# On linear fuzzy differential equations by differential inclusions' approach

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**Abstract**

In this paper, we study first order linear fuzzy differential equations under differential inclusions and strongly generalized differentiability approaches. We present some new results on the relation between their solutions. Finally, some examples are given to illustrate our results.

*Keywords:* First order fuzzy differential equation, Generalized differentiability, Differential Inclusion.

## 1. Introduction

As it is well-known, there exist several possibilities to define the derivative of a fuzzy function and, in consequence, to deal with fuzzy differential equations [7, 16, 29]. To refer to one of the first approaches to the topic, we mention the H-derivative of a fuzzy-number-valued function introduced in [29], which involves the consideration of the Hukuhara derivative on every $\alpha$-level set of the function, constituting a generalization of this concept of derivative defined for set-valued functions. With the first studies on the solutions to fuzzy differential equations interpreted from the point of view of the Hukuhara derivative, it was revealed its main drawback: it leads to solutions whose level sets have increasing length [15]. Under these circumstances, we cannot expect the properties of the crisp solution to be inherited by the fuzzy solution.

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The notion of strongly generalized differentiability was introduced in [3] and studied or applied in [4, 5, 9, 20, 21, 23]. This new definition allows to successfully overcome the difficulties detected in the use of Hukuhara derivative, expanding at the same time the family of fuzzy-number-valued functions which admit derivative in comparison with the previous approach.

On the other hand, Hüllermeier [17] considered a fuzzy differential inclusion (for differential inclusions, see [2]) and a solution to this inclusion to be a fuzzy R-solution. However, one shortcoming has also been ascribed to this formulation, consisting in the absence of a proper definition for the derivative of a fuzzy-number-valued function. In contrast, when the fuzzy differential equation is interpreted with the help of differential inclusions, we find the advantage that we only need the classical concept of differentiation.

For the Extension Principle's approach to fuzzy differential equations, we mention [26, 10, 1]. Generalized Hukuhara differentiability of interval-valued functions and interval differential equations are considered in [33] and the fuzzy-valued case is analyzed in [6]. Some other related works are: [30], where some algebraic and topological properties are obtained for the quotient space of fuzzy numbers with respect to the equivalence relation given by Mareš; [31] for fuzzy differential equations in this space; [12] on fuzzy differential equations with $\pi$-derivative; or [25] for granular differentiability of fuzzy-number-valued functions.

In [5], first order linear fuzzy differential equations under strongly generalized differentiability concept are considered and solutions to this problem in some especial cases are presented. Recently, a variation of constants formula for a first order linear fuzzy differential equation is provided in [20] under strongly generalized differentiability, in addition to the work [14]. See also [32] for comparison results on fuzzy differential equations.

In this paper, we consider first order linear fuzzy differential equations from the point of view of strongly generalized differentiability and differential inclusions and present some results on the relationship between the solutions corresponding to these two approaches. Several examples are presented to illustrate our results.

## 2. Preliminaries

In this section, we present some definitions and introduce the notation which is necessary for the rest of the paper. For details, see, for example, [15].

We denote by $\mathbb{R}_F$ the class of fuzzy subsets of the real axis (i.e., functions $u : \mathbb{R} \to [0, 1]$) which satisfy the following properties:

(i) $u$ is normal, i.e., there exists $s_0 \in \mathbb{R}$ such that $u(s_0) = 1$,
(ii) $u$ is fuzzy-convex (i.e., $u(ts+(1-t)r) \geq \min\{u(s), u(r)\}, \ \forall t \in [0,1], \ s, r \in \mathbb{R}$),
(iii) $u$ is upper semicontinuous on $\mathbb{R}$,
(iv) $cl\{s \in \mathbb{R} | u(s) > 0\}$ is compact, where $cl$ denotes the closure of a subset.

The set $\mathbb{R}_F$ previously defined is called the space of fuzzy numbers. For each $0 < \alpha \leq 1$, we set $[u]^\alpha = \{s \in \mathbb{R} | u(s) \geq \alpha\}$ and $[u]^0 = cl\{s \in \mathbb{R} | u(s) > 0\}$. Properties (i)-(iv) provide that, if $u$ belongs to $\mathbb{R}_F$, then the $\alpha$-level set of $u$, $[u]^\alpha$, is a non-empty compact interval for all $0 \leq \alpha \leq 1$. We use the notation

$$[u]^\alpha = [\underline{u}^\alpha, \overline{u}^\alpha],$$

to express the $\alpha$-level set of $u$. For $u \in \mathbb{R}_F$, we measure the length of the level sets of $u$ in the following form:

$$diam([u]^\alpha) = \overline{u}^\alpha - \underline{u}^\alpha, \ \alpha \in [0,1].$$

Note that the previous expression is nonincreasing in $\alpha$.

Given $u, v \in \mathbb{R}_F$ and $\lambda \in \mathbb{R}$, we define the sum $u + v$ and the product $\lambda u$ by the classical operations for real intervals $[u + v]^\alpha = [u]^\alpha + [v]^\alpha, \ [\lambda u]^\alpha = \lambda [u]^\alpha, \forall \alpha \in [0,1]$, where $[u]^\alpha + [v]^\alpha$ represents the usual addition of two subsets of $\mathbb{R}$ and $\lambda [u]^\alpha$ denotes the usual product between a scalar and a subset of $\mathbb{R}$.

It is well-known [15] that $(\mathbb{R}_F, D)$ is a complete metric space with the metric structure $D$ given by the Hausdorff distance

$$D : \mathbb{R}_F \times \mathbb{R}_F \to \mathbb{R}_+ \cup \{0\},$$

$$D(u, v) = \sup_{\alpha \in [0,1]} \max\{|\underline{u}^\alpha - \underline{v}^\alpha|, |\overline{u}^\alpha - \overline{v}^\alpha|\}.$$

**Definition 2.1.** *Given $x, y \in \mathbb{R}_F$, if there exists $z \in \mathbb{R}_F$ such that $x = y + z$ then $z$ is called the H-difference of $x, y$ and it is denoted $x \ominus y$.*

For the existence of the H-difference $x \ominus y$, we need the following conditions to be fulfilled:

- $diam([x]^\alpha) \geq diam([y]^\alpha), \ \forall \alpha \in [0,1]$.
- $\underline{x}^\alpha - \underline{y}^\alpha$ is nondecreasing in $\alpha \in [0,1]$.
- $\overline{x}^\alpha - \overline{y}^\alpha$ is nonincreasing in $\alpha \in [0,1]$.

Throughout the paper, we use the sign "$\ominus$" for the H-difference, in such a way that we usually denote $x + (-1)y$ by $x - y$, while $x \ominus y$ stands for the H-difference of $x, y$. Note that, in general, $x \ominus y \neq x + (-1)y$.

In the following notions and results, we consider the interval $I = (\xi, \eta)$, for $\xi < \eta \in \mathbb{R}$.

First, we recall the concept of strongly generalized differentiability, which was introduced in [3] and studied in [4, 5, 9, 19, 20, 33].

**Definition 2.2.** *[3] Let $F : I \to \mathbb{R}_F$ and fix $t_0 \in I$. We say that $F$ is differentiable at $t_0$, if there exists an element $F'(t_0) \in \mathbb{R}_F$ such that either:*
*(1) for all $h > 0$ sufficiently close to $0$, the H-differences $F(t_0 + h) \ominus F(t_0), F(t_0) \ominus F(t_0 - h)$ exist and the limits (in the metric $D$)*

$$\lim_{h \to 0^+} \frac{F(t_0 + h) \ominus F(t_0)}{h} = \lim_{h \to 0^+} \frac{F(t_0) \ominus F(t_0 - h)}{h} = F'(t_0),$$

*or*
*(2) for all $h > 0$ sufficiently close to $0$, the H-differences $F(t_0) \ominus F(t_0 + h), F(t_0 - h) \ominus F(t_0)$ exist and the limits (in the metric $D$)*

$$\lim_{h \to 0^+} \frac{F(t_0) \ominus F(t_0 + h)}{-h} = \lim_{h \to 0^+} \frac{F(t_0 - h) \ominus F(t_0)}{-h} = F'(t_0).$$

**Definition 2.3.** *Let $F : I \to \mathbb{R}_F$. We say that $F$ is (i)-differentiable on $I$ if $F$ is differentiable in the sense (1) of Definition 2.2 at every point of $I$, in which case its derivative is denoted by $D_1F$. Similarly, (ii)-differentiability of $F$ on $I$ consists in the differentiability of $F$ in the sense (2) of Definition 2.2 at every point of $I$, obtaining $D_2F$.*

References [4, 9, 20, 33] include several results on the essential properties in relation with strongly generalized differentiability. We recall some of them, which are useful to this paper.

