On Fuzzification Mechanisms for Unary Quantification

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Abstract We address a salient issue arising in Glöckner's two-step methodology for modeling vague quantifiers: the design of quantifier fuzzification mechanisms (QFMs). These are mechanisms to turn formal models of vague quantifiers operating only on crisp arguments (semi-fuzzy quantifiers) into vague quantifiers accepting vague arguments as well (fully-fuzzy quantifiers). We critically examine desiderata formulated by Glöckner for QFMs and also point out that previous approaches to quantifier fuzzification largely ignored the question whether the resulting quantifiers can be expressed in suitable extensions of t-norm based fuzzy logics, in particular in Łukasiewicz logic. We also introduce a new family of QFMs, and assess it fares with respect to the mentioned desiderata. We exclusively focus on unary quantifiers, in order to circumvent interference with vagueness related problems arising for all truth functional accounts of quantifiers that refer to more than one argument formula.

1 Introduction

Fuzzy quantifiers have been a major topic in the research on approximate reasoning, almost since the very inception of this field by Zadeh [37]. They model natural language expressions, like, many, about half, about 100, almost all, etc, which are vague, but useful for concise and effective communication. Formal models of such quantifier expressions within fuzzy logic have been first studied in detail in Zadeh's seminal [38]. Since then a huge amount of literature has been devoted to the subject. Given the many facets of this topic and its importance for applications like linguistic data summarization and fuzzy information retrieval, this continued interest should not come as a surprise. We refer to Glöckner's monograph [19] and the more recent survey paper [7] for references and extensive discussions of the state of the art. Glöckner 's approach is particularly interesting for us, since he aimed at obtaining both computationally and linguistically adequate models of fuzzy quantifiers, keeping a close connection with the well developed theory of generalized quantifiers

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[3,30] in the classical setting (see also the recent work [31] on the relation between fuzzy and classical generalized quantifiers).

The approach consists of the following two-steps for modeling quantifiers: (1) define so-called semi-fuzzy quantifiers, i.e. models of vague quantification where the scope formulas are crisp (bivalent), but the quantified statement may be evaluated to an intermediate degree of truth; (2) lift semi-fuzzy quantifiers to fully-fuzzy ones, where also the scope formulas are fuzzy, in a systematic manner. The two-step method is certainly very useful and has been followed also by other researchers (see, e.g., [7,9]). However, we think that not all of the manifold problems that may be encountered in the design of fully-fuzzy models of vague quantification are already adequately addressed in the current literature.

One of the challenges left open by Glöckner and by other researchers in this area consists in the fact that they define semi-fuzzy quantifier models for expressions like many, about half, almost all, etc, in an *ad hoc* fashion. While design principles and specific requirements for the lifting of semi-fuzzy to fully-fuzzy quantifiers are presented, no framework is offered that allows one to derive semi-fuzzy quantifier models in a systematic manner from first principles. An attempt to derive semi-fuzzy quantifier models from more basic reasoning principles using random sampling of witness elements in extensions of Giles's game for Łukasiewicz logic has been made in [13,14]. We will not be concerned with the derivation of semi-fuzzy quantifiers here, but rather focus on challenges arising for the second step of Glöckner's approach, i.e. fuzzification. To address those challenges, Glöckner devised a list of certain requirements or 'axioms' that any formal mechanism for fuzzification should satisfy. While this amounts to a laudably systematic approach to this problem, we argue that Glöckner's specific axioms are problematic from the point of view of contemporary Mathematical Fuzzy Logic (MFL), as documented, e.g., in the three already available volumes of the handbook [6]. Indeed, one of our main motivations for revisiting the topic of quantifier fuzzification is to try to bridge the current gap between MFL on the one side and research on fuzzy quantifiers on the other side. The latter is mostly motivated by various applications, whereas the first seeks to extend and deepen the logical foundations of reasoning with fuzzy propositions and predicates in general. Łukasiewicz logic turns out to play a central role in MFL. In contrast, Glöckner [19] explicitly dismisses it as a base logic for fuzzy quantifiers, since taking the residuum of Łukasiewicz t-norm as truth function for implication is not compatible with his specific way of relating propositional connectives and quantifiers. However, we think that this move might be too hasty. As shown in [13,14], certain types of semi-fuzzy quantifiers fit very well into the game semantic framework for Łukasiewicz logic provided by Giles [18]. Hence, it is natural to ask whether also fully fuzzy quantifiers that arise from applying different forms of fuzzification mechanism to semi-fuzzy quantifiers can be embedded into Łukasiewicz logic in a systematic manner. For this reason we will investigate for each of the quantifier fuzzification mechanisms considered in this paper how the resulting fuzzy quantifiers can be expressed in appropriate extensions of Łukasiewicz logic containing the corresponding semi-fuzzy quantifiers. Admittedly, this is largely of theoretical interest, in accordance with the overall research aims of MFL [6]. Nevertheless, these expressibility results may potentially also be useful for automated reasoning or query answering systems that support languages that feature fuzzy quantifiers not in isolation, but on top of an expressive deductive fuzzy logic (see, e.g., [15] for an attempt to enrich a logical query language with certain semi-fuzzy quantifiers, motivated by witness selection principles).

