) = \overline{C}(F_X(x), F_Y(y))$. Instead, in Proposition 4 we are also allowing for the existence of imprecision in the marginal distributions, that we model by means of p-boxes. Note that we have also established the coherence of the joint lower probability $\underline{P}_{(\underline{F},\overline{F})}$ and therefore of the p-box $(\underline{F}, \overline{F})$.

Proposition 4 generalizes to the imprecise case one of the implications in Sklar's theorem: if we combine two marginal p-boxes by means of a set of copulas, we obtain a coherent bivariate p-box, which is thus equivalent to a set of bivariate distribution functions. We focus now on the other implication: whether any bivariate p-box can be obtained as a function of its marginals.[2]

A partial result in this sense has been established in [23, Theorem 9]. In our language, it ensures that if the restriction on $\mathcal{D}$ of $\underline{P}_{(\underline{F},\overline{F})}$ is (a restriction of) a 2-monotone lower probability, then there exists a function $\underline{C} : [0, 1] \times [0, 1] \to [0, 1]$, which is component-wise increasing and satisfies (COP1) and (COP2), such that $\underline{F}(x, y) = \underline{C}(F_X(x), F_Y(y))$ for every (x, y) in $\overline{\mathbb{R}} \times \overline{\mathbb{R}}$. This has been used in the context of random sets in [1, 24].

---

[2]A similar study was made in [9, Theorem 2.4] in terms of capacities and semi-copulas, showing that the survival functions induced by a capacity can always be expressed as a semi-copula of their marginals. Here we investigate when the combination can be made in terms of an imprecise copula. Note moreover that our focus is on *coherent* bivariate p-boxes, which produces capacities that are most restrictive than those considered in [9] (they are closer to the precise case, so to speak). This is why we also consider the particular case where the semi-copulas constitute an imprecise copula.

Somewhat surprisingly, we show next that this result cannot be generalized to arbitrary p-boxes.

Example 1. Let P_1, P_2 be the discrete probability measures associated with the following masses on $\mathcal{X} \times \mathcal{Y} = \{1, 2, 3\} \times \{1, 2\}$:

	$(1,1)$	$(2,1)$	$(1,2)$	$(2,2)$	$(3,1)$	$(3,2)$
P_1	0.2	0	0.3	0	0	0.5
P_2	0.1	0.2	0.5	0.1	0	0.1

Let $\underline{P}$ be the lower envelope of $\{P_1, P_2\}$. Then, $\underline{P}$ is a coherent lower probability, and its associated p-box $(\underline{F}, \overline{F})$ satisfies

$$\underline{F}_X(1) = \underline{F}_X(2) = 0.5, \underline{F}_Y(1) = 0.2, \underline{F}(1,1) = 0.1 < \underline{F}(2,1) = 0.2.$$

If there was a function $\underline{C}$ such that $\underline{F}(x, y) = \underline{C}(\underline{F}_X(x), \underline{F}_Y(y))$ for every $(x, y) \in \overline{\mathbb{R}} \times \overline{\mathbb{R}}$, then we should have

$$\underline{F}(1,1) = \underline{C}(\underline{F}_X(1), \underline{F}_Y(1)) = \underline{C}(\underline{F}_X(2), \underline{F}_Y(1)) = \underline{F}(2,1).$$

This is a contradiction. As a consequence, the lower distribution in the bivariate p-box cannot be expressed as a function of its marginals. ♦

This shows that the direct implication of Sklar's theorem does not hold in the bivariate case: given a coherent bivariate p-box $(\underline{F}, \overline{F})$, there is not in general an imprecise copula $(\underline{C}, \overline{C})$ determining it by means of Eq. (10). The key point here is that the lower and upper distribution functions of a coherent bivariate p-box may not be distribution functions themselves, as showed in [22]; they need only be standardized functions. Indeed, if $\underline{F}, \overline{F}$ were distribution functions we could always apply Sklar's theorem to them, and we could express each of them as a copula of its marginals. What Example 1 shows is that this is no longer possible when $\underline{F}, \overline{F}$ are just standardized functions, nor in general when $(\underline{F}, \overline{F})$ is coherent. We can thus summarize the results of this section in the following theorem:

Theorem 2 (**Imprecise Sklar's Theorem**). *The following statements hold:*
  (a) *Given two marginal p-boxes $(\underline{F}_X, \overline{F}_X)$ and $(\underline{F}_Y, \overline{F}_Y)$ on $\overline{\mathbb{R}}$ and a set of copulas $\mathcal{C}$, the functions $\underline{F}, \overline{F}$ given by Eq. (10) determine a bivariate p-box on $\overline{\mathbb{R}} \times \overline{\mathbb{R}}$, whose associated lower probability is coherent.*
  (b) *Not every bivariate p-box can be expressed by means of its marginals and a set of copulas by Eq. (10), not even when its associated lower probability is coherent.*

3.2. **Natural extension of marginal p-boxes.** Next we consider two particular combinations of the marginal p-boxes into the bivariate one. First of all, we consider the case where there is no information about the copula that links the marginal distribution functions.

Lemma 2. *Consider the univariate p-boxes $(\underline{F}_X, \overline{F}_X)$ and $(\underline{F}_Y, \overline{F}_Y)$ on $\overline{\mathbb{R}}$, and let $\underline{P}$ be the lower probability defined on*

$$\mathcal{A}^* := \{A_{(x,+\infty)}, A^c_{(x,+\infty)}, A_{(+\infty,y)}, A^c_{(+\infty,y)} : x, y \in \overline{\mathbb{R}}\} \subseteq \mathcal{E}$$

by

(11) $\qquad \underline{P}(A_{(x,+\infty)}) = \underline{F}_X(x) \qquad \underline{P}(A^c_{(x,+\infty)}) = 1 - \overline{F}_X(x) \ \forall x \in \mathbb{R},$

(12) $\qquad \underline{P}(A_{(+\infty,y)}) = \underline{F}_Y(y) \qquad \underline{P}(A^c_{(+\infty,y)}) = 1 - \overline{F}_Y(y) \ \forall y \in \mathbb{R}.$

Then:

(1) $\underline{P}$ *is a coherent lower probability.*
(2) $\mathcal{M}(\underline{P}) = \mathcal{M}(C_{\mathrm{L}}, C_{\mathrm{M}})$, *where* $C_{\mathrm{L}}, C_{\mathrm{M}}$ *are the copulas given by Eq. (6) and*

$$\mathcal{M}(C_{\mathrm{L}}, C_{\mathrm{M}}) = \{P \text{ linear prevision} : C_{\mathrm{L}}(\underline{F}_{\mathrm{X}}, \underline{F}_{\mathrm{Y}}) \leq F_{\mathrm{P}} \leq C_{\mathrm{M}}(\overline{F}_{\mathrm{X}}, \overline{F}_{\mathrm{Y}})\}.$$

