Prelinear Hilbert algebras

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Abstract

In this paper we give an explicit description of the left adjoint of the forgetful functor from the algebraic category of Gödel algebras (i. e., prelinear Heyting algebras) to the algebraic category of bounded prelinear Hilbert algebras. We apply this result in order to study possible descriptions of the coproduct of two finite algebras in the algebraic category of prelinear Hilbert algebras.

1 Introduction and basic results

We assume that the reader is familiar with the theory of Heyting algebras [2], which is the algebraic counterpart of Intuitionistic Propositional Logic. Hilbert algebras represent the algebraic counterpart of the implicative fragment of Intuitionistic Propositional Logic, and they were introduced in early 50's by Henkin for some investigations of implication in intuicionistic and other non-classical logics ([26], pp. 16). In the 60's, Hilbert algebras were studied especially by Horn [17] and Diego [13].

Definition 1. A Hilbert algebra is an algebra $(H, \rightarrow, 1)$ of type (2, 0) which satisfies the following conditions for every $a, b, c \in H$:

- a) $a \to (b \to a) = 1$,
- b) $(a \to (b \to c)) \to ((a \to b) \to (a \to c)) = 1$,
- c) if $a \to b = b \to a = 1$ then a = b.

It is known that the class of Hilbert algebras is a variety. In every Hilbert algebra we have the partial order \leq given by

$$a \leq b$$
 if and only if $a \rightarrow b = 1$,

which is called *natural order*. Relative to the natural order on H we have that 1 is the greatest element.

We say that an algebra $(H, \rightarrow, 1, 0)$ of type (2, 0, 0) is a *bounded* Hilbert algebra if (H, 1) is a Hilbert algebra and $0 \le a$ for every $a \in H$.

The following lemma is part of the folklore of Hilbert algebras (see for example [5]).

Lemma 1. Let H be a Hilbert algebra and $a, b, c \in H$. Then the following conditions are satisfied:

- a) $a \to a = 1$,
- b) $1 \to a = a$,

c)
$$a \to (b \to c) = b \to (a \to c),$$

- d) $a \to (b \to c) = (a \to b) \to (a \to c),$
- e) if $a \leq b$ then $c \rightarrow a \leq c \rightarrow b$ and $b \rightarrow c \leq a \rightarrow c$.

Some additional properties of Hilbert algebras can be found in [5, 13].

For the general development of Hilbert algebras, the notion of implicative filter plays an important role. Let H be a Hilbert algebra. A subset $F \subseteq H$ is said to be an *implicative filter* if $1 \in F$ and $b \in F$ whenever $a \in F$ and $a \to b \in F$. We will denote the set of all implicative filters of H by $\operatorname{Fil}(H)$. Recall that if $X \subseteq H$, we define the filter generated by X as the least filter of H that contains the set X, which will be denoted by F(X). There is an explicit description for F(X) (see [4, Lemma 2.3]). More precisely, if $X \neq \emptyset$ we have that

$$F(X) = \{x \in H : a_1 \to (a_2 \to \cdots (a_n \to x) \dots) = 1 \text{ for some } a_1, \dots, a_n \in X\}$$

and $F(\emptyset) = \{1\}$. Let us consider a poset (X, \leq) . For each $Y \subseteq X$, the increasing set generated by Y is defined by $[Y) = \{x \in X : \text{ there is } y \in Y \text{ such that } y \leq x\}$. The decreasing set generated by Y is dually defined. If $Y = \{y\}$, then we will write [y) and (y] instead of $[\{y\})$ and $(\{y\}]$, respectively. We say that Y is an upset if Y = [Y), and a downset if Y = (Y]. If H is a Hilbert algebra then every filter of H is an upset and for every $a \in H$ we have that $F(\{a\}) = [a)$.

Let H be a Hilbert algebra and $F \in Fil(H)$. We say that F is *irreducible* if F is proper (i.e., $F \neq H$) and for any implicative filters F_1, F_2 such that $F = F_1 \cap F_2$ we have that $F = F_1$ or $F = F_2$. We write X(H) for the set of irreducible implicative filters of H. We also write X(H) for the poset of irreducible implicative filters of H where the order is given by the inclusion.

The proof of the following lemma can be found in [13].

Lemma 2. Let H be a Hilbert algebra and $F \in Fi(H)$. The following statements are equivalent:

- 1. $F \in X(H)$.
- 2. For every $a, b \in H$ such that $a, b \notin F$ there exists $c \notin F$ such that $a, b \leq c$.
- 3. For every $a, b \in H$ such that $a, b \notin F$ there exists $c \notin F$ such that $a \to c, b \to c \in F$.

Let *H* be a Hilbert algebra. An *order ideal* of *H* is a downset *I* of *H* such that for each $a, b \in I$, there exists $c \in I$ such that $a \leq c$ and $b \leq c$.

The following is [8, Theorem 2.6].

Theorem 1. Let H be a Hilbert algebra, $F \in Fi(H)$ and I an order ideal of H such that $F \cap I = \emptyset$. Then there exists $P \in X(H)$ such that $F \subseteq P$ and $P \cap I = \emptyset$.

The following corollaries are known in the literature, and can be obtained by using the previous theorem.

Corollary 3. Let H be a Hilbert algebra and $a, b \in H$ such that $a \notin b$. Then there exists $P \in X(H)$ such that $a \in P$ and $b \notin P$.

Corollary 4. Let H be a Hilbert algebra, $F \in Fil(H)$ and $a, b \in H$. Then $a \rightarrow b \notin F$ if and only if there is $P \in X(H)$ such that $F \subseteq P$, $a \in P$ and $b \notin P$.

We give a table with some of the categories we shall consider in this paper:

Category	Objects	Morphisms
Hey	Heyting algebras	Algebra homomorphisms
PHey	Prelinear Heyting algebras	Algebra homomorphisms
Hil	Hilbert algebras	Algebra homomorphisms
Hil ₀	Bounded Hilbert algebras	Algebra homomorphisms
PHil	Prelinear Hilbert algebras	Algebra homomorphisms
PHil ₀	Bounded Prelinear Hilbert algebras	Algebra homomorphisms
fPHil ₀	Bounded finite prelinear Hilbert algebras	Algebra homomorphisms
IS	Implicative semilattices	Algebra homomorphisms
Pos	Posets	Order preserving maps
PEs	Esakia spaces which are root systems	Continuous p-morphisms
HS	Hilbert spaces	Certain continuous maps
PHS	Hilbert spaces which are root systems	Certain morphisms of HS
PHS_0	Certain objects of PHS	Certain morphisms of PHS
$\mathrm{F_{fin}}$	Finite forests	Open maps
ChFor	Finite forests with a family of its subsets	Certain binary relations
hFor	Finite forests with a family of its subsets	Certain open maps

The paper is organized as follows. In Section 2 we study some properties of the variety of prelinear Hilbert algebras, which was considered by Monteiro in [21]. In Section 3 we present a categorical equivalence for the algebraic category of bounded prelinear Hilbert algebras. We use it in order to give an explicit description of the left adjoint of the forgetful functor from the algebraic category of prelinear Heyting algebras to the algebraic category of bounded prelinear Hilbert algebras. The ideas used in this section are similar to that developed in [7]. In Section 4 we apply results of the previous section in order to study some descriptions of the coproduct of two finite algebras in the algebraic category of bounded prelinear Hilbert algebras. In Section 5 we present a description of the product in some category of finite forests. In Section 6 we use results of Section 5 in order to give an explicit description of the product of two objects in certain category of finite forests F endowed with a distinguished family of subsets of F, which is equivalent to the category of finite bounded prelinear Hilbert algebras. This property allow us to obtain an explicit description of the coproduct of two finite algebras in the algebraic category of bounded prelinear Hilbert algebras. Finally, in Section 7 we give an explicit description of the coproduct of two finite algebras in the algebraic category of prelinear Hilbert algebras in terms of the coproduct in the algebraic category of bounded prelinear Hilbert algebras.

2 Prelinear Hilbert algebras

In this section we give some properties of prelinear Hilbert algebras, which were introduced and studied by Monteiro in [21].

Definition 2. We say that a Hilbert algebra is a Hilbert chain if its natural order is total. We call prelinear Hilbert algebra to the members of the variety generated by the class of Hilbert chains.

Let H be a Hilbert algebra. For every $a, b, c \in H$ we define

$$l(a, b, c) := ((a \to b) \to c) \to (((b \to a) \to c) \to c.$$

Considerer the following equation:

$$(\mathbf{P}) \quad l(a,b,c) = 1.$$

Let K be a class of algebras of the same type. As usual, we write V(K) for the variety generated by K, I(K) for the isomorphic members of K, H(K) for the homomorphic image of members of K, S(K) for the subalgebras of members of K and P(K) for the direct product of members of K. Recall that by Tarski's theorem we have that HSP(K) = V(K) (see [2]).

The following is [21, Theorem 2.2]. By clarity in the exposition of the present paper we think it is convenient to present a sketch of the proof.

Proposition 5. Let K be the class of Hilbert chains. Then the variety V(K) is characterized by an equational basis for the variety of Hilbert algebras with the additional equation (P).

Proof. Since every Hilbert chain satisfies the equation (P) then every member of V(K) satisfies (P). Conversely, let H be a Hilbert algebra which satisfies (P). If H is trivial then $H \in V(K)$. Assume that H is not trivial. Then it follows from [21, Theorem 7.1] that $H \in \mathsf{ISP}(K)$. Since $\mathsf{ISP}(K) \subseteq \mathsf{HSP}(K) = V(K)$ then $H \in V(K)$.

By Proposition 5 we can define a prelinear Hilbert algebra as a Hilbert algebra which satisfies the equation (P).

Let H be a Hilbert algebra and $a, b \in H$. If there exists the supremum of $\{a, b\}$ with respect to the natural order of H then we write $a \vee b$ for this element. In what follows we give some another characterizations for the variety of prelinear Hilbert algebras ([21, Theorem 5.1]).

Proposition 6. Let H be a Hilbert algebra. Then H is prelinear if and only if for every $a, b \in H$ it holds that $(a \rightarrow b) \lor (b \rightarrow a) = 1$.

The following definition will be used throughout the paper.

Definition 3. A poset (X, \leq) is said to be a root system if [x) is a chain for each $x \in X$.

The following proposition is [21, Theorem 4.5].

Proposition 7. Let H be a Hilbert algebra. Then H is prelinear if and only if X(H) is a root system.

Let H be a Hilbert algebra. It is part of the folklore of Hilbert algebras that there exists an order isomorphism between the lattice of congruences of H and the lattice of implicative filters of H. The isomorphism is established via the assignments $\theta \to 1/\theta$ and $F \to \theta_F = \{(a, b) \in H \times H : a \to b \in F \text{ and } b \to a \in F\}$. We write H/F in place of H/θ_F , where H/θ_F is the set of equivalence classes associated to the congruence θ_F .

Proposition 8. Let H be a Hilbert algebra. The following conditions are equivalent:

1. H is prelinear.

- 2. For every $a, b \in H$ and $P \in X(H)$, $a \to b \in P$ or $b \to a \in P$.
- 3. For every $P \in X(H)$, A/P is a chain.
- 4. For every $a, b, c \in H$ and $P \in X(H)$, $l(a, b, c) \in P$.

Proof. In order to show that 1. implies 2., let H be prelinear, $P \in X(H)$ and $a, b \in H$. By Proposition 6 we have that $(a \to b) \lor (b \to a) = 1$. Then we obtain that $(a \to b) \lor (b \to a) \in P$, so by [12, Theorem 3.2] we deduce that $a \to b \in P$ or $b \to a \in P$. Now we will see that 2. implies 1. Suppose that H is not prelinear, so by Proposition 7 we have that there exist $P, Q \in X(H)$ such that $P \subseteq Q, P \subseteq Z$, $Q \not\subseteq Z$ and $Z \not\subseteq Q$. Thus, there are $a, b \in H$ such that $a \in Q, a \notin Z, b \in Z$ and $b \notin Q$. This implies that $a \to b \notin P$ and $b \to a \notin P$, which is a contradiction.

The equivalence between 2. and 3. is immediate.

The fact that 1. implies 4. follows from Proposition 5. Now we will prove that 4. implies 1. Suppose that H is not prelinear. Hence, by Proposition 7 there exist $Q, Z \in X(H)$ such that $P \subseteq Q, P \subseteq Z, Q \not\subseteq Z$ and $Z \not\subseteq Q$. Hence, there exist $a, b \in H$ such that $a \in Q, a \notin Z, b \in Z$ and $b \notin Q$. Thus, $a, b \notin P$. Notice that $a \to b \notin P$, otherwise $a \to b \in Q$ and since $a \in Q$ then $b \in Q$, which is impossible. Similarly we deduce that $b \to a \notin P$. Since P is irreducible then by Lemma 2 we have that there exists $c \in A$ such that $(a \to b) \to c \in P, (b \to a) \to c \in P$ and $c \notin P$. Taking into account that $l(a, b, c) \in P$, applying modus ponens twice we get that $c \in P$, which is a contradiction. Thus, we have proved 4.