In this paper, we will not deal with the full realm of generalized quantifiers, but rather focus on *unary* quantification over *finite* domains. The second restriction is readily motivated by the intended application of modeling natural language semantics: in ordinary (non-mathematical) language the domain of quantification is usually assumed to consist of a finite number of concrete objects, determined by the context of discourse. Likewise, query answering systems usually deal with finite domains only. Correspondingly, we will ignore all worries that only arise for infinite domains, here. The first restriction, amounting to the assumption that the quantifier scope consists in a single formula (expressing a unary predicate), is less obvious and calls for a separate motivation. Typical natural language statements like Many Swedes are tall or Most athletes are healthy feature *binary* quantification. Consequently, following Glöckner [19], almost all papers on quantifier fuzzification consider frameworks that generalize from unary to binary, and even more generally to n-ary quantifiers. However, for the following three reasons we will focus on unary quantification, here: (1) Binary (and higher arity) quantification poses a specific challenge to fuzzy quantifier models that arises from possible semantic dependencies between vague range (i.e. domain restricting) predicates and scope predicates. Such dependencies cannot be modeled directly in the usual fuzzy set approach, as we will explain in more detail in Section 2. This vagueness related phenomenon is usually ignored in fuzzy logic, with noteworthy exceptions, e.g. [8]. In any case, the challenge is well recognized in linguistic literature (see, e.g., [2,22]). (2) Independently on whether one agrees on the significance of the phenomenon of vagueness-induced range/scope dependencies, this issue can be separated from the discussion of arguably more fundamental desiderata for generalizing semi-fuzzy to fully-fuzzy quantifiers in a systematic manner. The latter can be discussed most transparently for unary quantification, where, by definition, no range/scope dependencies can occur. (3) The third motive for the restriction is a purely pragmatic one. The paper is already quite long as it is. Since the issues that we want to address here already concern unary quantifiers, it is reasonable to focus on that basic case and defer more detailed discussions of specific problems regarding more general forms of quantification to another occasion. This has the additional benefit of admitting a leaner, more concrete formal framework.

The paper is organized as follows. In Section 2, we present a useful classification of quantifiers (or rather of quantification) with respect to the different components at which vagueness is involved and try to sort out some terminological issues in the field. Section 3 critically reviews Glöckner's axioms for quantifier fuzzification mechanisms (QFMs). In particular, we will discuss some additional and/or alternative desiderata for quantifier fuzzification, which are not covered by Glöckner. In Section 4 we briefly introduce the language and standard semantics of Łukasiewicz logic. In Section 5 we revisit some of the main QFMs in the literature, investigate them from the point of view of their expressibility in Łukasiewicz logic and study their relation. In Section 6 we introduce a new family of QFMs, and hint at some idea to go beyond Glöckner's QFM framework. The final Section 7 provides a compact summary of our findings and hints at directions for further research.

2 Types of Quantification

In its most general form, a *quantified sentence* $\mathsf{Q}x_1, \dots x_n(P_1, \dots, P_m)$ features a *quantifier* Q binding variables $x_1, \dots, x_n$ that may occur in several argument predicates $P_1, \dots, P_m$. Such quantifiers are often said to be of Type $\langle a_1, \dots, a_m \rangle$, where a_i is the arity of the predicate P_i $(1 \leq i \leq m)$.

In natural language, *binary* quantification[4] (Type $\langle 1, 1\rangle$ quantification) is the most common form. This term refers to sentences of the form $\mathsf{Q}x(R, S)$, like All girls are clever and Most boys are diligent, where the quantifier binds a single variable that occurs in both argument predicates, often called the *range (restricting)* predicate R and the *scope* predicate S. In traditional mathematical logic one usually focuses on *unary* quantification using the quantifiers $\forall$ (for all) and $\exists$ (there exists) only. As is well known, in classical logic – and in many other logics for that matter – one can reduce binary quantification for these quantifiers to unary quantification: $\forall x(R(x), S(x))$ can be expressed as $\forall x(R(x) \to S(x))$ and $\exists x(R(x), S(x))$ as $\exists x(R(x) \wedge S(x))$. Such a reduction of binary quantification to unary one by using propositional connectives is not possible for other quantifiers, in general. However, as long as the range predicate R denotes an ordinary (crisp) set R, binary quantification may still be reduced to unary quantification semantically by stipulating that $\mathsf{Q}x(R, S)$ is to be evaluated as $\mathsf{Q}xS$, where the domain of discourse is restricted to the set R.