Proof. (1) We use $\mathcal{C}^*$ to denote the set of all copulas. By Propositions 3 and 4 and Eq. (6),

$$\underline{F}(x, y) = \underline{C}(\underline{F}_{\mathrm{X}}(x), \underline{F}_{\mathrm{Y}}(y)) = \inf_{C \in \mathcal{C}^*} C(\underline{F}_{\mathrm{X}}(x), \underline{F}_{\mathrm{Y}}(y))$$
$$= C_{\mathrm{L}}(\underline{F}_{\mathrm{X}}(x), \underline{F}_{\mathrm{Y}}(y)),$$

and similarly $\overline{F}(x, y) = \overline{C}(\overline{F}_{\mathrm{X}}(x), \overline{F}_{\mathrm{Y}}(y)) = C_{\mathrm{M}}(\overline{F}_{\mathrm{X}}(x), \overline{F}_{\mathrm{Y}}(y))$. Let $\underline{P}_{(\underline{F},\overline{F})}$ the coherent lower probability induced by $(\underline{F}, \overline{F})$ by Eq. (3). Then

$$\underline{P}_{(\underline{F},\overline{F})}(A_{(x,+\infty)}) = \underline{F}(x, +\infty) = C_{\mathrm{L}}(\underline{F}_{\mathrm{X}}(x), \underline{F}_{\mathrm{Y}}(+\infty))$$
$$= \max\{\underline{F}_{\mathrm{X}}(x) + \underline{F}_{\mathrm{Y}}(+\infty) - 1, 0\}$$
$$= \max\{\underline{F}_{\mathrm{X}}(x), 0\} = \underline{F}_{\mathrm{X}}(x) = \underline{P}(A_{(x,+\infty)})$$

and also

$$\underline{P}_{(\underline{F},\overline{F})}(A^c_{(x,+\infty)}) = 1 - \overline{F}(x, +\infty) = 1 - C_{\mathrm{M}}(\overline{F}_{\mathrm{X}}(x), \overline{F}_{\mathrm{Y}}(+\infty))$$
$$= 1 - \min\{\overline{F}_{\mathrm{X}}(x), \overline{F}_{\mathrm{Y}}(+\infty)\} = 1 - \min\{\overline{F}_{\mathrm{X}}(x), 1\}$$
$$= 1 - \overline{F}_{\mathrm{X}}(x) = \underline{P}(A^c_{(x,+\infty)}).$$

With an analogous reasoning, we obtain $\underline{P}_{(\underline{F},\overline{F})}(A_{(+\infty,y)}) = \underline{P}(A_{(+\infty,y)})$ and $\underline{P}_{(\underline{F},\overline{F})}(A^c_{(+\infty,y)}) = \underline{P}(A^c_{(+\infty,y)})$. Therefore, $\underline{P}$ coincides with $\underline{P}_{(\underline{F},\overline{F})}$ in $\mathcal{A}^*$, and consequently $\underline{P}$ is coherent.

(2) Let $P \in \mathcal{M}(C_{\mathrm{L}}, C_{\mathrm{M}})$. Then, $P \geq \underline{P}_{(\underline{F},\overline{F})}$ on $\mathcal{E}$ by Lemma 1. Since $\underline{P}$ coincides with $\underline{P}_{(\underline{F},\overline{F})}$ on $\mathcal{A}^*$, $P \in \mathcal{M}(\underline{P})$.

Conversely, let $P \in \mathcal{M}(\underline{P})$, and let F_{P} be its associated distribution function. Then, Sklar's Theorem assures that there is $C \in \mathcal{C}^*$ such that $F_{\mathrm{P}}(x, y) = C(F_{\mathrm{P}}(x, +\infty), F_{\mathrm{P}}(+\infty, y))$ for every $(x, y) \in \overline{\mathbb{R}} \times \overline{\mathbb{R}}$. Hence,

$$\begin{aligned} C_{\mathrm{L}}(\underline{F}_X(x), \underline{F}_{\mathrm{Y}}(y)) &\leq& C_{\mathrm{L}}(F_{\mathrm{P}}(x, +\infty), F_{\mathrm{P}}(+\infty, y)) \\ &\leq& C(F_{\mathrm{P}}(x, +\infty), F_{\mathrm{P}}(+\infty, y)) \\ &\leq& C(\overline{F}_X(x), \overline{F}_Y(y)) \leq C_{\mathrm{M}}(\overline{F}_X(x), \overline{F}_Y(y)), \end{aligned}$$

taking into account that any copula is component-wise increasing and lies between C_{L} and C_{M}. Therefore, $P \in \mathcal{M}(C_{\mathrm{L}}, C_{\mathrm{M}})$ and as a consequence $\mathcal{M}(\underline{P}) = \mathcal{M}(C_{\mathrm{L}}, C_{\mathrm{M}})$. $\square$

From this result we can immediately derive the expression of the *natural extension* [28] of two marginal p-boxes, that is the least-committal (i.e., the most imprecise) coherent lower probability that extends $\underline{P}$ to a larger domain:

Proposition 5. *Let* $(\underline{F}_{\mathrm{X}}, \overline{F}_{\mathrm{X}})$ *and* $(\underline{F}_{\mathrm{Y}}, \overline{F}_{\mathrm{Y}})$ *be two univariate p-boxes. Let* $\underline{P}$ *be the lower probability defined on the set* $\mathcal{A}^*$ *by means of Eqs. (11) and (12). The natural extension* $\underline{E}$ *of* $\underline{P}$ *to* $\mathcal{E}$ *is given by*

$$\underline{E}(A_{(x,y)}) = C_{\mathrm{L}}(\underline{F}_{\mathrm{X}}(x), \underline{F}_{\mathrm{Y}}(y)) \text{ and } \underline{E}(A^c_{(x,y)}) = 1 - C_{\mathrm{M}}(\overline{F}_{\mathrm{X}}(x), \overline{F}_{\mathrm{Y}}(y)),$$

for every $x, y \in \overline{\mathbb{R}}$. *As a consequence, the bivariate p-box* $(\underline{F}, \overline{F})$ *associated with* $\underline{E}$ *is given by:*

$$\underline{F}(x,y) = C_{\mathrm{L}}(\underline{F}_{\mathrm{X}}(x), \underline{F}_{\mathrm{Y}}(y)) \ and \ \overline{F}(x,y) = C_{\mathrm{M}}(\overline{F}_{\mathrm{X}}(x), \overline{F}_{\mathrm{Y}}(y)).$$

