Conditions for the existence of maximal factorizations

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Abstract

Extending classical algorithms for ordinary weighted or string-to-string automata to automata with underlying more general algebraic structures is of significant practical and theoretical interest. However, the generalization of classical algorithms sets certain assumptions on the underlying structure. In this respect the *maximal factorization* turns out to be a sufficient condition for many practical problems, e.g. minimization and canonization. Recently, an axiomatic approach on monoid structures suggested that monoids with *most general equalizer* (mge-monoids) provide an alternative framework to achieve similar results. In this paper, we study the fundamental relation between monoids admitting a maximal factorization and mge-monoids. We describe necessary conditions for the existence of a maximal factorization and provide sufficient conditions for an mge-monoid to admit a maximal factorization.

Keywords: monoid, most general equalizer monoid, factorization, maximal factorization, fuzzy automata, weighted automata.

1. Introduction

During last decades, researchers in automata and languages theory have offered effective generalizations of ordinary automata using different algebraic structures in order to cope with different domains of practical applications. *Fuzzy automata* and *weighted automata* are ones of the best-known studied generalizations of automata [26][7]. For weighted automata, values on transitions, as well on initial and final states, are usually taken from

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semirings such as tropical semirings [24] or hemirings [6]. Other more general structures are also considered, as, for example, strong bimonoids [5][8]. For fuzzy automata, values on transitions are taken from certain ordered structures: lattice-ordered monoids [23], lattice-ordered structures [22], complete distributive lattices [3], general lattices [22], and complete residuated lattices [19][27][28]. For those extensions of the automata theory, traditional problems like determinization, minimization, and canonization of automata have been studied and, nowadays, there are several methods and algorithms to solve these problems efficiently. Mohri's article [24] is one of most representative works on presenting several practical algorithms to cope with those problems in weighted automata.

In [20] the authors introduced the notion of *factorization* (see Definition 2) as an appropriate framework to extend these algorithms to a class of more general semirings. In fact, previous minimization algorithms for weighted automata proposed in [24][9] apply factorization on tropical semirings and on *division semirings* respectively. Determinization methods based on factorizations have been also provided for weighted automata [20] and weighted tree-automata [2]. In the context of fuzzy automata, factorizations have been studied to obtain determinization methods [17][16][32], canonization of fuzzy automata [14][15][31], and a generalization of the Myhill-Nerode theorem [13].

Among the possible factorizations, the so-called *maximal factorization* possesses a property (see (2)) that makes those constructions more efficient for practical purposes. Kirsten and Mäurer [20] show that their determinization algorithm of weighted automata is optimal using maximal factorizations and the zero-divisor-free condition (Theorem 3.3 in [20]). This behaviour has been also corroborated in some determinization methods for fuzzy automata [16][32]. The original Mohri's minimization algorithm for weighted automata over tropical semiring applies a maximal factorization [24]. Other examples of applications of maximal factorizations are in [9][14][31][13].

Therefore, the motivation for this paper is to find out conditions for the existence of maximal factorizations when the underlying algebraic structure is a zero-divisor-free monoid[1]. Recent studies on the characterisation of

[1] We recognize that the zero-divisor-free condition is somewhat strong. In fact, Kirsten and Mäurer [20] formulate factorizations without such a condition. However, they use such a condition as we have indicated above. In fuzzy languages, the exclusion of such a condition turns out in complex proofs of important results as the *Pumping Lemma* for fuzzy regular languages [17]. In this paper, the relation between maximal factorizations and zero-divisor-free monoids is stated and clarified in Lemma 1

(sub)sequentinal rational functions for general class monoids [10][11] and characterization of fuzzy languages [13] suggest that there is a tight relation between the maximal factorization for zero-divisor-free monoids and infima of countable sets. In this paper, we study such a relation in order to obtain some conditions for existence of maximal factorizations. Our results can be applied to weighted/fuzzy automata and languages.

2. Preliminaries

In this paper we shall consider *monoids*. A monoid is an algebraic structure $\langle M, \cdot, 1\rangle$ where (i) M is an arbitrary non-empty set; (ii) $\cdot$ is a binary associative operation on M (called multiplication), and (iii) 1 is a unit element with respect to the multiplication, i.e. $a \cdot 1 = 1 \cdot a = a$ for every element $a \in M$.

In many applications, the monoid of interest $\langle M, \cdot, 1\rangle$ has a *zero* element, i.e., an element $0 \in M$ such that $a \cdot 0 = 0 \cdot a = 0$ for any $a \in M$. To indicate that a monoid has a zero element 0 we shall write $\langle M, \cdot, 1, 0\rangle$. In such a monoid, an element $b \in M$, $b \neq 0$, is called *zero-divisor* if there is $c \in M$, with $c \neq 0$, such that $b \cdot c = 0$ or $c \cdot b = 0$. If $\langle M, \cdot, 1, 0\rangle$ does not have zero-divisors, then it is called *zero-divisor-free*.

Given a subset S of M, and an element $a \in M$, $a \cdot S$ denotes the subset of M defined by $a \cdot S = \{a \cdot b \mid b \in S\}$.

Let $\langle M, \cdot, 1\rangle$ be a monoid. Given two elements $a, b \in M$, we define $a \leq b$ if and only if there is an element $c \in M$ with $a \cdot c = b$. In that case, we could say that a is a *divisor of* b. The relation *divisor of* defines a pre-order on M ($\leq$ is reflexive and transitive). Given a set $S \subseteq M$, we define

$$\begin{aligned}
\mathrm{up}(S) &= \{b \in M | \forall s \in S (s \leq b)\} \\
\mathrm{low}(S) &= \{a \in M | \forall s \in S (a \leq s)\} \\
\sup(S) &= \{u \in \mathrm{up}(S) | \forall m \in \mathrm{up}(S)(u \leq m)\} \\
\inf(S) &= \{l \in \mathrm{low}(S) | \forall m \in \mathrm{low}(S)(m \leq l)\}
\end{aligned}$$

In addition, we say that any element $i \in \inf(S)$ is an *infimum*[2] for S. An element $c \in M$ is *invertible* if and only if there exists $c' \in M$ with $c \cdot c' = 1 = c' \cdot c$.

Note also, that for any set $S \subseteq M$, we have $\sup(S) = \inf(\mathrm{up}(S))$ and $\inf(S) = \sup(\mathrm{low}(S))$.

