Hongliang Lai, Dexue Zhang^{*}, Gao Zhang

School of Mathematics, Sichuan University, Chengdu 610064, China

Abstract

This paper presents a comparative study of three kinds of ideals in fuzzy order theory: forward Cauchy ideals (generated by forward Cauchy nets), flat ideals and irreducible ideals, including their role in connecting fuzzy order with fuzzy topology.

Keywords: Fuzzy order, Fuzzy topology, Forward Cauchy ideal, Flat ideal, Irreducible ideal, Scott Q-topology, Scott Q-cotopology

1. Introduction

The notion of ideals (i.e., directed lower sets) in ordered sets is primitive in domain theory. Domains and Scott topology are both postulated in terms of ideals and their suprema. For a partially ordered set P, let Idl(P) denote the set of ideals in P with the inclusion order, and $\mathbf{y}: P \longrightarrow Idl(P)$ be the map that assigns each $x \in P$ to the principal ideal $\downarrow x$. Then P is directed complete if \mathbf{y} has a left adjoint sup : $Idl(P) \longrightarrow P$ (which sends each ideal to its supremum); P is a domain if it is directed complete and the left adjoint of \mathbf{y} also has a left adjoint. A Scott open set of P is an upper set U such that for each ideal I in P, if the supremum of I is in U then I intersects with U.

In order to establish a theory of fuzzy domains (or, quantitative domains), the first step is to find an appropriate notion of ideals for fuzzy orders (or, Q-orders, where Qis the truth-value quantale). The problem seems simple, but, it turns out to be a very intricate one because of the complication of the table of truth-values — the quantale Q. In fact, there are several natural extension of this notion to the fuzzy setting. This paper presents a comparative study of three kinds of them: forward Cauchy ideals, flat ideals and irreducible ideals.

Before summarizing related attempts in the literature and explaining what we will do in this paper, we recall some equivalent reformulations of ideals in a partially ordered set. Let P be a partially ordered set. A net $\{x_i\}$ in P is eventually monotone if there is some i such that $x_j \leq x_k$ whenever $i \leq j \leq k$ [9]. Let I be a non-empty lower set in P. The following are equivalent:

- I is an ideal, that is, for any x, y in I, there is some $z \in I$ such that $x, y \leq z$.
- There exists an eventually monotone net $\{x_i\}$ such that $I = \bigcup_i \bigcap_{i \ge i} \downarrow x_j$.

 $^{^{\}diamond}$ This work is supported by National Natural Science Foundation of China (No. 11771310)

^{*}Corresponding author.

Email addresses: hllai@scu.edu.cn (Hongliang Lai), dxzhang@scu.edu.cn (Dexue Zhang), gaozhang0810@hotmail.com (Gao Zhang)

- I is flat in the sense that for any upper sets G, H of P, if I intersects with both G and H, then I intersects with $G \cap H$.
- I is *irreducible* in the sense that for any lower sets B, C of P, if $I \subseteq B \cup C$ then either $I \subseteq B$ or $I \subseteq C$.

The net-approach is extended to fuzzy orders in [3, 36, 37], resulting in the notions of forward Cauchy net and Yoneda completeness (a.k.a liminf completeness). Fuzzy lower sets generated by forward Cauchy nets are called ideals in [6, 7]. They will be called forward Cauchy ideals in this paper, in order to distinguish them from flat ideals and irreducible ideals. Yoneda completeness, as a version of quantitative directed completeness, has received wide attention in the study of fuzzy orders, including generalized metric spaces as a special case, see e.g. [3, 6, 7, 8, 10, 15, 16, 21, 24, 31, 37].

The extension of the flat-approach to the fuzzy setting originates in the work of Vickers [34] in the case the truth-value quantale is Lawvere's quantale $([0,\infty]^{\text{op}},+)$ (which is isomorphic to the unit interval with the product t-norm). This approach results in the notions of flat ideal (called flat left module in [34]) and flat completeness of fuzzy orders. It is shown in [34] that for Lawvere's quantale, flat completeness is equivalent to Yoneda completeness.

The recent paper [44] extends the irreducible-approach to the fuzzy setting in the study of sobriety of fuzzy cotopological spaces, resulting in the notions of irreducible ideal and irreducible completeness of fuzzy orders.

Forward Cauchy ideals, flat ideals and irreducible ideals in a fuzzy ordered set are all natural generalizations of the notion of ideals in a partially ordered set; the resulting completeness for fuzzy orders are natural extensions of directed completeness in order theory. We note in passing that, from a category theory perspective, such completeness for fuzzy orders is an example of the theory of cocompleteness in enriched category theory with respect to a class of weights [1, 17, 18].

This paper aims to present a comparative study of forward Cauchy ideals, flat ideals and irreducible ideals, hence of the resulting completeness notions. Since all of them are intended to play the role of directed lower sets in fuzzy order theory, before comparing them with each other, we propose the following criteria for a class Φ of fuzzy sets that are meant for the role of ideals in fuzzy orders:

- (I1) If the truth-value quantale Q is the two-element Boolean algebra, then for each partially ordered set A, $\Phi(A)$ is the set of ideals in A. This is to require that Φ is a generalization of the class of ideals.
- (I2) Φ is saturated. Saturatedness of Φ guarantees that for each Q-ordered set A, $\Phi(A)$ is the free Φ -continuous Q-ordered set generated by A. So, for a saturated class Φ of fuzzy sets, there exist enough Φ -continuous Q-ordered sets.
- (I3) Φ generates a functor from the category of Q-ordered sets and Φ -cocontinuous maps to that of Q-topological spaces and/or Q-cotopological spaces. This functor is expected to play the role of the functor in domain theory that sends each partially ordered set to its Scott topology. As in the classical case, such functors are of fundamental importance in the theory of fuzzy domains.

Besides the interrelationship among the classes of forward Cauchy ideals, flat ideals and irreducible ideals, their connection to fuzzy topology will also be discussed.

The contents are arranged as follows.

Section 2 recalls some basic ideas that are needed in the subsequent sections.

Section 3 concerns the relationship among forward Cauchy ideals, flat ideals and irreducible ideals. The main results are: (i) Every forward Cauchy ideal is flat (irreducible, resp.) if and only if the truth-value quantale is meet continuous (dually meet continuous, resp.). (ii) For the quantale obtained by equipping [0, 1] with a left continuous t-norm, irreducible ideals coincide with forward Cauchy ideals. (iii) For a prelinear quantale, every irreducible ideal is flat. (iv) For a quantale that satisfies the law of double negation, flat ideals coincide with irreducible ideals.

Section 4 proves that for every quantale, both the class of flat ideals and that of irreducible ideals are saturated. As for forward Cauchy ideals, it is shown in [8] that for a completely distributive value quantale (see [7, 8] for definition), the class of forward Cauchy ideals is saturated. The conclusion is extended in [23] to the case that Q is a continuous and integral quantale.

Section 5 concerns the connection between fuzzy orders and fuzzy topological spaces. For each subclass Φ of flat ideals, a functor is constructed from the category of Q-ordered sets and Φ -cocontinuous maps to that of stratified Q-topological spaces. For each subclass Φ of irreducible ideals, a full functor is constructed from the category of Q-ordered sets and Φ -cocontinuous maps to that of stratified Q-cotopological spaces. This shows that irreducible ideals can be used to generate closed sets, hence Q-cotopologies, whereas flat ideals can be used to generate open sets, hence Q-topologies. We would like to remind the reader that, in general, there is no natural way to switch between closed sets and open sets in the fuzzy setting. This lack of "duality" between closed sets and open sets demands that we need different kinds of fuzzy ideals to connect fuzzy orders with fuzzy topological spaces and/or fuzzy cotopological spaces. This is the *raison d'etre* for flat ideals and irreducible ideals.

2. Preliminaries

In this preliminary section, we recall briefly some basic ideas of complete lattices [9], quantales [30], and Q-orders that will be needed.

A quantale Q is a monoid in the monoidal category of complete lattices and joinpreserving maps [30]. Explicitly, a quantale Q is a monoid (Q, &) such that Q is a complete lattice and

$$p\&\bigvee_{j\in J}q_j=\bigvee_{j\in J}p\&q_j,\ \Big(\bigvee_{j\in J}q_j\Big)\&p=\bigvee_{j\in J}q_j\&p.$$

for all $p \in Q$ and $\{q_j\}_{j \in J} \subseteq Q$. The unit 1 of the monoid (Q, &) is in general not the top element of Q. If it happens that the unit element coincides with the top element of Q, then we say that Q is *integral*. If the operation & is commutative then we say Q is a commutative quantale. A quantale (Q, &) is meet continuous if the underlying lattice Q is meet continuous.

Standing Assumption. Throughout this paper, if not otherwise specified, all quantales are assumed to be integral and commutative.

Since the semigroup operation & distributes over arbitrary joins, it determines a binary operation \rightarrow on Q via the adjoint property

$$p\&q \leq r \iff q \leq p \to r.$$

The binary operation \rightarrow is called the *implication*, or the *residuation*, corresponding to &.

Some basic properties of the binary operations & and \rightarrow are collected below, they can be found in many places, e.g. [2, 30].

Proposition 2.1. Let Q be a quantale. Then

(1)
$$1 \to p = p$$
.
(2) $p \le q \iff 1 = p \to q$.
(3) $p \to (q \to r) = (p\&q) \to r$.
(4) $p\&(p \to q) \le q$.
(5) $\left(\bigvee_{j\in J} p_j\right) \to q = \bigwedge_{j\in J} (p_j \to q)$.
(6) $p \to \left(\bigwedge_{j\in J} q_j\right) = \bigwedge_{j\in J} (p \to q_j)$.
(7) $p = \bigwedge_{q\in Q} ((p \to q) \to q)$.

We often write $\neg p$ for $p \to 0$ and call it the *negation* of p. Though it is true that $p \leq \neg \neg p$ for all $p \in Q$, the inequality $\neg \neg p \leq p$ does not always hold. A quantale Q satisfies the *law of double negation* if

$$(p \to 0) \to 0 = p$$

for all $p \in Q$.

Proposition 2.2. ([2]) Suppose that Q is a quantale that satisfies the law of double negation. Then

(1) $p \to q = \neg (p \& \neg q) = \neg q \to \neg p.$ (2) $p \& q = \neg (q \to \neg p) = \neg (p \to \neg q).$ (3) $\neg (\bigwedge_{i \in I} p_i) = \bigvee_{i \in I} \neg p_i.$

The quantales with the unit interval [0, 1] as underlying lattice are of particular interest in fuzzy set theory [11, 19]. In this case, the semigroup operation & is exactly a left continuous t-norm on [0, 1] [19]. A continuous t-norm on [0, 1] is a left continuous t-norm & that is continuous with respect to the usual topology.

Example 2.3. ([19]) Some basic t-norms:

(1) The t-norm min: $a\&b = a \land b = \min\{a, b\}$. The corresponding implication is given by

$$a \to b = \begin{cases} 1, & a \le b; \\ b, & a > b. \end{cases}$$

(2) The product t-norm: $a\&b = a \cdot b$. The corresponding implication is given by

$$a \to b = \begin{cases} 1, & a \le b; \\ b/a, & a > b. \end{cases}$$

(3) The Łukasiewicz t-norm: $a\&b = \max\{a+b-1,0\}$. The corresponding implication is given by

$$a \to b = \min\{1, 1 - a + b\}.$$

In this case, ([0, 1], &) satisfies the law of double negation.

(4) The nilpotent minimum t-norm:

$$a\&b = \begin{cases} 0, & a+b \le 1; \\ \min\{a,b\}, & a+b > 1. \end{cases}$$

The corresponding implication is given by

$$a \to b = \begin{cases} 1, & a \le b;\\ \max\{1-a, b\}, & a > b. \end{cases}$$

In this case, ([0, 1], &) satisfies the law of double negation.

The following theorem, known as the ordinal sum decomposition theorem, is of fundamental importance in the theory of continuous t-norms.

Theorem 2.4. ([5, 28]) For each continuous t-norm & on [0, 1], there is a set of disjoint open intervals $\{(a_i, b_i)\}$ of [0, 1] that satisfy the following conditions:

- (i) For each i, both a_i and b_i are idempotent and the restriction of & on [a_i, b_i] is either isomorphic to the Lukasiewicz t-norm or to the product t-norm;
- (ii) $x \& y = \min\{x, y\}$ if $(x, y) \notin \bigcup_i [a_i, b_i]^2$.

A Q-order (or an order valued in the quantale Q) [36, 42] on a set A is a reflexive and transitive Q-relation on A. Explicitly, a Q-order on A is a map $R : A \times A \longrightarrow Q$ such that R(x,x) = 1 and $R(y,z)\&R(x,y) \leq R(x,z)$ for any $x, y, z \in A$. The pair (A, R) is called a Q-ordered set. A Q-ordered set is also called a Q-category in the literature, since it is precisely a category enriched over the symmetric monoidal category Q. As usual, we write A for the pair (A, R) and A(x, y) for R(x, y) if no confusion would arise.

Two elements x, y in a Q-ordered set A are *isomorphic* if A(x, y) = A(y, x) = 1. We say that A is *separated* if isomorphic elements in A are equal, that is, A(x, y) = A(y, x) = 1 implies that x = y.