Proof. The lower probability $\underline{P}$ is coherent from the previous lemma, and in addition its associated credal set is $\mathcal{M}(\underline{P}) = \mathcal{M}(C_{\mathrm{L}}, C_{\mathrm{M}})$. The natural extension of $\underline{P}$ to the set $\mathcal{E}$ is given by:

$$
\begin{aligned}
\underline{E}(A_{(x,y)}) &= \inf_{P \in \mathcal{M}(\underline{P})} F_{\mathrm{P}}(x,y) \\
&= \inf_{P \in \mathcal{M}(C_{\mathrm{L}}, C_{\mathrm{M}})} F_{\mathrm{P}}(x,y) = C_{\mathrm{L}}(\underline{F}_{\mathrm{X}}(x), \underline{F}_{\mathrm{Y}}(y)). \\
\underline{E}(A^c_{(x,y)}) &= \inf_{P \in \mathcal{M}(\underline{P})}(1 - P(A_{(x,y)})) = 1 - \sup_{P \in \mathcal{M}(\underline{P})} F_{\mathrm{P}}(x,y) \\
&= 1 - \sup_{P \in \mathcal{M}(C_{\mathrm{L}}, C_{\mathrm{M}})} F_{\mathrm{P}}(x,y) = 1 - C_{\mathrm{M}}(\overline{F}_{\mathrm{X}}(x), \overline{F}_{\mathrm{Y}}(y)).
\end{aligned}
$$

The second part is an immediate consequence of the first.   □

The intuition of this result is clear: if we want to build the joint p-box $(\underline{F}, \overline{F})$ from two given marginals $(\underline{F}_{\mathrm{X}}, \overline{F}_{\mathrm{X}}), (\underline{F}_{\mathrm{Y}}, \overline{F}_{\mathrm{Y}})$, and we have no information about the interaction between the underlying variables X, Y, we should consider the largest, or most conservative, imprecise copula: $(C_{\mathrm{L}}, C_{\mathrm{M}})$. This corresponds to combining the compatible univariate distribution functions by means of all possible copulas, and then taking the envelopes of the resulting set of bivariate distribution functions. What Proposition 5 shows is that this procedure is equivalent to considering the *natural extension* of the associated coherent lower probabilities, and then take its associated bivariate p-box. In other words, the following diagram commutes:

$$
\begin{array}{ccc}
\underline{P}_{\mathrm{X}}, \underline{P}_{\mathrm{Y}} & \xrightarrow{\text{Natural extension}} & \underline{E} = \underline{P}_{(\underline{F}, \overline{F})} \\
\Big\uparrow{\scriptstyle \text{Eqs. (11), (12)}} & & \Big\uparrow{\scriptstyle \text{Eq. (3)}} \\
(\underline{F}_{\mathrm{X}}, \overline{F}_{\mathrm{X}}), (\underline{F}_{\mathrm{Y}}, \overline{F}_{\mathrm{Y}}) & \xrightarrow{(C_{\mathrm{L}}, C_{\mathrm{M}})} & (\underline{F}, \overline{F})
\end{array}
$$

3.3. Independent products of random variables. Next, we consider another case of interest: that where the variables X, Y are assumed to be independent. Under imprecise information, there is more than one way to model the notion of independence; see [2] for a survey on this topic. Because of this, there is more than one manner in which we can say that a coherent lower prevision $\underline{P}$ on the product space is an *independent product* of its marginals $\underline{P}_{\mathrm{X}}, \underline{P}_{\mathrm{Y}}$. Since the formalism considered in this paper can be embedded into the theory of coherent lower previsions, here we shall consider the notions of *epistemic irrelevance* and *independence*, which seem to be more sound under the behavioural interpretation that is at the core of this theory.

The study of independence under imprecision suffers from a number of drawbacks when the underlying possibility spaces are infinite [13]. Because of this fact, we shall consider that the variables X, Y under study take values in respective finite spaces $\mathcal{X}, \mathcal{Y}$. Then the available information about these variables is given by a coherent lower prevision $\underline{P}$ on $\mathcal{L}(\mathcal{X} \times \mathcal{Y})$. We shall denote by $\underline{P}_{\mathrm{X}}, \underline{P}_{\mathrm{Y}}$ its respective marginals on $\mathcal{L}(\mathcal{X}), \mathcal{L}(\mathcal{Y})$. Note that, similarly to Eq. (4), we can consider the bivariate p-box

$(\underline{F}, \overline{F})$ induced by $\underline{P}$ on $\mathcal{X} \times \mathcal{Y}$, and also the univariate p-boxes $(\underline{F}_X, \overline{F}_X), (\underline{F}_Y, \overline{F}_Y)$ induced by $\underline{P}_X, \underline{P}_Y$ on $\mathcal{X}, \mathcal{Y}$.

We say then that the random variable Y is *epistemically irrelevant* to X when

$$\underline{P}_X(f|y) := \underline{P}_X(f(\cdot, y)) \ \forall f \in \mathcal{L}(\mathcal{X} \times \mathcal{Y}), y \in \mathcal{Y}.$$

The variables X, Y are said to be *epistemically independent* when each of them is epistemically irrelevant to the other:

$$(13) \qquad \underline{P}_X(f|y) := \underline{P}_X(f(\cdot, y)) \text{ and } \underline{P}_Y(f|x) := \underline{P}_Y(f(x, \cdot))$$

for every $f \in \mathcal{L}(\mathcal{X} \times \mathcal{Y}), x \in \mathcal{X}, y \in \mathcal{Y}$.

Here a *conditional lower prevision* $\underline{P}(\cdot|\mathcal{X})$ on $\mathcal{X} \times \mathcal{Y}$ is a collection of coherent lower previsions $\{\underline{P}(\cdot|x) : x \in \mathcal{X}\}$, so that $\underline{P}(\cdot|x)$ models the available information about the outcome of (X, Y) when we know that X takes the value x.[3] Note that given $f \in \mathcal{L}(\mathcal{X} \times \mathcal{Y})$, $\underline{P}(f|\mathcal{X})$ is the gamble on $\mathcal{X} \times \mathcal{Y}$ that takes the value $\underline{P}(f|x)$ on the set $\{x\} \times \mathcal{Y}$. Analogous comments can be made with respect to $\underline{P}(\cdot|\mathcal{Y})$.

If we have a coherent lower prevision $\underline{P}$ and conditional lower previsions $\underline{P}(\cdot|\mathcal{X}), \underline{P}(\cdot|\mathcal{Y})$, we should check if the information they encompass is globally consistent. This can be done by means of the notion of *(joint) coherence* in [28, Def 7.1.4], and from this we can establish the following definition:

Definition 7. Let $\underline{P}$ be a coherent lower prevision on $\mathcal{L}(\mathcal{X} \times \mathcal{Y})$ with marginals $\underline{P}_X, \underline{P}_Y$. We say that $\underline{P}$ is an *independent product* when it is coherent with the conditional lower previsions $\underline{P}_X(\cdot|\mathcal{Y}), \underline{P}_Y(\cdot|\mathcal{X})$ derived from $\underline{P}_X, \underline{P}_Y$ by means of Eq. (13).