[2]Clearly, one can introduce the notion of *supremum* as an element in $\sup(S)$, but we do not need it and thus we do not use it in this paper.

Monoids with *most general equalizers*, mge-monoids, were introduced in [12]. Below we follow the equivalent definition from [10][11]:

Definition 1. *A monoid $\langle M, \cdot, 1\rangle$ satisifies*

$$\begin{array}{rll} \textit{Left Cancellation Axiom} & \textit{if} & c \cdot a = c \cdot b \Rightarrow a = b \\ \textit{Right Cancellation Axiom} & \textit{if} & a \cdot c = b \cdot c \Rightarrow a = b \\ \textit{Right Most General Equalizer Axiom} & \textit{if} & \mathrm{up}(\{a, b\}) \neq \emptyset \Rightarrow \sup(\{a, b\}) \neq \emptyset \end{array}$$

for any a, b, $c \in M$.

A monoid $\langle M, \cdot, 1\rangle$ is called an mge-monoid *if it satisfies the Left and Right Cancellation and the Right Most General Equalizer Axioms.*

Remark 1. *As mentioned above* $\sup(S) = \inf(\mathrm{up}(S))$ *for any set $S \subseteq M$. Thus, the Right Most General Equalizer Axiom actually corresponds to the existence of special kinds of infima in the monoid. More precisely, we have that:*

$$\sup(\{a, b\}) = \inf(\mathrm{up}(\{a, b\})).$$

Thus, the assumption is that if $\mathrm{up}(\{a, b\})$ *is not empty then so is* $\inf(\mathrm{up}(\{a, b\}))$.

Remark 2. *Note that in general, a set S may have zero, one, or more infima in a monoid $\mathcal{M} = \langle M, \cdot, 1\rangle$. Actually, if $a \in \inf(S)$ and c is invertible, then one can easily see that $a \cdot c \in \inf(S)$.*

For mge-monoids, in particular if the monoid satisfies the left cancellation axiom, the converse is also valid. Specifically, for any two elements $a, b \in \inf(S)$ there must[3] *be an invertible element c with $a = b \cdot c$.*

Remark 3. *Observe that either of the left and right cancellation axioms prohibits the existence of a zero element in the monoid. In particular, no mge-monoid admits a zero.*

3. Factorizations

In this section, we recall the notion of a (maximal) factorization. Originally, [20], these notions have been studied in the context of semirings and concern mostly the properties of the multiplicative monoid of the semiring

[3] Indeed, since $a, b \in \inf(S)$, $a \leq b$. Thus, there is some c with $a = b \cdot c$. Similarly, $b \leq a$, and there is some c' with $b = a \cdot c'$. Substituting, we get $a \cdot (c \cdot c') = a$ and $b \cdot (c' \cdot c) = b$. Now, by the left cancellation, c' is the inverse of c.

that admits a zero element. Thus, we start by recalling the basic definitions for the case of monoids with zero element. Next we show that under some additional, but in a way natural, assumptions for the monoid, maximal factorizations imply the zero-divisor-free property. Finally, in the end of the section, we show that the notions of (maximal) factorization can be easily adapted for arbitrary monoids. Furhermore, we point out the relation between both frameworks.

Throughout the rest of the paper we assume that U is an infinite countable set.

As mentioned in the introduction the main application that we have in mind are weighted and fuzzy automata. These are devices that represent functions from the set of words over a finite (non-empty) alphabet to a semiring or a monoid. Thus, the domain of these functions is a countable set. The purpose of the set U is to abstract the domain of these functions. Thus, even though most of the definitions and results, presented in the sequel, can be adapted to a more general framework by taking into account the cardinality of (an arbitrary set) U, we resort to the specific case where U is an infinite countable set. We also consider that this makes the outline easier to follow.

For notational convenience, elements of U are denoted by Greek letters. Let $\langle M, \cdot, 1, 0\rangle$ be a monoid with zero, $0 \in M$. $F(U, M)$ is defined as the set of all total functions of the form $X : U \to M$. We denote by $\overline{\mathbf{0}}$ the function $\overline{\mathbf{0}}(\gamma) = 0$ for every $\gamma \in U$. We also use $\overline{\mathbf{1}}$ to denote the function $\overline{\mathbf{1}}(\gamma) = 1$ for every element $\gamma \in U$. Given $X \in F(U, M)$ and $a \in M$, the function $a \cdot X \in F(U, M)$ is defined by $(a \cdot X)(\gamma) = a \cdot X(\gamma)$ for every $\gamma \in U$. In the following we introduce the notion of factorization as it was introduced in [16].

Definition 2. *Let $\langle M, \cdot, 1, 0\rangle$ be a monoid with zero. Let f be a function $f : F(U, M) \to F(U, M)$, and let g be a function $g : F(U, M) \to M$. The pair $\langle f, g\rangle$ is a factorization of $F(U, M)$ if for every function $X \in F(U, M)$ it holds:*

$$\begin{array}{l} \textit{(i)}\ X = g(X) \cdot f(X) \\ \textit{(ii)}\ g(\overline{\mathbf{0}}) = 1 \end{array} \tag{1}$$

It is simple to prove that $g(X) \neq 0$ for every $X \in F(U, M)$, and $X = \overline{\mathbf{0}} \Leftrightarrow f(X) = \overline{\mathbf{0}}$.

A factorization $\langle f, g\rangle$ of $F(U, M)$ is called *maximal* if

$$f(a \cdot X) = f(X) \tag{2}$$

for every $a \in M$ and every $X \in F(U, M)$ such that $a \cdot X \neq \overline{\mathbf{0}}$. Let us observe that the notion of a maximal factorization becomes meaningless without the restriction $a \cdot X \neq \overline{\mathbf{0}}$ [20].

As we have indicated in the introductory section, factorizations and maximal factorizations $\langle f, g \rangle$ have been studied in several papers for particular cases of algebraic structures: in [20] for semirings, in [14][32][31] for complete residuated lattices, and in [16] for continuous t-norms over the unit real interval $[0, 1]$. The next Lemma relates maximal factorizations and zero-divisor-free monoids.