**Theorem 2.4.** *Let $F : I \to \mathbb{R}_F$ and consider $[F(t)]^\alpha = [f_\alpha(t), g_\alpha(t)]$, for each $\alpha \in [0, 1]$.*
*(i) If $F$ is (i)-differentiable, then $f_\alpha$ and $g_\alpha$ are differentiable functions and $[D_1F(t)]^\alpha = [f'_\alpha(t), g'_\alpha(t)]$.*
*(ii) If $F$ is (ii)-differentiable, then $f_\alpha$ and $g_\alpha$ are differentiable functions and $[D_2F(t)]^\alpha = [g'_\alpha(t), f'_\alpha(t)]$.*

**Proof:** See [9].

When there is no possibility of confusion, we denote $D_1F$ and $D_2F$ simply by $F'$, making explicit reference to the type of differentiability ((i) or (ii), respectively).

**Theorem 2.5.** *Let $F$ be (ii)-differentiable on $I$ and assume that the derivative $F'$ is integrable over $I$. Then, for each $t \in I$, we have*

$$F(t) = F(\xi) \ominus \int_{\xi}^{t} -F'(\tau)d\tau.$$

**Proof:** See [20].

**Theorem 2.6.** *Let $F$ be a continuous fuzzy function on $I$ and define*

$$u(t) = \gamma \ominus \int_{\xi}^{t} -F(\tau)d\tau, \quad t \in I,$$

*where $\gamma \in \mathbb{R}_F$ is such that the previous H-difference exists for $t \in I$. Then $u$ is (ii)-differentiable and*

$$u'(t) = F(t), \quad t \in I.$$

**Proof:** See [20] for the proof with $\xi = 0$.

### 3. Differential inclusions' approach

We denote by $\mathcal{K}_C^n$ the family of all nonempty compact convex subsets of $\mathbb{R}^n$. Following [26], we consider the differential inclusion

$$\begin{cases} y'(t) \in F(t, y(t)), & t \in I, \\ y(0) = y_0 \in X_0, \end{cases} \tag{1}$$

where $0 \in I$, $F : I \times \mathbb{R}^n \to \mathcal{K}_C^n$ is a set-valued function and $X_0 \in \mathcal{K}_C^n$. In terms of [26], a function $y(t)$ with the initial condition $y_0 \in X_0$ is a solution of (1) on the interval $I$ if it is absolutely continuous and satisfies the equation in (1) for a.e. $t \in I$ [2, 26]. We refer to the subset of $\mathbb{R}^n$

$$\mathcal{A}_t(X_0) = \left\{ y(t) \;\middle|\; y \text{ is a solution of (1) with } y_0 \in X_0 \right\}$$

as the attainable set at time $t \in I$ related to problem (1). As it is remarked in [26] and the references therein, the choice of $F$ as a set-valued function is an excellent tool to deal with uncertainty since the equation in (1) represents that the derivative of $y$ at $t$ is not known exactly, the unique information is that it is an element of $F(t, y)$.

Next, we recall two different approaches to fuzzy dynamical systems. On one hand, the authors of [26] suggest, with the purpose of modeling fuzzy dynamical systems, to pass to a problem more general than (1) by replacing $F$ by a fuzzy set-valued function, as follows

$$\begin{cases} y'(t) \in \tilde{F}(t, y(t)), & t \in I, \\ y(0) = y_0 \in \tilde{X}_0, \end{cases} \tag{2}$$

for $\tilde{F} : I \times \mathbb{R}^n \to \mathbb{R}_F$ and $\tilde{X}_0 \in \mathbb{R}_F$, and interprete this fuzzy initial value problem (2) as a family of differential inclusions [17] at each $\alpha$-level, $0 \leq \alpha \leq 1$,

$$\begin{cases} y'(t) \in [\tilde{F}(t, y(t))]^\alpha, & t \in I, \\ y(0) = y_0 \in [\tilde{X}_0]^\alpha, \end{cases} \tag{3}$$

although, for simplicity, the fuzzy set taken as the initial condition will be represented just by $X_0$.

**Definition 3.1.** *[26] Given $\alpha \in [0,1]$, a mapping $y : I \to \mathbb{R}^n$ is said to be an $\alpha$-solution to (2) if it is a solution to problem (3). For each $t \in I$, we denote by $\mathcal{A}_t^\alpha := \mathcal{A}([X_0]^\alpha, t)$, the attainable set of the $\alpha$-solutions at time $t$, i.e.,*

$$\mathcal{A}_t^\alpha = \mathcal{A}([X_0]^\alpha, t) = \left\{ y(t) \;\middle|\; y \textit{ is a solution of } (3) \textit{ with } y_0 \in [X_0]^\alpha \right\}.$$

If, for each $t \in I$, the sets $\mathcal{A}_t^\alpha$ are the $\alpha$-level sets of a fuzzy set in $\mathbb{R}^n$, it will be denoted by $\mathcal{A}_t(X_0)$ or $\mathcal{A}(X_0, t)$ and referred to as the attainable set of problem (2) at time $t$, for $t \in I$. See [13] for sufficient conditions which guarantee that $\mathcal{A}_t^\alpha$ define a fuzzy set in $\mathbb{R}^n$, for $t \in I$.

On the other hand, concerning fuzzy differential equations, we first consider the particular case where function $f : [0, \eta] \times \mathbb{R}_F \to \mathbb{R}_F$ is the result of applying Zadeh's Extension Principle to a continuous function $h : [0, \eta] \times \mathbb{R} \to \mathbb{R}$. In this particular case, the level sets of $f(t, x)$ can be obtained in terms of function $h$ as follows:

$$[f(t, x)]^\alpha = h(t, [x]^\alpha),$$

for every $t \in [0, \eta]$, $x \in \mathbb{R}_F$ and $0 \leq \alpha \leq 1$. Following Hüllermeier [17], Diamond [13, 14] and Kaleva [18], we rewrite the fuzzy initial value problem

$$\begin{cases} y'(t) = f(t, y(t)), & t \in [0, \eta], \\ y(0) = y_0 \in \mathbb{R}_F, \end{cases} \tag{4}$$

as the following family of differential equations with set initial conditions

$$y'_\alpha(t) = h(t, y_\alpha(t)), \quad t \in [0, \eta], \qquad y_\alpha(0) \in [y_0]^\alpha, \quad 0 \le \alpha \le 1.$$

Imposing adequate hypotheses and using the Representation Theorem, it is possible to prove that the attainable sets of this family define a fuzzy funtion $y$ (see [18]), which is called a DI-solution to problem (4). It is known that, assuming the existence and uniqueness of solution for each initial value problem of the type $z'(t) = h(t, z(t)), z(0) = z_0 \in \mathbb{R}$, then it is deduced that $\mathcal{A}_t^\alpha = [z_1(t), z_2(t)]$, where $z_1'(t) = h(t, z_1(t)), z_1(0) = \underline{y_0}^\alpha$ and $z_2'(t) = h(t, z_2(t)), z_2(0) = \overline{y_0}^\alpha$. Here, $z_1$ and $z_2$ obviously depend on $\alpha$, but the level has been skipped in the expression of $\mathcal{A}_t^\alpha$ for simplicity.

Note that, in problem (4), $f$ is generated from $h$ and the right hand side of the fuzzy differential equation is independent of fuzzy parameters, so that the fuzziness is introduced only through the variable $y$. Introducing a fuzzy parameter $U$ in the equation (4), the problem adopts the form

$$\begin{cases} y'(t) = f(t, y(t), U), \quad t \in [0, \eta], \\ y(0) = y_0 \in \mathbb{R}_F, \end{cases}$$

where $f : [0, \eta] \times \mathbb{R}_F \times \mathbb{R}_F \to \mathbb{R}_F$. Assuming that $[f(t, x, U)]^\alpha = h(t, [x]^\alpha, [U]^\alpha)$, for every $\alpha \in [0, 1]$, where $h : [0, \eta] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, then the family of differential inclusions associated is (see [11])

$$y'_\alpha(t) \in h(t, y_\alpha(t), [U]^\alpha), \quad t \in [0, \eta], \qquad y_\alpha(0) \in [y_0]^\alpha, \quad 0 \le \alpha \le 1.$$