If $R : A \times A \longrightarrow Q$ is a Q-order on A, then $R^{\text{op}} : A \times A \longrightarrow Q$, given by $R^{\text{op}}(x, y) = R(y, x)$, is also a Q-order on A (by commutativity of &), called the opposite of R.

Example 2.5. This example belongs to the folklore in fuzzy order theory, see e.g. [2]. For all $p, q \in Q$, let

$$d_L(p,q) = p \to q.$$

Then (Q, d_L) is a separated Q-ordered set. The opposite of (Q, d_L) is (Q, d_R) , where

$$d_R(p,q) = q \to p.$$

Both (Q, d_L) and (Q, d_R) play important roles in the theory of Q-ordered sets.

Example 2.6. [2] Let X be a set. A map $\lambda : X \longrightarrow Q$ is called a fuzzy set (valued in Q) of X, the value $\lambda(x)$ is interpreted as the membership degree of x. The map

$$\operatorname{sub}_X: Q^X \times Q^X \longrightarrow Q,$$

given by

$$\operatorname{sub}_X(\lambda,\mu) = \bigwedge_{x \in X} \lambda(x) \to \mu(x),$$

defines a separated \mathcal{Q} -order on Q^X . Intuitively, the value $\operatorname{sub}_X(\lambda, \mu)$ measures the degree that λ is a subset of μ . Thus, sub_X is called the *fuzzy inclusion order* on Q^X . The opposite of sub_X is called the *converse fuzzy inclusion order* on Q^X . In particular, if Xis a singleton set then the \mathcal{Q} -ordered sets $(Q^X, \operatorname{sub}_X)$ and $(Q^X, \operatorname{sub}_X^{\operatorname{op}})$ degenerate to the \mathcal{Q} -ordered sets (Q, d_L) and (Q, d_R) , respectively.

A map $f: A \longrightarrow B$ between \mathcal{Q} -ordered sets is \mathcal{Q} -order preserving if

$$A(x_1, x_2) \le B(f(x_1), f(x_2))$$

for any $x_1, x_2 \in A$. We write

$\mathcal{Q} ext{-}\mathsf{Ord}$

for the category of Q-ordered sets and Q-order preserving maps.

Let $f : A \longrightarrow B$ and $g : B \longrightarrow A$ be Q-order preserving maps. We say f is left adjoint to g (or, g is right adjoint to f), $f \dashv g$ in symbols, if

$$A(x, g(y)) = B(f(x), y)$$

for all $x \in A$ and $y \in B$.

Let A, B be Q-ordered sets. A Q-distributor $\phi : A \longrightarrow B$ from A to B is a map $\phi : A \times B \longrightarrow Q$ such that

$$B(b,b')\&\phi(a,b)\&A(a',a) \le \phi(a',b')$$

for any $a, a' \in A$ and $b, b' \in B$. Roughly speaking, a \mathcal{Q} -distributor $\phi : A \longrightarrow B$ is a \mathcal{Q} -relation between A and B that is compatible with the \mathcal{Q} -orders on A and B.

It is easy to see that the set \mathcal{Q} -Dist(A, B) of all \mathcal{Q} -distributors from A to B form a complete lattice under the pointwise order.

Example 2.7. (Fuzzy lower sets as Q-distributors) A fuzzy lower set [22] of a Q-ordered set A is a map $\phi : A \longrightarrow Q$ such that

$$\phi(y)\&A(x,y) \le \phi(x).$$

It is obvious that $\phi : A \longrightarrow Q$ is a fuzzy lower set if and only if $\phi : A \longrightarrow (Q, d_R)$ preserves Q-order.

Dually, a fuzzy upper set [22] of A is a map $\psi: A \longrightarrow Q$ such that

$$A(x,y)\&\psi(x) \le \psi(y)$$

It is clear that $\psi : A \longrightarrow Q$ is a fuzzy upper set if and only if $\psi : A \longrightarrow (Q, d_L)$ preserves Q-order.

If we write * for the terminal object in the category \mathcal{Q} -Ord, namely, * is a \mathcal{Q} -ordered set with only one element, then for each fuzzy lower set $\phi : A \longrightarrow Q$ of A, the map

$$\phi^{\neg} : A \times \{*\} \longrightarrow Q, \quad \phi^{\neg}(x,*) = \phi(x)$$

is a \mathcal{Q} -distributor $\phi^{\neg} : A \longrightarrow *$. This establishes a bijection between fuzzy lower sets of A and \mathcal{Q} -distributors from A to *. For each fuzzy upper set ψ of A, the map

$$\ulcorner\psi: \{*\} \times A \longrightarrow Q, \quad \ulcorner\psi(*, x) = \psi(x)$$

is a \mathcal{Q} -distributor $\psi : * \longrightarrow A$. This establishes a bijection between fuzzy upper sets of A and \mathcal{Q} -distributors from * to A.

Lemma 2.8. Let ϕ be a fuzzy lower set (fuzzy upper set, resp.) of a Q-ordered set A, $p \in Q$.

- (1) Both $p\&\phi$ and $p \to \phi$ are fuzzy lower sets (fuzzy upper sets, resp.) of A.
- (2) $\phi \to p$ is a fuzzy upper set (fuzzy lower set, resp.) of A and $\phi = \bigwedge_{a \in O} (\phi \to q) \to q$).

Let $\mathcal{P}A$ denote the set of fuzzy lower sets of A endowed with the fuzzy inclusion order. Explicitly, elements in $\mathcal{P}A$ are \mathcal{Q} -order preserving maps $A \longrightarrow (Q, d_R)$, and

$$\mathcal{P}A(\phi_1,\phi_2) = \operatorname{sub}_A(\phi_1,\phi_2) = \bigwedge_{x \in A} (\phi_1(x) \to \phi_2(x)).$$

Dually, let $\mathcal{P}^{\dagger}A$ denote the set of fuzzy upper sets of A endowed with the *converse* fuzzy inclusion order. Explicitly, elements in $\mathcal{P}^{\dagger}A$ are \mathcal{Q} -order preserving maps $A \longrightarrow (Q, d_L)$, and

$$\mathcal{P}^{\dagger}A(\psi_1,\psi_2) = \operatorname{sub}_A(\psi_2,\psi_1) = \bigwedge_{x \in A} (\psi_2(x) \to \psi_1(x)).$$

It is clear that $(\mathcal{P}^{\dagger}A)^{\mathrm{op}} = \mathcal{P}(A^{\mathrm{op}})$ [32].

For each $a \in A$, A(-, a) is a fuzzy lower set of A. Moreover,

$$\mathcal{P}A(A(-,a),\phi) = \phi(a)$$

for all $a \in A$ and $\phi \in \mathcal{P}A$. This fact is indeed a special case of the Yoneda lemma in enriched category theory. The Yoneda lemma ensures that the assignment $a \mapsto A(-,a)$ defines an embedding $\mathbf{y} : A \longrightarrow \mathcal{P}A$, known as the Yoneda embedding.

The correspondence $A \mapsto \mathcal{P}A$ gives rise to a functor $\mathcal{P} : \mathcal{Q}\text{-}\mathsf{Ord} \longrightarrow \mathcal{Q}\text{-}\mathsf{Ord}$ that sends a $\mathcal{Q}\text{-}\mathsf{order}$ preserving map $f : A \longrightarrow B$ to $\mathcal{P}f = f^{\rightarrow} : \mathcal{P}A \longrightarrow \mathcal{P}B$, where

$$f^{\rightarrow}(\phi)(y) = \bigvee_{x \in A} \phi(x) \& B(y, f(x)).$$

Moreover, $f^{\rightarrow} : \mathcal{P}A \longrightarrow \mathcal{P}B$ has a right adjoint given by $f^{\leftarrow} : \mathcal{P}B \longrightarrow \mathcal{P}A$, where $f^{\leftarrow}(\psi) = \psi \circ f$. This means for all $\phi \in \mathcal{P}A$ and $\psi \in \mathcal{P}B$,

$$\operatorname{sub}_B(f^{\to}(\phi),\psi) = \operatorname{sub}_A(\phi,f^{\leftarrow}(\psi)).$$
(2.1)

The adjunction $f^{\rightarrow} \dashv f^{\leftarrow}$ is a special case of the enriched Kan extension in category theory [17, 25].

For \mathcal{Q} -distributors $\phi : A \longrightarrow B$ and $\psi : B \longrightarrow C$, the composite $\psi \circ \phi : A \longrightarrow C$ is given by

$$(\psi \circ \phi)(a,c) = \bigvee_{b \in B} \psi(b,c) \& \phi(a,b).$$

It is clear that $(\mathcal{Q}\text{-Dist}(*,*),\circ)$ is a quantale and is isomorphic to $\mathcal{Q} = (Q,\&)$. In this paper, we identify $(\mathcal{Q}\text{-Dist}(*,*),\circ)$ with \mathcal{Q} .

For a fuzzy lower set $\phi : A \longrightarrow Q$ and a fuzzy upper set $\psi : A \longrightarrow Q$ of a Q-ordered set A, the tensor product

$$\phi \otimes \psi$$

is defined as the composite of Q-distributors

$$\phi^{\neg} \circ \ulcorner \psi : * \longrightarrow A \longrightarrow * .$$

Explicitly, $\phi \otimes \psi$ is an element of the quantale \mathcal{Q} given by $\phi \otimes \psi = \bigvee_{x \in A} \phi(x) \& \psi(x)$. Intuitively, the value $\phi \otimes \psi$ measures the degree that the fuzzy lower set ϕ intersects with the fuzzy upper set ψ .

The correspondence

$$(\psi,\phi)\mapsto\phi\otimes\psi$$

defines a Q-distributor

 $\otimes: \mathcal{P}^{\dagger}A \longrightarrow \mathcal{P}A.$

In particular, for each fuzzy upper set ψ of A, the correspondence $\phi \mapsto \phi \otimes \psi$ defines a fuzzy upper set of $\mathcal{P}A$:

$$-\otimes \psi: \mathcal{P}A \longrightarrow Q.$$
 (2.2)

The following lemma exhibits a close relationship between the Q-distributor \otimes : $\mathcal{P}^{\dagger}A \longrightarrow \mathcal{P}A$ (intersection degree) and the fuzzy inclusion order (subset degree).

Lemma 2.9. Let A be a Q-ordered set.

(1) For each fuzzy lower set ϕ and each fuzzy upper set ψ of A,

$$\phi \otimes \psi = \bigwedge_{p \in Q} (\operatorname{sub}_A(\phi, \psi \to p) \to p).$$

In particular, if Q satisfies the law of double negation, then $\phi \otimes \psi = \neg(\operatorname{sub}_A(\phi, \neg \psi))$. (2) For any fuzzy lower sets ϕ_1, ϕ_2 of A,

$$\operatorname{sub}_A(\phi_1,\phi_2) = \bigwedge_{p \in Q} (\phi_1 \otimes (\phi_2 \to p) \to p).$$

In particular, if Q satisfies the law of double negation, then $\operatorname{sub}_A(\phi_1, \phi_2) = \neg(\phi_1 \otimes (\neg \phi_2))$.

Proof. (1) By Proposition 2.1(7), it holds that

$$\begin{split} \phi \otimes \psi &= \bigvee_{x \in A} \phi(x) \& \psi(x) \\ &= \bigwedge_{p \in Q} \left[\left(\left(\bigvee_{x \in A} \phi(x) \& \psi(x) \right) \to p \right) \to p \right] \\ &= \bigwedge_{p \in Q} \left[\bigwedge_{x \in A} (\phi(x) \to (\psi(x) \to p)) \to p \right] \\ &= \bigwedge_{p \in Q} (\operatorname{sub}_A(\phi, \psi \to p) \to p). \end{split}$$

(2) The proof is similar, so, we omit it here.

A supremum of a fuzzy lower set ϕ of a Q-ordered set A is an element of A, say sup ϕ , such that

$$A(\sup\phi, x) = \sup_A(\phi, \mathbf{y}(x))$$

for all $x \in A$. It is clear that, up to isomorphism, every fuzzy lower set has at most one supremum. So, we'll speak of the supremum of a fuzzy lower set. A Q-order preserving map $f : A \longrightarrow B$ preserves the supremum of a fuzzy lower set ϕ of A if, whenever $\sup \phi$ exists, $f(\sup \phi)$ is a supremum of $f^{\rightarrow}(\phi)$. It is well-known that left adjoints preserve suprema.

Example 2.10. [32] Let A be a Q-ordered set. Then every fuzzy lower set of $\mathcal{P}A$ has a supremum. Actually, for each fuzzy lower set Λ of $\mathcal{P}A$, $\sup \Lambda = \bigvee_{\phi \in \mathcal{P}A} \Lambda(\phi) \& \phi$.

Example 2.11. (Intersection degree as supremum) For each fuzzy lower set ϕ and each fuzzy upper set ψ of a \mathcal{Q} -ordered set A, the intersection degree of ϕ with ψ is the supremum of $\psi^{\rightarrow}(\phi)$ in (Q, d_L) (recall that $\psi : A \longrightarrow (Q, d_L)$ is a \mathcal{Q} -order preserving map), i.e., $\phi \otimes \psi = \sup \psi^{\rightarrow}(\phi)$. This is because for all $q \in Q$,

$$\operatorname{sub}_Q(\psi^{\to}(\phi), d_L(-, q)) = \operatorname{sub}_A(\phi, d_L(\psi(-), q))$$
$$= \bigwedge_{x \in A} (\phi(x) \to (\psi(x) \to q))$$
$$= d_L \Big(\bigvee_{x \in A} \phi(x) \& \psi(x), q\Big)$$
$$= d_L(\phi \otimes \psi, q).$$

In particular, letting ψ be the identity map on (Q, d_L) one obtains that for each fuzzy lower set ϕ of (Q, d_L) , $\sup \phi = \bigvee_{q \in Q} q \& \phi(q)$.