Lemma 1. *Let $\langle M, \cdot, 1, 0 \rangle$ be a monoid with zero. Assume that any four elements $g_1, g_2, c, d \in M$ satisfy the implication:*

$$\left| \begin{array}{lcl} g_1 \cdot c & = & 1 \\ g_1 \cdot d & = & 0 \\ g_2 \cdot c & = & 1 \end{array} \right. \Rightarrow g_2 \cdot d = 0. \tag{3}$$

If $\langle f, g \rangle$ is a maximal factorization of $F(U, M)$ then $\langle M, \cdot, 1, 0 \rangle$ is a zero-divisor-free monoid.

Proof: Let $\alpha, \beta \in U$ be distinct elements of U. Further assume that $\langle f, g \rangle$ is a maximal factorization of $F(U, M)$. For the sake of contradiction, suppose that $a \cdot b = 0$ for some $a \neq 0 \neq b$ in M.

We define the functions $X_a : U \to M$, $X_1 : U \to M$ as:

$$X_a(\gamma) = \begin{cases} 1, & \text{if } \gamma = \alpha \\ b, & \text{if } \gamma = \beta \\ 0, & \text{otherwise.} \end{cases} \qquad X_1(\gamma) = \begin{cases} 1, & \text{if } \gamma = \alpha \\ 0, & \text{otherwise.} \end{cases}$$

We have that $(a \cdot X_a)(\gamma) = a$ for $\gamma = \alpha$ and $(a \cdot X_a)(\gamma) = 0$ for $\gamma \neq \alpha$, in the case $\gamma = \beta$ this is due to the assumption $a \cdot b = 0$. This shows that $a \cdot X_a = a \cdot X_1$. Clearly, since $a \neq 0$, both functions are distinct from $\overline{\mathbf{0}}$. Consequently, by the maximal factorization property of $\langle f, g \rangle$ we get:

$$f(X_a) = f(a \cdot X_a) = f(a \cdot X_1) = f(X_1).$$

Therefore:

$$1 = X_a(\alpha) = g(X_a) \cdot f(X_a)(\alpha) = g(X_a) \cdot f(X_1)(\alpha),$$

where the first equality is by definition, the second by the property of factorization, and the last equality is by $f(X_a) = f(X_1)$. Similarly, we have

$$1 = X_1(\alpha) = g(X_1) \cdot f(X_1)(\alpha).$$

Let us put $c = f(X_1)(\alpha) = f(X_a)(\alpha)$ and $d = f(X_1)(\beta) = f(X_a)(\beta)$. Then we have:

$$\left| \begin{array}{lllll} g(X_1) \cdot c & = & g(X_1) \cdot f(X_1)(\alpha) & = & 1 \\ g(X_1) \cdot d & = & g(X_1) \cdot f(X_1)(\beta) & = & 0 \\ g(X_a) \cdot c & = & g(X_a) \cdot f(X_a)(\alpha) & = & 1 \\ g(X_a) \cdot d & = & g(X_a) \cdot f(X_a)(\beta) & = & b \neq 0. \end{array} \right.$$

However, this contradicts the Condition (3) of the lemma. $\square$

We will consider maximal factorizations and zero-divisor-free monoids by the following reasons:
(a) zero-divisor-free condition is a necessary condition for the existence of a maximal factorization in monoids with the additional Condition (3);
(b) maximal factorizations generate an optimal number of states in determinization algorithms for nondeterministic weighted/fuzzy automata over zero-divisor-free monoids, [20][16][32]; and
(c) algorithms for minimization of weighted/fuzzy deterministic automata are more efficient by using maximal factorizations [24][9][14][32].

Let us observe that if $\langle M, \cdot, 1, 0\rangle$ is a zero-divisor-free monoid, then removing 0 from M preserves the monoid structure, i.e., $\langle M \setminus \{0\}, \cdot, 1\rangle$ is a monoid. This transformations converts total functions $X : U \to M$ to (possibly) partial functions $\widetilde{X} : U \to M \setminus \{0\}$ when restricting their image to $M \setminus \{0\}$. Conversely, every monoid $\langle M, \cdot, 1\rangle$ can be extended to one with a zero by introducing a fresh element, 0, that acts as a zero and by extending appropriately the multiplication table. In this case the obtained monoid $\langle M \cup \{0\}, \cdot, 1, 0\rangle$ is trivially zero-divisor-free. Furthermore, every partial function $X : U \to M$ can be converted to a total function $\overline{X} : U \to M \cup \{0\}$, by simply replacing the undefined values with 0. Therefore, in order to study factorizations and maximal factorizations under partial functions, we introduce a slight modification in Definition 2.

Let $\langle M, \cdot, 1\rangle$ be a monoid without a zero element. We denote by $F_p(U, M)$ the set of all partial functions of the form $X : U \to M$. In addition, $\overline{\emptyset}$ denotes the nowhere defined function. Given $X \in F_p(U, M)$ and $a \in M$, the function $a \cdot X \in F_p(U, M)$ is defined by $(a \cdot X)(\gamma) = a \cdot X(\gamma)$ for every $\gamma \in U$ whenever $X(\gamma)$ is defined, and undefined otherwise.

Definition 3. *Let $\langle M, \cdot, 1\rangle$ be a monoid. Let f be a function $f : F_p(U, M) \to F_p(U, M)$, and let g be a function $g : F_p(U, M) \to M$. The pair $\langle f, g\rangle$ is a* *factorization of $F_p(U,M)$ if, for every function $X \in F_p(U,M)$, f and g satisfy the following conditions:*

$$\begin{array}{l} \textit{(i)}\ X = g(X) \cdot f(X) \\ \textit{(ii)}\ g(\overline{\emptyset}) = 1 \end{array} \tag{4}$$

Let $\langle f, g\rangle$ be a factorization of $F_p(U,M)$. Then, clearly, $f(X) = \overline{\emptyset}$ if and only if $X = \overline{\emptyset}$. A factorization $\langle f, g\rangle$ of $F_p(U,M)$ is called *maximal* if

$$f(a \cdot X) = f(X) \tag{5}$$

for every $a \in M$ and every $X \in F_p(U,M)$ with $X \neq \overline{\emptyset}$.

Following the above intuition, it is standard to establish the relation between maximal factorizations of $F_p(U,M)$ for arbitrary monoids (without zero) and maximal factorizations $F(U,M)$ when M contains a zero but is zero-divisor-free. Recall, that for a partial function $X : U \to M \setminus \{0\}$, we denote with $\overline{X}$ its totalization in M. Similarly, $\widetilde{X}$ is the restriction of a total function X with range in $M \cup \{0\}$ to $M \setminus \{0\}$.