Given $\underline{P}_Y$ and $\underline{P}_Y$, one example of independent product is the *strong product*, given by

$$(14) \qquad \underline{P}_X \boxtimes \underline{P}_Y := \inf\{P_X \times P_Y : P_X \geq \underline{P}_X, P_Y \geq \underline{P}_Y\},$$

where $P_X \times P_Y$ refers to the linear prevision uniquely determined by[4] the finitely additive probability such that $(P_X \times P_Y)(x, y) = P_X(x) \cdot P_Y(y) \ \forall x \in \mathcal{X}, y \in \mathcal{Y}$. The strong product is the joint model satisfying the notion of *strong independence*. However, it is not the only independent product, nor is it the smallest one. In fact, the smallest independent product of the marginal coherent lower previsions $\underline{P}_X, \underline{P}_Y$ is called their *independent natural extension*, and it is given, for every gamble f on $\mathcal{X} \times \mathcal{Y}$, by

$$(\underline{P}_X \otimes \underline{P}_Y)(f)$$
$$:= \sup\{\mu : f - \mu \geq g - \underline{P}_X(g|\mathcal{Y}) + h - \underline{P}_Y(h|\mathcal{X}) \text{ for some } g, h \in \mathcal{L}(\mathcal{X} \times \mathcal{Y})\}.$$

One way of building independent products is by means of the following condition:

Definition 8. A coherent lower prevision $\underline{P}$ on $\mathcal{L}(\mathcal{X} \times \mathcal{Y})$ is called *factorising* when

$$\underline{P}(fg) = \underline{P}(f\underline{P}(g)) \ \forall f \in \mathcal{L}^+(\mathcal{X}), g \in \mathcal{L}(\mathcal{Y})$$

and

$$\underline{P}(fg) = \underline{P}(g\underline{P}(f)) \ \forall f \in \mathcal{L}(\mathcal{X}), g \in \mathcal{L}^+(\mathcal{Y}).$$

---

[3]Strictly speaking, $\underline{P}(\cdot|\mathcal{X})$ refers to the lower prevision conditional on the partition $\{\{x\} \times \mathcal{Y} : x \in \mathcal{X}\}$ of $\mathcal{X} \times \mathcal{Y}$, and we use $\underline{P}(f|x)$ to denote $\underline{P}(f|\{x\} \times \mathcal{Y})$. The reason for this is that Walley's formalism defines lower previsions conditional on partitions of the possibility space [28, Chapter 6].

[4]Recall that this is possible because we are assuming that the possibility spaces $\mathcal{X}, \mathcal{Y}$ are finite; to see that the procedure above may not work with infinite spaces, we refer to [13].

Both the independent natural extension and the strong product are factorising. Indeed, it can be proven [4, Theorem 28] that any factorising $\underline{P}$ is an independent product of its marginals, but the converse is not true. Under factorisation, the following result holds:

Proposition 6. *Let* $(\underline{F}_X, \overline{F}_X), (\underline{F}_Y, \overline{F}_Y)$ *be marginal p-boxes, and let* $\underline{P}_X, \underline{P}_Y$ *be their associated coherent lower previsions. Let* $\underline{P}$ *be a factorising coherent lower prevision on* $\mathcal{L}(\mathcal{X} \times \mathcal{Y})$ *with these marginals. Then it induces the bivariate p-box* $(\underline{F}, \overline{F})$ *given by*

$$\underline{F}(x, y) = \underline{F}_X(x) \cdot \underline{F}_Y(y) \text{ and } \overline{F}(x, y) = \overline{F}_X(x) \cdot \overline{F}_Y(y) \ \forall (x, y) \in \mathcal{X} \times \mathcal{Y}.$$

Proof. Let x^*, y^* denote the maximum elements of $\mathcal{X}, \mathcal{Y}$, respectively. Since the indicator functions of $A_{(x,y^*)}, A_{(x^*,y)}$ are non-negative gambles such that $A_{(x,y)} = A_{(x,y^*)} \cdot A_{(x^*,y)}$ and taking also into account that $\underline{P}$ is factorising and positively homogeneous, we get

$$\underline{P}(A_{(x,y)}) = \underline{P}(A_{(x,y^*)} \cdot A_{(x^*,y)}) = \underline{P}(A_{(x,y^*)}) \cdot \underline{P}(A_{(x^*,y)}) = \underline{F}_X(x) \cdot \underline{F}_Y(y).$$

Similarly, if $\overline{P}$ is the conjugate upper prevision of $\underline{P}$, given by $\overline{P}(f) = -\underline{P}(-f)$ for every $f \in \mathcal{L}(\mathcal{X} \times \mathcal{Y})$, it holds that

$$\begin{aligned}
\overline{P}(A_{(x,y)}) &= \overline{P}(A_{(x,y^*)} \cdot A_{(x^*,y)}) \\
&= -\underline{P}(A_{(x,y^*)} \cdot (-A_{(x^*,y)})) = -\underline{P}(A_{(x,y^*)} \cdot (\underline{P}(-A_{(x^*,y)}))) \\
&= -\underline{P}(-A_{(x,y^*)} \cdot (\overline{P}(A_{(x^*,y)}))) = -\underline{P}(-A_{(x,y^*)}) \cdot \overline{P}(A_{(x^*,y)}) \\
&= \overline{P}(A_{(x,y^*)}) \cdot \overline{P}(A_{(x^*,y)}) = \overline{F}_X(x) \cdot \overline{F}_Y(y). \qquad \square
\end{aligned}$$

From this it is easy to deduce that the p-box $(\underline{F}, \overline{F})$ induced by a factorising $\underline{P}$ is the envelope of the set of bivariate distribution functions

$$\{F : F(x, y) = F_X(x) \cdot F_Y(y) \text{ for } F_X \in (\underline{F}_X, \overline{F}_X), F_Y \in (\underline{F}_Y, \overline{F}_Y)\}.$$

In other words, the bivariate p-box can be obtained by applying the imprecise version of Sklar's theorem (Proposition 4) with the product copula.

Further, it has been showed in [13] that a coherent lower prevision $\underline{P}$ with marginals $\underline{P}_X, \underline{P}_Y$ is factorising if and only if it lies between the independent natural extension and the strong product:

$$(15) \qquad\qquad\qquad \underline{P}_X \otimes \underline{P}_Y \leq \underline{P} \leq \underline{P}_X \boxtimes \underline{P}_Y;$$

as Walley showed in [28, Section 9.3.4], the independent natural extension and the strong product do not coincide in general, and this means that there may be an infinite number of factorising coherent lower previsions with marginals $\underline{P}_X, \underline{P}_Y$. What Proposition 6 tells us is that all these factorising coherent lower previsions induce the same bivariate p-box: the one determined by the product copula on the marginal p-boxes.