There is yet another challenge arising for binary quantification over vague predicates, that is usually ignored in the literature on fuzzy quantifiers (with some relevant exceptions, such as [8]), although linguists are certainly aware of it (see, e.g., [2,22]). If a vague predicate like heavy is joined with another predicate through a binary quantifier then the second predicate often influences the degrees of applicability of the first predicate. Consider, e.g., the following three sentences, occurring in the same context (e.g., a very comprehensive ontology, bringing together all kinds of facts for commonsense reasoning).

- Many coins are heavy.
- Many cars are heavy.
- Many ships are heavy.
- Many planets are heavy.

Obviously the range predicates (coin, car, ship, planet) create different standards for evaluating the scope predicate heavy adequately. Note that 'heavy' is not ambiguous (in the sense that, e.g., 'bank' is ambiguous, since it may either refer to a financial institution or to land next to a river). Rather it is the *vagueness* of 'heavy', which results in what Shapiro [35,34] calls *open-texture*. This property entails a specific form of context dependency that arises if a vague (scope) predicate is joined with another, not necessarily vague, (range) predicate through quantification. For fuzzy models of quantification this implies shifts in the interpretation of membership degrees of fuzzy sets representing a scope predicate, that influence the overall truth value of quantified formula. This phenomenon is absent for crisp predicates, like precisely defined legal or mathematical concepts. Consequently, any quantifier fuzzification mechanism that derives the truth function for binary fuzzy quantifiers *solely* from the truth function of a corresponding semi-fuzzy quantifier cannot be fully adequate, in the sense of properly taking into account a well-recognized feature of natural language. For our purposes, it suffices to point out that the outlined issue about vagueness-induced range/scope dependencies, by definition, does not arise for strictly unary quantification. A related fact is discussed in [11], where it is pointed out that vague range and scope predicates joined by a binary quantifier may give rise to either dependent or independent standards of precisification, which cannot be fully reflected by membership degrees, which proposes dependent and independent voting models. To understand this phenomenon, consider the following sentences:

[4] In [39] Zadeh speaks of a classification of quantifiers into the first kind, second kind, third kind, etc., rather than of unary, binary, ternary, etc., quantification. However, this is in conflict with Zadeh's own earlier terminology in [38], that we will mention later in this section.

– Every child is a child.
– Every child is poor.

The first sentence clearly should always be evaluated as true, independently of whether the universe of discourse contains borderline cases of children or not. The second sentence may well also be fully true in certain contexts of evaluation. However, widely shared intuitions on proper language use require that in such contexts the predicate poor has to apply fully to every individual that is considered to be a child to some positive degree. In other words: the second sentence should not be accepted as fully true if there are borderline cases of individuals that are poor as well as children. Consequently, the two sentences might well have to judged differently, even if the fuzzy sets representing the predicates child and poor are identical (e.g., by assigning degree 0.5 to all members of the universe).

One may summarize the linguist phenomena described above by saying that binary quantification over vague predicates often shows an *intensional* behavior that cannot be adequately reflected by a *extensional* (purely truth functional) framework, like that of fuzzy quantifiers. As already indicated, linguists are well aware of this fact when modeling vagueness, e.g., by contextually shifting standards of acceptance or rejection (see, e.g., [2]). For an assessment of this situation that is closer to the concerns of (deductive) fuzzy logic, we refer to [11]. Of course, this does not mean that the standard approach to binary fuzzy quantifiers is not useful. In fact, the models considered in the literature on fuzzy quantifiers are mainly motivated by computational concerns and may well be deemed adequate for certain types of applications. However, there is a trade off between computational efficiency and full linguistic adequateness.[5]

In any case, it seems fair to say that in order to fully take into account also the *intensional* aspects of vague quantification, one has to be prepared to consider more complex logical models that enrich the extensional framework by modal features, as featured in standard linguist accounts of vagueness; see, e.g., [2,16,21,22]. For the current purpose, it suffices to point out the indicated challenges arising from intensional dependencies between range and scope predicates only arise for binary and higher-arity quantifiers. Consequently, we will focus on *unary* quantification here, leaving a more thorough discussion of the challenges and desiderata arising specifically for other forms of quantification to future work.

While vagueness can be formally modeled in various different ways, in this paper we will follow the fuzzy logic approach to model vague predicates and statements. In particular, there is a considerable body of literature on *fuzzy quantifiers*, originating with Zadeh's [38], which spawned a wide range of follow up work. Most importantly, Glöckner devoted a comprehensive monograph [19] on this topic, which remains an important milestone. More recent work, not yet covered in [19], is surveyed in [7], in [8], and in [9]. Common to all models in this tradition is that a vague unary predicate P is modeled as a fuzzy set $\widetilde{\mathrm{P}} : D \to [0,1]$, where each element of the domain D (a non-empty crisp set) is mapped to a *degree of membership* from the unit interval $[0,1]$. Throughout this paper we assume that D is *finite*. This is justified by the intended applications, namely models of ordinary language statements, where the relevant domain of discourse is concrete (crisp)

[5] The situation is analogous to the well known dilemma arising at the propositional level for the conditional. Classical truth functional implication can hardly be viewed as a linguistically adequate model of the if-then-construction as used in natural language. Such models rather call for intensional logics that are capable of capturing causal or counter-factual aspects of the conditional. Nevertheless, there might be good pragmatic reasons for sticking with the simple classical truth table for material implication in many applications.

and finite. As we will see, this assumption supports a number of simplifications. For example, we will not have to worry about whether a given set of domain elements can be 'measured' in some suitable sense.