We assume that the reader is familiar with the theory of Heyting algebras [2]. Prelinear Heyting algebras were considered by Horn in [18] as an intermediate step between the classical calculus and intuitionistic one and they were studied also by Monteiro [20], G. Martínez [15] and others. This is the subvariety of Heyting algebras generated by the class of totally ordered Heyting algebras and can be axiomatized by the usual equations for Heyting algebras plus the prelinearity law $(x \to y) \lor (y \to x) = 1$. In ([2], ch. IX) and in [20] there are characterizations for prelinear Heyting algebras. Horn showed in [18] (although it was in fact proved before by Monteiro, see [20]) that prelinear Heyting algebras can be characterized among Heyting algebras in terms of the prime filters. More precisely, a Heyting algebra H is prelinear if and only if the poset of prime filters of H with the inclusion as order is a root system.

Prelinear Heyting algebras, under the name of Gödel algebras, are a particular class of t-norm based algebras of great interest for fuzzy logic [16].

We also assume that the reader is familiar with the theory of implicative semilattices [23]. It is known that every implicative element has a largest element with respect to the order associated to its unserlying semilattice. We denoted by 1 to this element. A *bounded implicative smilattice* is an algebra $(H, \land, \rightarrow, 0, 1)$ of type (2, 2, 0, 0) such that (H, \land, \rightarrow) is an implicative semilattice and 0 is the minimum element of (H, \land) . In what follows we establish some connections between implicative semilattices, prelinear Heyting algebras and prelinear Hilbert algebras.

The following result is known and can be deduced from the papers of W.C. Nemitz [23], A. Monteiro [21] and T. Katriňák [19]. Here we give a proof for completeness.

Proposition 9. Let $(H, \rightarrow, \land, 1)$ be an implicative semilattice such that $(H, \rightarrow, 1)$ is a prelinear Hilbert algebra. Then for every $a, b \in H$ there exist $a \lor b$ and it is given by

$$a \lor b = ((a \to b) \to b) \land ((b \to a) \to a).$$

Moreover, if $(H, \rightarrow, \land, 0, 1)$ is a bounded implicative semilattice then the algebra $(H, \rightarrow, \land, \lor, 0, 1)$ is a prelinear Heyting algebra.

Proof. Let $a, b \in H$. By Proposition 7 there exists the supremum of the set $\{a \to b, b \to a\}$. Moreover, $(a \to b) \lor (b \to a) = 1$. Consider the element $c = ((a \to b) \to b) \land ((b \to a) \to a)$. It is clear that $a \leq c$ and $b \leq c$. So, c is an upper bound of $\{a, b\}$. Let $z \in H$ such that $a \leq z$ and $b \leq z$. We will prove that $c \leq (a \to b) \to z$ and $c \leq (b \to a) \to z$. Suppose that $c \notin (a \to b) \to z$ or $c \notin (b \to a) \to z$. In the first case, it follows from Corollary 3 that there exists $P \in X(H)$ such that $c \in P$ and $(a \to b) \to z \notin P$. So, by Corollary 4 there exists $Q \in X(H)$ such that $a \to b \in Q, z \notin Q$ and $P \subseteq Q$. Since $c \in P$ then $(a \to b) \to b \in P \subseteq Q$. By modus ponens we have that $b \in Q$, and since $b \leq z$ then we get $z \in Q$, which is a contradiction. The other case is similar. Thus, $c \leq (a \to b) \to z$ and $c \leq (b \to a) \to z$. Therefore, $c \leq z$. Therefore, there exist $a \lor b$ and it is given by $a \lor b = ((a \to b) \to b) \land ((b \to a) \to a)$.

The rest of the proof follows from Proposition 6.

3 An adjunction

Let Hil be the algebraic category of Hilbert algebras, PHil the algebraic category of prelinear Hilbert algebras and $PHil_0$ the algebraic category of bounded prelinear

Hilbert algebras. In this section we give an explicit description of the left adjoint of the forgetful functor from PHey to $PHil_0$.

We start with some preliminary definitions and properties involving the duality developed in [6] for the algebraic category of Hilbert algebras (see also [11]).

If $f: H \to G$ is a function between Hilbert algebras, we define the relation $R_f \subseteq X(G) \times X(H)$ by

 $(P,Q) \in R_f$ if and only if $f^{-1}(P) \subseteq Q$.

If H is a Hilbert algebra and $a \in H$ we define

$$\varphi_H(a) := \{ P \in X(H) : a \in P \}.$$
(1)

If there is not ambiguity we write φ in place of φ_H . If X is a set then we define $Y^c := \{x \in X : x \notin Y\}.$

Remark 1. Let (X, \leq) be a poset. Write X^+ for the set of upsets of (X, \leq) . Define on X^+ the binary operation \Rightarrow by

$$U \Rightarrow V := (U \cap V^c)^c.$$
⁽²⁾

Then $(X^+, \cap, \cup, \Rightarrow, \emptyset, X)$ is a complete Heyting algebra. If there is not ambiguity, we also write X^+ for this Heyting algebra. In particular, X^+ can be seen as a Hilbert algebra.

Let (X, τ) be a topological space. An arbitrary non-empty subset Y of X is said to be *irreducible* if for any closed subsets Z and W such that $Y \subseteq Z \cup W$ we have that $Y \subseteq Z$ or $Y \subseteq W$. We say that (X, τ) is *sober* if for every irreducible closed set Y there exists a unique $x \in X$ such that $Y = \{x\}$, where $\{x\}$ denotes the clausure of $\{x\}$. A subset of X is *saturated* if it is an intersection of open sets. The *saturation* of a subset Y of X is defined as $\operatorname{sat}(Y) := \bigcap \{U \in \tau : Y \subseteq U\}$. Also recall that the *specialization order* of (X, τ) is defined by $x \leq y$ if and only if $x \in \{y\}$. The relation \leq is reflexive and transitive, i.e., a quasi-order. The relation \leq is a partial order if (X, τ) is T_0 . The dual quasi-order of \leq will be denoted by \leq_d . Hence,

 $x \leq_d y$ if and only if $y \in \overline{\{x\}}$.

Let (X, τ) be a topological space which is T_0 , and consider the order \leq_d . Let $x \in X$ and $Y \subseteq X$. Then $\overline{\{x\}} = [x)$ and $\operatorname{sat}(Y) = (Y]$.

Definition 4. A Hilbert space, or H-space for short, is a structure (X, τ, K) where (X, τ) is a topological space, K is a family of subsets of X and the following conditions are satisfied:

- (H1) \mathcal{K} is a base of open and compact subsets for the topology τ on X.
- (H2) For every $U, V \in \mathcal{K}$, sat $(U \cap V^c) \in \mathcal{K}$.

(H3) (X, τ) is sober.

In what follows, if (X, τ, \mathcal{K}) is an *H*-space we simply write (X, \mathcal{K}) .

Remark 2. 1. A sober topological space is T_0 .

2. Viewing any topological space as a poset, with the order \leq_d , condition (H2) of Definition 4 can be rewritten as: for every $U, V \in \mathcal{K}$, $(U \cap V^c] \in \mathcal{K}$.

If X, Y are sets and $R \subseteq X \times Y$, take $R(x) := \{y \in Y : (x, y) \in R\}$, and if $U \subseteq Y, R^{-1}(U) := \{x \in X : R(x) \cap U \neq \emptyset\}.$

Definition 5. Let $\mathbf{X}_1 = (X_1, \mathcal{K}_1)$ and $\mathbf{X}_2 = (X_2, \mathcal{K}_2)$ be two *H*-spaces. Let us consider a relation $R \subseteq X_1 \times X_2$. We say that *R* is an *H*-relation from \mathbf{X}_1 into \mathbf{X}_2 if it satisfies the following properties:

(HR1) $R^{-1}(U) \in \mathcal{K}_1$, for every $U \in \mathcal{K}_2$.

(HR2) R(x) is a closed subset of \mathbf{X}_2 , for all $x \in X_1$.

We say that R is an H-functional relation if it satisfies the following additional condition:

(HF) If $(x, y) \in R$ then there is $z \in X_1$ such that $z \in \overline{\{x\}}$ and $R(z) = \overline{\{y\}}$.

Remark 3. Condition (HF) from Definition 5 can also be given as follows: if $(x, y) \in R$ then there exists $z \in X_1$ such that $x \leq_d z$ and R(z) = [y).

If H is a Hilbert algebra then $\mathbf{X}(H) = (X(H), \mathcal{K}_H)$ is an H-space, where $\mathcal{K}_H := \{\varphi(a)^c : a \in H\}$. If f is a morphism in Hil then R_f is an H-functional relation. Write HS for the category whose objects are Hilbert spaces and whose morphisms are H-functional relations. The assignment $H \mapsto \mathbf{X}(H)$ can be extended to a functor $\mathbf{X} : \operatorname{Hil} \to \operatorname{HS}$.

Remark 4. If $H \in \text{Hil}$ and $P, Q \in \mathbf{X}(H)$, then $P \subseteq Q$ if and only if $P \leq_d Q$.

Let (X, \mathcal{K}) be an *H*-space. Define $D(X) := \{U \subseteq X : U^c \in \mathcal{K}\}$. Then $D(X) \subseteq X^+$. It follows from Definition 4 and Remark 2 that D(X) is closed under the operation \Rightarrow given in (2) of Remark 1. Since X^+ is a Heyting algebra then $\mathbf{D}(X) = (D(X), \Rightarrow, X)$ is a Hilbert algebra. If *R* is an *H*-functional relation from (X_1, \mathcal{K}_1) into (X_2, \mathcal{K}_2) , then the map $h_R : \mathbf{D}(X_2) \to \mathbf{D}(X_1)$ given by $h_R(U) = \{x \in X_1 : R(x) \subseteq U\}$ is a morphism in Hil. The assignment $X \mapsto \mathbf{D}(X)$ can be extended to a functor $\mathbf{D} : \mathsf{HS} \to \mathsf{Hil}$. Finally, if $H \in \mathsf{Hil}$ then the map $\varphi : H \to \mathbf{D}(\mathbf{X}(H))$ defined as in (1) is an isomorphism in Hil.

Recall that if (X, τ, \mathcal{K}) is an *H*-space then (X, \leq_d) is a poset, where \leq_d is the dual of the specialization order associated to the topological space (X, τ) . Also recall that if *H* is a Hilbert algebra then the dual of the specialization order associated to $\mathbf{X}(H)$ is the inclusion (Remark 4). We say that there is an order isomorphism between two *H*-spaces if and only if there is an order isomorphism between its associated posets obtained through the dual of the specialization order.

If $(X, \tau), (Y, \sigma)$ be topological spaces and $f : X \to Y$ a function. Then we define the binary relation f^R by

$$(x,y) \in f^R$$
 if and only if $f(x) \leq_d y$.

If (X, \mathcal{K}) is an *H*-space, then the map $\epsilon_X : X \to \mathbf{X}(\mathbf{D}(X))$ given by $\epsilon_X(x) = \{U \in D(X) : x \in U\}$ is an order isomorphism and a homeomorphism between the topological spaces X and $\mathbf{X}(\mathbf{D}(X))([6, \text{ Theorem 2.2}])$. If there is not ambiguity we will write ϵ in place of ϵ_X . Moreover, the relation $\epsilon^R \subseteq X \times X(D(X))$ is given by $(x, P) \in \epsilon^R$ if and only if $\epsilon(x) \subseteq P$. Moreover, ϵ^R is an *H*-functional relation which is an isomorphism in HS.

Theorem 2. The contravariant functors **X** and **D** define a dual equivalence between HiI and HS with natural equivalences ϵ^R and φ .

If *H* is a bounded Hilbert algebra then $\varphi(0) = \emptyset$, so $\mathbf{X}(H) \in \mathcal{K}_H$. Conversely, if (X, \mathcal{K}) is an *H*-space such that $X \in \mathcal{K}$ then $\mathbf{D}(X)$ is a bounded Hilbert algebra. If *H*, *G* are bounded Hilbert algebras and $f : H \to G$ is a morphism of Hilbert algebras such that f(0) = 0 then $R_f(P) \neq \emptyset$ for every $P \in \mathbf{X}(G)$. Conversely, if (X_1, \mathcal{K}_1) and (X_2, \mathcal{K}_2) are *H*-spaces such that $X_1 \in \mathcal{K}_1, X_2 \in \mathcal{K}_2$ and *R* is an *H*-functional relation from (X_1, \mathcal{K}_1) into (X_2, \mathcal{K}_2) such that $R(x) \neq \emptyset$ for every $x \in X_1$, then $h_R : \mathbf{D}(X_2) \to \mathbf{D}(X_1)$ satisfies that $h_R(\emptyset) = \emptyset$. Moreover, if (X, \mathcal{K}) is an *H*-space such that $X \in \mathcal{K}$ then $\epsilon(x) \neq \emptyset$ for every $x \in X$.

Let PHS_0 be the full subcategory of HS whose objects (X, \mathcal{K}) are such that (X, \leq_d) is a root system and $X \in \mathcal{K}$, and whose morphisms are *H*-functional relations *R* such that $R(x) \neq \emptyset$ for every *x*. Straightforward computations based in Theorem 2 and Proposition 7 proves the following result.