Example 2.12. (Inclusion degree as supremum) For any fuzzy lower sets ϕ, λ of a Q-ordered set A, the inclusion degree $\operatorname{sub}_A(\phi, \lambda)$ is the supremum of $\lambda^{\rightarrow}(\phi)$ in (Q, d_R) (recall that $\lambda : A \longrightarrow (Q, d_R)$ is a Q-order preserving map), i.e., $\operatorname{sub}_A(\phi, \lambda) = \sup \lambda^{\rightarrow}(\phi)$. This is because for all $q \in Q$,

$$sub_Q(\lambda^{\rightarrow}(\phi), d_R(-, q)) = sub_A(\phi, d_R(\lambda(-), q))$$
$$= \bigwedge_{x \in A} (\phi(x) \to (q \to \lambda(x)))$$
$$= \bigwedge_{x \in A} (q \to (\phi(x) \to \lambda(x)))$$
$$= d_R(sub_A(\phi, \lambda), q).$$

In particular, letting λ be the identity map on (Q, d_R) one obtains that for each fuzzy lower set ϕ of (Q, d_R) , the supremum of ϕ in (Q, d_R) is given by $\bigwedge_{q \in Q} (\phi(q) \to q)$.

3. Forward Cauchy ideals, flat ideals and irreducible ideals

A net $\{x_i\}$ in a \mathcal{Q} -ordered set A is forward Cauchy [37] if

$$\bigvee_{i} \bigwedge_{1 \le j \le k} A(x_j, x_k) = 1.$$

Forward Cauchy nets are clearly a Q-analogue of eventually monotone nets in partially ordered sets. A Yoneda limit (a.k.a liminf) [37] of a forward Cauchy net $\{x_i\}$ in A is an element a in A such that

$$A(a,y) = \bigvee_{i} \bigwedge_{i \le j} A(x_j,y)$$

for all $y \in A$. It is clear that Yoneda limit is a Q-version of *least eventual upper bound*. Yoneda limits of a forward Cauchy net, if exist, are unique up to isomorphism. **Lemma 3.1.** If $\{a_i\}$ is a forward Cauchy net in (Q, d_L) , then $\bigvee_i \bigwedge_{j\geq i} a_j$ is a Yoneda limit of $\{a_i\}$ and

$$\bigvee_i \bigwedge_{j \ge i} a_j = \bigwedge_i \bigvee_{j \ge i} a_j.$$

Proof. The first half is [37, Proposition 2.30]. It remains to check the equality

$$\bigvee_i \bigwedge_{j \ge i} a_j = \bigwedge_i \bigvee_{j \ge i} a_j.$$

Since $\bigvee_i \bigwedge_{j \ge i} a_j$ is a Yoneda limit of $\{a_i\}$, it follows that for all $x \in Q$,

$$\left(\bigvee_{i} \bigwedge_{j \ge i} a_{j}\right) \to x = d_{L}\left(\bigvee_{i} \bigwedge_{j \ge i} a_{j}, x\right)$$
$$= \bigvee_{i} \bigwedge_{j \ge i} d_{L}(x, a_{j})$$
$$= \bigvee_{i} \bigwedge_{j \ge i} (a_{j} \to x)$$
$$\leq \left(\bigwedge_{i} \bigvee_{j \ge i} a_{j}\right) \to x.$$

Letting $x = \bigvee_i \bigwedge_{j \ge i} a_j$ we obtain that

$$\bigvee_i \bigwedge_{j \ge i} a_j \ge \bigwedge_i \bigvee_{j \ge i} a_j.$$

The inequality ' \leq ' is trivial, so, the equality is valid.

Proposition 3.2. (A special case of [37, Theorem 3.1]) For each forward Cauchy net $\{\phi_i\}$ in $\mathcal{P}A$, the fuzzy lower set $\bigvee_i \bigwedge_{j\geq i} \phi_j$ is a Yoneda limit of $\{\phi_i\}$. That is, for each fuzzy lower set ϕ of A,

$$\operatorname{sub}_A\left(\bigvee_i \bigwedge_{j\geq i} \phi_j, \phi\right) = \bigvee_i \bigwedge_{j\geq i} \operatorname{sub}_A(\phi_j, \phi).$$

The following proposition says that every Yoneda limit of forward Cauchy net $\{x_i\}$ is a supremum of a fuzzy lower set generated by $\{x_i\}$.

Proposition 3.3. ([8, Lemma 46]) An element a in a \mathcal{Q} -ordered set A is a Yoneda limit of a forward Cauchy net $\{x_i\}$ if and only if a is a supremum of the fuzzy lower set $\bigvee_i \bigwedge_{i < j} A(-, x_j)$ generated by $\{x_i\}$.

A fuzzy set $\lambda : A \longrightarrow Q$ is *inhabited* if $\bigvee_{a \in A} \lambda(a) = 1$. Inhabited fuzzy sets are counterpart of non-empty sets in the fuzzy setting.

Definition 3.4. Let A be a \mathcal{Q} -ordered set, $\phi : A \longrightarrow Q$ a fuzzy lower set of A.

(1) ϕ is a forward Cauchy ideal if there exists a forward Cauchy net $\{x_i\}$ in A such that

$$\phi = \bigvee_{i} \bigwedge_{i \le j} A(-, x_j).$$

(2) ϕ is a flat ideal if it is inhabited and is flat in the sense that

$$\phi \otimes (\psi_1 \wedge \psi_2) = \phi \otimes \psi_1 \wedge \phi \otimes \psi_2$$

for all fuzzy upper sets ψ_1, ψ_2 of A.

(3) ϕ is an irreducible ideal if it is inhabited and is irreducible in the sense that

$$\operatorname{sub}_A(\phi, \phi_1 \lor \phi_2) = \operatorname{sub}_A(\phi, \phi_1) \lor \operatorname{sub}_A(\phi, \phi_2)$$

for all fuzzy lower sets ϕ_1, ϕ_2 of A.

Remark 3.5. Forward Cauchy ideals, flat ideals and irreducible ideals in Q-ordered sets are all natural extensions of ideals in a partially ordered set. The study of forward Cauchy ideals dates back to Wagner [36, 37]. For more information on forward Cauchy ideals the reader is referred to [6, 7, 8, 15, 16, 23, 45], besides the works of Wagner. The notion of flat ideals originates in the paper [34] of Vickers in the case that Q is Lawvere's quantale $([0, \infty]^{\text{op}}, +)$, under the name of *flat left module*. It is extended to the general case in [35]. It is shown in [35] that if the quantale Q = (Q, &) is a frame, i.e., $\& = \land$, then a fuzzy lower set ϕ of a Q-ordered set A is flat if and only if for any $x, y \in A$,

$$\phi(x) \land \phi(y) \le \bigvee_{z \in A} \phi(z) \land A(x, z) \land A(y, z).$$

Hence, in the case that Q = (Q, &) is a frame, flat ideals in a Q-ordered set A coincides with ideals of A in the sense of [23, Definition 5.1]. Irreducible ideals are introduced in [44] in the study of sobriety of Q-cotopological spaces.

Example 3.6. For each a in a Q-ordered set A, the fuzzy lower set A(-, a) is a forward Cauchy ideal, a flat ideal and an irreducible ideal.

Definition 3.7. ([1, 18, 23]) By a class of weights we mean a functor $\Phi : \mathcal{Q}$ -Ord $\longrightarrow \mathcal{Q}$ -Ord such that

- (1) for each Q-ordered set A, $\Phi(A)$ is a subset of $\mathcal{P}A$ with the Q-order inherited from $\mathcal{P}A$;
- (2) for all \mathcal{Q} -ordered set A and all $a \in A$, $\mathbf{y}(a) \in \Phi(A)$;
- (3) $\Phi(f) = \mathcal{P}f = f^{\rightarrow}$ for every \mathcal{Q} -order preserving map $f : A \longrightarrow B$.

The second condition ensures that A can be embedded in $\Phi(A)$ via the Yoneda embedding. We also write **y** for the embedding $A \longrightarrow \Phi(A)$ if no confusion will arise.

In category theory, a Q-distributor of the form $A \rightarrow *$ is called a *weight* or a *presheaf* [18, 32]. This accounts for the terminology *class of weights*.

Together with Example 3.6 the following conclusion asserts that forward Cauchy ideals, flat ideals and irreducible ideals are all examples of class of weights.

Proposition 3.8. If $f : A \longrightarrow B$ is Q-order preserving, then for each forward Cauchy ideal (flat ideal, irreducible ideal, resp.) ϕ of A, $f^{\rightarrow}(\phi)$ is a forward Cauchy ideal (flat ideal, irreducible ideal, resp.) of B.

Proof. We check, for example, that if ϕ is irreducible then so is $f^{\rightarrow}(\phi)$. For all fuzzy lower sets ϕ_1, ϕ_2 of B, thanks to Equation (2.1), we have

$$sub_B(f^{\rightarrow}(\phi), \phi_1 \lor \phi_2) = sub_A(\phi, (\phi_1 \lor \phi_2) \circ f)$$

= sub_A(\phi, \phi_1 \circ f) \bigstarrow sub_A(\phi, \phi_2 \circ f)
= sub_B(f^{\rightarrow}(\phi), \phi_1) \bigstarrow sub_B(f^{\rightarrow}(\phi), \phi_2),

hence $f^{\rightarrow}(\phi)$ is irreducible.

In the following, we write \mathcal{W}, \mathcal{I} and \mathcal{F} for the class of forward Cauchy ideals, irreducible ideals and flat ideals, respectively.

Definition 3.9. Let Φ be a class of weights. A Q-ordered set A is Φ -complete¹ if each $\phi \in \Phi(A)$ has a supremum. In particular, a Q-ordered set A is

- Yoneda complete (a.k.a liminf complete) if each forward Cauchy ideal of A has a supremum (which is equivalent to that every forward Cauchy net in A has a Yoneda limit);
- (2) irreducible complete if each irreducible ideal of A has a supremum;
- (3) flat complete if each flat ideal of A has a supremum.

Yoneda complete, irreducible complete, and flat complete are all natural extension of *directed complete* to the fuzzy setting. In the case that Q = (Q, &) is a frame, based on flat completeness (under the name of *fuzzy directed completeness*), a theory of frame-valued directed complete orders and frame-valued domains have been developed in [27, 39, 40, 41].

It is easily seen that A is Φ -complete if and only if $\mathbf{y} : A \longrightarrow \Phi(A)$ has a left adjoint. In this case, the left adjoint of \mathbf{y} sends each $\phi \in \Phi(A)$ to its supremum $\sup \phi$. A \mathcal{Q} -order preserving map $f : A \longrightarrow B$ is Φ -cocontinuous if for all $\phi \in \Phi(A)$, $f(\sup \phi) = \sup f^{\rightarrow}(\phi)$ whenever $\sup \phi$ exists.

This section mainly concerns the relationship among the class \mathcal{W} of forward Cauchy ideals, the class \mathcal{I} of irreducible ideals, and the class \mathcal{F} of flat ideals.

Given classes of weights Φ and Ψ , we say that Φ is a subclass of Ψ if $\Phi(A) \subseteq \Psi(A)$ for each Q-ordered set A. As we shall see, under some mild assumptions, \mathcal{W} is a subclass of both \mathcal{I} and \mathcal{F} .

A complete lattice L is meet continuous [9] if for all $a \in L$ and all directed subset D of L,

$$a \land \bigvee D = \bigvee_{d \in D} (a \land d).$$

A complete lattice is dually meet continuous if its opposite is meet continuous. A quantale Q = (Q, &) is (dually, resp.) meet continuous if the complete lattice Q is (dually, resp.) meet continuous.

Theorem 3.10. For a dually meet continuous quantale Q, every forward Cauchy ideal is irreducible.

¹ Φ -cocomplete would be a better terminology from the viewpoint of category theory. However, following the tradition in domain theory, we choose Φ -complete here.