Markov, in [24], considers the problem

$$X'(t) = F(t, X(t)),\ t \in [t_0, t_1] = I, \quad X(t_0) = X_0,$$

where $X_0$ is a fixed interval and $F$ is an interval-valued function that is continuous on $\mathcal{D} = \{(t, x) \,:\, t_0 \le t \le t_0 + r, \|x - x_0\| < \beta\}$, and analyzes the existence of an interval $T = [t_0, t_0 + r_1]$, with $r_1 \le r$, and an interval-valued function $X$ differentiable on $T$ and such that $\|X(t) - X_0\| < \beta$, for $t \in T$, satisfying that $X'(t) = F(t, X(t)),\ t \in T, \quad X(t_0) = X_0$ and $X$ is *diam*-increasing in $T$ (that is, such that $diam(X(t))$ is nondecreasing). As a particular case, the following initial value problem for interval differential equations is studied

$$X'(t) = a(t)X(t) + B(t),\ t \in [t_0, t_1] = I, \quad X(t_0) = X_0,$$

with $a : T = [t_0, \infty) \longrightarrow \mathbb{R}$ continuous, $B : T \longrightarrow \mathcal{K}_C$ continuous and $X_0 \in \mathcal{K}_C$ are given. Showing an application to control theory, Markov [24] also considers the problem

$$x'(t) = a(t)x(t) + b(t)u(t) + c(t), \quad x(t_0) = x_0, \tag{5}$$

where $u$ is the control variable, the coefficients $a$, $b$, $c$ are integrable on $T = [t_0, \infty)$ and $a > 0$. Considering $U(t)$ a continuous interval valued function on $T$, the set of admissible controls for $\tau \in [t_0, t]$, $\Omega_t$, is defined as

$$\Omega_t = \{u \,:\, u(\tau) \in U(\tau), t_0 \leq \tau \leq t\}.$$

The attainable set for control problem (5) is given by

$$\mathcal{A}(t) = \{x_u(t) \,:\, x_u \text{ satisfies (5), for } u \in \Omega_t\},$$

which represents an interval *diam*-increasing function of $t$ (since $a > 0$).

Markov in [24] ensures that the attainable set for (5) satisfies the interval differential equation

$$X'(t) = a(t)X(t) + b(t)U(t) + c(t), \quad X(t_0) = X_0, \tag{6}$$

whose right-hand side is the interval extension in $x$ and $u$ of the right-hand side of (5). The problem proposed by Markov [24] is to know which are the requirements imposed on $h$ and $U$, so that the attainable sets for the system $x'(t) = h(t, x(t), u(t))$, $x(t_0) = x_0$, is a solution to the interval differential equation

$$X'(t) = H(t, X(t), U(t)),\ X(t_0) = X_0,$$

where $H$ is the interval extension of $h$, that is,

$$H(t, X, U) = \bigcup_{x \in X,\, u \in U} h(t, x, u).$$

With our notation, we consider that $B : I \to \mathbb{R}_F$ is a continuous fuzzy function, then the admissible controls for $\tau \in [t_0, t]$ and $\alpha \in [0, 1]$,

$$\Omega_t^\alpha = \{\gamma : I \longrightarrow \mathbb{R},\ \gamma \text{ measurable},\ \gamma(\tau) \in [B(\tau)]^\alpha,\ t_0 \leq \tau \leq t\}.$$

Hence, the attainable set for the problem

$$x'(t) = a(t)x(t) + \gamma(t), \quad x(t_0) = x_0, \tag{7}$$

is

$$\mathcal{A}_\alpha(t) = \{x(t) \,:\, x \text{ satisfies (7) for } \gamma \in \Omega_t^\alpha\}.$$

This represents, for each $\alpha$, an interval whose diameter is monotonic increasing in $t$, if $a > 0$ (see [24]). Markov question's brought to the context of fuzzy differential equations is whether those attainable sets are coincident with the solution to the fuzzy differential equation

$$x'(t) = a(t)x(t) + B(t) = H(t, x(t), B(t)), \quad x(t_0) = x_0, \tag{8}$$

where $H$ is the fuzzy extension of $h(t, x, u) = a(t)x + u$ by using Zadeh' Extension Principle, that is, $H(t, X, U) = a(t)X + U$, from the properties of the sum and the multiplication by an scalar. Hence, $[H(t, X, U)]^\alpha = a(t)[X]^\alpha + [U]^\alpha = h(t, [X]^\alpha, [U]^\alpha)$. This problem can be formulated more generally as the equation

$$x'(t) = f(t, x, W(t)),\ x(0) = x_0, \tag{9}$$

where $f : I \times \mathbb{R}^n_F \times \mathbb{R}^m_F \longrightarrow \mathbb{R}^n_F$ is obtained by Zadeh' Extension Principle from a real-valued continuous function $h : I \times \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^n$, $x_0 \in \mathbb{R}^n_F$. The approach in [11], for solving this problem via differential inclusions, makes use of the identity

$$[f(t, x, U)]^\alpha = h(t, [x]^\alpha, [U]^\alpha),\ \alpha \in [0, 1],$$

for $h(t, C, D) = \{h(t, c, d) \,:\, c \in C, d \in D\}$. According to [11], the previous fuzzy problem can be written as a family of differential equations with set initial conditions

$$x'_\alpha(t) = \overline{f}_\alpha(t, x_\alpha(t)),\ x_\alpha(0) \in [x_0]^\alpha,\ 0 \le \alpha \le 1, \tag{10}$$

where the continuous function $\overline{f}_\alpha : I \times \mathbb{R} \longrightarrow \mathcal{K}^n_C$ is defined by

$$\overline{f}_\alpha(t, x_\alpha) = h(t, x_\alpha, [W(t)]^\alpha) = \{h(t, x_\alpha, w) \,:\, w \in [W(t)]^\alpha\}.$$

Hence, the attainable sets at time $t$ are given by

$$\mathcal{A}_\alpha(t, x_0^\alpha, [B(\cdot)]^\alpha) = \{x_\alpha(t) \,:\, x_\alpha(\cdot) \text{ is a solution to (10) on } I\}.$$

Assuming that, for every $t \in I$, the family $\{\mathcal{A}_\alpha(t, x_0^\alpha, [B_{|[0,t]}(\cdot)]^\alpha)\}_{\alpha \in [0,1]}$ represent the $\alpha$-levels of a compact fuzzy set $\mathcal{A}(t, x_0, B_{|[0,t]})$, then $X_{IW}(t) = \mathcal{A}(t, x_0, B_{|[0,t]})$ is called a weak fuzzy solution to (9) via differential inclusion. If, moreover, $\mathcal{A}(t, x_0, B_{|[0,t]})$ is also a convex fuzzy set, then $X_I(t) = \mathcal{A}(t, x_0, B_{|[0,t]})$ is called a fuzzy solution to (9) via differential inclusion, whose level sets are the attainable sets at time $t$, $\{\mathcal{A}_\alpha(t, x_0^\alpha, [B_{|[0,t]}(\cdot)]^\alpha)$ for all $\alpha \in [0, 1]$. Under some conditions on $f$, the family of differential inclusions (10) is equivalent to a family of dynamical systems controlled by parameters $\gamma_\alpha(t) \in [B(t)]^\alpha$, (see [11]),

$$x'_\alpha(t) = h(t, x_\alpha(t), \gamma_\alpha(t)),\ x_\alpha(0) = x_0^\alpha \in [x_0]^\alpha,\ 0 \le \alpha \le 1. \tag{11}$$

Denote by $\mathcal{M}(I, [B(\cdot)]^\alpha)$ the family of all measurable functions $\gamma_\alpha : I \longrightarrow \mathbb{R}^m$ such that $\gamma_\alpha(t) \in [B(t)]^\alpha$. For each $\gamma_\alpha \in \mathcal{M}(I, [B(\cdot)]^\alpha)$ and $x_0^\alpha \in [x_0]^\alpha$, under the existence and uniqueness hypotheses, we obtain the existence of a unique solution $x_\alpha(t, x_0^\alpha, \gamma_\alpha)$ to (11) and it is obviously a solution to the differential inclusion in (10). This justifies, similarly to [11], that

$$\mathcal{A}_\alpha(t, x_0^\alpha, [B_{|[0,t]}(\cdot)]^\alpha) = \bigcup_{\gamma_\alpha \in \mathcal{M}(I,[B(\cdot)]^\alpha),\, x_0^\alpha \in [x_0]^\alpha} x_\alpha(t, x_0^\alpha, \gamma_\alpha).$$

In our particular case, $x' = a(t)x + b(t)$, the existence and uniqueness of solution to each problem in (11) is guaranteed and $h(t, x, u) = a(t)x + u$, so that the family of differential inclusions in (10) is written as

$$x'_\alpha(t) = a(t)x_\alpha(t) + [B(t)]^\alpha,\ x_\alpha(0) \in [x_0]^\alpha,\ 0 \leq \alpha \leq 1, \tag{12}$$

and the family of dynamical systems controlled by parameters as

$$x'_\alpha(t) = a(t)x_\alpha(t) + \gamma_\alpha(t),\ x_\alpha(0) = x_0^\alpha \in [x_0]^\alpha,\ 0 \leq \alpha \leq 1, \tag{13}$$

where $\gamma_\alpha(t) \in [B(t)]^\alpha$, $t \in I$. Hence, the linear problem of interest can be considered such as in [11] but, in this case, the control $W$ is not necessarily constant in $t$, instead, it is a function of $t$.