Corollary 10. There exists a categorical equivalence between PHil_0 and $\mathsf{PHS}_0.$

The adjunction

Let PHey be the algebraic category of prelinear Heyting algebras.

Lemma 11. Let $H \in \mathsf{PHil}$. Then $\mathbf{X}(H)^+ \in \mathsf{PHey}$.

Proof. Let $U, V \in \mathbf{X}(H)^+$. We need to show that $(U \Rightarrow V) \cup (V \Rightarrow U) = \mathbf{X}(H)$, or, equivalently, that $(U \cap V^c] \cap (V \cap U^c] = \emptyset$. Suppose that there is $P \in (U \cap V^c] \cap (V \cap U^c]$. Thus, there exist $Q, Z \in \mathbf{X}(H)$ such that $P \subseteq Q, P \subseteq Z$, $Q \in U \cap V^c$ and $Z \in V \cap U^c$. By Proposition 7 we have that $\mathbf{X}(H)$ is a root system, so $Q \subseteq Z$ or $Z \subseteq Q$. Assume that $Q \subseteq Z$. Since $Q \in U$ then $Z \in U$, which is a contradiction because $Q \in U^c$. Analogously, we obtain a contradiction assuming that $Z \subseteq Q$. Therefore, $\mathbf{X}(H)^+$ is a prelinear Heyting algebra.

If S is a subset of a Heyting algebra H, write $\langle S \rangle_{\mathsf{Hey}}$ for the Heyting subalgebra of H generated by S. Let $H \in \mathsf{Hil}$. Since $\varphi(H)$ is a subset of the Heyting algebra $\mathbf{X}(H)^+$ then we define the following Heyting subalgebra of $X(H)^+$:

$$H^* := \langle \varphi(H) \rangle_{\mathsf{Hey}}.$$

Corollary 12. If $H \in \mathsf{PHil}$ then $H^* \in \mathsf{PHey}$.

Proof. It follows from Lemma 11 and the fact that the class of prelinear Heyting algebras is a variety. \Box

In what follows we study the link between morphisms in PHil_0 and morphisms in $\mathsf{PHey}.$

Lemma 13. Let $f : H \to G$ be a morphism in PHil_0 and $P \in \mathbf{X}(G)$. Then $f^{-1}(P) \in \mathbf{X}(H)$.

Proof. Let $a, b \in G$. By Proposition 8 we have that $f(a) \to f(b) \in P$ or $f(b) \to f(a) \in P$. Since $f(a \to b) = f(a) \to f(b)$ and $f(b \to a) = f(b) \to f(a)$ then $a \to b \in f^{-1}(P)$ or $b \to a \in f^{-1}(P)$. This property will be used in order to show that $f^{-1}(P) \in \mathbf{X}(H)$.

First note that since $0 \notin P$ then $f^{-1}(P)$ is a proper implicative filter. Now we will prove that $f^{-1}(P)$ is irreducible. Let $f^{-1}(P) = F_1 \cap F_2$ with $F_1, F_2 \in \mathbf{X}(H)$. Suppose that $F_1 \not\subseteq f^{-1}(P)$ and $F_2 \not\subseteq f^{-1}(P)$. Then there exist $a, b \in H$ such that $a \in F_1, f(a) \notin P, b \in F_2$ and $f(b) \notin P$. Since $a \to b \in F_1$ and $a \in F_1$ then $b \in F_1$, so $b \in F_1 \cap F_2 = f^{-1}(P)$, i.e., $f(b) \in P$, which is a contradiction. Thus, $f^{-1}(P) = F_1$ or $f^{-1}(P) = F_2$. Therefore, $f^{-1}(P) \in \mathbf{X}(H)$.

Let $f: H \to G$ be a morphism in PHil_0 . Taking into account the commutativity of the the following diagram



we obtain that $\varphi(f(a)) = \mathbf{D}(\mathbf{X}(f))(\varphi(a))$. Besides, in [6, Lemma 3.3] it was proved that if $P \in \mathbf{X}(H)$ then $f(a) \in P$ if and only if for all $Q \in \mathbf{X}(G)$, if $(P,Q) \in R_f$ then $a \in Q$, i.e., $f(a) \in P$ if and only if $R_f(P) \subseteq \varphi(a)$. Thus, for every $a \in H$ we have that

$$\mathbf{D}(\mathbf{X}(f))(\varphi(a)) = \{ P \in \mathbf{X}(G) : R_f(P) \subseteq \varphi(a) \}.$$
(3)

Corollary 14. Let $f : H \to G$ be a morphism in PHil_0 and $g = \mathbf{D}(\mathbf{X}(f))$. The morphism g in PHil_0 can be extended to the homomorphism of Heyting algebras $\hat{g} : \mathbf{X}(H)^+ \to \mathbf{X}(G)^+$ given by

$$\hat{g}(U) = \{ P \in \mathbf{X}(G) : R_f(P) \subseteq U \}.$$

Proof. The proof is similar to [7, Lemma 8] by considering Lemma 13.

The following remark is a well known fact from universal algebra [3].

Remark 5. Let A and B be algebras of the same type and $X \subseteq A$. Write Sg(X) for the subalgebra of A generated by X and Sg(f(X)) for the subalgebra of B generated by f(X). If $f: A \to B$ is a homomorphism then f(Sg(X)) = Sg(f(X)).

Lemma 15. The homomorphism of Heyting algebras \hat{g} defined in Corollary 14 satisfies $\hat{g}(H^*) \subseteq G^*$.

Proof. It follows from Lemma 14, Remark 5 and the equality $g(\varphi(a)) = \varphi(f(a))$ given in (3).

Let $f : H \to G$ be a morphism in PHil_0 . It follows from Corollary 14 and Lemma 15 that the map $f^* : H^* \to G^*$ given by $f^*(U) = \hat{g}(U)$ is a morphism in PHey.

Let Id be an identity morphism in PHil_0 . It is immediate that Id^* is an identity in PHey . Let $f: H \to G$ and $g: G \to K$ be morphisms in PHil_0 . It follows from [6, Theorem 3.3] that $R_{g \circ f} = R_g \circ R_f$. Hence, straightforward computations based in the above mentioned equality shows that

$$(g \circ f)^* = g^* \circ f^*.$$

Hence we have that the assignment $H \mapsto H^*$ and $f \mapsto f^*$ defines a functor $()^* : \mathsf{PHil}_0 \to \mathsf{PHey}.$

Let U be the forgetful functor from PHey to PHil_0 . Let $H \in \mathsf{PHil}_0$. Consider the injective morphism of Hilbert algebras $\psi : H \to \mathrm{U}(H^*)$ given by $\psi(a) = \varphi(a)$.

Proposition 16. Let $G \in \mathsf{PHey}$ and $f : H \to U(G^*) \in \mathsf{PHil}_0$. Then, there exists a unique morphism $h : H^* \to G$ such that $f = U(h) \circ \psi$.

Proof. The map $f^*: H^* \to G^*$ is a morphism in PHey. Since $G \in \mathsf{PHey}$ then for every $a, b \in G$ we have that $\varphi(a \land b) = \varphi(a) \cap \varphi(b)$, so we deduce that the map $\varphi: G \to G^*$ is an isomorphism in PHey. Hence, the map $h: H^* \to G$ given by $h = \varphi^{-1} \circ f^*$ is also a morphism in PHey. Finally, taking into account (3) we have that $f = U(h) \circ \psi$.

Let I be the identity functor in PHil_0 . It follows from (3) that $\Psi : I \to U \circ ()^*$ is a natural transformation. Here, the family of morphism associated to the natural transformation is given by the morphisms ψ .

In other words, to say that $\Psi : I \to U \circ ()^*$ is a natural transformation is equivalent to say that if $f : H \to G$ is a morphism in PHil_0 then the following diagram commutes:



Therefore we get the following result.

Theorem 3. The functor $()^* : \mathsf{PHil}_0 \to \mathsf{PHey}$ is left adjoint to U.

In [9] Celani and Jansana presented an explicit description for the left adjoint of the forgetful functor from the category of implicative semilattices to the category of Hilbert algebras. In particular, they proved that if H is a Hilbert algebra then

 $S(H) = \{ U : U = \varphi(a_1) \cap \dots \cap \varphi(a_n) \text{ for some } a_1, \dots, a_n \in H \}.$

is an implicative semilattice, which is called the *free implicative semilattice extension of the Hilbert algebra* H (see also [7]).

Proposition 17. Let $H \in \mathsf{PHil}_0$. Then $S(H) \in \mathsf{PHey}$ and $S(H) = H^*$.

Proof. By Proposition 7 and Proposition 9 in order to prove that $S(H) \in \mathsf{PHey}$ it is enough to see that $(U \Rightarrow V) \cup (V \Rightarrow U) = X(H)$ for every $U, V \in S(H)$. Let $U, V \in S(H)$. Then there exist two finite subsets $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_k\}$ of H such that $U = \varphi(a_1) \cap \ldots \cap \varphi(a_n)$ and $V = \varphi(b_1) \cap \ldots \cap \varphi(b_k)$. Suppose that there exists $P \in \mathbf{X}(H)$ such that $P \notin (U \Rightarrow V) \cup (V \Rightarrow U)$. Thus, there exist $Q, Z \in \mathbf{X}(H)$ such that $P \subseteq Q, P \subseteq Z, Q \in U \cap V^c$ and $Z \in V \cap U^c$. Hence, there exists $i = 1, \ldots, n$ and $j = 1, \ldots, k$ such that $Q \notin \varphi(b_j)$ and $Z \notin \varphi(a_i)$, i.e., $b_j \notin Q$ and $a_i \notin Z$. Since H is prelinear then it follows from Proposition 7 that $Q \subseteq Z$ or $Z \subseteq Q$. Then, $a_i \in Z$ or $b_j \in Q$, which is a contradiction. Thus, $S(H) \in \mathsf{PHey}$.

The fact that $S(H) = H^*$ follows from that S(H) is a Heyting algebra and $\varphi(H) \subseteq S(H) \subseteq H^*$.

Finally we will study some connections between $\mathbf{X}(H)$ and $\mathbf{X}(H^*)$ for H a finite Hilbert algebra.

Proposition 18. Let H be a finite Hilbert algebra. Then $H^* = \mathbf{X}(H)^+$.

Proof. In order to prove it, let $U \in \mathbf{X}(H)^+$ and $U \neq \emptyset$. Then there exist $P_1, \ldots, P_n \in \mathbf{X}(H)$ such that $U = \bigcup_{i=1}^n [P_i)$. For instance, we can choose P_1, \ldots, P_n as the minimal elements of U. For every $i = 1, \ldots, n$ there exist $a_{i1}, \ldots, a_{im} \in H$ such that $P_i = \{a_{i1}, \ldots, a_{im}\}$. So, $[P_i) = \bigcap_{j=1}^m \varphi(a_{ij})$. Notice that for every P_1, \ldots, P_n we can choose the same m in the above mentioned reasoning. In order to make it possible, for every $i = 1, \ldots, n$ let c_i be the cardinal of P_i . Write m for the maximum of the set $\{c_1, \ldots, c_n\}$. Then for each c_i such that $c_i < m$ define $a_{ij} = 1$ for every $j \in \{c_i + 1, \ldots, m\}$, which was our aim. Hence, for every

 $U \in \mathbf{X}(H)^+$ $(U \neq \emptyset)$ there exists a finite family $\{a_{ij}\}_{i,j}$ in H (with i = 1, ..., nand j = 1, ..., m) such that

$$U = \bigcup_{i=1}^{n} \bigcap_{j=1}^{m} \varphi(a_{ij}).$$
(4)

Therefore, $H^* = \mathbf{X}(H)^+$.

The following result will be used later.

Proposition 19. Let H be a finite algebra of PHil_0 . Then the map $\eta : \mathbf{X}(H^*) \to \mathbf{X}(H)$ given by $\eta(P) = \varphi^{-1}(P)$ is an order isomorphism.

Proof. It follows from [7, Theorem 11], [9, Lemma 4.6] and [9, Proposition 7.7]. \Box

4 Coproduct in $PHil_0$ for finite algebras

In this section we study some properties of the coproduct of two finite algebras in $PHil_0$ through the study of the product in PHS_0 of its associated finite *H*-spaces.

Let $H \in \mathsf{PHey}$ and $S \subseteq H$ with $0 \in S$. We write $\langle S \rangle_{\mathsf{PHil}_0}$ as the Hilbert subalgebra of H generated by S. Note that $\langle S \rangle_{\mathsf{PHil}_0} \in \mathsf{PHil}_0$.

Let $H, G \in \mathsf{PHil}_0$. Let $i_H : H^* \to H^* \coprod_{\mathsf{PHey}} G^*$ and $i_G : G^* \to H^* \coprod_{\mathsf{PHey}} G^*$ be the morphisms in PHey given by the definition of coproduct. Consider $\varphi_H : H \to H^*$ and $\varphi_G : G \to G^*$. Then we define

$$\eta_{GH} := i_H(\varphi_H(H)) \cup i_G(\varphi_G(G)).$$

Consider maps $j_H : H \to \langle \eta_{GH} \rangle_{\mathsf{PHil}_0}$ and $j_G : G \to \langle \eta_{GH} \rangle_{\mathsf{PHil}_0}$ defined by by $j_H = i_H \circ \varphi_H$ and $j_G = i_G \circ \varphi_G$.