Interestingly, this applies to other independence conditions that guarantee the factorisation, such as the Kuznetsov property [3, 4]. This would mean that any Kuznetsov product of the marginals $\underline{P}_X, \underline{P}_Y$ induces the bivariate p-box given by the product copula of the marginals.

However, not all independent products are factorising [4, Example 3], and those that do not may induce different p-boxes, as we show in the following example:

Example 2. Consider $\mathcal{X} = \mathcal{Y} = \{0,1\}$. Let P_1, P_2 be the linear previsions on $\mathcal{L}(\mathcal{X})$ given by

$$P_1(f) = 0.5f(0) + 0.5f(1), \quad P_2(f) = f(0) \; \forall f \in \mathcal{L}(\mathcal{X})$$

and let P_3, P_4 be the linear previsions on $\mathcal{L}(\mathcal{Y})$ given by

$$P_3(f) = 0.5f(0) + 0.5f(1), \quad P_4(f) = f(0) \; \forall f \in \mathcal{L}(\mathcal{Y}).$$

Consider the marginal lower previsions $\underline{P}_X := \min\{P_1, P_2\}, \underline{P}_Y := \min\{P_3, P_4\}$ on $\mathcal{L}(\mathcal{X}), \mathcal{L}(\mathcal{Y})$, respectively. Applying Eq. (14), their strong product is given by

$$\begin{aligned}
\underline{P}_X \boxtimes \underline{P}_Y &= \min\{P_1 \times P_3, P_1 \times P_4, P_2 \times P_3, P_2 \times P_4\} \\
&= \min\{(0.25, 0.25, 0.25, 0.25), (0.5, 0, 0.5, 0), (0.5, 0.5, 0, 0), (1, 0, 0, 0)\},
\end{aligned}$$

where in the equation above a vector (a, b, c, d) is used to denote the vector of probabilities $\{(P(0,0), P(0,1), P(1,0), P(1,1))\}$.

Let $\underline{P}$ be the coherent lower prevision determined by the mass functions

$$\begin{aligned}
\underline{P} :&= \min\{P_1 \times (0.5P_3 + 0.5P_4), (0.5P_1 + 0.5P_2) \times P_3, P_2 \times P_4\} \\
&= \min\{(0.375, 0.125, 0.375, 0.125), (0.375, 0.375, 0.125, 0.125), (1, 0, 0, 0)\},
\end{aligned}$$

where $(0.5P_3 + 0.5P_4)$ denotes the linear prevision on $\mathcal{L}(\mathcal{Y})$ given by

$$(0.5P_3 + 0.5P_4)(f) = 0.5P_3(f) + 0.5P_4(f) \; \forall f \in \mathcal{L}(\mathcal{Y}),$$

and similarly for $(0.5P_1 + 0.5P_2)$. Then the marginals of $\underline{P}$ are also $\underline{P}_X, \underline{P}_Y$. Moreover, since the extreme points of $\mathcal{M}(\underline{P})$ are convex combinations of those of $\mathcal{M}(\underline{P}_X \boxtimes \underline{P}_Y)$, we deduce that $\underline{P}$ dominates $\underline{P}_X \boxtimes \underline{P}_Y$. Applying [13, Proposition 5], we deduce that $\underline{P}$ is also an independent product of the marginal coherent lower previsions $\underline{P}_X, \underline{P}_Y$. Since it dominates strictly the strong product, we deduce from Eq. (15) that $\underline{P}$ is not factorising.

Now, since

$$\underline{P}(\{(0,0)\}) = 0.375 > 0.25 = (\underline{P}_X \boxtimes \underline{P}_Y)(\{(0,0)\}),$$

we see that the p-boxes associated with $\underline{P}$ and $\underline{P}_X \boxtimes \underline{P}_Y$ differ. We conclude thus that not all independent products induce the bivariate p-box that is the product copula of its marginals. ♦

Remark 2. Interestingly, we can somehow distinguish between the strong product and the independent natural extension in terms of bivariate p-boxes, in the following way: if we consider the set of bivariate distribution functions

$$\mathcal{F} := \{F_X \times F_Y : F_X \in (\underline{F}_X, \overline{F}_X), F_Y \in (\underline{F}_Y, \overline{F}_Y)\},$$

then it follows from Eq. (14) that

(16) $$\underline{P}_X \boxtimes \underline{P}_Y := \inf\{P : F_P \in \mathcal{F}\}.$$

This differs from the coherent lower prevision given by

$$\underline{P} := \min\{P : F_P \in (\underline{F}_X \cdot \underline{F}_Y, \overline{F}_X \cdot \overline{F}_Y)\},$$

which will be in general more imprecise than the independent natural extension $\underline{P}_X \otimes \underline{P}_Y$. Moreover, a characterisation similar to Eq. (16) cannot be made for the

independent natural extension, in the sense that there is no set of copulas $\mathcal{C}$ such that

$$\underline{P}_X \otimes \underline{P}_Y := \inf\{P : F_P = C(F_X, F_Y)$$
$$\text{for some } C \in \mathcal{C}, F_X \in (\underline{F}_X, \overline{F}_X), F_Y \in (\underline{F}_Y, \overline{F}_Y)\};$$

indeed, just by considering the precise case we see that $\mathcal{C}$ should consist just of the product copula, and this would give back the definition of the strong product. ♦

4. Stochastic orders and copulas

Next, we are going to apply the previous results to characterize the preferences encoded by p-boxes. To this end, let us first of all recall some basic notions on stochastic orders (see [12, 18, 25] for more information):

Definition 9. Given two univariate random variables X and Y with respective distribution functions F_X and F_Y, we say that X *stochastically dominates* Y, and denote it $X \succeq_{SD} Y$, when $F_X(t) \leq F_Y(t)$ for any t.

This is one of the most extensively used methods for the comparison of random variables. It is also called *first order stochastic dominance*, so as to distinguish it from the (weaker) notions of second, third, ..., n-th order stochastic dominance.

An alternative for the comparison of random variables is statistical preference.

Definition 10 ([5, 6]). Given two univariate random variables X and Y, X is said to be *statistically preferred* to Y if $P(X \geq Y) \geq P(Y \geq X)$. This is denoted by $X \succeq_{SP} Y$.

This notion is particularly interesting when the variables X, Y take values in a qualitative scale [8].

In addition to comparing pairs of random variables, or, more generally, couples of 'elements', with a preorder relation, we may be interested in comparing pairs of sets (of random variables or other 'elements') by means of the given order relation. We can consider several different possibilities:

Definition 11. Let $\succeq$ be a preorder over a set S. Given $A, B \subseteq S$, we say that:
  (1) $A \succeq_1 B$ if and only if for every $a \in A$, $b \in B$ it holds that $a \succeq b$.
  (2) $A \succeq_2 B$ if and only if there exists some $a \in A$ such that $a \succeq b$ for every $b \in B$.
  (3) $A \succeq_3 B$ if and only if for every $b \in B$ there is some $a \in A$ such that $a \succeq b$.
  (4) $A \succeq_4 B$ if and only if there are $a \in A$, $b \in B$ such that $a \succeq b$.
  (5) $A \succeq_5 B$ if and only if there is some $b \in B$ such that $a \succeq b$ for every $a \in A$.
  (6) $A \succeq_6 B$ if and only if for every $a \in A$ there is $b \in B$ such that $a \succeq b$.