Since we are working with continuous functions $a$ and $B$, to calculate $\mathcal{A}_\alpha(t, x_0^\alpha, [B(\cdot)]^\alpha)$, it is enough to consider $\gamma_\alpha$ continuous such that $\gamma_\alpha(t) \in [B(t)]^\alpha$. Also, since the solution to (13) exists always and it is increasing in $\gamma_\alpha$ and $x_0^\alpha$, then $\mathcal{A}_\alpha(t, x_0^\alpha, [B(\cdot)]^\alpha = [x_\alpha(t, \underline{x_0}^\alpha, \underline{B}^\alpha(\cdot)), x_\alpha(t, \overline{x_0}^\alpha, \overline{B}^\alpha(\cdot))]$, $\alpha \in [0, 1]$.

Concerning the fuzzy differential inclusion

$$x' \widehat{\in} \hat{f}(t, x(t)) = f(t, x(t), B(t)), \qquad x(0) \widehat{\in} X_0,$$

where $\hat{f} : I \times \mathbb{R}^n \longrightarrow \mathbb{R}^n_F$, $X_0 \in \mathbb{R}^n_F$, $\hat{f}(t, x) = f(t, x, B(t))$, where $f$ comes by Zadeh's Extension Principle from $h : I \times \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^n$ with respect to the third variable, then the family of differential inclusions associated is

$$x'_\alpha(t) \in [\hat{f}(t, x(t))]^\alpha = [f(t, x(t), B(t))]^\alpha = h(t, x_\alpha(t), [B(t)]^\alpha), x_\alpha(0) \in [X_0]^\alpha, \tag{14}$$

for $0 \leq \alpha \leq 1$, which coincides with that in (10), then both approaches coincide, similarly to the specifications in [11] for a fixed control $W \in \mathbb{R}_F$.

In the particular case where $\hat{f}(t, x) = a(t)x + B(t) = f(t, x, B(t))$, $h(t, x, u) = a(t)x + u$, $f(t, X, U) = a(t)X + U$, the differential inclusion $x' \in a(t)x + B(t)$, $x(0) \in X_0$ is written as (12).

The results in Section 3.4 [11] seem to be extensible to the case of non constant control function $B$ with minor adaptations. Hence, in the particular case of the linear problem, all those conditions hold: $h(t, x, u) = a(t)x + u$ is continuous, the boundedness assumption holds for all $x_0 \in [x_0]^\alpha$ and the inclusion

$$x' \in \overline{f_0}(t, x) = h(t, x, [B(t)]^0) = a(t)x + [B(t)]^0, \;\; x(0) \in [x_0]^0,$$

so that the attainable sets are nonempty and compact for each $t$. Besides, $\hat{f}(t, x) = f(t, x, B(t)) = a(t)x + B(t)$ is concave. Indeed, if $\alpha + \beta = 1$, then

$$\begin{aligned} &\alpha\hat{f}(t, x) + \beta\hat{f}(t, y) = \alpha(a(t)x + B(t)) + \beta(a(t)y + B(t)) \\ &= a(t)(\alpha x + \beta y) + \alpha B(t) + \beta B(t) = a(t)(\alpha x + \beta y) + B(t) \leq \hat{f}(t, \alpha x + \beta y), \end{aligned}$$

thus the attainable sets are convex (see [11] and the references therein). The hypotheses of Proposition 2 in [11] are also true replacing $W$ by $B(t)$. Besides,

$$\begin{aligned} D(f(t, X, B(t)), f(t, Y, B(t))) &= D(a(t)X + B(t), a(t)Y + B(t)) \\ &= D(a(t)X, a(t)Y) = |a(t)|D(X, Y) \leq LD(X, Y) \end{aligned}$$

on compact sets, so that $f$ is Lipschitzian on the compact sets of the type $I$, then the attainable sets are nonempty, convex and compact subsets of $\mathbb{R}$, for every $t \in I$. Concerning Theorem 4 in [11], $h$ is continuous, with the boundedness condition on

$$x' \in h(t, x, [B(t)]^0), \;\; x(0) \in [x_0]^0,$$

hence, there exists a unique solution to the problem on $I$. Since the attainable sets $\mathcal{A}_\alpha(t, x_0^\alpha, [B(\cdot)]^\alpha)$ are convex, for $\alpha \in [0, 1]$, there exists a unique fuzzy solution on $I$.

With respect to the relationship between (i)-solutions and the solution via differential inclusions, the following properties are already known. For $n = 1$, $a \geq 0$ and $F$ measurable and integrably bounded, Markov [24] ensures that the DI-solution coincides with the (i)-solution.

In [34], it is recalled the problem proposed by Markov [24] explicited for the problem $\dot{x} \in a(t)x + V(t)$, for $a$ real-valued and $V$ interval-valued. Tolstonogov gives a positive answer in the following case: if $a \geq 0$, $a$ summable on $T$ (resp., on compact sets of $\mathbb{R}^+$), and $V : T \to \operatorname{conv} X$ (resp., $V : \mathbb{R}^+ \to \operatorname{conv} X$) is measurable and integrably bounded on $T$ (resp., on compact sets of $\mathbb{R}^+$), then the integral funnel of the differential inclusion coincides with the Hukuhara-solution to the interval equation

$D_H U = a(t)U + V(t)$ on $T$ (resp., $\mathbb{R}^+$). Hence, in the case where the sets $\mathcal{A}_\alpha$ define a fuzzy number (which is true for linear equations), then we can affirm that the DI-solution coincides with the (i)-solution.

In the general case, consider $T$ a finite interval of the type $[0, b]$ or even $\mathbb{R}^+$, $X_0$ a convex subset of $\mathbb{R}$, $\Gamma : T \times \mathbb{R} \longrightarrow conv(\mathbb{R})$ and assume conditions which guarantee the existence of a unique solution for the differential inclusion (called $R$-solution) $F(t)$ with $F(0) = X_0$ defined on $T$ and the existence of a unique solution to the Hukuhara differential equation $U(t)$ with $U(0) = U_0$ defined on $T$. Under these conditions, in [34], it is deduced that $\Omega_\Gamma(t, X_0) = F(t) \subset U(t)$, for $t \in T$. Here, $\Phi_\Gamma(M) = \{(t, x(t)) : x(\cdot) \in H_\Gamma(M), t \in T\}$ is the integral funnel of the differential inclusion, $H_\Gamma(M)$ is the set of all Carathodory type solutions $x(t)$, $x(t) \in M$, and $\Omega_\Gamma(t, M) = \{x(t) : x(\cdot) \in H_\Gamma(M)\}$ is the integral funnel section at time $t$, that is, the attainable set at time $t$ (which we denote by $\mathcal{A}(t, X_0)$). Besides, due to the bijective correspondence between the integral funnel $\Phi_\Gamma(M)$ and the graph of the mapping $t \longrightarrow \Omega_\Gamma(t, M)$, the mapping $\Omega_\Gamma(\cdot, M)$ is understood as the integral funnel of the differential inclusion. In particular, Tolstonogov [34] affirms that, for a general linear problem, the DI-solution is 'included' in the (i)-solution. However, the identity is not necessarily valid, which is illustrated through the problem $\dot{x} = ax + [-m, m]$, $x(0) = 0$, where $m > 0$, for which (see [34, pages 209–210]) the DI-solution coincides with (i)-solution for $a > 0$, while the DI-solution is strictly included in the (i)-solution for $a < 0$.

In [11], it is shown that a fuzzy differential equation $x' = f(t, x(t))$, for $f : I \times \mathbb{R}^n_F \longrightarrow \mathbb{R}^n_F$, cannot always be written as a family of differential inclusions. In the example shown, the authors illustrate the possible lack of connection between the variable $x_\alpha$ and the endpoints of the $\alpha$-levels of $f(t, x)$, due to the dependence of all the level sets on a certain fixed level. However, if we can 'separate' levels, in such a way that $[f(t, x)]^\alpha = h_\alpha(t, [x]^\alpha)$, where $h_\alpha : I \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ continuous, then it is possible to consider a family of differential inclusions associated. As it is said in [11], a condition is that each $\alpha$-level of $f(t, x)$ is independent of the rest of levels $\beta$ with $\beta \neq \alpha$, such as it occurs for fuzzy functions generated through Zadeh' Extension Principle, but also for $f(t, x) = a(t)x + b(t)$, for a fuzzy $b$, which is our problem of interest.