Let $H, G \in \mathsf{PHil}_0$. In the following proposition we will show that for every $\alpha : H \to J \in \mathsf{PHil}_0$ and $\beta : G \to J \in \mathsf{PHil}_0$ there is an unique morphism $f : \langle \eta_{GH} \rangle_{\mathsf{PHil}_0} \to J$ in PHil_0 such that the following diagram commutes:



Proposition 20. Let $H, G \in \mathsf{PHil}_0$. Then $H \coprod_{\mathsf{PHil}_0} G \cong \langle \eta_{GH} \rangle_{\mathsf{PHil}_0}$.

Proof. Let $\alpha : H \to J \in \mathsf{PHil}_0$ and $\beta : G \to J \in \mathsf{PHil}_0$. Then $\alpha^* : H^* \to J^* \in \mathsf{PHey}$ and $\beta^* : G^* \to J^* \in \mathsf{PHey}$. Then there exists an unique morphism

 $F: H^* \coprod_{\mathsf{PHev}} G^* \to J^*$ in PHey such that the following diagram commutes:



In particular,

$$\alpha^* = F \circ i_H,\tag{5}$$

$$\beta^* = F \circ i_G. \tag{6}$$

In what follows we will see that $F(\langle \eta_{GH} \rangle_{\mathsf{PHil}_0}) \subseteq \varphi_J(J)$. First note that since F is a morphism of Heyting algebras then F is also a morphism of Hilbert algebras, so $F(\langle \eta_{GH} \rangle_{\mathsf{PHil}_0}) = \langle F(\eta_{GH}) \rangle_{\mathsf{PHil}_0}$. Taking into account that $\varphi_J \circ \alpha = \alpha^* \circ \varphi_H$, $\varphi_J \circ \beta = \beta^* \circ \varphi_G$, (5) and (6) we obtain that

$$F(\eta_{GH}) = F(i_H(\varphi_H(H))) \cup F(i_G(\varphi_G(G)))$$

= $(F \circ i_H)(\varphi_H(H)) \cup (F \circ i_G)(\varphi_G(G))$
= $(\alpha^* \circ \varphi_H)(H) \cup (\beta^* \circ \varphi_G)(G)$
= $\varphi_J(\alpha(H)) \cup \varphi_J(\beta(G))$
= $\varphi_J(\alpha(H) \cup \beta(G))$
 $\subseteq \varphi_J(J).$

Thus, $F(\langle \eta_{GH} \rangle_{\mathsf{PHil}_0}) = \langle F(\eta_{GH}) \rangle_{\mathsf{PHil}_0} \subseteq \langle \varphi_J(J) \rangle_{\mathsf{PHil}_0} = \varphi_J(J)$, i.e., $F(\langle \eta_{GH} \rangle) \subseteq \varphi_J(J)$, which was our aim.

We also write $\varphi_J : J \to \varphi_J(J)$. Since φ_J is an isomorphism in PHil_0 , we define $f : \langle \eta_{GH} \rangle_{\mathsf{PHil}_0} \to J$ by $f = \varphi_J^{-1} \circ F$. We will prove that f is the unique morphism in PHil_0 such that the following diagram commutes:



In order to show it, note that

$$\begin{aligned} f \circ j_H &= \varphi_J^{-1} \circ F \circ i_H \circ \varphi_H \\ &= \varphi_J^{-1} \circ \alpha^* \circ \varphi_H \\ &= \varphi_J^{-1} \circ \varphi_J \circ \alpha \\ &= \alpha. \end{aligned}$$

Thus, $f \circ j_H = \alpha$. Similarly it can be proved that $f \circ j_G = \beta$. Finally, let $g : \langle \eta_{GH} \rangle_{\mathsf{PHil}_0} \to J$ be a morphism in PHil_0 such that $g \circ j_H = \alpha$ and $g \circ j_G = \beta$. Since f = g in η_{GH} then f = g in $\langle \eta_{GH} \rangle_{\mathsf{PHil}_0}$. Therefore, $H \coprod_{\mathsf{PHil}_0} G \cong \langle \eta_{GH} \rangle_{\mathsf{PHil}_0}$.

Let Pos be the category of posets and Es the category of Esakia spaces. We assume that the reader is familiar with the theory of Esakia spaces and the fact that there exists a categorial equivalence between the algebraic category of Heyting algebras and the category whose objects are Esakia spaces and whose morphisms $f: (X, \leq, \tau) \rightarrow (Y, \sigma, \leq)$ are continuous order-preserving maps which satisfy that $f^{-1}([x)) = [f^{-1}(\{x\}))$ for every $x \in X$ [14, 22] (see also [24, 25]). In particular, there exists a categorical equivalence between PHey and PEs, where PEs is the full subcategory of Es whose objects are root systems.

It is part of the folklore of prelinear Heyting algebras that the coproduct of finite algebras in PHey is finite. This fact will be used in order to show that the coproduct of finite algebras in $PHil_0$ is also finite.

Lemma 21. Let H, G be finite algebras in $PHil_0$. Then the algebra $H \coprod_{PHil_0} G$ is finite.

Proof. Consider the monomorphism $i : H \coprod_{\mathsf{PHil}_0} G \to U((H \coprod_{\mathsf{PHil}_0} G)^*)$ in PHil_0 given by i(x) = x. Taking into account Theorem 3 we have that $(H \coprod_{\mathsf{PHil}_0} G)^* \cong H^* \coprod_{\mathsf{PHev}} G^*$ in PHey , so there exists a monomorphism

$$j: H\coprod_{\mathsf{PHil}_0}G \to \mathrm{U}(H^*\coprod_{\mathsf{PHey}}G^*)$$

in PHil_0 . Since H^* and G^* are finite then $H^* \coprod_{\mathsf{PHey}} G^*$ is finite. Therefore, $H \coprod_{\mathsf{PHil}_0} G$ is finite. \Box

Lemma 22. Let H, G be finite algebras in PHil_0 . Then there exists an order isomorphism between $\mathbf{X}(H) \prod_{\mathsf{PHS}_0} \mathbf{X}(G)$ and $\mathbf{X}(H^*) \prod_{\mathsf{PEs}} \mathbf{X}(G^*)$.

Proof. By Corollary 10 we have that

$$\mathbf{X}(H)\prod_{\mathsf{PHS}_0}\mathbf{X}(G)\cong\mathbf{X}(H\coprod_{\mathsf{PHil}_0}G)$$

in HS, so it follows from [6, Theorem 3.2] that

$$\mathbf{X}(H) \prod_{\mathsf{PHS}_0} \mathbf{X}(G) \cong \mathbf{X}(H \coprod_{\mathsf{PHil}_0} G) \text{ in Pos.}$$
(7)

Since H and G are finite then by Lemma 21 we have that $H \coprod_{\mathsf{PHil}} G$ is finite. Thus, it follows from Proposition 19 that

$$\mathbf{X}(H \coprod_{\mathsf{PHil}_0} G) \cong \mathbf{X}((H \coprod_{\mathsf{PHil}_0} G)^*) \text{ in Pos.}$$
(8)

By Theorem 3 we have that $(H \coprod_{\mathsf{PHil}_0} G)^* \cong H^* \coprod_{\mathsf{PHey}} G^*$ in PHey. Thus, we obtain that $\mathbf{X}((H \coprod_{\mathsf{PHil}_0} G)^*) \cong \mathbf{X}(H^*) \prod_{\mathsf{PEs}} \mathbf{X}(G^*)$ in PEs. In consequence,

$$\mathbf{X}((H \coprod_{\mathsf{PHil}_0} G)^*) \cong \mathbf{X}(H^*) \prod_{\mathsf{PEs}} \mathbf{X}(G^*) \text{ in Pos.}$$
(9)

Therefore, it follows from (7), (8) and (9) that there is an order isomorphism between $\mathbf{X}(H) \prod_{\mathsf{PHS}_0} \mathbf{X}(G)$ and $\mathbf{X}(H) \prod_{\mathsf{PEs}} \mathbf{X}(G)$.

Let H be a finite algebra of PHil_0 . Recall that \mathcal{K}_H denotes the associated base to the H-space $\mathbf{X}(H)$. We will prove that $\mathbf{X}(H^*) = \mathbf{X}(\mathbf{D}(\mathbf{X}(H)))$. In order to show it, first recall that it was proved in [6] that the map $\epsilon : \mathbf{X}(H) \to \mathbf{X}(\mathbf{D}(\mathbf{X}(H)))$ given by $\epsilon(x) = \{U \in \mathbf{D}(X) : x \in U\}$ is an order isomorphism. Then the map $\epsilon \circ \eta : \mathbf{X}(H^*) \to \mathbf{X}(\mathbf{D}(\mathbf{X}(H)))$ is an order isomorphism too, where $\eta : \mathbf{X}(H^*) \to \mathbf{X}(H)$ is the order isomorphism given in Proposition 19. Moreover,

$$\begin{aligned} \epsilon(\eta(P)) &= \{ U \in \mathbf{D}(\mathbf{X}(H)) : \eta(P) \in U \} \\ &= \{ \varphi(a) : \varphi^{-1}(P) \in \varphi(a) \} \\ &= \{ \varphi(a) : a \in \varphi^{-1}(P) \} \\ &= \{ \varphi(a) : \varphi(a) \in P \} \\ &= P. \end{aligned}$$

Thus, $\epsilon \circ \eta$ is the identity function. Hence,

$$\mathbf{X}(H^*) = \mathbf{X}(\mathbf{D}(\mathbf{X}(H))).$$

Since $(\mathbf{X}(\mathbf{D}(\mathbf{X}(H))), \subseteq, \mathcal{K}_{\mathbf{D}(\mathbf{X}(H))}) \in \mathsf{PHS}_0$ then $(\mathbf{X}(H^*), \subseteq, \mathcal{K}_{\mathbf{D}(\mathbf{X}(H))}) \in \mathsf{PHS}_0$. We denote by $\mathbf{X}(H^*)^{\dagger}$ to this object of PHS_0 .

Therefore, if H is a finite algebra of PHil_0 then $\mathbf{X}(H^*)^{\dagger} \in \mathsf{PHS}_0$. Let $\eta : \mathbf{X}(H^*) \to \mathbf{X}(H)$ be the order isomorphism given in Proposition 19. Note that the relation $\eta^R \subseteq \mathbf{X}(H^*) \times \mathbf{X}(H)$ is given by

$$(P,Q) \in \eta^R$$
 if and only if $\eta(P) \subseteq Q$.

Proposition 23. Let H and G be finite algebras of $PHil_0$. Then

- a) $U \in \mathcal{K}_H$ if and only if $\eta^{-1}(U) \in \mathcal{K}_{\mathbf{D}(\mathbf{X}(H))}$.
- b) η^R is an isomorphism in PHS₀.
- c) $\mathbf{X}(H) \prod_{\mathsf{PHS}_0} \mathbf{X}(G) \cong \mathbf{X}(H^*)^{\dagger} \prod_{\mathsf{PHS}_0} \mathbf{X}(G^*)^{\dagger}$ in PHS_0 .

Proof. a) Let $U \in \mathcal{K}_H$, so there exists $a \in H$ such that $U = \varphi_H(a)$. We will prove that

$$\eta^{-1}(U) = (\varphi_{\mathbf{D}(\mathbf{X}(H))}(\varphi_H(a)))^c.$$

Indeed,

$$\begin{split} \eta^{-1}(U) &= \eta^{-1}(\varphi_H(a)^c) \\ &= \{P \in \mathbf{X}(H^*) : \eta(P) \in \varphi_H(a)^c\} \\ &= \{P \in \mathbf{X}(H^*) : \varphi_H^{-1}(P) \in \varphi_H(a)^c\} \\ &= \{P \in \mathbf{X}(H^*) : a \notin \varphi_H^{-1}(P)\} \\ &= \{P \in \mathbf{X}(H^*) : \varphi_H(a) \notin P\} \\ &= \{P \in \mathbf{X}(\mathbf{D}(\mathbf{X}(H))) : \varphi_H(a) \notin P\} \\ &= (\varphi_{\mathbf{D}(\mathbf{X}(H))}(\varphi_H(a)))^c. \end{split}$$

Since $\eta^{-1}(U) = (\varphi_{\mathbf{D}(\mathbf{X}(H))}(\varphi_H(a)))^c$ then $\eta^{-1}(U) \in \mathcal{K}_{\mathbf{D}(\mathbf{X}(H))}$. Conversely, suppose that $\eta^{-1}(U) \in \mathcal{K}_{\mathbf{D}(\mathbf{X}(H))}$. Then there exists $a \in H$ such that $\eta^{-1}(U) = (\varphi_{\mathbf{D}(\mathbf{X}(H))}(\varphi_H(a)))^c$. Our aim is to show that $U = \varphi_H(a)^c$. First note that $P \in \mathbf{X}(H)$ if and only if there exists $Q_P \in \mathbf{X}(H^*)$ such that $P = \eta(Q_P)$. Thus, $P \in U$ if and only if $Q_P \in \eta^{-1}(U)$, i.e., $\varphi_H(a) \notin Q_P$, which is equivalent to $a \notin \varphi_H^{-1}(Q_P)$. But $\varphi_H^{-1}(Q_P) = \eta(Q_P) = P$. Hence, $P \in U$ if and only if $P \notin \varphi_H(a)$. Then $U = \varphi_H(a)^c$, so $U \in \mathcal{K}_H$.

b) It follows from item a) and [6, Lemma 3.2] that η^R is an *H*-relation from the *H*-space $\mathbf{X}(H^*)^{\dagger}$ to the *H*-space $\mathbf{X}(H)$. Moreover, since η is an order isomorphism it can be proved that η^R is an *H*-functional relation, so it is a morphism in HS. Finally, the fact that η^R is an isomorphism in HS follows again from item a) and [6, Theorem 3.2]. Therefore it follows from straightforward computations that η^R is an isomorphism in HS.