Lemma 2. *If $\langle M, \cdot, 1, 0\rangle$ is a zero-divisor-free monoid, and $\langle f, g\rangle$ is a (maximal) factorization of $F(U,M)$ then $\langle \widetilde{f}, \widetilde{g}\rangle$ is a (maximal) factorization of $F_p(U, M \setminus \{0\})$ where:*

$$\widetilde{f}(X) = \widetilde{f(\overline{X})} \text{ and } \widetilde{g}(X) = g(\overline{X}).$$

Lemma 3. *If $\langle M, \cdot, 1\rangle$ is a monoid, and $\langle f, g\rangle$ is a (maximal) factorization of $F_p(U,M)$ then $\langle \overline{f}, \overline{g}\rangle$ is a (maximal) factorization of $F(U, M \cup \{0\})$, where:*

$$\overline{f}(X) = \overline{f(\widetilde{X})} \text{ and } \overline{g}(X) = g(\widetilde{X}).$$

4. Results about maximal factorizations

In this section, we show that there is a tight relation between the existence of maximal factorizations for monoids and infima of non-empty countable sets. In view of Lemmas 2 and 3, it is equivalent whether we consider maximal factorizations of $F_p(U,M)$ and arbitrary monoids or maximal factorization $F(U,M)$ and zero-divisor-free monoids. However, for the statements of our results, it would be more convenient to consider $F_p(U,M)$. We shall obtain the corresponding results for $F(U,M)$ as corollaries by Lemmas 2 and 3.

Thus, we consider a monoid $\mathcal{M} = \langle M, \cdot, 1\rangle$ (without zero). As before U denotes an infinite countable set. Informally, the results we are presenting in this section are:

1. right cancellation in $\mathcal{M}$ and maximal factorization in $F_p(U, M)$ imply infima of countable sets in $\mathcal{M}$.
2. mge-monoid $\mathcal{M}$ and infima of countable sets in $\mathcal{M}$ imply a maximal factorization in $F_p(U, M)$.

The formal statements and proofs are given below.

Lemma 4. *Let $\mathcal{M} = \langle M, \cdot, 1\rangle$ be a monoid with the right cancellation property. If the set $F_p(U, M)$ admits a maximal factorization $\langle f, g\rangle$, then every non-empty countable (finite or infinite) subset $S \subseteq M$ admits an infimum in $\mathcal{M}$, i.e.:*

$$\inf(S) \neq \emptyset.$$

Proof: Let $\langle f, g\rangle$ be a maximal factorization of $F_p(U, M)$ and consider a non-empty countable set $S \subseteq M$. Since U is an infinite countable set, there is a function $X : U \to M$ such that:

$$\{X(\gamma) \,|\, \gamma \in U\} = S.$$

Clearly, $X \neq \overline{\emptyset}$.

We claim that $g(X) \in \inf(S)$. Indeed, by the equality

$$X(\gamma) = g(X) \cdot f(X)(\gamma)$$

we have that $g(X) \leq X(\gamma)$ for every γ in the domain of X. Since $S \neq \emptyset$, $X \neq \overline{\emptyset}$, there is at least one γ in the domain of X. Hence $g(X) \in \mathrm{low}(S)$.

Next, let $\ell \in \mathrm{low}(S)$ be arbitrary. Thus, in particular, $S = \ell \cdot S'$ for some $S' \subseteq M$ and therefore:

$$X = \ell \cdot X'.$$

for some X'. Obviously, $\ell \cdot X' \neq \overline{\emptyset}$ and, consequently, $X' \neq \overline{\emptyset}$. Now, by the maximality (see (5)) of $\langle f, g\rangle$ we have that $f(X') = f(X)$ and also $X' = g(X') \cdot f(X')$. To conclude the proof, we use that S is non-empty and therefore there is some $\gamma \in U$ for which $X(\gamma)$ is defined. Hence:

$$\begin{aligned} g(X) \cdot f(X)(\gamma) &= X(\gamma) \\ &= \ell \cdot X'(\gamma) \\ &= \ell \cdot g(X') \cdot f(X')(\gamma) \\ &= \ell \cdot g(X') \cdot f(X)(\gamma) \end{aligned}$$

where the last equality follows by $f(X) = f(X')$. Finally, by the right cancellation property, we obtain that:

$$g(X) = \ell \cdot g(X').$$

The last equality implies, by definition, that $\ell \leq g(X)$. Since $\ell \in \text{low}(S)$ was arbitrary, we deduce that $g(X) \in \inf(S)$. $\square$

For mge-monoids, the converse of Lemma 4 is also true. Before stepping to the formal details, we recall the following important property of mge-monoids that we need for the proof:

Lemma 5 (Lemma 7 in [10]). *Let $\mathcal{M} = \langle M, \cdot, 1\rangle$ be an mge-monoid. If $\emptyset \subsetneq S \subseteq M$ and $m \in M$ are arbitrary, then:*

$$\inf(m \cdot S) = m \cdot \inf(S).$$

Lemma 6. *Let $\mathcal{M} = \langle M, \cdot, 1\rangle$ be an mge-monoid. If every non-empty countable subset $S \subseteq M$ admits an infimum, then $F_p(U, M)$ admits a maximal factorization.*

Proof: Consider the relation $\equiv \subseteq \mathcal{F}_p(U, M) \times \mathcal{F}_p(U, M)$ defined as:

$$X \equiv Y \iff \exists m \in M (m \text{ is invertible and } Y = m \cdot X).$$

Clearly, $\equiv$ is an equivalence relation on $F_p(U, M)$. Assuming the Axiom of Choice, for every equivalence class C of $\equiv$ we fix an element $\rho(C) \in C$. Now we can construct the factorization $\langle f, g\rangle$ as follows. For every $X : U \to M$ we consider the set:

$$S_X = \{X(\gamma) \,|\, \gamma \in U \text{ and } X(\gamma) \text{ is defined}\}.$$

First assume that $X \neq \overline{\emptyset}$. Thus, $S_X \neq \emptyset$. Clearly, S_X is (possibly finite) countable and by the assumption of the lemma there is an element:

$$s_X \in \inf(S_X).$$

Therefore $X = s_X \cdot X'$ for some X'. Let m_X be the invertible element that witnesses for $\rho([X']_\equiv) \equiv X'$. Then we set:

$$f(X) = \rho([X']_\equiv) \text{ and } g(X) = s_X \cdot m_X$$

Note that according to Remark 2, $g(X) \in \inf(S_X)$.