The linear equation $y' = a(t)y + b(t)$ is not necessarily of the type $y' = f(t, y, U)$, where $U$ is fuzzy.

**Theorem 3.2.** *If $b : I \longrightarrow \mathbb{R}$, then we can compare the differential inclusion solution with generalized solution and the switching points are the points* *where $a$ changes its sign. However this is not always possible for $b$ nonreal.*

For problem (4), we use the following concepts of solution from the point of view of strongly generalized differentiability.

**Definition 3.3.** *Let $y : I = [0, \eta] \to \mathbb{R}_F$ be a fuzzy function. We say that $y$ is a (i)-solution to problem (4) if $D_1 y$ exists on $I$ and $y$ satisfies problem (4) taking $D_1 y$ as $y'$. Similarly, we say that $y$ is a (ii)-solution to problem (4) if $D_2 y$ exists on $I$ and $y$ satisfies problem (4) taking $D_2 y$ as $y'$.*

Again, we use the notation $y'$ both for $D_1 y$ and $D_2 y$, making explicit the type of differentiability used.

The following results, included respectively in [18, 9], allow to compare the solutions obtained from differential inclusions' approach and using the strongly generalized differentiability, distinguishing the cases where (i)-differentiability (Theorem 3.4) or (ii)-differentiability (Theorem 3.5) is used.

**Theorem 3.4.** *(Theorem 3 [18]) If $h$ is nondecreasing with respect to the second argument, then the (i)-solution and the DI-solution are identical.*

**Proof:** See [18].

**Theorem 3.5.** *(Theorem 8 [9]) If $h$ is nonincreasing with respect to the second argument, then the (ii)-solution and the DI-solution are identical.*

**Proof:** See [9].

However, these results are restricted to the case of fuzzy functions obtained by Zadeh's Extension Principle from a real continuous function. The purpose of the following section is to consider a class of initial value problems for fuzzy differential equations where Theorems 3.4–3.5 are not applicable and compare the solutions from the different approaches in order to detect their possible coincidence or distinction.

## 4. Solving linear fuzzy differential equations

In this section, we first work with a simple homogeneus equation to illustrate the procedure and then we take a more general linear equation. We start with the simple fuzzy initial value problem

$$\begin{cases} y'(t) = ay(t), \quad t \in I, \\ y(0) = y_0, \end{cases} \tag{15}$$

where $a \in \mathbb{R}$, $0 \in I$ and $y_0 \in \mathbb{R}_F$. For simplicity, we work in intervals of the type $I = [0, A]$, for some $A > 0$, or $I = [0, \infty)$.

Considering Eq. (15) under the differential inclusion's approach, since function $F(t, y) = ay$ is obtained by Zadeh's extension principle from the continuous function

$$f : I \times \mathbb{R} \to \mathbb{R}$$

$$f(t, y) = ay,$$

then, for each $\alpha \in [0, 1]$ and $t \in I$, we have

$$[y'(t)]^\alpha = [F(t, y(t))]^\alpha = f(t, [y(t)]^\alpha) = a[y(t)]^\alpha.$$

Hence, the corresponding family of differential equations with a set of initial conditions is given by

$$\begin{cases} y'_\alpha(t) = ay_\alpha(t), & t \in I, \\ y_\alpha(0) \in [y_0]^\alpha, \end{cases}$$

whose attainable sets are the $\alpha$-level sets of the fuzzy function

$$y(t) = y_0 e^{at}.$$

Now, if $a > 0$, then $f$ is nondecreasing with respect to the second variable and, according to Theorem 3.4, the (i)-solution of (15) is exactly the same as the DI-solution (see [5, 20])

$$y(t) = y_0 e^{at}.$$

On the other hand, if $a < 0$, then $f$ is nonincreasing with respect to the second variable and, consistently with Theorem 3.5, (ii)-solution is equal to the solution by differential inclusions $y(t) = y_0 e^{at}$ (see [5, 20]).

We note that, in the differential inclusion's approach, the solution has a unique expression independently of the sign of $a$. In this case of homogeneus linear equations, the diameter of the DI-solution is nondecreasing for $a > 0$ and nonincreasing for $a < 0$, which is consistent with the properties of (i)- and (ii)-solutions by virtue of Theorems 3.4–3.5.

Now, we consider the more general case of first order nonhomogeneous linear fuzzy differential equations with initial value conditions of the type:

$$\begin{cases} y'(t) = a(t)y(t) + b(t), & t \in I, \\ y(0) = y_0, \end{cases} \tag{16}$$

where $a : I \to \mathbb{R}, b : I \to \mathbb{R}_F$ are continuous functions and $y_0 \in \mathbb{R}_F$.
Our aim is to investigate the connection between the solutions under differential inclusion's approach and strongly generalized differentiable solutions of Eq. (16).

**Theorem 4.1.** *For $a > 0$, the (i)-solution of problem (16) is given by*

$$y(t) = e^{\int_0^t a(u)du} \left( y_0 + \int_0^t b(s) e^{-\int_0^s a(u)du} ds \right),$$

*and (ii)-solution is as follows*

$$y(t) = \cosh(\int_0^t a(u)du) \left( y_0 \ominus \int_0^t \left[ b(s)\sinh(\int_0^s a(u)du) - b(s)\cosh(\int_0^s a(u)du) \right] ds \right)$$

$$\ominus - \sinh(\int_0^t a(u)du) \left( y_0 \ominus \int_0^t \left[ b(s)\sinh(\int_0^s a(u)du) - b(s)\cosh(\int_0^s a(u)du) \right] ds \right),$$

*where we assume that the H-differences exist and the minus sign '−' represents the classical addition of the opposite.*

**Proof:** See Theorem 6 in [5] for the expression of the (i)-solution and [20] for (ii)-solution.

**Theorem 4.2.** *For $a < 0$, the (ii)-solution of problem (16) is given by*

$$y(t) = e^{\int_0^t a(u)du} \left( y_0 \ominus \int_0^t (-b(s)) e^{-\int_0^s a(u)du} ds \right),$$

*provided that the H-differences exist, and (i)-solution is as follows*

$$y(t) = \cosh(\int_0^t a(u)du) \left( y_0 + \int_0^t \left[ b(s)\cosh(\int_0^s a(u)du) \ominus b(s)\sinh(\int_0^s a(u)du) \right] ds \right)$$

$$+ \sinh(\int_0^t a(u)du) \left( y_0 + \int_0^t \left[ b(s)\cosh(\int_0^s a(u)du) \ominus b(s)\sinh(\int_0^s a(u)du) \right] ds \right),$$

*provided that the H-differences in the integral terms exist.*

**Proof:** See Theorem 6 in [5] for the expression of the (ii)-solution and [20] for (i)-solution.

The conditions that provide the existence of the solutions given in Theorems 4.1, 4.2 are reduced to the existence of the corresponding $H$-differences. For more details, see [8, 22].

**Remark 4.3.** *In the special case where* $b(t) \in \mathbb{R}$*, for every* $t \in I$*, we have that the fuzzy function*

$$F(t,y) = a(t)y + b(t), \quad y \in \mathbb{R}_F,$$

*is obtained from the real continuous function*

$$f(t,y) = a(t)y + b(t), \quad y \in \mathbb{R},$$

*by Zadeh's extension principle. Indeed, for* $(t,y) \in I \times \mathbb{R}_F$ *and each* $\alpha \in [0,1]$*, we have*

$$[F(t,y)]^\alpha = [a(t)y + b(t)]^\alpha = a(t)[y]^\alpha + \{b(t)\} = f(t,[y]^\alpha).$$

*Then, according to Theorems 3.4 and 3.5, if* $a > 0$ *(i.e.,* $a(t) > 0$ *for every* $t \in I$*), then the (i)-solution of problem (16) is the same as the solution under differential inclusion's approach and, analogously, when* $a < 0$ *the (ii)-solution is the same as differential inclusion's solution.*