Lemma 24. Let (X,τ) be a finite topological space and \mathcal{K} a family of subsets of Χ.

- a) If (X, τ, \mathcal{K}) is an H-space then τ is the set of downsets of (X, \leq_d) .
- b) (X,τ,\mathcal{K}) is an H-space if and only if (X,τ) is T_0 and \mathcal{K} is a base of (X,τ) such that $\operatorname{sat}(B_1 \cap B_2^c) \in \mathcal{K}$ for every $B_1, B_2 \in \mathcal{K}$.

Proof. a) We know that there is an order isomorphism and a homemorphism between (X, τ, \mathcal{K}) and $\mathbf{X}(\mathbf{D}(X))$. Then it is enough to prove that given a Hilbert algebra H we have that U is an open in $\mathbf{X}(H)$ if and only if U is a downset in $\mathbf{X}(H)$. Let U be a downset in $\mathbf{X}(H)$. Then $U^c \in \mathbf{X}(H)^+$. Thus it follows from (4) that U^c is a closed set of $\mathbf{X}(H)$, i.e., U is an open of $\mathbf{X}(H)$. It is immediate the fact that if U is an open in $\mathbf{X}(H)$ then U is a downset in $\mathbf{X}(H)$.

b) Suppose that \mathcal{K} is a base of (X, τ) such that $\operatorname{sat}(B_1 \cap B_2^c) \in \mathcal{K}$ for every $B_1, B_2 \in \mathcal{K}$. Since X is finite then the elements of \mathcal{K} are compact. Hence, \mathcal{K} is a base of open compact sets of (X, τ) . Since X is finite and (X, τ) is T_0 then (X, τ) is sober.

Let $(X, \tau_X, \mathcal{K}_X)$ be a finite *H*-space and (Y, \leq) a finite poset. Suppose that there exists an order isomorphism $i: (X, \leq_d) \to (Y, \leq)$. Define the family $\mathcal{K}_Y :=$ $\{i(B): B \in \mathcal{K}_X\}$, where $i(B) := \{i(x): x \in B\}$. Also write τ_Y for the family of downsets of (Y, \leq) .

Lemma 25. The structure $(Y, \tau_Y, \mathcal{K}_Y)$ is an H-space and $(X, \tau_X, \mathcal{K}_X) \cong (Y, \tau_Y, \mathcal{K}_Y)$ in HS. Moreover, if $(X, \tau_X, \mathcal{K}_X) \in \mathsf{PHS}_0$ then $(Y, \tau_Y, \mathcal{K}_Y) \in \mathsf{PHS}_0$ and $(X, \tau_X, \mathcal{K}_X) \cong (Y, \tau_Y, \mathcal{K}_Y)$ in PHS_0 .

Proof. By a) of Lemma 30 we have that τ is the set of downsets of (X, \leq_d) . Thus, straightforward computations based in b) of Lemma 30 and the fact that $i: (X, \leq_d) \to (Y, \leq)$ is an order isomorphism show that $(Y, \tau_Y, \mathcal{K}_Y)$ is an *H*-space.

Now we will prove that $(X, \tau_X, \mathcal{K}_X) \cong (Y, \tau_Y, \mathcal{K}_Y)$ in HS. Note that if $B \in \mathcal{K}_X$ then $i^{-1}(i(B)) = B$. Hence, $i(B) \in \mathcal{K}_Y$ if and only if $i^{-1}(i(B)) \in \mathcal{K}_X$. By [6, Lemma 3.2] we have that i^R is an *H*-relation, so by [6, Theorem 3.2] we have that $(X, \tau_X, \mathcal{K}_X) \cong (Y, \tau_Y, \mathcal{K}_Y)$ in the category of *H*-spaces and *H*-relations. Since *i* is an order isomorphism straightforward computations prove that $(X, \tau_X, \mathcal{K}_X) \cong (Y, \tau_Y, \mathcal{K}_Y)$ in HS. The rest of the proof follows from straightforward computations.

Theorem 4. Let H, G finite algebras in PHil_0 . Then there exists τ and \mathcal{K} families of subsets of $\mathbf{X}(H^*) \prod_{\mathsf{PEs}} \mathbf{X}(G^*)$ such that

$$P(X,Y) := (\mathbf{X}(H^*) \prod_{\text{PEs}} \mathbf{X}(G^*), \tau, \mathcal{K}) \in \mathsf{PHS}_0$$

and $\mathbf{X}(H) \prod_{\mathsf{PHS}_0} \mathbf{X}(G) \cong P(X,Y)$ in PHS_0 .

Proof. It follows from lemmas 22 and 25.

5 The product in the category F_{fin} of finite forests and open maps

A (finite) forest is a (finite) poset F such that for every $x \in F$ the set (x] is a chain. An order preserving map $f: F \to G$ between forests is open if for every $x \in F$ and $y \in G$, if $y \leq f(x)$ then there exists $z \in X$ such that $z \leq x$ and f(z) = y. Notice that an order preserving map $f: F \to G$ between forests is open if for every $x \in X$ it holds that f((x)) = (f(x)). In this case we also have that if x is minimal in F then f(x) is minimal in G.

We write $F_{\rm fin}$ for the category of finite forests and order-preserving open maps between them. In this section we give an explicit description of the product of two objects in the category $F_{\rm fin}$.

Definition 6. Let $F \in F_{fin}$. An u-succession f is a set $\{f_0, \ldots, f_n\}$ of elements of F such that f_0 is minimal in F and $f_0 < \cdots < f_n$. We write US(F) for the set of u-successions of F.

Remark 6. Let $F \in F_{\text{fin}}$. Let f_0, \ldots, f_n elements of F such that f_0 is minimal in F and $f_0 \leq f_1 \leq \cdots \leq f_n$. In this case we also say that the set $\{f_0, \ldots, f_n\}$ is an u-succession because $\{f_0, \ldots, f_n\} = \{g_0, \ldots, g_m\}$ for some $g_0, \ldots, g_m \in F$ with $g_0 = f_0$ and $g_0 < g_1 < \cdots < g_m$. Let $F \in F_{\text{fin}}$. Let $f = \{f_0, \ldots, f_n\}$ and $g = \{g_0, \ldots, g_m\}$ elements of US(F). We say that $f \leq g$ if and only $n \leq m$ and $f_i = g_i$ for every $i = 1, \ldots, n$. It is immediate that \leq is an order. Moreover, $(US(F), \leq) \in F_{\text{fin}}$. If there is not ambiguity we write US(F) in place of $(US(F), \leq)$. Note that the order \leq defined on US(F) is exactly the same order defined on the set of paths of an arbitrary poset. See [1] for details about it.

Let $S, T \in F_{\text{fin}}$. We define $\pi_1 : S \times T \to S$ by $\pi_1(s, t) = s$ and $\pi_2 : S \times T \to S$ by $\pi_2(s, t) = t$. Let $p \in \text{US}(S \times T)$, i.e., $p = \{p_0, \ldots, p_k\}$ where $p_0, \ldots, p_k \in S \times T$, p_0 is minimal in $S \times T$ and $p_0 < \cdots < p_k$. We also define maps $\hat{\pi}_1 : \text{US}(S \times T) \to \text{U}(S)$ and $\hat{\pi}_2 : \text{US}(S \times T) \to \text{U}(T)$ by $\hat{\pi}_1(p) = \{\pi_1(p_0), \ldots, \pi_1(p_k)\}$ and $\hat{\pi}_2(p) = \{\pi_2(p_0), \ldots, \pi_2(p_k)\}$. It is clear that $\hat{\pi}_1$ and $\hat{\pi}_2$ are well defined maps. Finally we define the set

$$S \otimes T = \{ p \in \mathrm{US}(S \times T) : \hat{\pi}_1(p) = (\pi_1(p_k)] \text{ and } \hat{\pi}_2(p) = (\pi_2(p_k)) \}.$$

Lemma 26. Let $S, T \in F_{\text{fin}}$. The maps $\overline{\pi_1} : S \otimes T \to S$ and $\overline{\pi_2} : S \otimes T \to T$ given by $\overline{\pi_1}(p) = \max \pi_1(p)$ and $\overline{\pi_2}(p) = \max \pi_2(p)$ are morphisms in F_{fin} .

Proof. It is immediate that $\overline{\pi_1}$ preserves the order. In order to prove that $\overline{\pi_1}$ is open, let $s \in S$ and $p \in S \otimes T$ such that $s \leq \overline{\pi_1}(p)$. Since $p \in S \otimes T$ then $p = \{p_0, \ldots, p_k\}$ where $p_0, \ldots, p_k \in S \times T$, p_0 is minimal in $S \times T$, $p_0 < \cdots < p_k$ and $\widehat{\pi_1}(p) = (\pi_1(p_k)]$. Thus, $s \in (\pi_1(p_k)]$, so there exists $i = 0, \ldots, k$ such that $s = \pi_1(p_i)$. Let $q = \{p_0, \ldots, p_i\}$. We have that $q \in S \otimes T$, $q \leq p$ and $\overline{\pi_1}(q) = s$. Hence, $\overline{\pi_1}$ is a morphism in F_{fin} . In a similar way it can be proved that $\overline{\pi_2}$ is a morphism in F_{fin} .

Lemma 27. Let $\alpha: F \to S$ and $\beta: F \to T$ be morphisms in F_{fin} . Then the map $h: F \to S \otimes T$ given by $h(f) = \{(\alpha(g), \beta(g)) : g \leq f\}$ is a morphism in F_{fin} such that $\overline{\pi_1} \circ h = \alpha$ and $\overline{\pi_2} \circ h = \beta$.

Proof. First we will prove the well definition of h. Let $f \in F$. Since F is finite and (f] is a forest then there exists $f_0, \ldots, f_n \in F$ such that f_0 is minimal in F and $f_0 < f_1 < \cdots < f_n = f$. Thus,

$$h(f) = \{ (\alpha(f_0), \beta(f_0)), \dots, (\alpha(f_n), \beta(f_n)) \}.$$

Since $\alpha(f_0)$ is minimal in S and $\beta(f_0)$ is minimal in T then $(\alpha(f_0), \beta(f_0))$ is minimal in $S \times T$. Moreover, since α and β are order-preserving maps then $(\alpha(f_i), \beta(f_i)) \leq (\alpha(f_{i+1}), \beta(f_{i+1}))$ for every $i = 0, \ldots, n$. Hence, $h(f) \in \text{US}(S \times T)$. Besides, $\pi_1(h(f)) = \{\alpha(f_0), \ldots, \alpha(f_n)\} = (\alpha(f_n)]$ because $\alpha((f_n)) = (\alpha(f_n))$, so $h(f) \in S \otimes T$. Then h is a well defined map.

The fact that h preserves the order follows from that α and β preserve the order. Now we will show that h is a morphism in F_{fin} . Let $f \in F$ and $p \in S \otimes T$ such that $p \leq h(f)$. Then there exist $k \leq n$ such that $p = \{(\alpha(f_0), \beta(f_0)), \ldots, (\alpha(f_k), \beta(f_k))\}$. Hence, $f_k \leq f$ and $h(f_k) = p$. Thus, h is a morphism in F_{fin} . It is immediate that $\overline{\pi_1} \circ h = \alpha$ and $\overline{\pi_2} \circ h = \beta$. **Lemma 28.** Let $\alpha : F \to S$, $\beta : F \to T$ and $g : F \to S \otimes T$ morphisms in F_{fin} such that $\overline{\pi_1} \circ g = \alpha$ and $\overline{\pi_2} \circ g = \beta$. Then g = h.