Lemma 3.11. If $\{a_i\}$ is a forward Cauchy net in the Q-ordered set (Q, d_R) , then $\bigwedge_i \bigvee_{j \ge i} a_j$ is a Yoneda limit of $\{a_i\}$ and

$$\bigvee_i \bigwedge_{j \ge i} a_j = \bigwedge_i \bigvee_{j \ge i} a_j.$$

Proof. First, we show that $\bigwedge_i \bigvee_{j \ge i} a_j$ is a Yoneda limit of $\{a_i\}$. That is, for all $x \in Q$,

$$d_R\left(\bigwedge_i\bigvee_{j\geq i}a_j,x\right)=\bigvee_i\bigwedge_{j\geq i}d_R(a_j,x)$$

On one hand, since $\{a_i\}$ is a forward Cauchy net in (Q, d_R) ,

$$\bigvee_{i} \bigwedge_{1 \le j \le l} (a_l \to a_j) = \bigvee_{i} \bigwedge_{1 \le j \le l} d_R(a_j, a_l) = 1,$$

then

$$\bigvee_{i} \bigwedge_{j \ge i} \left[\bigvee_{k} \bigwedge_{l \ge k} (a_{l} \to a_{j}) \right] = 1,$$

hence

$$\bigvee_{i} \bigwedge_{j \ge i} \left[\left(\bigwedge_{k} \bigvee_{l \ge k} a_{l} \right) \to a_{j} \right] = 1.$$

Thus,

$$d_R\Big(\bigwedge_i \bigvee_{j \ge i} a_j, x\Big) = \Big[x \to \bigwedge_k \bigvee_{l \ge k} a_l\Big] \& \bigvee_i \bigwedge_{j \ge i} \Big[\Big(\bigwedge_k \bigvee_{l \ge k} a_l\Big) \to a_j\Big)\Big]$$
$$\leq \bigvee_i \bigwedge_{j \ge i} (x \to a_j)$$
$$= \bigvee_i \bigwedge_{j \ge i} d_R(a_j, x).$$

On the other hand, since for each i we always have

$$x \to \bigvee_{j \ge i} a_j \ge \bigvee_k \bigwedge_{l \ge k} (x \to a_l),$$

it follows that

$$d_R\Big(\bigwedge_i \bigvee_{j \ge i} a_j, x\Big) = \bigwedge_i \Big(x \to \bigvee_{j \ge i} a_j\Big) \ge \bigvee_k \bigwedge_{l \ge k} (x \to a_l) = \bigvee_i \bigwedge_{j \ge i} d_R(a_j, x).$$

Therefore, $\bigwedge_i \bigvee_{j \ge i} a_j$ is a Yoneda limit of $\{a_i\}$. Next, we prove the equality

$$\bigvee_{i} \bigwedge_{j \ge i} a_j = \bigwedge_{i} \bigvee_{j \ge i} a_j.$$

Since $\bigwedge_i \bigvee_{j \ge i} a_j$ is a Yoneda limit of $\{a_i\}$ in (Q, d_R) , it follows that for all $x \in Q$,

$$x \to \bigwedge_{i} \bigvee_{j \ge i} a_{j} = d_{R} \Big(\bigwedge_{i} \bigvee_{j \ge i} a_{j}, x \Big)$$
$$= \bigvee_{i} \bigwedge_{j \ge i} d_{R}(a_{j}, x)$$
$$= \bigvee_{i} \bigwedge_{j \ge i} (x \to a_{j})$$
$$\leq x \to \bigvee_{i} \bigwedge_{j \ge i} a_{j}.$$

Letting x = 1, we obtain that

$$\bigwedge_i \bigvee_{j \ge i} a_j \le \bigvee_i \bigwedge_{j \ge i} a_j.$$

The converse inequality is trivial, so the equality is valid.

Lemma 3.1 and Lemma 3.11 imply that every forward Cauchy net in the Q-ordered sets (Q, d_L) and (Q, d_R) is order convergent. But, (Q, d_L) and (Q, d_R) may have different forward Cauchy nets. For example, the sequence $\{n\}$ is forward Cauchy in $([0, \infty], d_R)$ but not in $([0, \infty], d_L)$.

Proof of Theorem 3.10. Let $\{x_i\}$ be a forward Cauchy net in a \mathcal{Q} -ordered set A and $\varphi = \bigvee_i \bigwedge_{j \ge i} A(-, x_j)$. We show that φ is an irreducible ideal.

Step 1. φ is inhabited. This is easy since

$$\bigvee_{x \in A} \varphi(x) \ge \bigvee_{i} \varphi(x_{i}) = \bigvee_{i} \bigvee_{j} \bigwedge_{k \ge j} A(x_{i}, x_{k}) \ge \bigvee_{i} \bigwedge_{k \ge i} A(x_{i}, x_{k}) = 1.$$

Step 2. For each fuzzy lower set ϕ of A,

$$\operatorname{sub}_A(\varphi, \phi) = \bigwedge_i \bigvee_{j \ge i} \phi(x_j).$$

Since ϕ is a fuzzy lower set, $\{\phi(x_i)\}$ is a forward Cauchy net in (Q, d_R) . Then,

$$sub_{A}(\varphi, \phi) = sub_{A}\left(\bigvee_{i} \bigwedge_{j \ge i} A(-, x_{j}), \phi\right)$$

$$= \bigvee_{i} \bigwedge_{j \ge i} sub_{A}(A(-, x_{j}), \phi) \qquad (Proposition 3.2)$$

$$= \bigvee_{i} \bigwedge_{j \ge i} \phi(x_{j}) \qquad (Yoneda \ lemma)$$

$$= \bigwedge_{i} \bigvee_{j \ge i} \phi(x_{j}). \qquad (Lemma 3.11)$$

Step 3. For all fuzzy lower sets ϕ_1, ϕ_2 of A, $\operatorname{sub}_A(\varphi, \phi_1 \lor \phi_2) = \operatorname{sub}_A(\varphi, \phi_1) \lor \operatorname{sub}_A(\varphi, \phi_2)$.

This is easy since

$$\operatorname{sub}_{A}(\varphi,\phi_{1}) \vee \operatorname{sub}_{A}(\varphi,\phi_{2}) = \bigwedge_{i} \bigvee_{j \ge i} \phi_{1}(x_{j}) \vee \bigwedge_{i} \bigvee_{j \ge i} \phi_{2}(x_{j})$$
(Step 2)
$$= \bigwedge \bigvee (\phi_{1}(x_{j}) \vee \phi_{2}(x_{j}))$$
(Q is dually meet continuous)

$$= \bigwedge_{i} \bigvee_{j \ge i} (\phi_1(x_j) \lor \phi_2(x_j)) \qquad (\mathcal{Q} \text{ is dually meet continuous})$$

$$= \operatorname{sub}_A(\varphi, \phi_1 \lor \phi_2). \tag{Step 2}$$

The proof is completed.

Interestingly, the dual meet continuity of \mathcal{Q} is also necessary for Theorem 3.10.

Proposition 3.12. If all forward Cauchy ideals are irreducible, then the quantale Q is dually meet continuous.

Proof. We show that for each $a \in Q$ and each filtered set F in Q,

$$a \lor \bigwedge_{x \in F} x = \bigwedge_{x \in F} (a \lor x)$$

Consider the fuzzy lower set $\phi = \bigvee_{x \in F} d_R(-, x)$ of the \mathcal{Q} -ordered set (Q, d_R) . Since

$$\phi = \bigvee_{x \in F} \bigwedge_{y \in F, y \le x} d_R(-, y),$$

it follows that ϕ is a forward Cauchy ideal, hence an irreducible ideal by assumption.

Since both the identity map id_Q on Q and the constant map $\underline{a}: Q \longrightarrow Q$ with value a are fuzzy lower sets of (Q, d_R) , then

$$\begin{aligned} a \lor \bigwedge_{x \in F} x &= \mathrm{sub}_Q(\phi, \underline{a}) \lor \mathrm{sub}_Q(\phi, \mathrm{id}_Q) \\ &= \mathrm{sub}_Q(\phi, \underline{a} \lor \mathrm{id}_Q) \\ &= \bigwedge_{x \in F} (a \lor x). \end{aligned}$$

This finishes the proof.

Irreducible ideals need not be forward Cauchy in general. Let $\mathcal{Q} = \{0, a, b, 1\}$ be the Boolean algebra with four elements. Assume that A is the \mathcal{Q} -ordered set with points x, y and

$$A(x, x) = A(y, y) = 1, \quad A(x, y) = A(y, x) = 0.$$

Then the map ϕ , given by $\phi(x) = a$ and $\phi(y) = b$, is an irreducible ideal in A. But, ϕ cannot be generated by any forward Cauchy net in A. This example is essentially [44, Note 3.12].

The following conclusion is very useful in the theory of fuzzy orders based on left continuous t-norms, it says every irreducible ideal is forward Cauchy in this case.

Theorem 3.13. If the quantale Q is the unit interval equipped with a left continuous t-norm &, then irreducible ideals coincide with forward Cauchy ideals.

Proof. By Theorem 3.10, we only need to prove that every irreducible ideal ϕ of a Q-ordered set A is forward Cauchy.

Let

$$C\phi = \{ (x, r) \in X \times [0, 1) \mid \phi(x) > r \}.$$

Define a relation \sqsubseteq on $C\phi$ by

$$(x,r) \sqsubseteq (y,s) \iff A(x,y) \to r \le s.$$

We claim that $(C\phi, \sqsubseteq)$ is a directed set. Before proving this, we note that if $(x, r) \sqsubseteq (y, s)$ then r < A(x, y) and $r \leq s$. That \sqsubseteq is reflexive and transitive is easy, it remains to check that it is directed. For any $(x, r), (y, s) \in C\phi$, consider the fuzzy lower sets $\psi_1 = A(x, -) \rightarrow r$ and $\psi_2 = A(y, -) \rightarrow s$. Since ϕ is an irreducible ideal,

$$sub_X(\phi, \psi_1 \lor \psi_2) = sub_X(\phi, A(x, -) \to r) \lor sub_X(\phi, A(y, -) \to s)$$
$$= sub_X(A(x, -), \phi \to r) \lor sub_X(A(y, -), \phi \to s)$$
$$= (\phi(x) \to r) \lor (\phi(y) \to s).$$

Since $(\phi(x) \to r) \lor (\phi(y) \to s) < 1$, there exists some z such that

$$\phi(z) \to \left[(A(x,z) \to r) \lor (A(y,z) \to s) \right] < 1.$$

Let $t = (A(x, z) \to r) \lor (A(y, z) \to s)$, then $(z, t) \in C\phi$ and $(x, r) \sqsubseteq (z, t), (y, s) \sqsubseteq (z, t)$. Hence $(C\phi, \sqsubseteq)$ is a directed set.

From now on, we also write an element in $C\phi$ as a pair (x_i, r_i) . Define a net

$$\mathfrak{x}: \mathbf{C}\phi \longrightarrow A$$

by $\mathfrak{x}(x_i, r_i) = x_i$. We prove in two steps that \mathfrak{x} is a forward Cauchy net and it generates ϕ , hence ϕ is a forward Cauchy ideal.

Step 1. r is forward Cauchy.

Let t < 1. Since ϕ is inhabited, there is some $(x_i, r_i) \in C\phi$ such that $t \leq r_i$. Then $A(x_j, x_k) \to r_j \leq r_k < 1$ whenever $(x_k, r_k) \supseteq (x_j, r_j) \supseteq (x_i, r_i)$, hence

$$A(x_j, x_k) > r_j \ge r_i \ge t.$$

By arbitrariness of t we obtain that \mathfrak{x} is forward Cauchy.

Step 2. ϕ is generated by \mathfrak{x} , i.e.,

$$\phi(x) = \bigvee_{(x_i, r_i)} \bigwedge_{(x_j, r_j) \supseteq (x_i, r_i)} A(x, x_j)$$

for all $x \in A$.

Take $x \in A$ and $r < \phi(x)$. For all $(x_j, r_j) \in C\phi$, if $(x, r) \sqsubseteq (x_j, r_j)$, then $A(x, x_j) > r$, hence, by arbitrariness of r,

$$\phi(x) \leq \bigvee_{r < \phi(x)} \bigwedge_{(x_j, r_j) \sqsupseteq (x, r)} A(x, x_j) \leq \bigvee_{(x_i, r_i)} \bigwedge_{(x_j, r_j) \sqsupseteq (x_i, r_i)} A(x, x_j).$$

For the converse inequality, we show that for each $(x_i, r_i) \in C\phi$,

$$\phi(x) \ge \bigwedge_{(x_j, r_j) \supseteq (x_i, r_i)} A(x, x_j).$$

Let t be an arbitrary number that is strictly smaller than

$$\bigwedge_{(x_j,r_j) \supseteq (x_i,r_i)} A(x,x_j).$$

Since & is left continuous and ϕ is inhabited, there is some $(x_k, r_k) \in C\phi$ such that

$$t \leq r_k \& \bigwedge_{(x_j,r_j) \supseteq (x_i,r_i)} A(x,x_j)$$

Take some $(x_l, r_l) \in C\phi$ such that $(x_i, r_i), (x_k, r_k) \sqsubseteq (x_l, r_l)$. Then

$$\phi(x) \ge \phi(x_l) \& A(x, x_l) \ge r_k \& A(x, x_l) \ge t.$$

Therefore, by arbitrariness of t,

$$\phi(x) \ge \bigwedge_{(x_j,r_j) \sqsupseteq (x_i,r_i)} A(x,x_j)$$

The proof is completed.

A slight improvement of the argument shows that the above theorem is valid for all linearly ordered quantales. That is, if Q is a linearly ordered quantale, then irreducible ideals coincide with forward Cauchy ideals.

As an application, the following corollary characterizes, for the quantale $\mathcal{Q} = ([0, 1], \&)$ with & being a left continuous t-norm, the irreducible ideals in the \mathcal{Q} -ordered sets $([0, 1], d_L)$ and $([0, 1], d_R)$.

Corollary 3.14. Let & be a left continuous t-norm and $\mathcal{Q} = ([0,1], \&)$.

- (1) A fuzzy lower set ϕ of the Q-ordered set $([0,1], d_L)$ is an irreducible ideal if and only if either $\phi(x) = x \to a$ for some $a \in [0,1]$ or $\phi(x) = \bigvee_{b < a} (x \to b)$ for some a > 0.
- (2) A fuzzy lower set ψ of the Q-ordered set $([0,1], d_R)$ is an irreducible ideal if and only if either $\psi(x) = a \to x$ for some $a \in [0,1]$ or $\psi(x) = \bigvee_{b>a} (b \to x)$ for some a < 1.