An easy computation shows that

$$X = s_X \cdot X' = s_X \cdot m_x \cdot \rho([X']_\equiv) = g(X) \cdot f(X).$$

Furthermore, note that $f(X)$ and $g(X)$ do not depend on the specific choice of s_X. Indeed, let $s'_X \in \inf(S_X)$ be arbitrary, not necessarily s_X, and let

$X = s'_X \cdot X''$. Since s'_X and s_X are both in $\inf(S_X)$ it follows that $s'_X \cdot m' = s_X$ and $s_X \cdot m'' = s'_X$ for some elements $m', m'' \in M$. Therefore $s'_X = s'_X \cdot m' \cdot m''$ and, by the left cancellation property, $m' \cdot m'' = 1$. Interchanging s_X and s'_X, we see that $m'' \cdot m' = 1$ and hence m'' and m' are invertible. Now, for every $\gamma \in U$ such that $X(\gamma)$ is defined, we have that $X(\gamma) = s'_X \cdot X''(\gamma)$ and $X(\gamma) = s_X \cdot X'(\gamma)$. Substituting $s_X = s'_X \cdot m'$ and using the left cancellation property we obtain that $X''(\gamma) = m' \cdot X'(\gamma)$. Since X', X'', and X have the same domain, we conclude $X'' = m' \cdot X'$ and, since m' is invertible, we deduce $X' \equiv X''$. Thus, $\rho([X']_\equiv) = \rho([X'']_\equiv)$ is independent of the choice of s_X. Now, since $X = g(X) \cdot f(X)$ and $X \neq \overline{\emptyset}$, by the right cancellation property for mge-monoids we conclude that $g(X)$ is also independent of the choice of s_X.

Next, assume that $Y = a \cdot X$ for some $a \in M$, then $S_Y = a \cdot S_X$ and by Lemma 5 we get:

$$\inf(S_Y) = a \cdot \inf(S_X).$$

Therefore, $a \cdot s_X \in \inf Y$. Let $Y = a \cdot s_X \cdot Y'$. Hence, for every γ in the domain of Y, we have $Y(\gamma) = a \cdot X(\gamma) = a \cdot s_X \cdot X'(\gamma)$ and $Y(\gamma) = a \cdot s_X \cdot Y'(\gamma)$. By the left cancellation property, we conclude $Y'(\gamma) = X'(\gamma)$ for every γ in the domain of Y. Since the domains of Y, Y', X, and X' are the same, we deduce that $Y' = X'$. Therefore, by the above argument, we have $f(X) = \rho([X']_\equiv) = \rho([Y']_\equiv) = f(Y)$.

Finally, in the special case where $S_X = \emptyset$, which is equivalent to $X = \overline{\emptyset}$, we set $g(X) = 1$ and $f(X) = X$. In conclusion, $\langle f, g\rangle$ is a maximal factorization of $F_p(U, M)$ □

Lemma 7. *Let $\mathcal{M} = \langle M, \cdot, 1\rangle$ be a monoid. Further, assume that every element $x \in M$ that has a right inverse has also a left inverse, i.e.:*

$$\exists z(x \cdot z = 1) \Rightarrow \exists y(y \cdot x = 1).$$

If $\mathcal{F}_p(U, M)$ admits a maximal factorization, then $\mathcal{M}$ satisfies the left cancellation property.

Proof: Let $\langle f, g\rangle$ be a maximal factorization for $F_p(U, M)$. First, let us consider the constant function $\overline{\mathbf{1}}$, i.e. $\overline{\mathbf{1}}(\gamma) = 1$ for all $\gamma \in U$. Then, we have that:

$$\overline{\mathbf{1}} = g(\overline{\mathbf{1}}) \cdot f(\overline{\mathbf{1}}).$$

We claim that $Z = f(\overline{\mathbf{1}})$ is also a constant function. Indeed let $\alpha \in U \neq \emptyset$. Then:

$$g(\overline{\mathbf{1}}) \cdot Z(\alpha) = 1.$$

It follows that $g(\overline{\mathbf{1}})$ admits a right inverse $Z(\alpha)$. Therefore, by the assumptions of the lemma, it admits also a left inverse, $g' \in M$ such that $g' \cdot g(\overline{\mathbf{1}}) = 1$. Now it is clear that:

$$Z(\alpha) = 1 \cdot Z(\alpha) = g' \cdot g(\overline{\mathbf{1}}) \cdot Z(\alpha) = g' \cdot 1 = g'.$$

Since this equality is independent of α, we get that $Z(\alpha) = g'$ for all $\alpha \in U$.

Now we are ready to establish the left cancellation property of $\mathcal{M}$. Let $a, b, c \in M$ be such that:

$$a \cdot b = a \cdot c.$$

We prove that $b = c$. Let $\alpha \in U$ be arbitrary and let us consider the functions $X, X' \in F_p(U, M)$ defined as:

$$X(\gamma) = a \cdot b \text{ and } X'(\gamma) = \begin{cases} b \text{ if } \gamma = \alpha \\ c \text{ otherwise.} \end{cases}$$

Now, since $a \cdot b = a \cdot c$, $X = a \cdot X'$ and therefore $f(X) = f(X')$. On the other hand $X = (a \cdot b) \cdot \overline{\mathbf{1}}$. We get that $f(X') = f(X) = f(\overline{\mathbf{1}})$ is a constant function. We conclude that:

$$X' = g(X') \cdot f(X')$$

is constant function. Since $|U| \geq 2$, we conclude that $b = c$. $\square$

Corollary 1. *Let $\mathcal{M} = \langle M, \cdot, 1 \rangle$ be a commutative monoid. If $\mathcal{F}_p(U, M)$ admits a maximal factorization, then $\mathcal{M}$ is an mge-monoid.*