However, if, in Eq. (16), we consider

$$F : I \times \mathbb{R}_F \to \mathbb{R}_F,$$

$$F(t,y) = a(t)y + b(t),$$

where $b$ is a nonreal fuzzy function, i.e., $b(t) \in \mathbb{R}_F$ (or even $b(t) \in \mathcal{K}_C^1$, but excluding the case where $b(t) \in \mathbb{R}$ for every $t \in I$), then the fuzzy function $F$ is not the extension of any real continuous function $f : I \times \mathbb{R} \to \mathbb{R}$. Indeed, there is no such a real function $f$ satisfying that

$$[F(t,y)]^\alpha = a(t)[y]^\alpha + [b(t)]^\alpha = f(t,[y]^\alpha), \text{ for every } \alpha \in [0,1],$$

if, for at least some $t \in I$ and $\alpha \in [0,1]$, $[b(t)]^\alpha$ has positive length. Therefore, we cannot apply Theorems 3.4 and 3.5 to compare the different types of solutions.
According to Hüllermeier's interpretation [17], we can write problem (16) as a family of differential inclusions

$$\begin{cases} y'_\alpha(t) \in a(t)y_\alpha(t) + [b(t)]^\alpha, & t \in I, \\ y_\alpha(0) \in [y_0]^\alpha, \end{cases}$$

or

$$\begin{cases} y'_\alpha(t) \in F_\alpha(t, y_\alpha(t)), & t \in I, \\ y(0) \in [y_0]^\alpha, \end{cases}$$

where $F_\alpha(t,y) = a(t)y + [b(t)]^\alpha$, $F_\alpha : I \times \mathbb{R} \longrightarrow \mathcal{K}_C^1$, and solve the collection of problems

$$\begin{cases} y'_\alpha(t) = a(t)y_\alpha(t) + \gamma(t), \quad t \in I, \quad \gamma(t) \in [b(t)]^\alpha, \\ y_\alpha(0) \in [y_0]^\alpha. \end{cases} \tag{17}$$

**Definition 4.4.** *We call a function $y : I \to \mathbb{R}$ an $\alpha$-solution to (16) if it is continuous and satisfies (17) for some admissible function $\gamma$. The set of all $\alpha$-solutions to (16) is denoted by $X_\alpha$ and the attainable set at $t$, $\mathcal{A}_\alpha(t)$, are given by*

$$\mathcal{A}_\alpha(t) = \{y(t)|y(\cdot) \in X_\alpha\}.$$

Using classical results and the theory of differential inclusions [2], since $a, \gamma : I \to \mathbb{R}$ are continuous functions and the solutions to

$$\begin{cases} y'_\alpha(t) = a(t)y_\alpha(t) + \gamma(t), \quad t \in I, \\ y_\alpha(0) = C \end{cases}$$

are nondecreasing in the initial condition $C$ and also in the independent term $\gamma$, then the solution of the differential inclusion (17) is obtained as:

$$\mathcal{A}_\alpha(t) = e^{\int_0^t a(u)du}\left([y_0]^\alpha + \int_0^t [b(s)]^\alpha e^{-\int_0^s a(u)du} ds\right), \quad \forall \alpha \in [0,1]. \tag{18}$$

## 5. Some relations between solutions

In the following, we study the connection between the solutions corresponding to the strongly generalized differentiability concept and those of differential inclusion's approach. We consider problem (16) in three cases: $a > 0, a < 0$ and $a \equiv 0$. It is clear that the most interesting case corresponds to a nonreal function $b$.

**Case 1.** $a > 0$**:**
According to Theorem 4.1 and Eq. (18), for $a > 0$, the DI-solution and the (i)-solution are equal. Then, in this case, any relation between the (i)-solution and the (ii)-solution is also valid between the DI-solution and the (ii)-solution. The diameter of each level set of the (ii)-solution is nonincreasing and the diameter of each level set of the (i)-solution (or DI-solution) is nondecreasing.

**Case 2.** $a < 0$**:**
Comparing Eq. (18) and Theorem 4.2, DI-solution appears to be different

from (i)-solution and (ii)-solution. In the rest of the section, we study the relationship between the DI-solution and the strongly generalized differentiable solutions to problem (16).

**Theorem 5.1.** *Suppose that $a < 0$ and let $y^{(*)}$ be the DI-solution of problem (16). Then, as far as the (ii)-solution $y^{(ii)}$ to (16) exists, it is less fuzzy than (or has the same fuzziness as) the DI-solution $y^{(*)}$, i.e., we have*

$$\underline{y}^{(*)}(t) \leq \underline{y}^{(ii)}(t) \quad \text{and} \quad \overline{y}^{(ii)}(t) \leq \overline{y}^{(*)}(t), \quad \forall t \in J, \tag{19}$$

*where $J \subseteq I$ is the interval where the (ii)-solution $y^{(ii)}$ is well-defined.*

**Proof:** First, we prove that $\underline{y}^{(*)}(t) \leq \underline{y}^{(ii)}(t), \quad \forall t \in J$. For this, we need to show, for every $t \in J$,

$$e^{\int_0^t a(u)du}\left(\underline{y_0} + \int_0^t \overline{b}(s)e^{-\int_0^s a(u)du}ds\right) \geq e^{\int_0^t a(u)du}\left(\underline{y_0} + \int_0^t \underline{b}(s)e^{-\int_0^s a(u)du}ds\right),$$

which is correct, since it is equivalent to

$$e^{\int_0^t a(u)du}\left(\int_0^t (\overline{b}(s) - \underline{b}(s))e^{-\int_0^s a(u)du}ds\right) \geq 0,\ t \in J.$$

For the other inequality, it is obvious that, for $t \in J$,

$$e^{\int_0^t a(u)du}\left(\overline{y_0} + \int_0^t \overline{b}(s)e^{-\int_0^s a(u)du}ds\right) \geq e^{\int_0^t a(u)du}\left(\overline{y_0} + \int_0^t \underline{b}(s)e^{-\int_0^s a(u)du}ds\right),$$

therefore $\overline{y}^{(ii)}(t) \leq \overline{y}^{(*)}(t)$, $\forall t \in J$.

**Remark 5.2.** *Note that the inequalities in (19) are valid on the interval $I$, even in the case where the (ii)-solution is not well-defined in the whole interval $I$. This situation corresponds to the interchanging of position of the branches which are not defining a proper (ii)-solution anymore, but both functions still lie in the region determined by the branches of the DI-solution.*

In the statement of Theorem 5.1, the solutions are identical just if $b$ is a real-valued function. That is why we affirm that, for $b$ nonreal, the (ii)-solution $y^{(ii)}$ is strictly less fuzzy than the DI-solution $y^{(*)}$. This is explained in detail in the following Remark.

**Remark 5.3.** *Another consequence of the proof of Theorem 5.1 is the following: under the assumptions of Theorem 5.1 and for b nonreal, we have*

$$\underline{y}^{(*)}(t) \prec \underline{y}^{(ii)}(t) \quad \text{and} \quad \overline{y}^{(ii)}(t) \prec \overline{y}^{(*)}(t), \quad \text{for } t \in I \text{ with } t > \mu, \tag{20}$$

*where $\mu \in I$ is such that $b(\mu) \notin \mathbb{R}$. In the previous inequalities, $\underline{u} \prec \underline{v}$ means that $\underline{u} \leq \underline{v}$ but $\underline{u}(\alpha) < \underline{v}(\alpha)$ for some $\alpha \in [0,1]$ (and, necessarily, for $\alpha = 0$).*

*This comes from the fact that $b(\mu) \notin \mathbb{R}$ implies that $\overline{b}(\mu;\alpha) > \underline{b}(\mu;\alpha)$ for some $\alpha \in [0,1]$ and, by continuity of $b$, $\overline{b}(s;\alpha) > \underline{b}(s;\alpha)$ for $s$ in a neighborhood of $\mu$, so that*

$$e^{\int_0^t a(u)du}\left(\int_0^t (\overline{b}(s;\alpha) - \underline{b}(s;\alpha))e^{-\int_0^s a(u)du}ds\right) > 0, \ t > \mu,$$

*which leads to*

$$\underline{y}^{(*)}(t;\alpha) < \underline{y}^{(ii)}(t;\alpha) \quad \text{and} \quad \overline{y}^{(ii)}(t;\alpha) < \overline{y}^{(*)}(t;\alpha), \quad \text{for } t \in I \text{ with } t > \mu,$$

*and similarly for the rest of levels $\beta \in [0,\alpha]$. In fact, (20) holds for $t > \delta$, where $\delta = \inf\{\mu \in I \ : \ b(\mu) \notin \mathbb{R}\}$. If $b(0) \notin \mathbb{R}$, then (20) holds for $t > 0$, since, at $t = 0$, the value of both solutions is $y_0$, but the same happens if $b(t) \notin \mathbb{R}$, for $t > 0$.*

The conclusion of the following result is well-known (see [27, 28, 34]). In [34], Tolstonogov proves that the DI-solution is less fuzzy than the (i)-solution for every linear problem and independently of the sign of the coefficient $a$. However, the following proof gives the clue to determine under which circumstances the comparison between the different solutions is strict.