Proof. (1) Sufficiency is easy since the fuzzy lower set $\phi(x) = \bigvee_{b < a} (x \to b)$ is generated by the forward Cauchy sequence $\{a - 1/n\}$. As for necessity, suppose that ϕ is an irreducible ideal of $([0, 1], d_L)$. Then there is a forward Cauchy net $\{x_i\}$ in $([0, 1], d_L)$ such that

$$\phi(x) = \bigvee_{i} \bigwedge_{j \ge i} (x \to x_j).$$

Let $a = \bigvee_i \bigwedge_{j \ge i} x_j$. Then

$$\phi(x) = \bigvee_{i} \bigwedge_{j \ge i} (x \to x_j) = \bigvee_{i} \left(x \to \bigwedge_{j \ge i} x_j \right),$$

hence either $\phi(x) = x \to a$ or $\phi(x) = \bigvee_{b < a} (x \to b)$.

(2) Similar to (1).

Theorem 3.15. For a meet continuous quantale Q, every forward Cauchy ideal is flat.

	1	

Proof. We only need to show that if $\{x_i\}$ is a forward Cauchy net in a Q-ordered set A and

$$\varphi = \bigvee_{i} \bigwedge_{j \ge i} A(-, x_j),$$

then φ is flat. We do this in two steps.

Step 1. For each fuzzy upper set ψ of A,

$$\varphi \otimes \psi = \bigvee_i \bigwedge_{j \ge i} \psi(x_j).$$

Since ψ is a fuzzy upper set, $\{\psi(x_i)\}$ is a forward Cauchy net in (Q, d_L) , hence

$$\begin{split} \varphi \otimes \psi &= \bigwedge_{p \in Q} (\operatorname{sub}_X(\varphi, \psi \to p) \to p) \qquad \text{(Lemma 2.9)} \\ &= \bigwedge_{p \in Q} \left(\operatorname{sub}_X \left(\bigvee_i \bigwedge_{j \ge i} A(-, x_j), \psi \to p \right) \to p \right) \\ &= \bigwedge_{p \in Q} \left(\left(\bigvee_i \bigwedge_{j \ge i} \operatorname{sub}_A (A(-, x_j), \psi \to p) \right) \to p \right) \qquad \text{(Proposition 3.2)} \\ &= \bigwedge_{p \in Q} \left(\left(\bigvee_i \bigwedge_{j \ge i} (\psi(x_j) \to p) \right) \to p \right) \qquad \text{(Yoneda lemma)} \\ &= \bigwedge_{p \in Q} \left(\left(\left(\bigvee_i \bigwedge_{j \ge i} \psi(x_j) \to p \right) \to p \right) \to p \right) \qquad \text{(Lemma 3.1)} \\ &= \bigvee_i \bigwedge_{j \ge i} \psi(x_j). \end{split}$$

Step 2. φ is flat. For any fuzzy upper sets ψ_1 and ψ_2 ,

$$\varphi \otimes \psi_1 \wedge \phi \otimes \psi_2 = \bigvee_i \bigwedge_{j \ge i} \psi_1(x_j) \wedge \bigvee_i \bigwedge_{j \ge i} \psi_2(x_j)$$
(Step 1)
$$= \bigvee_i \bigwedge_{j \ge i} (\psi_1(x_j) \wedge \psi_2(x_j))$$
(Q is meet continuous)
$$= \varphi \otimes (\psi_1 \wedge \psi_2).$$
(Step 1)

The proof is completed.

Similar to Proposition 3.12, it can be shown that the meet continuity of Q is also necessary in Theorem 3.15.

Proposition 3.16. If all forward Cauchy ideals are flat, then the quantale Q is meet continuous.

Example 3.17. Consider the quantale $Q = ([0,1], \wedge)$ and the Q-ordered set $([0,1], d_L)$. By linearity of [0,1], every fuzzy lower set ϕ of $([0,1], d_L)$ satisfies

$$\phi(x) \land \phi(y) \le \bigvee_{z \in [0,1]} \phi(z) \land d_L(x,z) \land d_L(y,z),$$

hence every inhabited fuzzy lower set ϕ of $([0,1], d_L)$ is a flat ideal by Remark 3.5. In particular, the map $\phi : [0,1] \longrightarrow [0,1]$, given by

$$\phi(x) = \begin{cases} 1 - x, & x \le 1/2, \\ 1/2, & x > 1/2, \end{cases}$$

is a flat ideal of $([0, 1], d_L)$. But, it is not irreducible by Corollary 3.14, hence not forward Cauchy by Theorem 3.10.

In the case that Q is the interval [0, 1] equipped with a continuous t-norm, we are able to present a sufficient and necessary condition for flat ideals to be forward Cauchy.

Theorem 3.18. Let Q be the unit interval equipped with a continuous t-norm &. The following are equivalent:

- (1) & is Archimedean, i.e., & has no non-trivial idempotent elements.
- (2) Every flat ideal is forward Cauchy.
- (3) Every flat ideal is irreducible.

We prove a lemma first. A quantale Q = (Q, &) is called *divisible* [11, 12] if

$$x\&(x \to y) = x \land y$$

for all $x, y \in Q$. It is known that the underlying lattice of a divisible quantale is a frame, hence a distributive lattice, see e.g. [12]. Let \mathcal{Q} be a divisible quantale and b an idempotent element. Then for all $x \in Q$,

$$b \wedge x = b\&(b \to x) = b\&b\&(b \to x) \le b\&x \le b \wedge x,$$

hence $b \wedge x = b\&x$.

Lemma 3.19. Suppose Q = (Q, &) is a divisible quantale, b is an idempotent element of &. Then for each $a \in Q$, the map $\phi(x) = b \lor (x \to a)$ is a flat ideal of the Q-ordered set (Q, d_L) .

Proof. It is clear that ϕ is a fuzzy lower set of (Q, d_L) and $\bigvee_{x \in Q} \phi(x) = 1$. It remains to check that $\phi \otimes (\psi_1 \wedge \psi_2) = (\phi \otimes \psi_1) \wedge (\phi \otimes \psi_2)$ for all upper sets ψ_1, ψ_2 of (Q, d_L) .

Because for i = 1, 2,

$$\begin{split} \phi \otimes \psi_i &= \bigvee_{x \in Q} \left((b \& \psi_i(x)) \lor ((x \to a) \& \psi_i(x)) \right) \\ &= (b \land \psi_i(1)) \lor \psi_i(a) \qquad (b \text{ is idempotent}) \\ &= (b \lor \psi_i(a)) \land \psi_i(1), \qquad (Q \text{ is distributive}) \end{split}$$

therefore

$$(\phi \otimes \psi_1) \wedge (\phi \otimes \psi_2) = (b \lor \psi_1(a)) \wedge \psi_1(1) \wedge (b \lor \psi_2(a)) \wedge \psi_2(1)$$
$$= (b \lor (\psi_1(a) \land \psi_2(a)) \wedge (\psi_1(1) \land \psi_2(1))$$
$$= \phi \otimes (\psi_1 \land \psi_2).$$

This completes the proof.

Proof of Theorem 3.18. (1) \Rightarrow (2) This is contained in [26, Proposition 7.9]. If & is isomorphic to the product t-norm, an equivalent version of this implication can also be found in Vickers [34].

 $(2) \Rightarrow (3)$ This follows immediately from Theorem 3.10.

 $(3) \Rightarrow (1)$ Suppose b is a non-trivial idempotent element of &. Take some $a \in (0, b)$. Since [0, 1] together with a continuous t-norm is a divisible quantale [2, 11], it follows from Lemma 3.19 that $\phi(x) = b \lor (x \to a)$ is a flat ideal of $([0, 1], d_L)$. But, ϕ is not irreducible, because neither $\phi(x) \leq b$ for all x nor $\phi(x) \leq x \to a$ for all x, a contradiction.

Finally, we discuss the relationship between irreducible ideals and flat ideals.

A quantale \mathcal{Q} is prelinear if $(p \to q) \lor (q \to p) = 1$ for all $p, q \in Q$. It is known that \mathcal{Q} is prelinear if and only if $(p \land q) \to r = (p \to r) \lor (q \to r)$ for all $p, q, r \in Q$, see e.g. [2].

Proposition 3.20. If Q is prelinear, then every irreducible ideal is a flat ideal.

Proof. Assume that ϕ is an irreducible ideal. Then for all fuzzy upper sets ψ_1, ψ_2 ,

$$\phi \otimes (\psi_1 \wedge \psi_2) = \bigwedge_{p \in Q} (\operatorname{sub}_A(\phi, (\psi_1 \wedge \psi_2) \to p) \to p)$$

=
$$\bigwedge_{p \in Q} (\operatorname{sub}_A(\phi, (\psi_1 \to p) \lor (\psi_2 \to p)) \to p)$$

=
$$\bigwedge_{p \in Q} \left((\operatorname{sub}_A(\phi, \psi_1 \to p) \to p) \land (\operatorname{sub}_A(\phi, \psi_2 \to p) \to p) \right)$$

=
$$(\phi \otimes \psi_1) \land (\phi \otimes \psi_2),$$

hence ϕ is flat.

Proposition 3.21. If Q satisfies the law of double negation then flat ideals coincide with *irreducible ideals*.

Proof. If \mathcal{Q} satisfies the law of double negation, then, by Lemma 2.9, for all fuzzy lower sets ϕ, φ and fuzzy upper set ψ ,

$$\operatorname{sub}_A(\phi,\varphi) = \phi \otimes (\varphi \to 0) \to 0$$

and

$$\phi \otimes \psi = \operatorname{sub}_A(\phi, \psi \to 0) \to 0.$$

The conclusion follows easily from these equations.

Corollary 3.22. Let Q be the unit interval equipped with a left continuous t-norm &. If Q satisfies the law of double negation, then for each fuzzy lower set ϕ of a Q-ordered set, the following are equivalent:

- (1) ϕ is a forward Cauchy ideal.
- (2) ϕ is an irreducible ideal.
- (3) ϕ is a flat ideal.

4. Saturatedness

Let Φ be a class of weights. A Q-ordered set A is Φ -continuous if it is Φ -complete and the left adjoint sup : $\Phi(A) \longrightarrow A$ of $\mathbf{y} : A \longrightarrow \Phi(A)$ has a left adjoint. This kind of postulation is standard in order theory [38]. In the case that $\Phi = \mathcal{P}$ (the largest class of weights), Φ -continuous Q-ordered sets are the completely distributive (or, totally continuous) Q-categories in [29, 33]. We write Φ -Cont for the category of Φ -continuous Q-ordered sets and Φ -cocontinuous maps. The category Φ -Cont is the subject of fuzzy domain theory. So, a natural question is whether such Q-ordered sets exist. As we will see, saturatedness of Φ guarantees that there exist enough such things.

A class of weights Φ is *saturated* [18, 23] if for all Q-ordered set A and $\Lambda \in \Phi(\Phi(A))$,

$$\bigvee_{\phi \in \Phi(A)} \Lambda(\phi) \& \phi \in \Phi(A)$$

A category-minded reader will recognize soon that a saturated class of weights is an example of KZ-monads [20, 46].

Theorem 4.1. Let Φ be a saturated class of weights.

- (1) For each Q-ordered set A, $\Phi(A)$ is Φ -continuous.
- (2) For each Q-order preserving map $f: A \longrightarrow B, \Phi(f)$ is Φ -cocontinuous.
- (3) The functor $\Phi : \mathcal{Q}\text{-}\mathsf{Ord} \longrightarrow \Phi\text{-}\mathsf{Cont}$, which sends each $\mathcal{Q}\text{-}order$ preserving map f to $\Phi(f)$, is a left adjoint of the forgetful functor $\Phi\text{-}\mathsf{Cont} \longrightarrow \mathcal{Q}\text{-}\mathsf{Ord}$.

Proof. (1) For each $\Lambda \in \Phi(\Phi(A))$, since Φ is saturated, $\bigvee_{\phi \in \Phi(A)} \Lambda(\phi) \& \phi$ belongs to $\Phi(A)$. It is easy to verify that for all $\psi \in \Phi(A)$,

$$\mathcal{P}\Phi(A)(\Lambda,\Phi(A)(-,\psi)) = \Phi(A)\Big(\bigvee_{\phi\in\Phi(A)}\Lambda(\phi)\&\phi,\psi\Big)$$

hence $\bigvee_{\phi \in \Phi(A)} \Lambda(\phi) \& \phi$ is a supremum of Λ in $\Phi(A)$, i.e.,

$$\sup_{\Phi(A)} \Lambda = \bigvee_{\phi \in \Phi(A)} \Lambda(\phi) \& \phi.$$

This shows that $\Phi(A)$ is Φ -complete.