Proof: Indeed, if the monoid is commutative then $x \cdot z = 1$ is equivalent to $z \cdot x = 1$. Thus, the assumptions of Lemma 7 are met. Therefore $\mathcal{M}$ satisfies the left cancellation property. By the commutativity, we get the right cancellation property for $\mathcal{M}$ as well. Now $\mathcal{M}$ has the right cancellation property and $F_p(U, M)$ admits a maximal factorization. By Lemma 4 every non-empty countable (finite or infinite) subset of M admits an infimum. By the Remark 1, we get that $\mathcal{M}$ satisfies also the right most general equalizer axiom. Therefore $\mathcal{M}$ is an mge-monoid. $\square$

In view of Lemma 2 and Lemma 3 the results for $F_p(U, M)$ translate immediately to $F(U, M)$ where $\langle M, \cdot, 1, 0 \rangle$ is a zero-divisor-free monoid. Since the latter situation often occurs in practical applications, we state them separately:

Corollary 2. *Let $\mathcal{M} = \langle M, \cdot, 1, 0 \rangle$ be a zero-divisor-free monoid. Let $\mathcal{M}' = \langle M \setminus \{0\}, \cdot, 1 \rangle$, then:*

1. *If $F(U, M)$ admits a maximal factorization, then every non-empty countable subset $S \subseteq M \setminus \{0\}$ admits an infimum in $\mathcal{M}'$.*
2. *If $\mathcal{M}'$ is an mge-monoid and every non-empty countable set $S \subseteq M \setminus \{0\}$ admits an infimum in $\mathcal{M}'$, then $F(U, M)$ admits a maximal factorization.*

Proof: The proof follows immediately by Lemma 2 and Lemma 3, Lemma 4 and Lemma 6 □

5. Applications to residuated lattices and *t*-norms

In this section, we consider the fuzzy automata and languages area to study some implications of the results obtained in the previous sections. In particular, we consider *complete residuated lattices* as the algebraic structures for modelling truth values and operations on fuzzy sets.

A complete residuated lattice is an algebra $\mathcal{L} = \langle L, \vee, \wedge, \otimes, \rightarrow, 1, 0\rangle$ such that (i) $\langle L, \vee, \wedge, 1, 0\rangle$ is a complete lattice with the least element 0 and the greatest element 1; (ii) $\langle L, \otimes, 1\rangle$ is a commutative monoid with the unit 1; and (iii) $\otimes$ and $\rightarrow$ form and *adjoint pair*, i.e., they satisfy the *adjunction property*: for all a, b, $c \in L$, $a \otimes b \leq c \Leftrightarrow a \leq b \rightarrow c$.

In the following, $\leq$ denotes the order relation in L, and $\leq_\otimes$ denotes the relation *divisor of* defined in the previous section, i.e., $a \leq_\otimes b$ if $a \otimes c = b$ for some c. The operations $\otimes$ (*multiplication*) and $\rightarrow$ (*residuum*) are intended for modeling the conjunction and implication of the corresponding logical calculus, and supremum ($\bigvee$) and infimum ($\bigwedge$) are intended for modeling the existential and general quantifier, respectively. It can be verified that with respect to $\leq$, multiplication $\otimes$ is isotonic in both arguments, and $\rightarrow$ is isotonic in the second and antitonic in the first argument. For any a, $b \in L$, the following basic property is satisfied:

$$a \otimes (a \rightarrow b) = b \Leftrightarrow a \leq_\otimes b \tag{6}$$

In addition, for any $a \in L$ and any $\{b_i\}_{i \in I} \subseteq L$,

$$(\bigvee_{i \in I} b_i) \otimes a = \bigvee_{i \in I} (b_i \otimes a). \tag{7}$$

The reader is referred to [1, 4] for further properties on complete residuated lattices. Let us observe that $0 \in L$ and $0 \otimes a = 0$ for any $a \in L$, i.e., $\langle L, \otimes, 1, 0\rangle$ is a monoid with a zero element.

Given a complete residuated lattice $\mathcal{L}$, a *fuzzy subset* of a set U over $\mathcal{L}$, or simply a fuzzy subset of U, is any function from U to L [4]. Thus,

$F(U, L)$ denotes the set of all possible fuzzy subsets of U. As in previous sections, we assume that U is an infinite countable set.

Let us notice that by monotonicity of $\otimes$, if, for a, $b \in L$, $a \leq_{\otimes} b$ then $b \leq a$. A complete residuated lattice $\mathcal{L}$ satisfies the *divisibility property* if the converse is also true, i.e., for any a, $b \in L$,

$$b \leq a \Rightarrow a \leq_{\otimes} b \tag{8}$$

If $\mathcal{L}$ satisfies the divisibility property then it is possible to define a non-trivial factorization[4] (Definition 2) of $F(U, L)$ [16].

Let us define $f_M : F(U, L) \to F(U, L)$ and $g_M : F(U, L) \to L$ as follows:

$$g_M(X) = \begin{cases} \bigvee_{\gamma \in U} X(\gamma) \text{ if } X \neq \overline{\mathbf{0}} \\ 1 \text{ if } X = \overline{\mathbf{0}}. \end{cases}$$

and

$$f_M(X)(\gamma) = g_M(X) \to X(\gamma)$$

for any $X \in F(U, M)$ and $\gamma \in U$.

Let us observe that, for any $X \in F(U, L)$ and $\gamma \in U$, $g_M(X) \geq X(\gamma)$. By the divisibility property (8) and property (6), $X(\gamma) = g_M(X) \otimes f_M(X)(\gamma)$ holds for any $\gamma \in U$. Therefore, $\langle f_M, g_M \rangle$ is a factorization of $F(U, L)$.

For historical reasons, $\langle f_M, g_M \rangle$ is called Mohri's factorization [20]. Mohri's factorization is not the unique possible non-trivial factorization of $F(U, L)$. In fact, it is possible to define multiple non-trivial factorizations as it was proved in [13]. Here, we study the necessary and sufficient conditions when $F(U, L)$ admits a maximal factorization given that $\mathcal{L}$ is a complete residuated lattice.