Proof. Let $f \in F$. Since F is finite and (f] is a forest then there exists $f_0, \ldots, f_n \in F$ such that f_0 is minimal in F and $f_0 < f_1 < \cdots < f_n = f$. Thus,

$$h(f) = \{ (\alpha(f_0), \beta(f_0)), \dots, (\alpha(f_n), \beta(f_n)) \}.$$

Besides, there exists $(s_0, t_0) < \cdots < (s_k, t_k) \in S \times T$ such that (s_0, t_0) is minimal in $S \times T$, $(s_0, t_0) < (s_1, t_1) < \cdots < (s_k, t_k)$ and

$$g(f) = \{(s_0, t_0), \dots, (s_k, t_k)\}.$$

We will prove that g(f) = h(f). Consider (s_i, t_i) for some $i = 0, \ldots, k$ and $p = \{(s_0, t_0), \ldots, (s_k, t_k)\}$. Then $p \in S \otimes T$ and $p \preceq g(f)$. Thus, there exists $f' \leq f$ such that g(f') = p, so there is $j = 0, \ldots, n$ such that $f' = f_j$. In consequence we obtain that $\alpha(f_j) = \overline{\pi_1}(g(f_j)) = s_i$ and $\beta(f_j) = \overline{\pi_2}(g(f_j)) = t_i$, so $(s_i, t_i) = (\alpha(f_j), \beta(f_j)) \in h(f)$. Then, $g(f) \subseteq h(f)$. Conversely, consider $(\alpha(f_i), \beta(f_i))$ for some $i = 0, \ldots, n$. Since $f_i \leq f$ then $g(f_i) \preceq g(f)$. Thus, there exists $j = 0, \ldots, k$ such that $g(f_i) = \{(s_0, t_0), \ldots, (s_j, t_j)\}$. Since $\alpha(f_i) = \overline{\pi_1}(g(f_i)) = s_j$ and $\beta(f_i) = \overline{\pi_2}(g(f_i)) = t_j$, we conclude that $(\alpha(f_i), \beta(f_i)) = (s_j, t_j) \in g(f)$. Then, $h(f) \subseteq g(f)$. Therefore, g(f) = h(f).

Proposition 29. Let $S, T \in F_{fin}$. Then $S \otimes T$ is the product between S and T in F_{fin} .

6 The product in hFor

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In this section we use results of Section 5 in order to give an explicit description of the product of two objects in certain category of finite forests F endowed with a distinguished family of subsets of F, which is equivalent to the category of finite bounded prelinear Hilbert algebras. This property allow us to obtain an explicit description of the coproduct of two finite algebras in PHil₀. We also give an explicit description of the coproduct of two finite algebras in PHil₀. We also give an explicit in PHil₀.

We start with the following lemma.

Lemma 30. Let (X, τ) be a finite topological space and \mathcal{K} a base for the topology τ on X. Then (X, τ, \mathcal{K}) is an H-space if and only if (X, \leq_d) is a poset, τ is the set of downsets of (X, \leq_d) and $(B_1 \cap B_2^c] \in \mathcal{K}$ for every $B_1, B_2 \in \mathcal{K}$.

Proof. Let (X, τ, \mathcal{K}) be an *H*-space. It follows from Lemma 30 that (X, \leq_d) is a poset, τ is the set of downsets of (X, \leq_d) and $(B_1 \cap B_2^c] \in \mathcal{K}$ for every $B_1, B_2 \in \mathcal{K}$.

Conversely, suppose that (X, \leq_d) is a poset, τ is the set of downsets of (X, \leq_d) and $(B_1 \cap B_2^c] \in \mathcal{K}$ for every $B_1, B_2 \in \mathcal{K}$. We will prove that (X, τ) is T_0 . Let $x, y \in X$ such that $x \nleq_d y$. Since (X, \leq_d) is a poset we can assume that $x \nleq_d y$, so $y \notin \overline{\{x\}}$. Thus, there is $U \in \tau$ such that $y \in U$ and $U \cap \{x\} = \emptyset$, i.e., $x \notin U$. Hence, (X, τ) is T_0 . Since X is finite then (X, τ) is sober. Therefore, (X, τ, \mathcal{K}) is an *H*-space.

Remark 7. Let X be a finite set. Then it follows from Lemma 30 that (X, τ, κ) is an H-space if and only if (X, \leq) is a poset, τ is the set of upsets of (X, \leq) and κ is a base of (X, τ) such that $[B_1 \cap B_2^c) \in \kappa$ for every $B_1, B_2 \in \kappa$.

If (X, \leq) is a poset and $U \subseteq X$, we write U^m for the set of minimal elements of U.

Definition 7. Let F be a finite forest. An h-base for F is a family \mathcal{B} of upsets of F such that satisfies the following conditions:

- 1) $[x) \in \mathcal{B}$ for every $x \in F$,
- 2) if $B \in \mathcal{B}$ and $M \subseteq B^m$ then $[M] \in \mathcal{B}$.

Notice that if F be a finite forest and \mathcal{B} is an h-base of F then $\emptyset \in \mathcal{B}$.

Lemma 31. Let F be a finite forest and B a family of upsets of F. Then the following conditions are equivalent:

- a) \mathcal{B} is an h-base.
- b) \mathcal{B} is a base of F with the topology given by the upsets of the poset F such that $[B_1 \cap B_2^c) \in \mathcal{B}$ for every $B_1, B_2 \in \mathcal{B}$.

Proof. Assume the condition a).

First we will prove that \mathcal{B} is a base of F. For every $x \in X$ it is immediate that $x \in [x)$ and $[x) \in \mathcal{B}$. Let now $x \in B_1 \cap B_2$ with $B_1, B_2 \in \mathcal{B}$. Then $x \in [x) \subseteq B_1 \cap B_2$ and $[x) \in \mathcal{B}$. Thus, \mathcal{B} is a base of F. We also have that the topology generated by \mathcal{B} is equal to the set of the upsets of F. In order to show it, let U be in the topology generated by \mathcal{B} . Since the elements of \mathcal{B} are upsets then U is an upset. Conversely, let U be an upset. Then $U = \bigcup_{x \in U} [x)$. Since $[x) \in \mathcal{B}$ then U is in the topology generated by \mathcal{B} .

Let $B_1, B_2 \in \mathcal{B}$. In what follows we will see that $[B_1 \cap B_2^c) \in \mathcal{B}$. Notice that $B_1^m \cap B_2^c \subseteq B_1^m$. Since $B_1 \in \mathcal{B}$ then $[B_1^m \cap B_2^c) \in \mathcal{B}$. Motivated by this fact, in order to prove that $[B_1 \cap B_2^c) \in \mathcal{B}$ we will see that $[B_1^m \cap B_2^c) = [B_1 \cap B_2^c)$. The inclusion $[B_1^m \cap B_2^c) \subseteq [B_1 \cap B_2^c)$ is immediate. Conversely, let $x \in [B_1 \cap B_2^c)$. Hence, there exists $y \in B_1 \cap B_2^c$ such that $y \leq x$. Consider $z \in B_1^m$ such that $z \leq y$. Since $y \notin B_2 = [B_2)$ then $z \notin B_2$, so $z \in B_1^m \cap B_2^c$ and $z \leq x$. Thus, $x \in [B_1^m \cap B_2^c)$. Then $[B_1 \cap B_2^c) \subseteq [B_1^m \cap B_2^c)$, so $[B_1^m \cap B_2^c) = [B_1 \cap B_2^c)$, which was our aim.

Finally we have that Remark 7 and [6, Lemma 4.1] proves that the condition b) implies the condition a). $\hfill \Box$

Let ChFor be the category whose objects are structures (X, \leq, \mathcal{B}_X) such that (X, \leq) is a finite forest, $X \in \mathcal{B}_X$ and \mathcal{B}_X is an h-base for (X, \leq) . The morphisms in ChFor are defined as binary relations $R : (X, \leq, \mathcal{B}_X) \to (Y, \leq, \mathcal{B}_Y)$ which satisfy the following conditions:

- 1. R(x) is a non empty downset for every $x \in X$,
- 2. if $(x, y) \in R$ then there exists $z \in X$ such that $z \leq x$ and R(z) = (y),
- 3. $R^{-1}(U) \in \mathcal{B}_X$ for every $U \in \mathcal{B}_Y$.

The identity in ChFor is the binary relation \geq .

Let $fPHil_0$ be the full subcategory of $PHil_0$ whose objects are finite. In Corollary 10 it was proved that there exists a categorical equivalence between $PHil_0$ and PHS_0 . The explicit construction of this equivalence allow us to set that there exists an equivalence between $fPHil_0$ and the full subcategory of PHS_0 whose objects are also finite. By the results of this section we have that the last mentioned category is isomorphic to ChFor. Therefore, there exists a categorical equivalence between $fPHil_0$ and ChFor.

Remark 8. Let $R : (X, \leq, \mathcal{B}_X) \to (Y, \leq, \mathcal{B}_Y)$ be a morphism in ChFor and $x, y \in X$ such that $x \leq y$. Then $R(x) \subseteq R(y)$. In order to see it, let $z \in R(x)$, i.e., $(x,y) \in R$. Besides $y \geq x$, so $(y,z) \in (\geq \circ R)$. It follows from Remark 7 and [6, Theorem 3.1] that $\geq \circ R = R$, so $(y,z) \in R$, i.e., $z \in R(y)$. Therefore, $R(x) \subseteq R(y)$.

Lemma 32. Let (X, \leq, \mathcal{B}_X) and (Y, \leq, \mathcal{B}_Y) objects in the category ChFor.

Suppose that $f : (X, \leq) \to (Y, \leq)$ is an open map such that $f^{-1}(U) \in \mathcal{B}_X$ for every $U \in \mathcal{B}_Y$. Then the binary relation $R(f) : (X, \leq, \mathcal{B}_X) \to (Y, \leq, \mathcal{B}_Y)$ given by

 $(x, y) \in R(f)$ if and only if $y \leq f(x)$

is a morphism in ChFor.

Conversely, suppose that $R: (X, \leq, \mathcal{B}_X) \to (Y, \leq, \mathcal{B}_Y)$ is a morphism in ChFor. Then for every $x \in X$ there exists the maximum of R(x). Moreover, the map $f_R: (X, \leq) \to (Y, \leq)$ is an open map such that $f_R^{-1}(U) \in \mathcal{B}_X$ for every $U \in \mathcal{B}_Y$.

Proof. Suppose that $f: (X, \leq) \to (Y, \leq)$ is an open map such that $f^{-1}(U) \in \mathcal{B}_X$ for every $U \in \mathcal{B}_Y$. Let $x \in X$. Then R(f)(x) = (f(x)], so we have that R(f)(x) is a downset. Since $f(x) \in R(f)(x)$ then $R(f)(x) \neq \emptyset$. Now consider $(x, y) \in R(f)$, i.e., $y \leq f(x)$. Taking into account that f is an open map we have that there exists $z \leq x$ such that f(z) = y, so R(z) = (y]. Let $U \in \mathcal{B}_Y$. Straightforward computations show that $R(f)^{-1}(U) = f^{-1}(U)$. Since $f^{-1}(U) \in \mathcal{B}_X$ then $R(f)^{-1}(U) \in \mathcal{B}_X$.

Conversely, suppose that $R: (X, \leq, \mathcal{B}_X) \to (Y, \leq, \mathcal{B}_Y)$ is a morphism in ChFor. Let $x \in X$. Since $R(x) \neq \emptyset$ and R(x) is finite then the set of maximal elements of R(x) is non empty. In what follows we will see that R(x) has a maximum element. Let y_1, y_2 maximal elements in R(x). Since $(x, y_1), (x, y_2) \in R$ then there exists $z_1, z_2 \in X$ such that $z_1 \leq x, z_2 \leq x, R(z_1) = (y_1]$ and $R(z_2) = (y_2]$. Taking into account that X is a forest we deduce that $z_1 \leq z_2$ or $z_2 \leq z_1$. We can assume that $z_1 \leq z_2$. It follows from Remark 8 that $R(z_1) \subseteq R(z_2)$. Since $y_1 \in R(z_1) \subseteq R(z_2) = (y_2]$ we have that $y_1 \leq y_2$, so $y_1 = y_2$. Hence, the set R(x)has a maximum element.

Now we will prove that the map $f_R : (X, \leq) \to (Y, \leq)$ is an open map such that $f_R^{-1}(U) \in \mathcal{B}_X$ for every $U \in \mathcal{B}_Y$. Let $x \leq y$. By Remark 8 we have that $R(x) \subseteq R(y)$, so $f_R(x) \leq f_R(y)$. Hence, f_R is a monotone map. In order to see that f_R is an open map, let $y \leq f_R(x)$, i.e., $(x, y) \in R$. Then there exists $z \in X$ such that $z \leq x$ and R(z) = (y]. Thus, $f_R(z) = y$. Thus, f_R is an open map. Finally, let $U \in \mathcal{B}_Y$. Straightforward computations show that $f_R^{-1}(U) = R^{-1}(U)$. Since $R^{-1}(U) \in \mathcal{B}_X$ then $f_R^{-1}(U) \in \mathcal{B}_X$, which was our aim.

Corollary 33. Let (X, \leq, \mathcal{B}_X) and (Y, \leq, \mathcal{B}_Y) be objects in ChFor. Then there exists a bijection between the set of open maps from (X, \leq) to (Y, \leq) which satisfy that $f^{-1}(U) \in \mathcal{B}_X$ for every $U \in \mathcal{B}_Y$ and the set of morphisms of ChFor from (X, \leq, \mathcal{B}_X) to (Y, \leq, \mathcal{B}_Y) .