**Theorem 5.4.** *Suppose that $a < 0$ and let $y^{(*)}$ be the DI-solution of (16). Then the DI-solution is less fuzzy than (or has the same fuzziness as) the (i)-solution $y^{(i)}$ of (16), i.e., we have*

$$\underline{y}^{(i)}(t) \leq \underline{y}^{(*)}(t) \quad \text{and} \quad \overline{y}^{(*)}(t) \leq \overline{y}^{(i)}(t), \quad \forall t \in I.$$

**Proof:** We prove the first part and the other part can be deduced similarly. By expression (18) and Theorem 4.2, we need to show

$$\underline{y}^{(i)}(t) = \underline{y_0}\cosh z(t) + \int_0^t \left\{\cosh z(t)\cosh z(s)\underline{b}(s) - \cosh z(t)\sinh z(s)\overline{b}(s)\right\} ds$$

$$+\overline{y_0}\sinh z(t) + \int_0^t \left\{\sinh z(t)\cosh z(s)\overline{b}(s) - \sinh z(t)\sinh z(s)\underline{b}(s)\right\} ds$$

$$\leq \underline{y_0} e^{z(t)} + \int_0^t \underline{b}(s) e^{z(t)-z(s)} ds,$$

where $z(t) = \int_0^t a(u)du$. This inequality is equivalent to the following

$$\underline{y}^{(i)}(t) = \underline{y_0} \cosh z(t) + \overline{y_0} \sinh z(t) + \int_0^t \underline{b}(s) \{\cosh z(t) \cosh z(s) - \sinh z(t) \sinh z(s)\} \, ds$$

$$+ \int_0^t \overline{b}(s) \{\sinh z(t) \cosh z(s) - \cosh z(t) \sinh z(s)\} \, ds$$

$$= \underline{y_0} \cosh z(t) + \overline{y_0} \sinh z(t) + \int_0^t \left\{\underline{b}(s) \cosh(z(t) - z(s)) + \overline{b}(s) \sinh(z(t) - z(s))\right\} ds$$

$$\leq \underline{y_0} e^{z(t)} + \int_0^t \underline{b}(s) e^{z(t)-z(s)} ds,$$

or also

$$\underline{y_0}(-\cosh z(t) + e^{z(t)}) - \overline{y_0} \sinh z(t)$$

$$+ \int_0^t \left\{\underline{b}(s)(e^{z(t)-z(s)} - \cosh(z(t) - z(s))) - \overline{b}(s) \sinh(z(t) - z(s))\right\} ds \geq 0.$$

The previous inequality can also be written as

$$(\underline{y_0} - \overline{y_0}) \sinh z(t) + \int_0^t \sinh(z(t) - z(s))(\underline{b}(s) - \overline{b}(s)) \, ds \geq 0,$$

or, equivalently,

$$\sinh z(t) \; diam[y_0]^\alpha + \int_0^t \sinh(z(t) - z(s)) diam[b(s)]^\alpha \, ds \leq 0,$$

for every $\alpha \in [0, 1]$, which is trivially valid. Then the proof is complete.

**Remark 5.5.** *Under the assumptions of Theorem 5.4, since $z(t) = \int_0^t a(u)du < 0$, for every $t > 0$, then $\sinh z(t) < 0$, for $t > 0$, and also $\sinh(z(t) - z(s)) = \sinh(\int_s^t a(u)du) < 0$, for $t > 0$ and $s \in (0, t)$. Hence, the unique possibility of coincidence between $\underline{y}^{(i)}(t)$ and $\underline{y}^{(*)}(t)$ is*

$$diam[y_0]^\alpha = 0, \; \forall \alpha \in [0, 1] \quad \text{and} \quad diam[b(t)]^\alpha = 0, \; \forall \alpha \in [0, 1], \; t \in I,$$

*that is, $y_0$ real and $b$ real function simultaneously, and similarly for the upper branch. If this is not the case, then the DI-solution is less fuzzy than the (i)-solution.*

**Case 3.** $a \equiv 0$**:**
In this case, problem (16) is reduced to

$$y'(t) = b(t),\ t \in I, \qquad y(0) = y_0.$$

The (i)-solution of this problem is

$$y_0 + \int_0^t b(s)ds,$$

and the (ii)-solution is given by

$$y_0 \ominus \int_0^t (-b(s))ds,$$

provided that the H-differences exist (see, for instance, [20]). The DI-solution is the same as the (i)-solution and the results of comparison between solutions are identical to the case $a > 0$.

*5.1. Monotonicity of the diameter of the solutions*

Independently of the sign of $a(t)$, the diameter of the level sets of the solutions corresponding to strongly generalized differentiability is monotonic: the diameter of the level sets of the (i)-solution is nondecreasing and the diameter of the level sets of the (ii)-solution is nonincreasing.
According to the specifications made in the case $a > 0$, we see that the diameter of the level sets of the DI-solution ((i)-solution) is nondecreasing. However, this property is not true in general for the case $a < 0$, where most DI-solutions are fuzzier than the corresponding (ii)-solutions and less fuzzy than (i)-solutions. To this purpose, we have to define the concept of diameter of the level sets of the DI-solution. For simplicity, let $y(t)$ denote the DI-solution of problem (16). Set $\phi_\alpha(t) = diam[y(t)]^\alpha := diam(\mathcal{A}_t^\alpha) = diam(\mathcal{A}_\alpha(t))$, for $t \in I$ and $\alpha \in [0, 1]$. Then we have

$$\phi_\alpha(t) = e^{z(t)} diam[y_0]^\alpha + e^{z(t)} \int_0^t diam[b(s)]^\alpha e^{-z(s)} ds,$$

where $z(t) = \int_0^t a(u)du$.

**Remark 5.6.** *It is obvious that, if $a < 0$, the diameter of the level sets of the DI-solution is not monotonic in general. Indeed, since $a < 0$, the term $e^{z(t)}$ in $\phi_\alpha(t)$ is nonincreasing and $diam[y_0]^\alpha + \int_0^t diam[b(s)]^\alpha e^{-z(s)}ds$ is nondecreasing. Hence, the monotonicity of $\phi_\alpha(t)$ in the variable $t$ depends on $a$, $b$ and the initial value $y_0$.*

**Remark 5.7.** *Since $a(t)$ and $b(t)$ are continuous functions, then we have*

$$\phi'_\alpha(t) = e^{z(t)}a(t)\left(diam[y_0]^\alpha + \int_0^t diam[b(s)]^\alpha e^{-z(s)}ds\right) + diam[b(t)]^\alpha.$$

*In the particular case where $b$ is a real function, then $diam[b(t)]^\alpha = 0$, for every $t$ and $\alpha$, so that the diameter of the level sets of the DI-solution is nonincreasing, which is consistent with Remark 4.3 (it is (ii)-solution).*

If the initial condition is $y(0) = y_0$, when $t$ increases, the monotonicity of the diameter of the level sets of the DI-solution in a neighborhood of $t = 0$ depends on $a$, $b$ and $y_0$. In the following section, we present Examples 6.1 and 6.2, where the diameter of the level sets of the DI-solution first decreases and then increases, and Example 6.3, where it increases and then decreases, alternating its monotonicity indefinitely. The number of changes in the monotonicity of the solution itself also depends on these parameters $a$, $b$ and $y_0$.

In Examples 6.1 and 6.2, we take $b(t) = \gamma t$, thus $[b(0)]^\alpha$ is a singleton and hence, for every $\alpha \in [0, 1]$, $diam[b(0)]^\alpha = 0$. Then, by Remark 5.7, the diameter of the $\alpha$-level set of the DI-solution is decreasing near $t = 0$, for every $\alpha \in [0, 1]$ with $diam[y_0]^\alpha > 0$ (the initial condition is taken such that $y_0 \notin \mathbb{R}$). However, in Example 6.3, the behavior is different.

## 6. Examples

**Example 6.1.** *Let us consider the initial value problem for first order fuzzy differential equations*

$$y'(t) = -y(t) + \gamma t, \quad t \geq 0, \quad y(0) = \delta, \tag{21}$$

*where $\gamma, \delta \in \mathbb{R}_F$ and $[\gamma]^\alpha = [\frac{\alpha}{2} + 1, 2 - \frac{\alpha}{2}]$, $[\delta]^\alpha = [\frac{\alpha}{2}, 1 - \frac{\alpha}{2}]$, $\forall \alpha \in [0, 1]$. Fig. 1 shows the plots of the supports of (i)-solution and DI-solution on the interval $[0, 2]$ and (ii)-solution on the interval $[0, 1]$. It is easy to check that this problem satisfies the requirements in Theorems 5.1, 5.4 and 5.6, as well as Remarks 5.3 and 5.5. Concerning Remark 5.7, we have*

$$\phi'_\alpha(t) = -e^{-t}(1-\alpha)\left(1 + \int_0^t se^s ds\right) + t(1-\alpha) = (1-\alpha)(1-2e^{-t}),$$