Lemma 8. *Let $\mathcal{L} = \langle L, \vee, \wedge, \otimes, \to, 1, 0 \rangle$ be a complete residuated lattice that satisfies the divisibility property. $F(U, L)$ admits a maximal factorization if and only if $\langle L \setminus \{0\}, \otimes, 1 \rangle$ is an mge-monoid.*

Proof: ($\Rightarrow$). As $\otimes$ is isotonic in both arguments, then, L satisfies Condition (3). Indeed, assume that $g_1, g_2, c, d \in L$ and $g_1 \otimes c = g_2 \otimes c = 1$ and $g_1 \otimes d = 0$. The first two equalities imply that $g_1 = g_2 = 1$. Thus, by the third, we get $d = 0$ and therefore Condition (3) holds. Thus, by Lemma 1, $\langle L, \otimes, 1, 0 \rangle$ is a zero-divisor-free monoid. In addition, $\langle L, \otimes, 1, 0 \rangle$ is a commutative monoid. Therefore, $\langle L \setminus \{0\}, \otimes, 1 \rangle$ is a commutative monoid. By

[4] The trivial factorization is simply defined by $g(X) = 1$ and $f(X) = X$ for every fuzzy subset X.

Lemma 2, $F_p(U, L \setminus \{0\})$ admits a maximal factorization. By Corollary 1, $\langle L \setminus \{0\}, \otimes, 1\rangle$ is an mge-monoid.

($\Leftarrow$). Let $\langle L\setminus\{0\}, \otimes, 1\rangle$ be an mge-monoid. We will prove that any countable subset $S \subseteq L \setminus \{0\}$ admits an infimum (with respect to $\leq_\otimes$). Let us recall that $\mathrm{low}(S) = \{a \in L | \forall s \in S(a \leq_\otimes s)\}$ and $\inf(S) = \{l \in \mathrm{low}(S) | \forall m \in \mathrm{low}(S)(m \leq_\otimes l)\}$.
By the divisibility property (8) and monotonicity of $\otimes$, $v \geq s$ if and only if $v \leq_\otimes s$. Thus, $\mathrm{low}(S) = \{v \in L | \forall s \in S(v \geq s)\}$. The value $v_S = \bigvee_{s\in S} s$ is well defined for S, and, obviously, $v_S \in \mathrm{low}(S)$. In addition, any value $v \in \mathrm{low}(S)$ satisfies $v \geq v_S$, i.e., $v \leq_\otimes v_S$. Therefore, $v_S \in \inf(S)$. By Corollary 2, $F(U, L)$ admits a maximal factorization. □

In fuzzy automata and languages, many practical applications and examples have been developed using monoids based on triangular norms (t-norms), $\langle[0, 1], \otimes, 0, 1\rangle$ where $[0, 1]$ is the real unit interval and $\otimes$ denotes a t-norm. The reader is referred to the monographs on t-norms [18, 21] for further details. More specifically, the common $[0, 1]$-valued-semantics for fuzzy sets, are algebras defined by a *left-continuous* t-norm, i.e., algebras of the form $[0, 1]_\otimes = \langle[0, 1], \vee, \wedge, \otimes, \rightarrow, 0, 1\rangle$ where $\vee$ denotes *max* operation, $\wedge$ denotes *min* operation, and $\rightarrow$ denotes the *R-implication* [33].

These algebras $[0, 1]_\otimes$ are particular cases of *complete residuated lattices* and the operation $\rightarrow$ is also called *residuum*. The importance of left-continuity is due to the fact that $\otimes$ has a residuum $\rightarrow$ if and only if $\otimes$ is left-continuous. In addition, a relevant result of t-norms establishes that a left-continuous t-norm is *continuous* if and only if it satisfies the divisibility property. Equivalently, a t-norm is continuous if and only if $\otimes$ and $\rightarrow$ satisfies:

$$a \otimes (a \rightarrow b) = a \wedge b \tag{9}$$

for any a, $b \in [0, 1]$.

The main examples of continuous t-norms are the Łukasiewicz t-norm ($a \otimes_L b = max(a + b - 1, 0)$, and $a \rightarrow_L b = min(1 - a + b, 1)$), the Product t-norm ($a \otimes_\Pi b = ab$, and $a \rightarrow_\Pi b = 1$ if $a \leq b$ and $= b/a$ otherwise), and the Gödel t-norm ($a \otimes_G b = min(a, b)$, and $a \rightarrow_G b = 1$ if $a \leq b$ and $= b$ otherwise). They are the most prominent examples of continuous t-norms because *a t-norm is continuous if and only if it is isomorphic to an ordinal sum of the Gödel, Łukasiewicz, and Product t-norm* [21].

For these algebraic structures Mohri's factorization is also a factorization of $F(U, [0, 1]_\otimes)$. However, as the Gödel t-norm does not satisfy cancellation properties and Łukasiewicz t-norm is not a zero-divisor-free monoid, then, by Lemma 8, they do not admit a maximal factorization. Let us observe that the Product t-norm is a continuous t-norm, zero-divisor-free and satisfies the cancellation properties. In addition, a Product t-norm based monoid is an mge-monoid. Therefore, by Lemma 8, monoids based on continuous t-norms, that are isomorphic to the Product t-norm, admit maximal factorizations.

6. Conclusions

We conclude that monoids of complete residuated lattices that satisfy the divisibility property, and in particular, monoids based on continuous t-norms, induce mge-monoids satisfying the existence of an infimum (with respect to the relation 'divisor of' $\leq_{\otimes}$) for any countable subset of $L \setminus \{0\}$. By the results provided in this paper those monoids admit a maximal factorization.

Furthermore, certain constructions, provided for weighted automata with values in an mge-monoid, could be adapted to fuzzy automata with truth-values on the monoids considered in this section. In these scenarios U is typically the set of words over a non-empty alphabet, thus it is an *infinite countable* set. In [12] the authors describe a test for unambiguity for automata with values in an mge-monoid. In practice, this is the test whether a non-deterministic transducer recognizes a function. Thus, this construction can be immediately transferred to fuzzy automata with truth-values. Additionally, following [12], we can construct a bimachine, i.e. a deterministic representation preserving the linear traversal, for such transducer. Further examples are provided in [11] where constructions for canonization, also known as *early-on* or *frank* according to [30] normal form, and minimization are presented. Essentially, the canonization strives at *pushing forward* the outputs so that at soon as we have processed a given prefix of a word we know the best estimate of its possible extensions. This is important for practical applications, [24, 25]. These constructions depend on an mge-monoid with two further axioms[5] that are trivially satisfied for commutative complete residuated lattices. Thus, we can translate them to complete residuated lattices without complications. Finally, the general framework provided by mge-monoids and maximal factorizations might be of interest to other fields in automata theory, e.g. symbolic transducers, [29], where minimization, canonization, and unambiguity play an important role.