Next, we show that $\Phi(A)$ is Φ -continuous, that is, $\sup_{\Phi(A)} : \Phi(\Phi(A)) \longrightarrow \Phi(A)$ has a left adjoint. To this end, write \mathbf{y}_A for the Yoneda embedding $A \longrightarrow \Phi(A)$. For each $\Lambda \in \Phi(\Phi(A))$, since $\Lambda : \Phi(A) \longrightarrow (Q, d_R)$ preserves \mathcal{Q} -order, it follows that for all $x \in A$ and $\phi \in \Phi(A)$,

$$\phi(x) = \operatorname{sub}_A(\mathbf{y}_A(x), \phi) \le \Lambda(\phi) \to \Lambda(\mathbf{y}_A(x)),$$

hence

$$\sup_{\Phi(A)}(\Lambda)(x) = \bigvee_{\phi \in \Phi(A)} \Lambda(\phi) \& \phi(x) = \Lambda \circ \mathbf{y}_A(x).$$

This means that $\sup_{\Phi(A)}$ is the map obtained by restricting the domain and codomain of

$$\mathbf{y}_A^{\leftarrow}: \mathcal{P}\Phi(A) \longrightarrow \mathcal{P}A$$

to $\Phi(\Phi(A))$ and $\Phi(A)$, respectively. Therefore, $\sup_{\Phi(A)}$ has a left adjoint, given by restricting the domain and codomain of $\mathbf{y}_A^{\rightarrow} : \mathcal{P}A \longrightarrow \mathcal{P}\Phi(A)$ to $\Phi(A)$ and $\Phi(\Phi(A))$, respectively.

(2) and (3) are a special case of [23, Theorem 4.7], which is again a special case of a general result in category theory [1, 17, 18].

The above theorem shows that if Φ is a saturated class of weights, then for each Q-ordered set A, $\Phi(A)$ is the free Φ -continuous Q-ordered set generated by A.

This section concerns the saturatedness of the classes of forward Cauchy ideals, irreducible ideals, and flat ideals.

A quantale Q = (Q, &) is completely distributive (continuous, resp.) if the complete lattice Q is a completely distributive lattice (a continuous lattice, resp.). So, each completely distributive quantale is a continuous quantale and each continuous quantale is a meet continuous quantale. For continuity and completely distributivity of complete lattices, the reader is referred to the monograph [9].

The following proposition was first proved in [8] when Q is a *completely distributive* value quantale, the version presented below was proved in [22] making use of Lemma 4.3.

Proposition 4.2. If Q is a continuous quantale, then the class of forward Cauchy ideals is saturated.

Lemma 4.3. ([22]) Let Q be a continuous quantale and ϕ be an inhabited fuzzy lower set of a Q-ordered set A. The following are equivalent:

- (1) ϕ is a forward Cauchy ideal.
- (2) If $r \ll \phi(x)$ and $s \ll \phi(y)$, then for every $t \ll 1$, there is some $z \in A$ such that $t \ll \phi(z), r \ll A(x, z)$ and $s \ll A(y, z)$.

The saturatedness of the classes of flat ideals and irreducible ideals is a special case of a general result in enriched category theory, namely, [18, Proposition 5.4]. However, in order to make this paper self-contained, a direct verification in this special case is included here.

Proposition 4.4. The class of irreducible ideals is saturated.

Proof. It suffices to show that for each \mathcal{Q} -ordered set A and each irreducible ideal Λ : $\mathcal{I}A \longrightarrow \mathcal{Q}$ of $(\mathcal{I}A, \mathrm{sub}_A)$, the map $\mathrm{sup} \Lambda : A \longrightarrow \mathcal{Q}$, given by

$$\sup \Lambda(x) = \bigvee_{\phi \in \mathcal{I}A} \Lambda(\phi) \& \phi(x),$$

is an irreducible ideal of A.

Step 1. $\bigvee_{x \in A} \sup \Lambda(x) = 1$. This is easy since

$$\bigvee_{x \in A} \sup \Lambda(x) = \bigvee_{x \in A} \bigvee_{\phi \in \mathcal{I}A} \Lambda(\phi) \& \phi(x) = \bigvee_{\phi \in \mathcal{I}A} \bigvee_{x \in A} \Lambda(\phi) \& \phi(x) = \bigvee_{\phi \in \mathcal{I}A} \Lambda(\phi) = 1.$$

Step 2. For any fuzzy lower sets ϕ_1, ϕ_2 of A,

$$\operatorname{sub}_A(\operatorname{sup}\Lambda,\phi_1\vee\phi_2) = \operatorname{sub}_A(\operatorname{sup}\Lambda,\phi_1)\vee\operatorname{sub}_A(\operatorname{sup}\Lambda,\phi_2)$$

To see this, for a fuzzy lower set ϕ of A, consider the fuzzy lower set of $(\mathcal{I}A, \mathrm{sub}_A)$:

$$\operatorname{sub}_A(-,\phi):\mathcal{I}A\longrightarrow \mathcal{Q}.$$

Then

$$\operatorname{sub}_{\mathcal{I}A}(\Lambda, \operatorname{sub}_A(-, \phi)) = \bigwedge_{\psi \in \mathcal{I}A} (\Lambda(\psi) \to \operatorname{sub}_A(\psi, \phi))$$
$$= \bigwedge_{\psi \in \mathcal{I}A} \left(\Lambda(\psi) \to \bigwedge_{x \in A} (\psi(x) \to \phi(x)) \right)$$
$$= \bigwedge_{x \in A} \left(\left(\bigvee_{\psi \in \mathcal{I}A} \Lambda(\psi) \& \psi(x) \right) \to \phi(x) \right)$$
$$= \operatorname{sub}_A(\operatorname{sup}\Lambda, \phi).$$

Therefore,

$$\begin{aligned} \operatorname{sub}_{A}(\operatorname{sup}\Lambda,\phi_{1}\vee\phi_{2}) &= \operatorname{sub}_{\mathcal{I}A}(\Lambda,\operatorname{sub}_{A}(-,\phi_{1}\vee\phi_{2})) \\ &= \operatorname{sub}_{\mathcal{I}A}(\Lambda,\operatorname{sub}_{A}(-,\phi_{1})\vee\operatorname{sub}_{A}(-,\phi_{2})) \\ &= \operatorname{sub}_{\mathcal{I}A}(\Lambda,\operatorname{sub}_{A}(-,\phi_{1}))\vee\operatorname{sub}_{\mathcal{I}A}(\Lambda,\operatorname{sub}_{A}(-,\phi_{2})) \\ &= \operatorname{sub}_{A}(\operatorname{sup}\Lambda,\phi_{1})\vee\operatorname{sub}_{A}(\operatorname{sup}\Lambda,\phi_{2}), \end{aligned}$$

where the second equality holds since each element in $\mathcal{I}A$ is irreducible; the reason for the third equality is that Λ is irreducible.

Proposition 4.5. The class of flat ideals is saturated.

Proof. We only need to show that for each \mathcal{Q} -ordered set A and each flat ideal $\Lambda : \mathcal{F}A \longrightarrow \mathcal{Q}$ of $(\mathcal{F}A, \operatorname{sub}_A)$, the map $\operatorname{sup} \Lambda : A \longrightarrow \mathcal{Q}$, given by

$$\sup \Lambda(x) = \bigvee_{\phi \in \mathcal{F}A} \Lambda(\phi) \& \phi(x),$$

is a flat ideal of A.

Step 1. $\bigvee_{x \in A} \sup \Lambda(x) = 1$. This is easy since

$$\bigvee_{x \in A} \sup \Lambda(x) = \bigvee_{x \in A} \bigvee_{\phi \in \mathcal{F}A} \Lambda(\phi) \& \phi(x) = \bigvee_{\phi \in \mathcal{F}A} \bigvee_{x \in A} \Lambda(\phi) \& \phi(x) = \bigvee_{\phi \in \mathcal{F}A} \Lambda(\phi) = 1.$$

Step 2. For all fuzzy upper sets ψ_1, ψ_2 of A,

$$\sup\Lambda\otimes(\psi_1\wedge\psi_2)=(\sup\Lambda\otimes\psi_1)\wedge(\sup\Lambda\otimes\psi_2).$$

To see this, for each fuzzy upper set ψ on A, consider the fuzzy upper set of $(\mathcal{F}A, \operatorname{sub}_A)$ (see Equation (2.2)):

$$-\otimes \psi: \mathcal{F}A \longrightarrow \mathcal{Q}.$$

Then

$$\begin{split} \Lambda \otimes (- \otimes \psi) &= \bigvee_{\phi \in \mathcal{F}A} \left(\Lambda(\phi) \& \bigvee_{x \in A} (\phi(x) \& \psi(x)) \right) \\ &= \bigvee_{x \in A} \bigvee_{\phi \in \mathcal{F}A} (\Lambda(\phi) \& \phi(x)) \& \psi(x) \\ &= \bigvee_{x \in A} \sup \Lambda(x) \& \psi(x) \\ &= \sup \Lambda \otimes \psi. \end{split}$$

Therefore,

$$\begin{split} \sup \Lambda \otimes (\psi_1 \wedge \psi_2) &= \Lambda \otimes (- \otimes (\psi_1 \wedge \psi_2)) \\ &= \Lambda \otimes ((- \otimes \psi_1) \wedge (- \otimes \psi_2)) \\ &= (\Lambda \otimes (- \otimes \psi_1)) \wedge (\Lambda \otimes (- \otimes \psi_2)) \\ &= (\sup \Lambda \otimes \psi_1) \wedge (\sup \Lambda \otimes \psi_2). \end{split}$$

The proof is completed.

5. Scott Q-topology and Scott Q-cotopology

The connection between partially ordered sets and topological spaces is the essence of domain theory. The fuzzy version of Alexandroff topology has been investigated in [22]. This section concerns the extension of Scott topology to the fuzzy setting.

We recall some basic definitions first. A Q-topology on a set X is a subset τ of Q^X subject to the following conditions:

(O1) $p_X \in \tau$ for all $p \in Q$;

- (O2) $\lambda \wedge \mu \in \tau$ for all $\lambda, \mu \in \tau$;
- (O3) $\bigvee_{i \in J} \lambda_j \in \tau$ for each subset $\{\lambda_j\}_{j \in J}$ of τ .

For a Q-topological space (X, τ) , elements in τ are said to be open. A Q-topology τ is *stratified* [13, 14] if

- (O4) $p\&\lambda \in \tau$ for all $p \in Q$ and $\lambda \in \tau$.
- A Q-topology τ is co-stratified [4] if
- (O5) $p \to \lambda \in \tau$ for all $p \in Q$ and $\lambda \in \tau$.
- A Q-topology is strong [4, 43] if it is both stratified and co-stratified.

A Q-cotopology on a set X is a subset τ of Q^X subject to the following conditions:

- (C1) $p_X \in \tau$ for all $p \in Q$;
- (C2) $\lambda \lor \mu \in \tau$ for all $\lambda, \mu \in \tau$;
- (C3) $\bigwedge_{j \in J} \lambda_j \in \tau$ for each subset $\{\lambda_j\}_{j \in J}$ of τ .

For a Q-cotopological space (X, τ) , elements in τ are said to be closed. A Q-cotopology τ is *stratified* if

- (C4) $p \to \lambda \in \tau$ for all $p \in Q$ and $\lambda \in \tau$.
- A Q-cotopology τ is *co-stratified* if
- (C5) $p\&\lambda \in \tau$ for all $p \in Q$ and $\lambda \in \tau$.

A Q-cotopology τ is strong [4, 43] if it is both stratified and co-stratified.

Let \mathcal{Q} be a quantale that satisfies the law of double negation. If τ is a stratified (co-stratified, resp.) \mathcal{Q} -cotopology on a set X, then

$$\neg(\tau) = \{\neg\lambda \mid \lambda \in \tau\}$$

is a stratified (co-stratified, resp.) \mathcal{Q} -topology on X, where $\neg \lambda(x) = \neg(\lambda(x))$ for all $x \in X$. Conversely, if τ is a stratified (co-stratified, resp.) \mathcal{Q} -topology on X, then

$$\neg(\tau) = \{\neg\lambda \mid \lambda \in \tau\}$$

is a stratified (co-stratified, resp.) Q-cotopology on X.

In general, there does not exist a natural way to switch between closed sets and open sets, so, we need to consider the open-set version and the closed-set version when generalizing Scott topology to Q-ordered sets. As demonstrated below, flat ideals and irreducible ideals are related to the open-set and the closed-set version, respectively.

Definition 5.1. Let Φ be a class of weights and A be a Q-ordered set.

(1) A fuzzy set $\psi: A \longrightarrow Q$ is Φ -open if it is a fuzzy upper set and for all $\phi \in \Phi(A)$,

$$\psi(\sup\phi) \le \phi \otimes \psi$$

whenever $\sup \phi$ exists.

(2) A fuzzy set $\lambda : A \longrightarrow Q$ is Φ -closed if it is a fuzzy lower set and for all $\phi \in \Phi(A)$,

$$\operatorname{sub}_A(\phi, \lambda) \leq \lambda(\sup \phi)$$

whenever $\sup \phi$ exists.

Let ψ be a fuzzy upper set of A. Since $\phi \otimes \psi = \sup \psi^{\rightarrow}(\phi)$ by Example 2.11, then $\phi \otimes \psi \leq \psi(\sup \phi)$, hence a fuzzy upper set ψ is Φ -open if and only if

$$\psi(\sup\phi) = \phi \otimes \psi$$

for all $\phi \in \Phi(A)$, if and only if $\psi : A \longrightarrow (Q, d_L)$ is Φ -cocontinuous in the sense that $\psi(\sup \phi) = \sup \psi^{\rightarrow}(\phi)$ for all $\phi \in \Phi(A)$. Similarly, a fuzzy lower set λ is Φ -closed if and only if

$$\operatorname{sub}_A(\phi, \lambda) = \lambda(\sup \phi) \tag{5.1}$$

for all $\phi \in \Phi(A)$, if and only if $\lambda : A \longrightarrow (Q, d_R)$ is Φ -cocontinuous.