Proof. Let Γ be the set of open maps from (X, \leq) to (Y, \leq) which satisfy that $f^{-1}(U) \in \mathcal{B}_X$ for every $U \in \mathcal{B}_Y$ and let Δ be the set of morphisms of ChFor from (X, \leq, \mathcal{B}_X) to (Y, \leq, \mathcal{B}_Y) . Let $F: \Gamma \to \Delta$ given by F(f) = R(f). It follows from Lemma 32 that it is a well defined map. The injectivity of F is immediate. In order to prove that F is a surjective map, let $R \in \Delta$. It follows from Lemma 32 that $f_R \in \Gamma$. We will prove that $F(f_R) = R$, i.e., $R(f_R) = R$. Let $(x, y) \in R$, i.e., $y \in R(x)$. In particular, $y \leq f_R(x)$. Thus, $y \in R(f_R)(x)$, so $(x, y) \in R(f_R)$. Then we have proved that $R \subseteq R(f_R)$. Conversely, let $(x, y) \in R(f_R)$, i.e., $y \leq f_R(x)$. Suppose that $y \notin R(x)$. Since $R(x)^c$ is an upset we have that $R(x)^c = \bigcup_{i=1}^n U_i$, for some $U_1, \ldots, U_n \in \mathcal{B}_Y$. So there exists $i = 1, \ldots, n$ such that $y \in U_i$. It is immediate that $R(x) \subseteq U_i^c$, i.e., that $x \notin R^{-1}(U_i)$. However, $R^{-1}(U_i) = f_R^{-1}(U_i)$, so $f_R(x) \notin U_i$. But $y \in U_i$ and $y \leq f_R(x)$, so since U_i is an upset we deduce that $f_R(x) \in U_i$, which is a contradiction. Then $y \in R(x)$. Thus, $R(f_R) \subseteq R$.

Let hFor the category whose objects are the objects of ChFor and whose morphisms are open maps $f : (X, \leq, \mathcal{B}_X) \to (Y, \leq, \mathcal{B}_Y)$ such that $f^{-1}(U) \in \mathcal{B}_X$ for every $U \in \mathcal{B}_Y$.

Lemma 34. Let $(X, \leq, \mathcal{B}_X), (Y, \leq, \mathcal{B}_Y), (Z, \leq, \mathcal{B}_Z)$ be objects of ChFor. If $f : (X, \leq, \mathcal{B}_X) \to (Y, \leq, \mathcal{B}_Y)$ and $g : (Y, \leq, \mathcal{B}_Y) \to (Z, \leq, \mathcal{B}_Z)$ are morphisms in hFor then $R(g \circ f) = R(g) \circ R(f)$. Conversely, if $R : (X, \leq, \mathcal{B}_X) \to (Y, \leq, \mathcal{B}_Y)$ and $S : (Y, \leq, \mathcal{B}_Y) \to (Z, \leq, \mathcal{B}_Z)$ are morphisms in hFor then $f_{S \circ R} = f_S \circ f_R$.

Proof. Let $f: (X, \leq, \mathcal{B}_X) \to (Y, \leq, \mathcal{B}_Y)$ and $g: (Y, \leq, \mathcal{B}_Y) \to (Z, \leq, \mathcal{B}_Z)$ be morphisms in hFor. Let $y \in R(g \circ f)(x)$, i.e., $y \leq g(f(x))$. Since $f(x) \leq f(x)$ we have

that $y \in [R(g) \circ R(f)](x)$. Let $y \in [R(g) \circ R(f)](x)$, i.e., $(x, y) \in R(g) \circ R(f)$. Then there exists z such that $z \leq g(x)$ and $y \leq f(z)$. Thus, $y \leq f(z) \leq f(g(x))$, so $y \leq f(g(x))$. Hence, $(x, y) \in R(g \circ f)$. Then $R(g \circ f) = R(g) \circ R(f)$.

Let $R : (X, \leq, \mathcal{B}_X) \to (Y, \leq, \mathcal{B}_Y)$ and $S : (Y, \leq, \mathcal{B}_Y) \to (Z, \leq, \mathcal{B}_Z)$ be morphisms in hFor. Let $x \in X$. Then $f_{S \circ R}(x) = \max(S \circ R)(x)$ and $(f_S \circ f_R)(x) = \max(S(\max R(x)))$. We have that $y \leq \max(x)$ for every $y \in R(x)$, so $S(y) \subseteq S(\max R(x))$ for every $y \in R(x)$. Thus, $\max(S(y)) \leq \max(S(\max R(x)))$ for every $y \in R(x)$, so $\max(S \circ R)(x) \leq \max(S(\max R(x)))$. On the other hand, $\max(x) \in R(x)$, so $S(\max R(x)) \subseteq (S \circ R)(x)$. Then, $\max(S(\max R(x))) \leq \max(S \circ R)(x)$. Therefore, $\max(S \circ R)(x) \leq \max(S(\max R(x)))$.

Straightforward computations based in Corollary 33 and Lemma 34 proves the following result.

Proposition 35. The categories ChFor and hFor are isomorphic. In particular, there exists a categorical equivalence between hFor and $fPHil_0$.

The following definition will be useful to obtain a description of the product in hFor.

Definition 8. Let (X, \leq, \mathcal{B}_X) and (Y, \leq, \mathcal{B}_Y) be objects in hFor. Consider the tern $(X \otimes Y, \preceq, \mathcal{B}_X \otimes \mathcal{B}_Y)$, where $\mathcal{B}_X \otimes \mathcal{B}_Y$ is defined as follows:

 $U \in \mathcal{B}_X \otimes \mathcal{B}_Y$ if and only if some of the following conditions are satisfied:

- 1. U = [u] for some $u \in X \otimes Y$.
- 2. U = [T] for some $T \subseteq [(\overline{\pi_1})^{-1}(V)]^m$ with $V \in \mathcal{B}_X$.
- 3. U = [T] for some $T \subseteq [(\overline{\pi_2})^{-1}(V)]^m$ with $V \in \mathcal{B}_Y$.

Our aim is to prove that $(X \otimes Y, \preceq, \mathcal{B}_X \otimes \mathcal{B}_Y)$ is the product in hFor. In order to make it possible we will use Proposition 29.

Let $(X, \leq, \mathcal{B}_X), (Y, \leq, \mathcal{B}_Y) \in h$ For. Note that by definition $\mathcal{B}_X \otimes \mathcal{B}_Y$ is a family of upsets of $(X \otimes Y, \preceq)$. Besides, $X \in \mathcal{B}_X$ and $(\overline{\pi_1})^{-1}(X) = X \otimes Y$. Let Mbe the set of minimal elements of $X \otimes Y$. Since $X \otimes Y = [M)$ we have that $X \otimes Y \in \mathcal{B}_X \otimes \mathcal{B}_Y$.

In order to prove that $\mathcal{B}_X \otimes \mathcal{B}_Y$ is an *h*-base of $(X \otimes Y, \preceq)$, let $u \in X \otimes Y$. Then $[u) \in \mathcal{B}_X \otimes \mathcal{B}_Y$. Let $U \in \mathcal{B}_X \otimes \mathcal{B}_Y$ and $M \subseteq U^m$. We need to prove that $[M) \in \mathcal{B}_X \otimes \mathcal{B}_Y$. If $M = \emptyset$ then $[M) = M \in \mathcal{B}_X \otimes \mathcal{B}_Y$ by definition of $\mathcal{B}_X \otimes \mathcal{B}_Y$. Suppose that $M \neq \emptyset$. If U = [u) for some u then [M) = U, so $[M) \in \mathcal{B}_X \otimes \mathcal{B}_Y$. Assume that U = [T), where $T \subseteq [(\overline{\pi_1})^{-1}(V)]^m$ and $V \in \mathcal{B}_X$. Since $M \subseteq U^m \subseteq T$ then $M \subseteq [(\overline{\pi_1})^{-1}(V)]^m$. Hence, $[M) \in \mathcal{B}_X \otimes \mathcal{B}_Y$. The case U = [T), where $T \subseteq [(\overline{\pi_1})^{-1}(V)]^m$ and $V \in \mathcal{B}_Y$ is similar. So, $\mathcal{B}_X \otimes \mathcal{B}_Y$ is an *h*-base of $X \otimes Y$. Therefore, $(X \otimes Y, \preceq, \mathcal{B}_X \otimes \mathcal{B}_Y) \in h$ For.

Lemma 36. Let (X, \leq, \mathcal{B}_X) and (Y, \leq, \mathcal{B}_Y) be objects in hFor. Then the maps $\overline{\pi_1}: (X \otimes Y, \leq, \mathcal{B}_X \otimes \mathcal{B}_Y) \to (X, \leq, \mathcal{B}_X)$ and $\overline{\pi_2}: (X \otimes Y, \leq, \mathcal{B}_X \otimes \mathcal{B}_Y) \to (Y, \leq, \mathcal{B}_Y)$ are morphisms in hFor.

Proof. We know that $\overline{\pi_1}$ and $\overline{\pi_2}$ are open maps. Let $U \in \mathcal{B}_X$ and let $M = [(\overline{\pi_1})^{-1}(U)]^m$. In particular, $M \subseteq [(\overline{\pi_1})^{-1}(U)]^m$ and $[M) = (\overline{\pi_1})^{-1}(U)$ because $(\overline{\pi_1})^{-1}(U)$ is an upset. Since $[M) \in \mathcal{B}_X \otimes \mathcal{B}_Y$ we have that $(\overline{\pi_1})^{-1}(U) \in \mathcal{B}_X \otimes \mathcal{B}_Y$. In the same way can be proved that $(\overline{\pi_2})^{-1}(U) \in \mathcal{B}_X \otimes \mathcal{B}_Y$. Thus, $\overline{\pi_1}$ and $\overline{\pi_2}$ are morphisms in hFor.

Let $(X, \leq, \mathcal{B}_X), (Y, \leq, \mathcal{B}_Y) \in h$ For. Let $h: (Z, \leq, \mathcal{B}_Z) \to (X \otimes Y, \leq, \mathcal{B}_X \otimes \mathcal{B}_Y)$ be the map given in Lemma 27. Consider $\alpha: (Z, \leq, \mathcal{B}_Z) \to (X, \leq, \mathcal{B}_X)$ and $\beta: (Z, \leq, \mathcal{B}_Z) \to (Y, \leq, \mathcal{B}_Y)$ morphisms in hFor such that $\overline{\pi_1} \circ h = \alpha$ and $\overline{\pi_2} \circ h = \beta$. Let $u \in X \otimes Y$. The following technical lemma will be used later.

Lemma 37. Then $[h^{-1}(u)]^m \subseteq [\alpha^{-1}(\overline{\pi_1}(u))]^m$ or $[h^{-1}(u)]^m \subseteq [\beta^{-1}(\overline{\pi_2}(u))]^m$.

Proof. Let $z, w \in [h^{-1}(u)]^m$. It is enough to prove that $z, w \in [\alpha^{-1}(\overline{\pi_1}(u))]^m$ or $z, w \in [\beta^{-1}(\overline{\pi_2}(u))]^m$. Since $z, w \in h^{-1}(u)$ then h(z) = h(w) = u, so $\overline{\pi_1}(u) = \alpha(z) = \alpha(w)$ and $\overline{\pi_2}(u) = \beta(z) = \beta(w)$. Thus, $z, w \in \alpha^{-1}(\overline{\pi_1}(u))$ and $z, w \in \beta^{-1}(\overline{\pi_2}(u))$. We will prove that $z, w \in [\alpha^{-1}(\overline{\pi_1}(u))]^m$ or $z, w \in [\beta^{-1}(\overline{\pi_2}(u))]^m$.

Let $(z] = \{z_0, z_1, \ldots, z_n\}$ and $(w] = \{w_0, w_1, \ldots, w_k\}$, where $z_0 < z_1 < \ldots < z_n = z$ and $w_0 < w_1 < \ldots < w_k = w$. By definition of h we have that $h(z) = \{(\alpha(z_0), \beta(z_0)), \ldots, (\alpha(z_n), \beta(z_n))\}$. Notice that $(\alpha(z_0), \beta(z_0)) \preceq \ldots \preceq (\alpha(z_{n-1}), \beta(z_{n-1}) \preceq (\alpha(z_n), \beta(z_n))\}$. Suppose $(\alpha(z_{n-1}), \beta(z_{n-1})) = (\alpha(z_n), \beta(z_n))$. Thus, $h(z_{n-1}) = h(z_n) = u$. But $z_{n-1} < z_n, z_{n-1} \in h^{-1}(u)$ and $z_n \in [h^{-1}(u)]^m$, which is a contradiction. Hence, $(\alpha(z_{n-1}), \beta(z_{n-1})) \prec (\alpha(z_n), \beta(z_n))$. Analogously it can be proved that $(\alpha(w_{k-1}), \beta(w_{k-1})) \prec (\alpha(w_k), \beta(w_k))$. We write (P) for the properties $(\alpha(z_{n-1}), \beta(z_{n-1})) \prec (\alpha(z_n), \beta(z_n))$ and $(\alpha(w_{k-1}), \beta(w_{k-1})) \prec (\alpha(w_k), \beta(w_k))$. Also note that $\alpha(z) = \alpha(w)$ and $\beta(z) = \beta(w)$.