In the literature on fuzzy quantifiers, one often conflates syntax and semantics: no explicit distinction between a quantifier expression and its corresponding truth function is made. Likewise, one does not distinguish between the formula (over some given object language) and the set or relation that the formula denotes under a given interpretation. Since one of our main aims is to embed quantifier models into formal deductive fuzzy logics — in particular into appropriate extensions of Łukasiewicz logic — we prefer to make the syntax/semantics distinction explicit, as usual in Mathematical Fuzzy Logic.

An *interpretation* $\mathcal{I}$ assigns a fuzzy set $\widetilde{\mathrm{P}}$ to each unary predicate symbol P. Consequently, the corresponding *truth function* $v_{\mathcal{I}}(\cdot)$ assigns the truth value (degree of truth) $v_{\mathcal{I}}(P(\overline{c})) = \widetilde{\mathrm{P}}(c)$ to the sentence (closed atomic formula) $P(\overline{c})$, where $\overline{c}$ is a constant symbol that $\mathcal{I}$ maps to the element c of the domain D. (In the rest of the paper we will identify domain elements with corresponding constant symbols.) More generally, fuzzy *relations* are assigned to predicate symbols of corresponding arity. Moreover, the predicate that serves as argument of a unary quantifier may be presented by a compound formula F of some given logic. (We will formally introduce Łukasiewicz logic with the $\triangle$ operator for this purpose in Section 4). In the special case where all predicates are crisp, we speak of a *classical interpretation.* For a quantified sentence $\mathsf{Q}xF(x)$, we obtain a truth degree $v_{\mathcal{I}}(\mathsf{Q}xF(x))$ by associating the quantifier Q with a function that maps every fuzzy set, i.e. every denotation of the argument formula $F(x)$) into the set of truth values $[0, 1]$. Note that, even if the formula F is classical, i.e. $v_{\mathcal{I}}(F(\overline{c})) \in \{0, 1\}$ for all $c \in D$, $v_{\mathcal{I}}(\mathsf{Q}xF(x))$ may still be an intermediate truth value: neither 0 nor 1. Below, we will write $\hat{F}$ instead of F, whenever we want to emphasize that the formula is classical.

Let us recall a useful classification, introduced by Liu and Kerre [24]:

Type I: the quantifier is crisp and its arguments are crisp;
Type II: the quantifier is crisp, but the arguments may be fuzzy;
Type III: the quantifier is fuzzy, but its arguments are crisp (*semi-fuzzy*);
Type IV: the quantifier as well as its arguments are fuzzy (*fully-fuzzy*).

These types are considered in a cumulative fashion: crispness is understood as a special case of fuzziness (arising from restricting $[0, 1]$ to $\{0, 1\}$) and thus every Type I sentence is of Type II and Type III as well; moreover the latter are also of Type IV. However Type II and III are incomparable and only share Type I as special cases. Strictly speaking, the types refer to quantified sentences and not just to the used quantifier expressions. Nevertheless, we may speak of semi-fuzzy or (fully-)fuzzy quantifiers, thus indicating whether the arguments are implicitly restricted to crisp predicates or not.

Of particular importance are natural language expressions like almost all, about half, at least (about) a third, at most (about) 10%, etc, which refer to the proportion of domain elements satisfying the scope predicate. In contrast, quantifier expressions like about five, at least about hundred, much more than ten, refer not to proportions, but to absolute numbers. We will speak of *proportional* and *absolute* quantification, respectively.[6]

[6] In [38] Zadeh speaks of quantifiers of the first kind and of the second kind, referring to 'absolute and relative counts', respectively. As pointed out above, this is in conflict with Zadeh's own (later) terminology in [39]. We therefore follow Glöckner [19] in using the terms 'absolute' and 'proportional', instead.

Note that proportion is only clearly defined for finite domains — corresponding to our setting, as specified above — and for crisp predicates, i.e., for semi-fuzzy quantification. Similarly, (absolute) cardinality is unambiguous only for predicates that correspond to crisp sets.