*which implies that the diameter of the core of the DI-solution is constant ($\alpha = 1$) and the diameter of the $\alpha$-level sets ($\alpha < 1$) of the DI-solution decreases near $t = 0$ and it is increasing after $t = \ln 2$. Besides, the (ii)-differentiable solution exists only on $[0, 1]$ (other approach to continue the solution consists in taking into account the switching points in generalized differentiability [33]). If we consider the critical point of the diameter of the level sets of the DI-solution, $\phi'_\alpha(t) = 0$, we get $t = \ln 2$, which is different from the switching point $t = 1$ corresponding to the strongly generalized differentiability concept.*

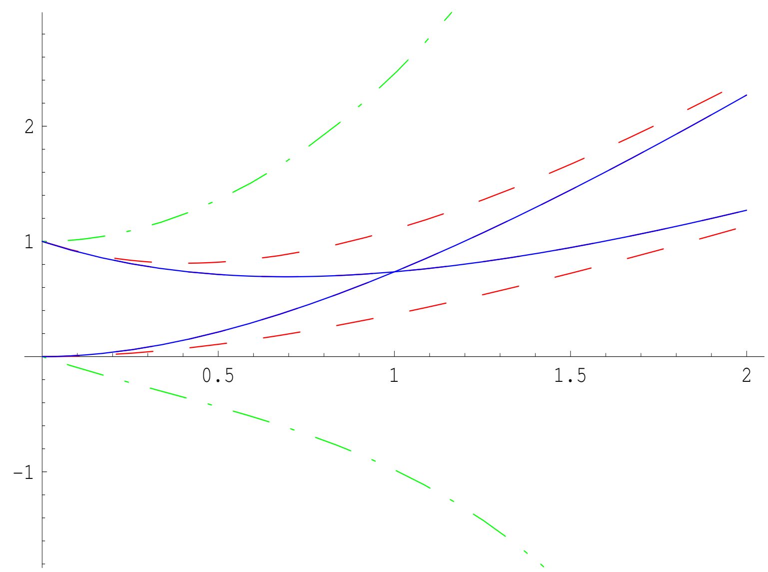


Figure 1: Endpoints of the support of (i)-solution (green dot-dashed curves) and DI-solution (red dashed curves) to problem (21) on $[0, 2]$ and support of (ii)-solution (blue solid curves) only on $[0, 1]$.

What can also be observed in Example 6.1 is that we can not guarantee the existence of the (ii)-solution in the interval where $\phi_\alpha$ is decreasing.

**Example 6.2.** *Consider the first order fuzzy differential equation with initial condition*

$$y'(t) = -y(t) - \gamma t, \quad t \geq 0, \quad y(0) = \delta, \tag{22}$$

*where $\gamma, \delta$ are the same as in Example 6.1. In Fig. 2, we show the plots of the supports of (i)-solution and DI-solution on the interval $[0, 2]$ and (ii)-solution on $[0, 1]$. The study of the monotonicity of the diameter of the level sets of the DI-solution is similar to Example 6.2.*

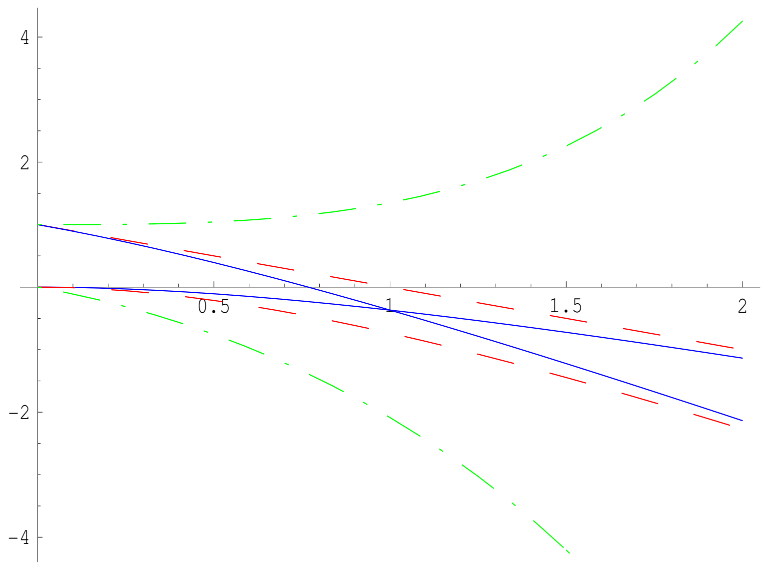


Figure 2: Endpoints of the support of (i)-solution (green dot-dashed curves), (ii)-solution (blue solid curves) and DI-solution (red dashed curves) to problem (22) ((ii)-solution defined only on $[0,1]$).

**Example 6.3.** *Consider the first order fuzzy differential equation with initial condition*

$$y'(t) = -y(t) + \gamma(1+\cos t), \quad t \geq 0, \quad y(0) = \delta, \tag{23}$$

*where* $\gamma, \delta$ *are the same as in Example 6.1. In this case,*

$$\phi'_\alpha(t) = -e^{-t}(1-\alpha)\left(1 + \int_0^t (1+\cos s)e^s ds\right) + (1+\cos t)(1-\alpha)$$
$$= (1-\alpha)\frac{1}{2}(\cos t - \sin t),$$

*which implies that the diameter of the core of the DI-solution is constant* ($\alpha = 1$) *and the diameter of the* $\alpha$*-level sets* ($\alpha < 1$) *of the DI-solution is increasing in* $[0, \frac{\pi}{4}) \cup \{(\frac{5\pi}{4} + 2k\pi, \frac{9\pi}{4} + 2k\pi) \ : \ k \in \mathbb{N}\}$ *and decreasing in* $\{(\frac{\pi}{4} + 2k\pi, \frac{5\pi}{4} + 2k\pi) \ : \ k \in \mathbb{N}\}$.

*In Fig. 3, we show the plots of the supports of (i)-solution and DI-solution on* $[0,2]$ *and (ii)-solution on* $[0,\nu]$, *where* $\nu$ *is the solution of* $e^t(1 + \frac{1}{2}(\sin t + \cos t)) = \frac{5}{2}$, *that is,* $\nu \simeq 0.41$. *Again,* $\nu < \frac{\pi}{4}$.

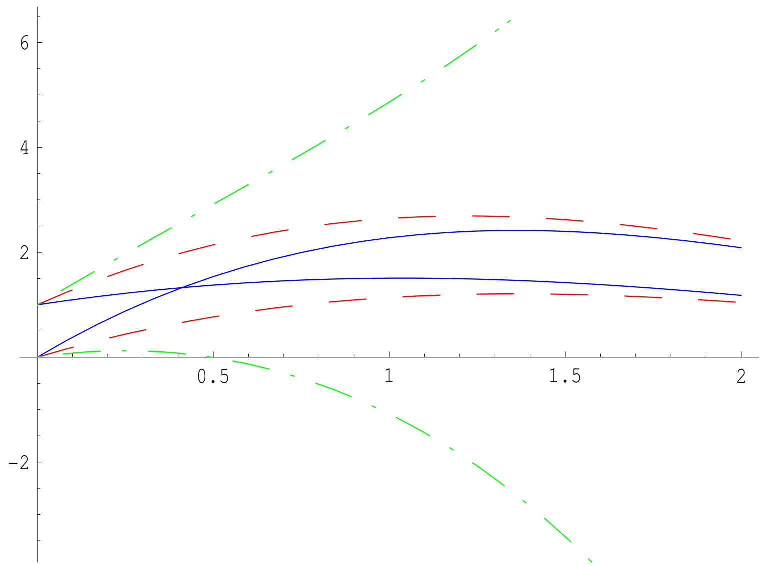


Figure 3: Endpoints of the support of (i)-solution (green dot-dashed curves), (ii)-solution (blue solid curves) and DI-solution (red dashed curves) to problem (23) ((ii)-solution defined on $[0,\nu]$).

## 7. Conclusions

In this paper, we have considered first order linear fuzzy differential equations under strongly generalized differentiability concept and differential inclusion's approach and some relations between the solutions corresponding to different methods have been presented. It is observed that the solution under differential inclusion's approach exists always and lies between (i)-solution and (ii)-solution. If the independent term $b$ is nonreal, then the DI-solution is fuzzier than (ii)-solution but it is less fuzzy than (i)-solution. We have also studied the behavior of the diameter of the level sets of the DI-solution, in contrast with the context of strongly generalized differentiability. Considering fuzzy differential equations under differential inclusions' approach, we can study some properties already important for the solutions to ordinary differential equations. In future works, we will consider some properties of DI-solutions related to the theory of ODEs.

## Acknowledgements

The authors are grateful to the Editor and the anonymous Referees for their interesting comments and suggestions towards the improvement of this

manuscript.

The research has been partially supported by AEI of Spain under grant MTM2016-75140-P, and Xunta de Galicia under grants GRC2015/004 and R2016-022.

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