The relations $\succeq_i$ in Definition 11 have been discussed in [17] in the case that $\succeq$ is the stochastic dominance relation $\succeq_{SD}$ and in [16] in the case of statistical preference, showing that several of them are related to decision criteria explored in the literature of imprecise probabilities.

Figure 1 illustrates some of these extensions. In Figure 1a, $A \succeq_1 B$ because all the alternatives in A are better than all the alternatives in B; in Figure 1b, $A \succeq_2 B$ because there is an optimal element in A, a_1, that is preferred to all the alternatives in B; Figure 1c shows an example of $A \succeq_4 B$ because there are alternatives $a_1 \in A$ and $b_2 \in B$ such that $a_1 \succeq b_2$; finally, Figure 1d shows an example of $A \succeq_5 B$

FIGURE 1. Examples of the extensions of $\succeq_i$. In this picture $a_i \to b_j$ means $a_i \succeq b_j$.

because there is a worst element in B, b_1 that is dominated by all the elements in A. The difference between the second and the third extensions (resp., fifth and sixth) lies in the existence of a maximum (resp., minimum) or a supremum (resp., infimum) element in A (resp., B).

4.1. **Univariate orders.** Although stochastic dominance does not imply statistical preference in general[5], in the univariate case a number of sufficient conditions have been established for the implication, in terms of the copula that determines the joint distribution from the marginal ones. This is for instance the case when:

(SD-SP1): X and Y are stochastically independent random variables, i.e., they are linked by the product copula (see [5, 7, 15]);

(SD-SP2): X and Y are absolutely continuous random variables and they are coupled by an Archimedean copula (see [14]).

(SD-SP3): X, Y are either comonotonic or countercomonotonic, and they are both either simple or absolutely continuous.

In such cases, the implication transfers to the relations comparing sets of random variables, by means of the following lemma. Its proof is immediate and therefore omitted.

Lemma 3. *Let $\succeq$ be a preorder in a set S and $A, B \subseteq S$. Let also $\sqsupseteq$ be a preorder that extends $\succeq$, i.e. $x \succeq y \Rightarrow x \sqsupseteq y \; \forall x, y \in S$. Then, $A \succeq_i B \Rightarrow A \sqsupseteq_i B$ for all $i = 1, \ldots, 6$.*

Here A, B are sets of random variables, denoted $\mathcal{V}_X$, $\mathcal{V}_Y$. The following special case of Lemma 3 is an instance.

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[5]Consider for instance the case where the joint distribution is given by $P(X = 0, Y = 0.5) = 0.2, P(X = 0.5, Y = 0) = P(X = 1, Y = 0) = P(X = 0.5, Y = 1) = 0.1$ and $P(X = 1, Y = 1) = 0.5$. Then X and Y are equivalent with respect to stochastic dominance because their cumulative distribution functions coincide; however, Y is strictly statistically preferred to X.

Proposition 7. *Consider two sets of random variables $\mathcal{V}_X, \mathcal{V}_Y$. Assume that any $X \in \mathcal{V}_X, Y \in \mathcal{V}_Y$ satisfy one of the conditions (SD-SP1)÷(SD-SP3) above. Then, for all $i = 1, \ldots, 6$:*

$$\mathcal{V}_X \succeq_{SD_i} \mathcal{V}_Y \Rightarrow \mathcal{V}_X \succeq_{SP_i} \mathcal{V}_Y.$$

Proof. As we have remarked, conditions (SD-SP1)÷(SD-SP3) above ensure that the statistical preference relation is an extension of stochastic dominance. The result follows from Lemma 3. □

4.2. **Bivariate orders.** Next we consider the following extension of stochastic dominance to the bivariate case:

Definition 12. Let $X = (X_1, X_2)$ and $Y = (Y_1, Y_2)$ be two random vectors with respective bivariate distribution functions F_{X_1, X_2} and F_{Y_1, Y_2}. We say that (X_1, X_2) *stochastically dominates* (Y_1, Y_2), and denote it $(X_1, X_2) \succeq_{SD} (Y_1, Y_2)$, if $F_{X_1, X_2}(s, t) \leq F_{Y_1, Y_2}(s, t)$ for all $(s, t) \in \mathbb{R}^2$.

This definition establishes a way of comparing two bivariate vectors $X = (X_1, X_2)$, $Y = (Y_1, Y_2)$ in case their associated distribution functions are precisely known. However, it is not uncommon to have uncertain information about these distribution functions, that we can model by means of respective sets of distribution functions $\mathcal{F}_X, \mathcal{F}_Y$. If we now take Definition 11 into account, we can propose a generalisation of Definition 12 to the imprecise case:

Definition 13. Let $X = (X_1, X_2)$ and $Y = (Y_1, Y_2)$ be two random vectors with respective sets of bivariate distribution functions $\mathcal{F}_X, \mathcal{F}_Y$. We say that (X_1, X_2) *i-stochastically dominates* (Y_1, Y_2), and denote it $(X_1, X_2) \succeq_{SD_i} (Y_1, Y_2)$, if $\mathcal{F}_X \leq_i \mathcal{F}_Y$.

Since by Remark 1 copulas can be interpreted as bivariate distribution functions, the extensions $\leq_i$ are also applicable to them.

Note that the sets of distribution functions $\mathcal{F}_X, \mathcal{F}_Y$ may be obtained by combining two respective marginal p-boxes by means of a set of copulas. In that case, we may study to which extent the relationships between the sets $\mathcal{F}_X, \mathcal{F}_Y$ can be determined by means of the relationships between their marginal univariate p-boxes. In other words, if we have information stating that X_1 stochastically dominates Y_1 and X_2 stochastically dominates Y_2, we may wonder in which cases the pair (X_1, X_2) *i*-stochastically dominates (Y_1, Y_2). The following result gives an answer to this question:

Proposition 8. *Given two random vectors $X = (X_1, X_2)$, $Y = (Y_1, Y_2)$, let $(\underline{F}_{X_1}, \overline{F}_{X_1}), (\underline{F}_{X_2}, \overline{F}_{X_2}), (\underline{F}_{Y_1}, \overline{F}_{Y_1}), (\underline{F}_{Y_2}, \overline{F}_{Y_2})$ be the marginal p-boxes associated with X_1, X_2, Y_1, Y_2 respectively. Let $\mathcal{C}_X$ and $\mathcal{C}_Y$ be two sets of copulas. Define the following sets of bivariate distribution functions $\mathcal{F}_X, \mathcal{F}_Y$:*

$$\mathcal{F}_X := \{C(F_{X_1}, F_{X_2}) : C \in \mathcal{C}_X, F_{X_1} \in (\underline{F}_{X_1}, \overline{F}_{X_1}), F_{X_2} \in (\underline{F}_{X_2}, \overline{F}_{X_2})\},$$
$$\mathcal{F}_Y := \{C(F_{Y_1}, F_{Y_2}) : C \in \mathcal{C}_Y, F_{Y_1} \in (\underline{F}_{Y_1}, \overline{F}_{Y_1}), F_{Y_2} \in (\underline{F}_{Y_2}, \overline{F}_{Y_2})\}.$$

Consider $i \in \{1, \ldots, 6\}$ and assume that $(\underline{F}_{X_j}, \overline{F}_{X_j}) \leq_i (\underline{F}_{Y_j}, \overline{F}_{Y_j})$ for $j = 1, 2$. Then:

$$\mathcal{C}_X \leq_i \mathcal{C}_Y \Rightarrow (X_1, X_2) \succeq_{SD_i} (Y_1, Y_2).$$

Proof. $(i = 1)$ We know that:

$$\forall F_{X_j} \in (\underline{F}_{X_j}, \overline{F}_{X_j}), F_{Y_j} \in (\underline{F}_{Y_j}, \overline{F}_{Y_j}), F_{X_j} \le F_{Y_j}, (j = 1, 2);$$
$$\forall C_X \in \mathcal{C}_X, C_Y \in \mathcal{C}_Y, C_X \le C_Y,$$

Consider $F_X \in \mathcal{F}_X$ and $F_Y \in \mathcal{F}_Y$. They can be expressed in the following way: $F_X(x, y) = C_X(F_{X_1}(x), F_{X_2}(y))$ and $F_Y(x, y) = C_Y(F_{Y_1}(x), F_{Y_2}(y))$, where $C_X \le C_Y$. Then:

$$\begin{aligned} F_X(x, y) &= C_X(F_{X_1}(x), F_{X_2}(y)) \le C_X(F_{Y_1}(x), F_{Y_2}(y)) \\ &\le C_Y(F_{Y_1}(x), F_{Y_2}(y)) = F_Y(x, y), \end{aligned}$$

where the inequalities hold because copulas are component-wise increasing.

$(i = 2)$ We know that:

$$\exists F^*_{X_j} \in (\underline{F}_{X_j}, \overline{F}_{X_j}) \text{ s.t. } F^*_{X_j} \le F_{Y_j} \quad \forall F_{Y_j} \in (\underline{F}_{Y_j}, \overline{F}_{Y_j}), (j = 1, 2).$$
$$\exists C^*_X \in \mathcal{C}_X \text{ s.t. } C^*_X \le C_Y \quad \forall C_Y \in \mathcal{C}_Y.$$

Consider $F_X(x, y) := C^*_X(F^*_{X_1}(x), F^*_{X_2}(y))$, and let us see that $F_X \le F_Y$ for any $F_Y = C_Y(F_{Y_1}, F_{Y_2})$ in $\mathcal{F}_Y$:

$$\begin{aligned} F_X(x, y) &= C^*_X(F^*_{X_1}(x), F^*_{X_2}(y)) \le C^*_X(F_{Y_1}(x), F_{X_2}(y)) \\ &\le C_Y(F_{Y_1}(x), F_{X_2}(y)) = F_Y(x, y). \end{aligned}$$

$(i = 3)$ We know that:

$$\forall F_{Y_j} \in (\underline{F}_{Y_j}, \overline{F}_{Y_j}), \exists F^*_{X_j} \in (\underline{F}_{X_j}, \overline{F}_{X_j}) \text{ s.t. } F^*_{X_j} \le F_{Y_j}, (j = 1, 2).$$
$$\forall C_Y \in \mathcal{C}_Y \exists C^*_X \in \mathcal{C}_X \text{ s.t. } C^*_X \le C_Y.$$

Let $F_Y \in \mathcal{F}_Y$. Then, there are $C_Y \in \mathcal{C}_Y, F_{Y_1} \in (\underline{F}_{Y_1}, \overline{F}_{Y_1})$ and $F_{Y_2} \in (\underline{F}_{Y_2}, \overline{F}_{Y_2})$ such that $F_Y(x, y) = C_Y(F_{Y_1}(x), F_{Y_2}(y))$. Let us check that there is F_X in $\mathcal{F}_X$ such that $F_X \le F_Y$. Let $F_X(x, y) = C^*_X(F^*_{X_1}(x), F^*_{X_2}(y))$. Then:

$$\begin{aligned} F_X(x, y) &= C^*_X(F^*_{X_1}(x), F^*_{X_2}(y)) \le C^*_X(F_{Y_1}(x), F_{Y_2}(y)) \\ &\le C_Y(F_{Y_1}(x), F_{Y_2}(y)) = F_Y(x, y). \end{aligned}$$

$(i = 4)$ We know that:

$$\exists F^*_{X_j} \in (\underline{F}_{X_j}, \overline{F}_{X_j}), F^*_{Y_j} \in (\underline{F}_{Y_j}, \overline{F}_{Y_j}) \text{ s.t. } F^*_{X_j} \le F^*_{Y_j}, (j = 1, 2).$$
$$\exists C^*_X \in \mathcal{C}_X, C^*_Y \in \mathcal{C}_Y \text{ s.t. } C^*_X \le C^*_Y.$$

Let us consider the distribution functions $F_X(x, y) = C^*_X(F^*_{X_1}(x), F^*_{X_2}(y))$ and $F_Y(x, y) = C^*_Y(F^*_{Y_1}(x), F^*_{Y_2}(y))$. It holds that $F_X \le F_Y$:

$$\begin{aligned} F_X(x, y) &= C^*_X(F^*_{X_1}(x), F^*_{X_2}(y)) \le C^*_X(F^*_{Y_1}(x), F^*_{Y_2}(y)) \\ &\le C^*_Y(F^*_{Y_1}(x), F^*_{Y_2}(y)) = F_Y(x, y). \end{aligned}$$

$(i = 5, i = 6)$ The proof of these two cases is analogous to that of $i = 2$ and $i = 3$, respectively. $\square$

4.3. Natural extension and independent products.
To conclude this section, we consider the particular cases discussed in Sections 3.2 and 3.3: those where the bivariate p-box is the natural extension or a factorising product.