Definition 1. *Given a classical interpretation $\mathcal{I}$ over the domain D and a formula F, we define*

$$Prop_{\mathcal{I}}(F) = \frac{|\{c \in D : v_{\mathcal{I}}(F(c)) = 1\}|}{|D|} = \frac{\sum_{c \in D} v_{\mathcal{I}}(F(c))}{|D|}.$$

We denote by Π the semi-fuzzy quantifier whose truth function is $Prop_{\mathcal{I}}$, i.e. such that $v_{\mathcal{I}}(\Pi x F(x)) = Prop_{\mathcal{I}}(F)$. The quantifier has been originally introduced in [13,14] as a random choice quantifier, motivated in a game semantic framework via a non-strategic player, called Nature*, that samples witnessing elements for statements uniformly randomly, see also [5] for a more comprehensive analysis.*

In the following, we call a semi-fuzzy quantifier Q proportional *if there exists a function $g_{\mathsf{Q}} \colon [0,1] \to [0,1]$ such that, for any classical interpretation $\mathcal{I}$, $v_{\mathcal{I}}(\mathsf{Q}xF(x)) = g_{\mathsf{Q}}(v_{\mathcal{I}}(\Pi x F(x)) = g_{\mathsf{Q}}(Prop_{\mathcal{I}}(F))$.*

Similarly, we call a semi-fuzzy quantifier Q absolute *if there exists a function g_{Q} such that for any classical interpretation $\mathcal{I}$, $v_{\mathcal{I}}(\mathsf{Q}xF(x)) = g_{\mathsf{Q}}(|\{c \in D : v_{\mathcal{I}}(F(c)) = 1\}|$.*

The main challenge addressed in this paper is to find suitable generalizations of semi-fuzzy (Type III) proportional and absolute quantification to a fully-fuzzy (Type IV) setting. At a first glance, one might suppose that it suffices to generalize cardinality in the usual way by using the Σ-count, i.e. $\sum_{c \in D} v_{\mathcal{I}}(F(c))$ instead of $|\{c \in D : v_{\mathcal{I}}(F(c)) = 1\}|$. Such measure induces for instance a fully-fuzzy version of $Prop_{\mathcal{I}}(F) = \frac{\sum_{c \in D} v_{\mathcal{I}}(F(c))}{|D|}$ and of the quantifier Π, which we denote by the same symbol and interpret by $v_{\mathcal{I}}(\Pi x F(x)) = Prop_{\mathcal{I}}$. While this form of 'lifting' semi-fuzzy to fuzzy quantification is technically straightforward, it is, unfortunately, *linguistically inadequate*, in general. To see this, consider, e.g., the absolute quantifier exactly five applied to a fuzzy predicate F over 10 domain elements. Replacing cardinality by the Σ-count in Definition 1, forces us to evaluate the sentence Exactly five [of the 10 domain elements] are F as perfectly true in an interpretation where $v_{\mathcal{I}}(F(c)) = 0.5$ for every $c \in D$. This is clearly inadequate. The phenomenon has been generally called *aggregative behavior of low truth values* [19,9]. It occurs whenever many low truth degrees are attributed to each element of the domain, but their aggregation in the evaluation of a quantified sentence gives rise, counterintuitively, to a high truth degree.

A similar phenomenon occurs also for proportional quantification, as the case for the quantifier expression half reveals: we would obtain the same truth value for Half [of the domain elements] are F in an interpretation where $v_{\mathcal{I}}(F(c)) = 0.5$ for every $c \in D$ as in an interpretation where $v_{\mathcal{I}}(F(c)) = 1$ for exactly half of the domain elements and $v_{\mathcal{I}}(F(c)) = 0$ for the other half.

3 Desiderata for Quantifier Fuzzification Mechanisms

As already indicated in the introduction, Glöckner's approach [19] to the challenge of designing adequate and useful fuzzy models of vague natural language quantifiers remains of central importance. In this section we review his desiderata for lifting semi-fuzzy quantifiers to fully-fuzzy quantifiers, but also critically discuss some potential problems and alternative principles. We will occasionally also suggest more suitable names for important postulates. The central notion is the following.

Definition 2. *A* quantifier fuzzification mechanism (QFM) $\mathcal{F}$ *assigns to each semi-fuzzy quantifier* Q *a corresponding fully-fuzzy quantifier* $\mathcal{F}(\mathsf{Q})$*.*

Glöckner imposes six requirements ('axioms') for QFMs, two of which concern the relation between unary quantifiers and related quantifiers with more than one argument formula. As already pointed out in the introduction, we think that Glöckner's specific way of extending unary quantifiers to quantifiers with more than one arguments, at least potentially, runs into serious problems with respect to adequately capturing vagueness related issues arising for higher-arity quantification due to intensional relations between different argument predicates. In this paper we put those issues aside and solely discuss various options for generalizing unary Type III (and, later, also Type II) quantification to unary fully-fuzzy (Type IV) quantification. However, Glöckner aimed at an axiomatic characterization of the preferred QFMs, where the axioms are pairwise independent. Consequently, some of his own desiderata are not directly reflected in the axioms, since corresponding properties can be derived from the six stated axioms. Since we also want to bring alternative lifting principles in view, we will have to discuss also some of these derived properties.