[5]These are $\inf\{a, b\} \neq \emptyset$ and $b \leq a \cdot b \cdot c \Rightarrow b \leq a \cdot b$.

In this regard, the relation between mge-monoids and maximal factorizations described in this paper can be used as a theoretical basis to indicate whether those problems in the specific domain can be solved by the means of the classical constructions, [24, 25], or their appropriate modifications [11].

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References

[1] G. Birkhoff. *Lattice Theory.* American Mathematical Society, 1948.

[2] M. Büchse, J. May, and H. Vogler. Determinization of weighted tree automata using factorizations. *Journal of Automata, Languages and Combinatorics*, 15:229–254, 2010.

[3] R. Bělohlávek. Determinism and fuzzy automata. *Information Sciences*, 143:205–209, 2002.

[4] R. Bělohlávek. *Fuzzy Relational Systems: Foundations and Principles.* Springer Science & Business Media, Berlin, 2002.

[5] M. Ćirić, M. Droste, J. Ignjatović, and H. Vogler. Determinization of weighted finite automata over strong bimonoids. *Information Sciences*, 180:3497–3520, 2010.

[6] M. Droste and W. Kuich. Weighted finite automata over hemirings. *Theoretical Computer Science*, 485:38–48, 2018.

[7] M. Droste, W. Kuich, and H. Vogler, editors. *Handbook of Weighted Automata.* Springer-Verlag, Berlin, 2009.

[8] M. Droste, T. Stüber, and H. Vogler. Weighted finite automata over strong bimonoids. *Information Sciences*, 180:156–166, 2010.

[9] J. Eisner. Simpler and more general minimization for weighted finite-state automata. In *Proceedings of HLT-NAACL2003*, pages 64–71, 2003.

[10] S. Gerdjikov. Characterisation of (sub)sequential rational function over a general class monoids. *CoRR*, abs/1801.10063, 2018.

[11] S. Gerdjikov. A general class of monoids supporting canonisation and minimisation of (sub)sequential transducers. In S. T. Klein, C. Martín-Vide, and D. Shapira, editors, *Language and Automata Theory and Application, 12th International Conference LATA2018, LNCS*, volume 10792, pages 143–155, Cham, 2018. Springer International Publishing.

[12] S. Gerdjikov, S. Mihov, and K. U. Schulz. Space-efficient bimachine construction based on the equalizer accumulation principle. *CoRR*, abs/1803.04312, 2018.

[13] J. R. Gonzalez de Mendivil. A generalization of Myhill-Nerode theorem for fuzzy languages. *Fuzzy Sets and Systems*, 301:103–115, 2016.

[14] J. R. Gonzalez de Mendivil. Conditions for minimal fuzzy deterministic finite automata via Brzozowski's procedure. *IEEE Transactions on Fuzzy Systems*, 26(4):2409–2420, 2018.

[15] J. R. Gonzalez de Mendivil and F. Fariña Figueredo. Canonization of max-min fuzzy automata. *Fuzzy Sets and Systems*, (https://doi.org/10.1016/j.fss.2019.03.009), 2019.

[16] J. R. Gonzalez de Mendivil and J. R. Garitagoitia. Determinization of fuzzy automata via factorization of fuzzy states. *Information Sciences*, 283:165–179, 2014.

[17] J. R. Gonzalez de Mendivil and J. R. Garitagoitia. Fuzzy languages of infinite range: pumping lemmas and determinization procedure. *Fuzzy Sets and Systems*, 249:1–26, 2014.

[18] S. Gottwald. *Fuzzy Sets and Fuzzy Logic. The Foundations of Application – from a Mathematical Point of View.* Vieweg+Teubner Verlag, 1993.

[19] J. Ignjatović, M. Ćirić, and S. Bogdanović. Determinization of fuzzy automata with membership values in complete residuated lattices. *Information Sciences*, 178:164–180, 2008.

[20] D. Kirsten and I. Mäurer. On the determinization of weighted automata. *Journal of Automata, Languages and Combinatorics*, 10:287–312, 2005.

[21] E. P. Klement, R. Mesiar, and E. Pap. *Triangular Norms*. Springer Netherlands, 2000.

[22] Y. M. Li. Finite automata theory with membership values in lattices. *Information Sciences*, 181:1003–1017, 2011.

[23] Y. M. Li and W. Pedrycz. Fuzzy finite automata and fuzzy regular expressions with membership values in lattice ordered monoids. *Fuzzy Sets and Systems*, 156:68–92, 2005.

[24] M. Mohri. Finite-state transducers in language and speech processing. *Computational Linguistics*, 23:269–311, 1997.

[25] M. Mohri. Minimization algorithms for sequential transducers. *Theoretical Computer Science*, 234:177–201, March 2000.

[26] J. Mordeson and D. Malik. *Fuzzy Automata and Languages: Theory and Applications*. Chapman & Hall, CRC Press, London, Boca Raton, FL., 2002.

[27] D. W. Qiu. Automata theory based on completed residuated lattice-valued logic (i). *Science in China, Ser. F*, 44(6):419–429, 2001.

[28] D. W. Qiu. Automata theory based on completed residuated lattice-valued logic (ii). *Science in China, Ser. F*, 45(6):442–452, 2002.

[29] O. Saarikivi and M. Veanes. Minimization of symbolic transducers. In *International Conference on Computer Aided Verification*, pages 176–196, 2017.

[30] J. Sakarovitch. *Elements of Automata Theory*. Cambridge University Press, 2009.

[31] S. Stanimirović, M. Ćirić, and J. Ignjatović. A double reverse canonization method for fuzzy automata. Internal communication, 2017.

[32] S. Stanimirović, M. Ćirić, and J. Ignjatović. Determinization of fuzzy automata by factorizations of fuzzy states and right invariant fuzzy quasi-orders. *Information Sciences*, 469:79–100, 2018.

[33] E. Trillas and L. Valverde. On some functionally expressible implications for fuzzy set theory. In E. P. Klement, editor, *Proceedings of the 3rd International Seminar on Fuzzy Set Theory*, pages 173–190, 1981.