By Proposition 5, the natural extension of two marginal p-boxes $(\underline{F}_X, \overline{F}_X)$ and $(\underline{F}_Y, \overline{F}_Y)$ is given by:

$$\tag{17} \underline{F}(x, y) = C_L(\underline{F}_X(x), \underline{F}_Y(y)) \text{ and } \overline{F}(x, y) = C_M(\overline{F}_X(x), \overline{F}_Y(y)).$$

This allows us to prove the following result:

Corollary 1. *Consider marginal p-boxes* $(\underline{F}_{X_1}, \overline{F}_{X_1}), (\underline{F}_{X_2}, \overline{F}_{X_2}), (\underline{F}_{Y_1}, \overline{F}_{Y_1})$ *and* $(\underline{F}_{Y_2}, \overline{F}_{Y_2})$. *Let* $(\underline{F}_X, \overline{F}_X)$ *(resp.,* $(\underline{F}_Y, \overline{F}_Y))$ *be the natural extension of the p-boxes* $(\underline{F}_{X_1}, \overline{F}_{X_1}), (\underline{F}_{X_2}, \overline{F}_{X_2})$ *(resp.,* $(\underline{F}_{Y_1}, \overline{F}_{Y_1}), (\underline{F}_{Y_2}, \overline{F}_{Y_2}))$ *by means of Eq. (17). Then for* $i = 2, \ldots, 6$,

$$(\underline{F}_{X_j}, \overline{F}_{X_j}) \leq_i (\underline{F}_{Y_j}, \overline{F}_{Y_j}), j = 1, 2 \Rightarrow (X_1, X_2) \succeq_{SD_i} (Y_1, Y_2).$$

Proof. Take $\mathcal{C}_X = \mathcal{C}_Y = \{C_L, C_M\}$ in Proposition 8. Since $C_L \leq C_L$, $C_M \leq C_M$ and $C_L \leq C_M$, we get $\mathcal{C}_X \leq_i \mathcal{C}_Y$, $i = 2, \ldots, 6$. Then, Proposition 8 ensures $\{\underline{F}_X, \overline{F}_X\} \leq_i \{\underline{F}_Y, \overline{F}_Y\}$ $(i = 2, \ldots, 6)$. It is not difficult to check then that this implies also $(\underline{F}_X, \overline{F}_X) \leq_i (\underline{F}_Y, \overline{F}_Y)$ $(i = 2, \ldots, 6)$, because of the special form of $\mathcal{C}_X, \mathcal{C}_Y$. $\square$

To see that the result does not hold for $\leq_1$, consider the following example:

Example 3. For $j = 1, 2$, let $\underline{F}_{X_j} = \overline{F}_{X_j} = \underline{F}_{Y_j} = \overline{F}_{Y_j}$ be the distribution function associated with the uniform probability distribution on $[0, 1]$, given by $F(x) = x$ for every $x \in [0, 1]$. Then trivially

$$F = (\underline{F}_{X_j}, \overline{F}_{X_j}) \leq_1 (\underline{F}_{Y_j}, \overline{F}_{Y_j}) = F \ \forall j = 1, 2.$$

However, $(\underline{F}_X, \overline{F}_X) \not\leq_1 (\underline{F}_Y, \overline{F}_Y)$, since $C_M(F, F) \in (\underline{F}_X, \overline{F}_X), C_L(F, F) \in (\underline{F}_Y, \overline{F}_Y)$ and

$$C_M(F, F)(0.5, 0.5) = C_M(F(0.5), F(0.5)) = C_M(0.5, 0.5) = 0.5 > 0$$
$$= C_L(0.5, 0.5) = C_L(F(0.5), F(0.5)) = C_L(F, F)(0.5, 0.5). \blacklozenge$$

On the other hand, Proposition 6 implies that, given two finite spaces $\mathcal{X}, \mathcal{Y}$, any factorising coherent lower prevision $\underline{P}$ on $\mathcal{L}(\mathcal{X} \times \mathcal{Y})$ determines a bivariate p-box that is the product of its marginal p-boxes by means of the product copula. Taking this property into account, we can compare two factorising independent products in terms of the relationships between their marginals. From Proposition 8, we deduce the following:

Corollary 2. *Consider marginal p-boxes* $(\underline{F}_{X_1}, \overline{F}_{X_1}), (\underline{F}_{Y_1}, \overline{F}_{Y_1}), (\underline{F}_{X_2}, \overline{F}_{X_2})$ *and* $(\underline{F}_{Y_2}, \overline{F}_{Y_2})$, *and let us define the following sets of bivariate distribution functions* $\mathcal{F}_X, \mathcal{F}_Y$ *by*

$$\mathcal{F}_X := \{F_{X_1} \cdot F_{X_2} : F_{X_1} \in (\underline{F}_{X_1}, \overline{F}_{X_1}), F_{X_2} \in (\underline{F}_{X_2}, \overline{F}_{X_2})\},$$
$$\mathcal{F}_Y := \{F_{Y_1} \cdot F_{Y_2} : F_{Y_1} \in (\underline{F}_{Y_1}, \overline{F}_{Y_1}), F_{Y_2} \in (\underline{F}_{Y_2}, \overline{F}_{Y_2})\}.$$

Then, for $i = 1, \ldots, 6$,

$$(\underline{F}_{X_j}, \overline{F}_{X_j}) \leq_i (\underline{F}_{Y_j}, \overline{F}_{Y_j}), j = 1, 2 \Rightarrow (X_1, X_2) \succeq_{SD_i} (Y_1, Y_2).$$

Proof. The result is the particular case of Proposition 8 where $\mathcal{C}_X = \mathcal{C}_Y = \{C_P\}$. $\square$

5. Conclusions and open problems

In this work we have studied the extension of Sklar's theorem to an imprecise framework, where instead of random variables precisely described by their distribution functions, we have considered the case when they are imprecisely described by p-boxes. For this aim, we have introduced the notion of imprecise copula, and have proven that if we link two marginal p-boxes by means of a set of copulas we obtain a bivariate p-box whose associated lower probability is coherent. Unfortunately,

the main implication of Sklar's theorem does not hold in the imprecise framework: there exist coherent bivariate p-boxes that are not uniquely determined by their marginals.

We have investigated two particular cases: on the one hand, we considered the absence of information about the copula that links the marginals. In that case, we end up with the natural extension of the marginal p-boxes, that can be expressed in terms of the Łukasiewicz and the minimum copulas. On the other hand, we looked upon the case where the marginal distributions satisfy the condition of epistemic independence, and showed that the joint p-box can be obtained in most, but not all cases, by means of the product copula.

There are a few open problems that arise from our work in this paper: on the one hand, we should deepen the study of the properties of imprecise copulas from the point of view of aggregation operators. With respect to Sklar's theorem, we intend to look for sufficient conditions for a bivariate p-box to be determined as an imprecise copula of its marginals. A third open problem would be the study in the imprecise case of the other extensions of stochastic dominance to the bivariate case, based on the comparisons of survival functions or expectations. Finally, it would be interesting to generalize our results to the n-variate case. An interesting work in this respect was carried out by Durante and Spizzichino in [9].

Acknowledgements

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