As already mentioned in Section 2 Glöckner obliterates the distinction between a quantifier expression (a syntactic object) and the corresponding truth function (semantics). Accordingly, there also is no explicit distinction between predicate symbols and their denotations. As long as one does not aim at embedding the quantifier models in deductive logic, this syntax/semantic conflation is harmless and leads to shorter formulas. When speaking of a 'quantifier', we will also leave it to the context to disambiguate between a quantifier expression and its corresponding truth function. Similarly, we will simply speak of a (fuzzy or crisp) predicate or argument, without always making explicit whether we refer to an (atomic or compound) formula of some formal logic or to its denotation. However — keeping in mind the embeddability of quantifier models in full-fledged deductive fuzzy logics — we will usually make the reference to an interpretation $\mathcal{I}$ and the corresponding evaluation function $v_{\mathcal{I}}(\cdot)$ explicit.

In the following, $\mathcal{F}$ will always denote the QFM in question. Glöckner calls the following principle "perhaps the most important axiom":

Correct Generalization: For every crisp formula $\hat{F}$: $v_{\mathcal{I}}(\mathcal{F}(\mathsf{Q})x\hat{F}(x)) = v_{\mathcal{I}}(\mathsf{Q}x\hat{F}(x))$

This principle expresses that the fuzzified quantifier $\mathcal{F}(\mathsf{Q})$ should coincide with the semi-fuzzy quantifier Q on crisp arguments. We will see later, in Section 6.1, that even this seemingly innocent principle may have to be dropped for certain models that actually arise not from directly lifting semi-fuzzy quantifiers to fully-fuzzy ones, but that rather take Type I quantification as a starting point for devising a fuzzy quantification model.

For the following definition, remember that we identify domain elements with constants.

Definition 3. *For any crisp formula* $\hat{F}$ *and any* $c \in D$*, we define the Type I* projection quantifier $\triangle_c$ *by* $v_{\mathcal{I}}(\triangle_c x\hat{F}(x)) = v_{\mathcal{I}}(\hat{F}(c))$*.*

Note that $\triangle_c x\hat{F}(x)$ is classical (bivalent). Glöckner postulates the following axiom for lifting $\triangle_c$ to fuzzy predicates.

Projection Quantifiers: For every formula F: $v_{\mathcal{I}}(\mathcal{F}(\triangle_c)xF(x)) = v_{\mathcal{I}}(F(c))$.

While this stipulation is certainly natural, it may be questioned whether there is a need to introduce projection quantifiers at all. Obviously, Glöckner's corresponding axiom only makes sense if we insist on representations of vague predicates as fuzzy formulas. While this is certainly open to discussion from a linguistic point of view, we will take it for granted here.

Except for the projection quantifier, all quantifiers that we want to discuss here are *logical*, in the sense that the truth value of a quantified statement does not depend on the order of domain elements, but only on quantitative aspects of the argument predicate. This can be expressed in various ways. Following [30], we will take the following property as the hallmark of logicality.

Definition 4. *A semi-fuzzy*[7] *or fuzzy quantifier* Q *is called* quantitative *if for every automorphism*[8] $\xi : D \to D$ *and every formula* F, $v_{\mathcal{I}}(\mathsf{Q}xF(x)) = v_{\mathcal{I}^{\xi}}(\mathsf{Q}xF(x))$, *where the interpretation* $\mathcal{I}^{\xi}$ *results from the interpretation* $\mathcal{I}$ *by mapping every* $c \in D$ *into* $\xi(c)$.

Preservation of Quantitativity: If Q is quantitative then $\mathcal{F}(\mathsf{Q})$ is quantitative, too.

Another basic desideratum for quantifier fuzzification is called 'extensionality' in [19]. It refers to the fact that the interpretation of quantifiers usually shows some kind of context insensitivity: embedding a smaller model into a larger one does not affect evaluations. Clearly, this property should be preserved under fuzzification. Our logic based approach guarantees that this principle is already built into the formal setting and does not even have to be formulated explicitly.

Let us now formulate a principle that can easily be confused with extensionality, but actually refers to a quite different property. It is useful to first introduce the following notions.

Definition 5. *We call the interpretation* $\mathcal{I}'$ *a* conservative extension *of* $\mathcal{I}$ *and say that* $\mathcal{I}'$ conservatively extends $\mathcal{I}$ *if* $\mathcal{I}'$ *results from* $\mathcal{I}$ *by (possibly) adding further elements to the domain* D *without changing the interpretation of predicates over* D *itself (i.e.,* $v_{\mathcal{I}'}(\cdot)$ *and* $v_{\mathcal{I}}(\cdot)$ *agree on* D*). Moreover, we call* D-based variants *of* $\mathcal{I}$, *any two conservative extensions of* $\mathcal{I}$.