Proposition 5.2. Let Φ be a class of weights.

- (1) Each constant fuzzy set is Φ -open.
- (2) If ψ is a Φ -open, then so is $p\&\psi$ for all $p \in Q$.
- (3) The join of a set of Φ -open fuzzy sets is Φ -open.
- (4) If Φ is a subclass of flat ideals, then the meet of two Φ -open fuzzy sets is Φ -open.

Thus, if Φ is a subclass of flat ideals, then for each Q-ordered set A, the Φ -open fuzzy sets of A form a stratified Q-topology on A, called the Φ -Scott Q-topology and denoted by $\sigma_{\Phi}(A)$.

Proposition 5.3. Let Φ be a class of weights.

- (1) Each constant fuzzy set is Φ -closed.
- (2) If ψ is a Φ -closed fuzzy set of A, then so is $p \to \psi$ for all $p \in Q$.
- (3) The meet of a set of Φ -closed fuzzy sets is Φ -closed.

(4) If Φ is a subclass of irreducible ideals, then the join of two Φ -closed fuzzy sets is Φ -closed.

Thus, if Φ is a subclass of irreducible ideals, then for each Q-ordered set A, the Φ closed fuzzy sets form a stratified Q-cotopology on A, called the Φ -Scott Q-cotopology on A and denoted by $\sigma_{\Phi}^{co}(A)$.

Convention. For the class \mathcal{F} of all flat ideals, we say fuzzy Scott open sets (Scott \mathcal{Q} -topology, resp.) instead of \mathcal{F} -open fuzzy sets (\mathcal{F} -Scott \mathcal{Q} -topology, resp.), and write $\sigma(A)$ instead of $\sigma_{\mathcal{F}}(A)$. Dually, For the class \mathcal{I} of all irreducible ideals, we say fuzzy Scott closed sets (Scott \mathcal{Q} -cotopology, resp.) instead of \mathcal{I} -closed fuzzy sets (\mathcal{I} -Scott \mathcal{Q} -cotopology, resp.), and write $\sigma^{co}(A)$ instead of $\sigma_{\mathcal{I}}^{co}(A)$.

Remark 5.4. (1) Let $\{x_i\}$ be a forward Cauchy net in a \mathcal{Q} -ordered set A and

$$\varphi = \bigvee_i \bigwedge_{j \ge i} A(-, x_j)$$

For each fuzzy upper set ψ of A, by the argument of Theorem 3.15, we have

$$\varphi \otimes \psi = \bigvee_{i} \bigwedge_{j \ge i} \psi(x_j).$$

Thus, a fuzzy upper set ψ of A is W-open if and only if

$$\bigvee_i \bigwedge_{j \ge i} \psi(x_j) \ge \psi(x)$$

for every forward Cauchy net $\{x_i\}$ with a Yoneda limit x. This shows that \mathcal{W} -open fuzzy sets are the Scott open fuzzy sets in the sense of Wagner [37, Definition 4.1]. In particular, if \mathcal{Q} is meet continuous, then \mathcal{W} -open fuzzy sets of A form a stratified \mathcal{Q} -topology on A.

(2) Lemma 4.6 in [37] claims that if ϕ, ψ are \mathcal{W} -open fuzzy sets of a \mathcal{Q} -ordered set A, then so is the fuzzy set $\phi \& \psi : A \longrightarrow Q$ given by $(\phi \& \psi)(x) = \phi(x) \& \psi(x)$. This is not true in general. Let \mathcal{Q} be the unit interval [0, 1] equipped with the product t-norm &. For each class of weights Φ , the identity map id is clearly Φ -open in the \mathcal{Q} -ordered set ([0, 1], d_L), but, id&id is not a fuzzy upper set of ([0, 1], d_L).

(3) If $\mathcal{Q} = (Q, \&)$ is a frame, i.e., $\& = \land$, then, as noted in Remark 3.5, the fuzzy ideals considered in [40] are exactly the flat ideals, hence a fuzzy set ψ of a \mathcal{Q} -ordered set A is fuzzy Scott open if and only if it is so in the sense of Yao [40, Definition 2.10].

Remark 5.5. Let $\{x_i\}$ be a forward Cauchy net in a \mathcal{Q} -ordered set A and

$$\phi = \bigvee_{i} \bigwedge_{j \ge i} A(-, x_j).$$

By Proposition 3.2 (or, **Step 2** in the argument of Theorem 3.10),

$$\operatorname{sub}_A(\phi,\psi) = \bigvee_i \bigwedge_{j \ge i} \psi(x_j)$$

for each fuzzy lower set ψ of A. By Proposition 3.3, Yoneda limits of $\{x_i\}$ are exactly the suprema of the fuzzy lower set

$$\phi = \bigvee_{i} \bigwedge_{j \ge i} A(-, x_j).$$

So, a fuzzy lower set ψ of A is W-closed if and only if

$$\bigvee_{i} \bigwedge_{j \ge i} \psi(x_j) \le \psi(x)$$

for every forward Cauchy net $\{x_i\}$ with a Yoneda limit x. This shows that \mathcal{W} -closed fuzzy sets are exactly the Scott closed fuzzy sets in the sense of Wagner ([37], Definition 4.4).

Proposition 5.6. Let Φ be a subclass of flat ideals. Then for each Φ -cocontinuous map $f: A \longrightarrow B$ between Q-ordered sets, $f: (A, \sigma_{\Phi}(A)) \longrightarrow (B, \sigma_{\Phi}(B))$ is continuous.

Therefore, for a subclass Φ of flat ideals, assigning each Q-ordered set A to the Qtopological space $(A, \sigma_{\Phi}(A))$ defines a functor Σ_{Φ} from the category of Q-ordered sets and Φ -cocontinuous maps to that of stratified Q-topological spaces. It is known in domain theory that the functor sending each ordered set to its Scott topology is a full functor, but, it is not clear whether Σ_{Φ} is a full functor.

The situation with Scott Q-cotopology looks more promising. The following conclusion implies that for every subclass Φ of irreducible ideals, assigning each Q-ordered set A to the Q-cotopological space $(A, \sigma_{\Phi}^{co}(A))$ gives a full functor Σ_{Φ}^{co} from the category of Q-ordered sets and Φ -cocontinuous maps to that of stratified Q-cotopological spaces.

Proposition 5.7. ([37, Proposition 4.15] for the class of forward Cauchy ideals) Let Φ be a class of weights. For each map $f : A \longrightarrow B$ between Q-ordered sets, the following are equivalent:

- (1) $f: A \longrightarrow B$ is Φ -cocontinuous.
- (2) For each Φ -closed fuzzy set ϕ of B, $\phi \circ f$ is a Φ -closed fuzzy set of A.

Proof. (1) \Rightarrow (2) This is easy since the composite of Φ -cocontinuous maps is Φ -cocontinuous. (2) \Rightarrow (1) First, we show that f preserves \mathcal{Q} -order. For all $a_1, a_2 \in A$, since $\psi = B(-, f(a_2))$ is a Φ -closed fuzzy set of B, then $\psi \circ f = B(f(-), f(a_2))$ is Φ -closed, hence

$$A(a_1, a_2) = \psi \circ f(a_2) \& A(a_1, a_2) \le \psi \circ f(a_1) = B(f(a_1), f(a_2)),$$

showing that f preserves Q-order.

Second, we show that for each $\phi \in \Phi(A)$, if $\sup \phi$ exists, then for all $b \in B$,

$$\operatorname{sub}_B(f^{\to}(\phi), B(-, b)) = B(f(\operatorname{sup}\phi), b),$$

hence $f(\sup \phi)$ is a supremum of $f^{\rightarrow}(\phi)$. Since B(-,b) is a Φ -closed fuzzy set of B, $B(-,b) \circ f$ is a Φ -closed fuzzy set of A, hence, by Eq. (5.1),

$$\operatorname{sub}_B(f^{\to}(\phi), B(-, b)) = \operatorname{sub}_A(\phi, B(-, b) \circ f) = B(f(\operatorname{sup}\phi), b).$$

This completes the proof.

Example 5.8. Let $\mathcal{Q} = ([0,1], \&)$ with & being a left continuous t-norm. Then ϕ is a fuzzy Scott closed set in $([0,1], d_R)$ if and only if $\phi : ([0,1], d_L) \longrightarrow ([0,1], d_L)$ is right continuous and \mathcal{Q} -order preserving.

By Corollary 3.14, a fuzzy lower set ψ of $([0,1], d_R)$ is an irreducible ideal if and only if either $\psi(x) = a \to x$ for some $a \in [0,1]$ or $\psi(x) = \bigvee_{b>a} (b \to x)$ for some a < 1. Since the supremum of $\bigvee_{b>a} (b \to x)$ in $([0,1], d_R)$ is (see Example 2.12)

$$\bigwedge_{x\in[0,1]} \Big(\bigvee_{b>a} (b\to x)\to x\Big) = \bigwedge_{b>a} \bigwedge_{x\in[0,1]} ((b\to x)\to x) = a,$$

it follows that a fuzzy lower set ϕ of $([0,1], d_R)$ is fuzzy Scott closed if and only if for all a < 1,

$$\operatorname{sub}_{[0,1]}\Big(\bigvee_{b>a}(b\to x),\phi\Big) = \bigwedge_{b>a}\phi(b) \le \phi(a).$$

The conclusion thus follows.

If \mathcal{Q} is the unit interval [0, 1] equipped with a continuous t-norm &, we have a bit more: the fuzzy Scott \mathcal{Q} -cotopology on each \mathcal{Q} -ordered set is a strong \mathcal{Q} -cotopology.

Proposition 5.9. Let Q = ([0, 1], &) with & being a left continuous t-norm. The following are equivalent:

(1) & is a continuous t-norm.

(2) The Scott Q-cotopology on each Q-ordered set is a strong Q-cotopology.

Proof. (1) \Rightarrow (2) We only need to check that if ϕ is a fuzzy Scott closed fuzzy set of a Q-ordered set A, then so is $a\&\phi$ for all $a \in [0,1]$. Since ϕ is a fuzzy Scott closed set of A and a&id is a fuzzy Scott closed set of $([0,1],d_R)$, both $\phi : A \longrightarrow ([0,1],d_R)$ and a&id : $([0,1],d_R) \longrightarrow ([0,1],d_R)$ preserve suprema of irreducible ideals, then $a\&\phi =$ (a&id) $\circ \phi : A \longrightarrow ([0,1],d_R)$ preserve suprema of irreducible ideals, hence $a\&\phi$ is fuzzy Scott closed.

 $(2) \Rightarrow (1)$ If & is not continuous, by [19, Proposition 1.19] there is some $a \in [0, 1]$ such that a&id : $[0, 1] \longrightarrow [0, 1]$ is not right continuous, hence not a fuzzy Scott closed set in $([0, 1], d_R)$. Since the identity map on [0, 1] is fuzzy Scott closed in $([0, 1], d_R)$ by Example 5.8, the Scott \mathcal{Q} -cotopology on $([0, 1], d_R)$ cannot be a strong one.

Example 5.10. This example shows that if $\mathcal{Q} = ([0,1], \&)$ with & being a continuous t-norm, then the Scott \mathcal{Q} -cotopology on $([0,1], d_R)$ is the strong \mathcal{Q} -cotopology on [0,1] generated by the identity map.

Let τ denote the strong \mathcal{Q} -cotopology on [0,1] generated by the identity map. By Example 5.8, a fuzzy Scott closed set in $([0,1], d_R)$ is exactly a right continuous and \mathcal{Q} -order preserving map $\phi : ([0,1], d_L) \longrightarrow ([0,1], d_L)$. So, the conclusion has already been proved in [44] in the case that & is the t-norm min, the product t-norm, and the Lukasiewicz t-norm. Here we prove it in the general case by help of the ordinal sum decomposition of continuous t-norms. Since the Scott \mathcal{Q} -cotopology on $([0,1], d_R)$ is strong and contains the identity map as a closed set, it suffices to show that if ϕ is a fuzzy Scott closed set in $([0,1], d_R)$ then $\phi \in \tau$. We do this in two steps.

Step 1. If ϕ is a fuzzy Scott closed set in $([0,1], d_R)$ and $\phi \ge id$, then $\phi \in \tau$.

Since & is a continuous t-norm, there is a set of disjoint open intervals $\{(a_i, b_i)\}$ such that

- for each *i*, both a_i and b_i are idempotent and the restriction of & on $[a_i, b_i]$ is either isomorphic to the Lukasiewicz t-norm or to the product t-norm;
- $x \& y = \min\{x, y\}$ if $(x, y) \notin \bigcup_i [a_i, b_i]^2$.