Suppose that it is not true that $z, w \in [\alpha^{-1}(\overline{\pi_1}(u))]^m$. We can assume that $z \notin [\alpha^{-1}(\overline{\pi_1}(u))]^m$ (the case $w \notin [\alpha^{-1}(\overline{\pi_1}(u))]^m$ is similar). Let $z' \leq z$ with $z' \in \beta^{-1}(\overline{\pi_2}(u))$. We will see that z = z'. Since $z \notin [\alpha^{-1}(\overline{\pi_1}(u))]^m$ then there exists $z'' \in [\alpha^{-1}(\overline{\pi_1}(u))]^m$ such that z'' < z. In particular, $\alpha(z) = \alpha(z'')$. Then $z' \leq z$ and $z'' \leq z$. Since Z is a forest we have that $z' \leq z''$ or $z'' \leq z'$. Suppose that $z' \leq z''$. Then $\beta(z) = \beta(z') \leq \beta(z'') \leq \beta(z)$, so $\beta(z) = \beta(z'')$. Since $\alpha(z) = \alpha(z'')$ then $(\alpha(z), \beta(z'') \text{ and } z'' < z$ then we obtain a contradiction by (P). Hence, we have proved that $z'' \leq z$. Then $\alpha(z) = \alpha(z'') \leq \alpha(z)$, so $\alpha(z) = \alpha(z')$. Besides $\beta(z) = \beta(z')$. Since $(\alpha(z), \beta(z)) = (\alpha(z', \beta(z') \text{ and } z' \leq z \text{ then by (P) we have that <math>z = z'$. Thus, $z \in [\beta^{-1}(\overline{\pi_2}(u))]^m$. Finally we need to prove that $w \in [\beta^{-1}(\overline{\pi_2}(u))]^m$. If $w \notin [\alpha^{-1}(\overline{\pi_1}(u))]^m$ then $w \in [\beta^{-1}(\overline{\pi_2}(u))]^m$, so there exists w' such that w' < w and $w' \in \beta^{-1}(\overline{\pi_2}(u))$. In particular, $\beta(w') = \beta(w)$. Since h(z) = h(w) and w' < w then by (P) there exists z' such that z' < z and h(z') = h(w'). Then $\alpha(z') = \alpha(w')$ and $\beta(z') = \beta(w')$. Taking into account that $w' \in \beta^{-1}(\overline{\pi_2}(u))$, so $z' \in \beta^{-1}(\overline{\pi_2}(u))$ too. Since z' < z and $z \in [\beta^{-1}(\overline{\pi_2}(u))]^m$ we have a contradiction. Therefore, $w \in [\beta^{-1}(\overline{\pi_2}(u))^m$, which was our aim.

Corollary 38. The map h is a morphism in hFor.

Proof. Let $u \in X \otimes Y$. First we will see that $h^{-1}([u)) \in \mathcal{B}_X \otimes \mathcal{B}_Y$. Note that $h^{-1}([u)) = [h^{-1}(u)) = ([h^{-1}(u)]^m]$. Besides, it follows from Lemma 37 that $[h^{-1}(u)]^m \subseteq [\alpha^{-1}(\overline{\pi_1}(u))]^m$ or $[h^{-1}(u)]^m \subseteq [\beta^{-1}(\overline{\pi_2}(u))]^m$. Also note that we have $\alpha^{-1}((\overline{\pi_1})^{-1}(u)) \in \mathcal{B}_Z$ and $\beta^{-1}((\overline{\pi_1})^{-1}(u)) \in \mathcal{B}_Z$ because α and β are morphisms in hFor and $[u) \in \mathcal{B}_X \otimes \mathcal{B}_Y$. Thus, by definition of *h*-base we conclude that $h^{-1}([u)) \in \mathcal{B}_X \otimes \mathcal{B}_Y$.

Let U = [T) for some $T \subseteq X \otimes Y$ and suppose that $T \subseteq [(\overline{\pi_1})^{-1}(V)]^m$ for some $V \in \mathcal{B}_X$. In particular, $T \subseteq (\overline{\pi_1})^{-1}(V)$. Since $\overline{\pi_1} \circ h = \alpha$ then $\alpha^{-1}(V) = h^{-1}[(\overline{\pi_1})^{-1}(V)]$. Thus, $h^{-1}([T)) = [h^{-1}(T)) \subseteq [\alpha^{-1}(V))$, so $h^{-1}([T)) \subseteq [\alpha^{-1}(V))$. We will prove that $[h^{-1}(T)]^m \subseteq [\alpha^{-1}(T)]^m$. Let $z \in [h^{-1}(T)]^m$, so $h(z) \in T$. Then $\overline{\pi_1}(h(z)) = \alpha(z) \in V$, i.e., $z \in \alpha^{-1}(V)$. Let $w \leq z$ with $w \in \alpha^{-1}(T)$. Since $\overline{\pi_1}(h(w)) = \alpha(w)$ then $h(w) \in (\overline{\pi_1})^{-1}(V)$. Since $h(z) \in T \subseteq [(\overline{\pi_1})^{-1}(V)]^m$ and $h(w) \leq h(z)$ then h(z) = h(w). But $z \leq w$ and $z \in [h^{-1}(T)]^m$, so z = w. Hence, $[h^{-1}(T)]^m \subseteq [\alpha^{-1}(T)]^m$. Taking into account that α is a morphism in hFor and that $V \in \mathcal{B}_X$ we have that $\alpha^{-1}(V) \in \mathcal{B}_Z$. Thus, by definition of h-base and the equality $[h^{-1}(T)]^m \subseteq [\alpha^{-1}(T)]^m$ we have that $[[h^{-1}(T)]^m) \in \mathcal{B}_Z$.

The following theorem follows from the previous results of this section and Proposition 29.

Theorem 5. Let (X, \leq, \mathcal{B}_X) and (Y, \leq, \mathcal{B}_Y) be objects in the category hFor. Then $(X \otimes Y, \leq, \mathcal{B}_X \otimes \mathcal{B}_Y)$ is the product in hFor.

7 Some remarks on the coproduct of finite prelinear Hilbert algebras

In this section we apply the just developed construction of the product in the category hFor, together with the duality between hFor and $PHil_0$, in order to build explicit constructions for some finite coproducts in both $PHil_0$ and in PHil.

Recall, from Lemma 21, that $H \coprod_{\mathsf{PHil}_0} G$ is finite whenever H and G are finite algebras in PHil_0 . A straightforward verification shows that the free prelinear Hilbert algebra with cero in one generator p coincides with the free Gödel algebra in one generator. Hence, its underlying lattice is as depicted below.



In particular, it is a finite algebra and in consequence, any finitely generated free algebra is also finite. That is to say, the variety PHil_0 is locally finite.

Now, given finite algebras H, G in PHil_0 , write $H \coprod_f G$ for the bounded prelinear Hilbert algebra obtained by dualizing the construction of Theorem 5, applying Proposition 35.

Write $j_H : H \to H \coprod_f G$ and $j_G : G \to H \coprod_f G$ to the morphisms induced by the proyections by the duality of Proposition 35, and assume there are $f : H \to K$ and $g : G \to K$, morphisms in PHil₀. Let L be the subalgebra of K generated by $f(H) \cup g(G)$, which is finite because $f(H) \cup g(G)$ is finite and PHil₀ is locally finite. Since L is a finite, the aforementioned duality together with the universal property of the product in hFor guarantee the existence of a unique morphism $h : H \coprod_f G \to K$ such that $f = h \circ j_G$ and $g = h \circ j_H$, proving that $H \coprod_f G \cong$ $H \coprod_{\mathsf{PHil}_0} G$. Hence, Theorem 5 allows us to give and explicit calculation of the finite coproduct of finite algebras in PHil₀.

Let us finally see how to adapt the construction for the coproduct in $PHil_0$, to get an explicit construction for the coproduct of two finite (non necessarily bounded) prelinear Hilbert algebras in PHil. Note that prelinear property does not play any essential role in the following argument, so, if we would have an explicit construction for the coproduct in the category of bounded Hilbert algebras, we had been able to get one in the category of Hilbert algebras too.

Let $(H, \to, 1) \in \mathsf{PHil}$. Consider the correspondence $H \mapsto H^0$ that assigns to H the algebra $(H^0, \to_0, 1, 0)$ of type (2,0,0), whose universe is the disjoint union $H \sqcup \{0\}$ and whose binary operation is given by $h \to_0 k = h \to k$, if $h, k \in H$, $0 \to_0 h = 1$, if $h \in H$ or h = 0 and $h \to_0 0 = 0$, if $h \in H$. It can be checked that H^0 , as defined above, is a bounded prelinear Hilbert algebra and that this correspondence extends to a functor $(\)^0$: $\mathsf{PHil} \to \mathsf{PHil}_0$ by assigning to $f : H \to K \in \mathsf{PHil}$ the morphism

$$f^{0}(h) := \begin{cases} f(h), \text{ if } h \in H, \\ 0, \text{ if } h = 0, \end{cases}$$

in PHil_0 . For any $H \in \mathsf{PHil}$, let us write $i_H : H \to H^0$ for the inclusion of H into H^0 , which can be seen to be a morphism in PHil .

Let us now proceed to the construction of the coproduct in PHil of two finite algebras $G, H \in \mathsf{PHil}$. Let us first compute $H^0 \coprod_{\mathsf{PHil}} G^0 \in \mathsf{PHil}_0$, as before, and write j_{G^0} and j_{H^0} for the natural inclusions¹ Put $k_G = j_{G^0} \circ i_G$ and $k_H = j_{H^0} \circ i_H$ for the inclusions of G and H into $G^0 \coprod_{\mathsf{PHil}} H^0$. Write G * H for the Hilbert subalgebra of $G^0 \coprod_{\mathsf{PHil}} H^0$ generated by $k_G(G) \cup k_H(H)$, and $j_G = k_{G|_G}$ and $j_H = k_{H|_H}$, respectively. Let us check that G * H with the injections k_G and k_H is the coproduct of G and H in PHil. In order to do that, take $K \in \mathsf{PHil}$ endowed with two morphisms $g: G \to K$ and $h: H \to K$, also in PHil. Since $g^0: G^0 \to K^0$ and $h^0: H^0 \to K^0$ are in PHil₀, by the universal property of $G^0 \coprod_{\mathsf{PHil}} H^0$, we have that there is a unique $k: G^0 \coprod_{\mathsf{PHil}} H^0 \to K^0$ such that $k \circ j_{G^0} = g^0$ and $k \circ j_{H^0} = h^0$. Writing $i: G * H \to G^0 \coprod_{\mathsf{PHil}} H^0$ for the inclusion morphism (in PHil), we have that $(k \circ i)(G * H) \subseteq i_K(K)$. Hence, the restriction of k to i(G * H), is a morphism $\hat{k}: G * H \to K \in \mathsf{PHil}$, that makes the following diagram commute.



To see the uniqueness of the morphism \hat{k} . Suppose that $\delta : G * H \to K \in \mathsf{PHil}$ also renders the diagram commutative. Then $\delta(j_G(a)) = g(a) = \hat{k}(j_G(a))$ for every $a \in G$ and $\delta(j_H(b)) = h(b) = \hat{k}(j_H(b))$ for every $b \in H$. Since $j_G(G) \cup j_H(H)$ generates G * H then $\delta = \hat{k}$.

Let us end this section calculating a coproduct both in PHil_0 and in PHil with the procedures described above.

Example 1. Let G = H = 2, the implicative reduct of the boolean algebra with two elements. Since 2 is bounded, let us compute their coproduct in PHil₀. Since $\mathbf{X}(2) \cong 1$, and $1 \times 1 = 1$, we get that $G \coprod_f H \cong 2$.

Example 2. Let us consider again G = H = 2, but now compute their coproduct in PHil. Let us start by noticing that $G^0 = H^0 = 2^0 = 3$, the implicative reduct of the Gödel chain with 3 elements. Hence, $\mathbf{X}(G^0) \cong \mathbf{X}(H^0) \cong \mathbf{X}(3) \cong 2$, here, we write 2 for the two elements chain. A direct computation of their product in hFor

¹It can be seen that in this case, natural inclusions are in fact injections. Let us see that j_{H^0} is injective. Take $t: G^0 \to H^0$, the morphism in PHil_0 given by t(g) = 1, if $g \in G$ and t(g) = 0 if g = 0, and consider the cocone in PHil_0 $id_{H^0}: H^0 \to H^0 \leftarrow G^0: t$. By the universal property of the coproduct, there is a unique $h: H^0 \coprod_{\mathsf{PHil}_0} G^0 \to H^0$ in PHil_0 , such that $h \circ j_{H^0} = id_{H^0}$. Since, id_{H^0} is injective, so is j_{H^0} .

shows that $\mathbf{X}(G^0 \coprod_f H^0)$ is as depicted below.



Here, the empty bullets correspond to one of the two nontrivial basic sets. The other is symmetrical. Hence, $G^0 \coprod_f H^0$ is the Hilbert algebra, generated by a and b, whose underlying algebra is depicted bellow.



Finally, G * H is the Hilbert subalgebra $(G^0 \coprod_f H^0) - \{0\}$, which is the free prelinear Hilbert algebra in two generators.

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