Definition 6. *A (semi-fuzzy or fuzzy) quantifier* Q *is called* non-decreasing under extension *if for every formula* F, $v_{\mathcal{I}}(\mathsf{Q}xF(x)) \leq v_{\mathcal{I}'}(\mathsf{Q}xF(x))$, *whenever* $\mathcal{I}'$ *conservatively extends* $\mathcal{I}$. *It is called* non-increasing under extension *if, under the same condition,* $v_{\mathcal{I}}(\mathsf{Q}xF(x)) \geq v_{\mathcal{I}'}(\mathsf{Q}xF(x))$.

Note that neither proportional nor absolute quantifiers are non-decreasing or non-increasing under extension, in general. However, the classical existential quantifier and, more generally, Type I quantifiers expressing at least k ($k > 0$) are non-decreasing under extension, while the quantifiers expressing none or at most k ($k > 0$) are non-increasing under extension. Although neither is considered by Glöckner [19] nor by Delgado *et al.* [7], it is not unreasonable to ask for the preservation of these properties under fuzzification.

Preservation of Monotonicity under Conservative Extension: If Q is non-decreasing (non-increasing) under extension then $\mathcal{F}(\mathsf{Q})$ is non-decreasing (non-increasing) under extension, too.

Glöckner wants to generalize the dualities of syllogistic reasoning expressed in Aristotle's square from classical to fuzzy logic (see Section 3.5 of [19]). This requires not only the presence of a connective for negation ($\neg$), but also of antonymic quantifiers. Since we are only interested in unary quantifiers, the corresponding definition is straightforward.

[7] In the following we will tacitly assume that whenever Q is a semi-fuzzy quantifier then the corresponding scope formula is crisp, even if that is not indicated explicitly by our notation.

[8] In our case this is just a bijection of D into itself, i.e., a permutation of the domain.

Definition 7. *For any quantifier* Q *its* antonym $\mathsf{Q}^{\neg}$ *is given by* $v_{\mathcal{I}}(\mathsf{Q}^{\neg}xF(x)) = v_{\mathcal{I}}(\mathsf{Q}x\neg F(x))$. *The* negated quantifier $^{\neg}\mathsf{Q}$ *is given by* $v_{\mathcal{I}}(^{\neg}\mathsf{Q}xF(x)) = v_{\mathcal{I}}(\neg\mathsf{Q}xF(x))$. *Moreover, the* dual quantifier Q^{d} *is defined by* $v_{\mathcal{I}}(\mathsf{Q}^{d}xF(x)) = v_{\mathcal{I}}(\neg\mathsf{Q}x\neg F(x))$. *In other words, the dual quantifier is the negated antonym of the given quantifier.*

The following corresponding desiderata arise for a QFM $\mathcal{F}$ and semi-fuzzy Q:

Internal Negation: For any formula F: $v_{\mathcal{I}}(\mathcal{F}(\mathsf{Q}^{\neg})xF(x)) = v_{\mathcal{I}}(\mathcal{F}(\mathsf{Q})x\neg F(x))$.
External Negation: For any formula F: $v_{\mathcal{I}}(\mathcal{F}(^{\neg}\mathsf{Q})xF(x)) = v_{\mathcal{I}}(\neg\mathcal{F}(\mathsf{Q})xF(x))$.

In the context of his other axioms, Glöckner only needs to formulate the following desideratum explicitly:

Dualization: For every formula F: $v_{\mathcal{I}}(\mathcal{F}(\mathsf{Q}^{d})xF(x)) = v_{\mathcal{I}}(\mathcal{F}(\mathsf{Q})^{d}xF(x))$.

As already indicated, Dualization presupposes the existence of a unique negation operator. Glöckner tackles this problem by introducing a mechanism for deriving truth functions for propositional connectives from QFMs. While he speaks of a 'canonical construction' we emphasize that the set of truth functions preferred by Glöckner for negation, disjunction, conjunction and implication are incompatible with those of either Łukasiewicz, Gödel, or Product logic.

More generally, Glöckner's approach to propositional connectives is incompatible with a more recent approach to deductive fuzzy logics [20,6,5], where one starts with a (left-)continuous t-norm for conjunction, uses its residuum for implication and derives all other connectives from these in a canonical fashion.

The next principle focuses on an important subclass of quantifiers.

Definition 8. *A quantifier* Q *is called* non-increasing *if* $v_{\mathcal{I}}(F(c)) \leq v_{\mathcal{I}}(F'(c))$ *for every* $c \in D$ *implies* $v_{\mathcal{I}}(\mathsf{Q}xF(x)) \geq v_{\mathcal{I}}(\mathsf{Q}xF'(x))$.