For each $x \in [0,1]$, define $g_x : [0,1] \longrightarrow [0,1]$ by

$$g_x(y) = \begin{cases} \phi(x) \lor ((\phi(x) \to x) \to y), & (x, \phi(x)) \in (a_i, b_i)^2 \text{ for some } i \text{ and } \phi(x) > x, \\ \phi(x) \lor (b_i \to y), & (x, \phi(x)) \in (a_i, b_i)^2 \text{ for some } i \text{ and } \phi(x) = x, \\ \phi(x) \lor (x \to y), & (x, \phi(x)) \notin (a_i, b_i)^2 \text{ for any } i. \end{cases}$$

Each g_x is clearly a member of τ , so, in order to see that $\phi \in \tau$, it suffices to show that for all $y \in [0, 1]$,

$$\phi(y) = \bigwedge_{x \in [0,1]} g_x(y).$$

Before proving this equality, we list here some facts about the maps g_x , the verifications are left to the reader.

- (M1) $\phi(y) \leq g_x(y)$ whenever $y \leq x$.
- (M2) If $(x, \phi(x)) \in (a_i, b_i)^2$ for some i and $\phi(x) > x$, then $g_x(x) = (\phi(x) \to x) \to x = \phi(x)$ and $g_x(y) = (\phi(x) \to x) \to y \ge \phi(y)$ for all y > x.
- (M3) If $(x, \phi(x)) \in (a_i, b_i)^2$ for some i and $\phi(x) = x$, then $\phi(y) = y = g_x(y)$ whenever $x \le y < b_i$ and $g_x(y) = 1 \ge \phi(y)$ for all $y \ge b_i$.
- (M4) If $(x, \phi(x)) \notin (a_i, b_i)^2$ for any i, then for all $y \ge x$, $g_x(y) = \phi(x) \lor (x \to y) = 1 \ge \phi(y)$.

It follows immediately from these facts that for all $y \in [0, 1]$, $\phi(y) \leq \bigwedge_{x \in [0, 1]} g_x(y)$. For the converse inequality, we distinguish three cases.

Case 1. $(y, \phi(y)) \in (a_i, b_i)^2$ for some *i* and $\phi(y) > y$. Then by fact (M2), $g_y(y) = \phi(y)$, hence $\phi(y) \ge \bigwedge_{x \in [0,1]} g_x(y)$.

Case 2. $(y, \phi(y)) \in (a_i, b_i)^2$ for some *i* and $\phi(y) = y$. Then by fact (M3), $g_y(y) = \phi(y)$, hence $\phi(y) \ge \bigwedge_{x \in [0,1]} g_x(y)$.

Case 3. $(y, \phi(y)) \notin (a_i, b_i)^2$ for any *i*. In this case, if we can show that $g_x(y) = \phi(x)$ for all x > y, then we will obtain that $\phi(y) = \bigwedge_{x > y} \phi(x) \ge \bigwedge_{x \in [0,1]} g_x(y)$ by right continuity of ϕ . The proof is divided into four subcases.

Subcase 1. $y \in (a_i, b_i)$ for some *i* and $\phi(y) \ge b_i$. If $x \le b_i$, then $x \to y \le b_i$ and $\phi(x) \ge \phi(y) \ge b_i$, hence $g_x(y) = \phi(x) \lor (x \to y) = \phi(x)$. For $x > b_i$,

• if $(x, \phi(x)) \in (a_j, b_j)^2$ for some j and $\phi(x) > x$, then

$$g_x(y) = \phi(x) \lor ((\phi(x) \to x) \to y) = \phi(x) \lor y = \phi(x);$$

• if $(x, \phi(x)) \in (a_j, b_j)^2$ for some j and $\phi(x) = x$, then

$$g_x(y) = \phi(x) \lor (b_j \to y) = \phi(x) \lor y = \phi(x);$$

• if $(x, \phi(x)) \not\in (a_j, b_j)^2$ for any j, then

$$g_x(y) = \phi(x) \lor (x \to y) = \phi(x) \lor y = \phi(x).$$

Subcase 2. $y \notin [a_i, b_i]$ for any *i*. In this case, since $t \to y = y$ for all t > y, it follows that $g_x(y) = \phi(x) \lor y = \phi(x)$.

Subcase 3. $y = a_i$. For $x < b_i$,

- if $x < \phi(x) < b_i$, then $(x, \phi(x)) \in (a_i, b_i)^2$, hence $g_x(y) = \phi(x) \lor ((\phi(x) \to x) \to a_i) = \phi(x)$;
- if $\phi(x) \ge b_i$, then $g_x(y) = \phi(x) \lor (x \to y) = \phi(x)$ since $x \to y = x \to a_i \le b_i$;
- if $\phi(x) = x$, then $g_x(y) = \phi(x) \lor (b_i \to y) = \phi(x) \lor (b_i \to a_i) = \phi(x)$.

For $x > b_i$,

• if $(x, \phi(x)) \in (a_j, b_j)^2$ for some j and $\phi(x) > x$, then $a_i < a_j$, hence

$$g_x(y) = \phi(x) \lor ((\phi(x) \to x) \to a_i) = \phi(x) \lor a_i = \phi(x);$$

- if $(x, \phi(x)) \in (a_j, b_j)^2$ for some j and $\phi(x) = x$, then $g_x(y) = \phi(x) \vee (b_j \to a_i) = \phi(x)$;
- if $(x, \phi(x)) \notin (a_j, b_j)^2$ for any j, then $g_x(y) = \phi(x) \lor (x \to a_i) = \phi(x)$.

Subcase 4. $y = b_i$. If $a_j = b_i$ for some j, then the conclusion holds by Subcase 3. Otherwise, the argument for Subcase 2 can be applied to show that $g_x(y) = \phi(x)$.

Step 2. If ϕ is a fuzzy Scott closed set in $([0,1], d_R)$, then $\phi \in \tau$.

Since $\phi(1) \to \phi$ is fuzzy Scott closed and $id \leq \phi(1) \to \phi$, it follows that $\phi(1) \to \phi \in \tau$ by Step 1. Since τ is strong and & is continuous, then $\phi = \phi(1)\&(\phi(1) \to \phi) \in \tau$.

References

- M.H. Albert, G.M. Kelly, The closure of a class of colimits, Journal of Pure and Applied Algebra 51 (1998) 1-17.
- [2] R. Bělohlávek, Fuzzy Relational Systems: Foundations and Principles, Kluwer Academic Publishers, Dordrecht, 2002.
- [3] M.M. Bonsangue, F. van Breugel, J.J.M.M. Rutten, Generalized metric space: completion, topology, and powerdomains via the Yoneda embedding, Theoretical Computer Science 193 (1998) 1-51.
- [4] P. Chen, H. Lai, D. Zhang, Coreflective hull of finite strong L-topological spaces, Fuzzy Sets and Systems 182 (2011) 79-92.
- [5] W.M. Faucett, Compact semigroups irreducibly connected between two idempotents, Proceedings of the American Mathematical Society 6 (1955) 741-747.
- [6] R.C. Flagg, R. Kopperman, Continuity spaces: Reconciling domains and metric spaces, Theoretical Computer Science 177 (1997) 111-138.
- [7] R.C. Flagg, P. Sünderhauf, The essence of ideal completion in quantitative form, Theoretical Computer Science 278 (2002) 141-158.
- [8] R.C. Flagg, P. Sünderhauf, K.R. Wagner, A logical approach to quantitative domain theory, Topology Atlas Preprint No. 23, 1996. http://at.yorku.ca/e/a/p/p/23.htm
- [9] G. Gierz, K.H. Hofmann, K. Keimel, J.D. Lawson, M. Mislove, D.S. Scott, Continuous Lattices and Domains, Encyclopedia of Mathematics and its Applications, Vol. 93, Cambridge University Press, Cambridge, 2003.

- [10] J. Goubault-Larrecq, Non-Hausdorff Topology and Domain Theory, Cambridge University Press, Cambridge, 2013.
- [11] P. Hájek, Metamathematics of Fuzzy Logic, Kluwer Academic Publishers, Dordrecht, 1998.
- [12] U. Höhle, Commutative, residuated ℓ-monoids, in: U. Höhle, E.P. Klement (eds.), Non-classical Logics and Their Applications to Fuzzy Subsets: A Handbook on the Mathematical Foundations of Fuzzy Set Theory, Kluwer, Dordrecht, 1995, pp.53-105.
- [13] U. Höhle, A.P. Šostak, A general theory of fuzzy topological spaces, Fuzzy Sets and Systems 73 (1995) 131-149.
- [14] U. Höhle, A.P. Šostak, Axiomatic foundition of fixed-basis fuzzy topology, in: U. Höhle, S.E. Rodabaugh, (Eds.), Mathematics of Fuzzy Sets: Logic, Topology and Measure Theory, Springer, 1999, pp. 123-272.
- [15] D. Hofmann, P. Waszkiewicz, Approximation in quantale-enriched categories, Topology and its Applications 158 (2011) 963-977.
- [16] D. Hofmann, P. Waszkiewicz, A duality of quantale-enriched categories, Journal of Pure and Applied Algebra 216 (2012) 1866-1878.
- [17] G.M. Kelly, Basic Concepts of Enriched Category Theory, London Mathematical Society Lecture Notes Series, Vol. 64, Cambridge University Press, Cambridge, 1982.
- [18] G.M. Kelly, V. Schmitt, Notes on enriched categories with colimits of some class, Theory and Applications of Categories 14 (2005) 399-423.
- [19] E.P. Klement, R. Mesiar, E. Pap, Triangular Norms, Trends in Logic, Vol. 8, Kluwer Academic Publishers, Dordrecht, 2000.
- [20] A. Kock, Monads for which structures are adjoint to units, Journal of Pure and Applied Algebra 104 (1995) 41-59.
- [21] H.P. Künzi, M.P. Schellekens, On the Yoneda completion of a quasi-metric space, Theoretical Computer Science 278 (2002) 159-194.
- [22] H. Lai, D. Zhang, Fuzzy preorder and fuzzy topology, Fuzzy Sets and Systems 157 (2006) 1865-1885.
- [23] H. Lai, D. Zhang, Complete and directed complete Ω-categories, Theoretical Computer Science 388 (2007) 1-25.
- [24] H. Lai, D. Zhang, Closedness of the category of liminf complete fuzzy orders, Fuzzy Sets and Systems 282 (2016) 86-98.
- [25] F.W. Lawevere, Metric spaces, genetalized logic, and closed categories, Rendiconti del Seminario Matématico e Fisico di Milano 43 (1973) 135-166.
- [26] W. Li, H. Lai, D. Zhang, Yoneda completeness and flat completeness of ordered fuzzy sets, Fuzzy Sets and Systems 313 (2017) 1-24.
- [27] M. Liu, B. Zhao, Two cartesian closed subcategories of fuzzy domains, Fuzzy Sets and Systems 238 (2014) 102-112.
- [28] P.S. Mostert, A.L. Shields, On the structure of semigroups on a compact manifold with boundary, Annals of Mathematics 65 (1957) 117-143.
- [29] Q. Pu, D. Zhang, Categories enriched over a quantaloid: Algebras, Theory and Applications of Categories 30 (2015) 751-774.

- [30] K.I. Rosenthal, Quantales and Their Applications, Longman, Essex, 1990.
- [31] J.J.M.M. Rutten, Weighted colimits and formal balls in generalized metric spaces, Topology and its Applications 89 (1998) 179-202.
- [32] I. Stubbe, Categorical structures enriched in a quantaloid: categories, distributors and functors, Theory and Applications of Categories 14 (2005) 1-45.
- [33] I. Stubbe, Towards "dynamic domains": totally continuous cocomplete Q-categories. Theoretical Computer Science 373 (2007) 142-160.
- [34] S. Vickers, Localic completion of generalized metric spaces, Theory and Application of Categories 14 (2005) 328-356.
- [35] Y. Tao, H. Lai, D. Zhang, Quantale-valued preorders: Globalization and cocompleteness, Fuzzy Sets and Systems 256 (2014) 236-251.
- [36] K.R. Wagner, Solving recursive domain equations with enriched categories, Ph.D. Thesis, Carnegie Mellon University, Tech. Report CMU-CS-94-159, July, 1994.
- [37] K.R. Wagner, Liminf convergence in Ω-categories, Theoretical Computer Science 184 (1997) 61-104.
- [38] R.J. Wood, Ordered sets via adjunction, in: Categorical Foundations, Encyclopedia of Mathematics and its Applications, Vol. 97, Cambridge University Press, Cambridge, 2004, pp. 5-47.
- [39] W. Yao, Quantitative domains via fuzzy sets: Part I: Continuity of fuzzy directed complete posets, Fuzzy Sets and Systems 161 (2010) 973-987.
- [40] W. Yao, A categorical isomorphism between injective fuzzy T_0 -spaces and fuzzy continuous lattices, IEEE Transactions on Fuzzy Systems 24 (2016) 131-139.
- [41] W. Yao, F. Shi, Quantitative domains via fuzzy sets: Part II: Fuzzy Scott topology on fuzzy directed complete posets, Fuzzy Sets and Systems 173 (2011) 60-80.
- [42] L.A. Zadeh, Similarity relations and fuzzy orderings, Information Sciences 3 (1971) 177-200.
- [43] D. Zhang, An enriched category approach to many valued topology, Fuzzy Sets and Systems 158 (2007) 349-366.
- [44] D. Zhang, Sobriety of quantale-valued cotopological spaces, Fuzzy Sets and Systems, http://dx.doi.org/10.1016/j.fss.2017.09.005
- [45] Q. Zhang, L. Fan, Continuity in quantitative domains, Fuzzy Sets and Systems 154 (2005) 118-131.
- [46] V. Zöberlein, Doctrines on 2-categories, Mathematische Zeitschrift 148 (1976) 267-279.