Note that in the case of crisp formulas $\hat{F}$ and $\hat{F}'$, the condition that $v_{\mathcal{I}}(\hat{F}(c)) \leq v_{\mathcal{I}}(\hat{F}'(c))$ for every $c \in D$ expresses that the extension of $\hat{F}$ is a subset of the extension of $\hat{F}'$. In this restricted form, monotonicity of quantifiers is often discussed in linguistic literature (see, e.g., [30]). For example, for any constant k the Type I quantifier at most k is non-increasing; but also the vague quantifiers few and less than about half are non-increasing. Thus it is understandable that Glöckner wants to preserve this property in lifting from semi-fuzzy to fully-fuzzy quantifiers.

Preservation of Monotonicity ($\geq$): If Q is non-increasing then $\mathcal{F}(\mathsf{Q})$ is non-increasing, too.

Note that, following Glöckner, monotonicity is only formulated for one direction of the inequality here. However the following property is just as important.

Definition 9. *A quantifier* Q *is called* non-decreasing *if* $v_{\mathcal{I}}(F(c)) \leq v_{\mathcal{I}}(F'(c))$ *for every* $c \in D$ *implies* $v_{\mathcal{I}}(\mathsf{Q}xF(x)) \leq v_{\mathcal{I}}(\mathsf{Q}xF'(x))$.

Accordingly, one should augment Preservation of Monotonicity as follows.

Preservation of Monotonicity ($\leq$): If Q is non-decreasing then $\mathcal{F}(\mathsf{Q})$ is non-decreasing, too.

Glöckner does not include the latter postulate among his axioms because it is entailed by the Preservation of Monotonicity for non-increasing quantifiers and Dualization. In contrast, we argue that these monotonicity conditions should be kept independent of the particular choice of the negation connective.

For semi-fuzzy quantifiers, we may consider the following alternative definition of monotonicity that only refers to proportions (see Definition 1).

Definition 10. *A semi-fuzzy quantifier* Q *is called* non-decreasing in proportion*, if* $v_{\mathcal{I}}(\mathsf{Q}x\hat{F}(x)) \leq v_{\mathcal{I}}(\mathsf{Q}x\hat{G}(x))$*, whenever* $Prop_{\mathcal{I}}(\hat{F}) \leq Prop_{\mathcal{I}}(\hat{G})$*. Analogously, we call* Q non-increasing in proportion*, if* $v_{\mathcal{I}}(\mathsf{Q}x\hat{F}(x)) \geq v_{\mathcal{I}}(\mathsf{Q}x\hat{G}(x))$*, under the same condition.*

The following lemma follows from the fact that the condition for monotonicity in proportion is weaker than the one for ordinary monotonicity.

Lemma 1. *If a semi-fuzzy quantifier* Q *is non-increasing (non-decreasing) in proportion, then it is also non-increasing (non-decreasing).*

The converse direction holds for all logical quantifiers, i.e. for quantitative quantifiers in the sense of Definition 4, as the following lemma demonstrates.

Lemma 2. *If a quantitative semi-fuzzy quantifier* Q *is non-increasing (non-decreasing), then it is also non-increasing (non-decreasing) in proportion.*

Proof. Let G_1 and G_2 be two formulas that fulfill $Prop_{\mathcal{I}}(G_1) \leq Prop_{\mathcal{I}}(G_2)$. We define two formulas H_1 and H_2 such that $Prop_{\mathcal{I}}(H_1) = Prop_{\mathcal{I}}(G_1)$ and $Prop_{\mathcal{I}}(H_2) = Prop_{\mathcal{I}}(G_2)$ and moreover $v_{\mathcal{I}}(H_1(c)) \leq v_{\mathcal{I}}(H_2(c))$ for every $c \in D$. Note this is always possible by introducing new monadic predicate symbols for H_1 and H_2.

Since Q is quantitative and non-decreasing, it follows that $v_{\mathcal{I}}(\mathsf{Q}xH_1(x)) \leq v_{\mathcal{I}}(\mathsf{Q}xH_2(x))$. It therefore remains to observe, that $v_{\mathcal{I}}(\mathsf{Q}xH_1(x)) = v_{\mathcal{I}}(\mathsf{Q}xG_1(x))$ and $v_{\mathcal{I}}(\mathsf{Q}xH_2(x)) = v_{\mathcal{I}}(\mathsf{Q}xG_2(x))$, which is clear since Q is quantitative and $Prop_{\mathcal{I}}(H_1) = Prop_{\mathcal{I}}(G_1)$, as well as $Prop_{\mathcal{I}}(H_2) = Prop_{\mathcal{I}}(G_2)$.

The case for non-increasing quantifiers is analogous. □

Glöckner singles out QFMs that fulfill the four above mentioned postulates ('axioms' in his terminology): Correct Generalization, Projection Quantifiers, Dualization, Preservation of Montonicity ($\geq$). Moreover, he formulates axioms called 'Internal Joins' and 'Functional Application', which are only relevant for quantifiers with more than one argument position. A QFM that satisfies these six postulates is called a *determiner fuzzification scheme (DFS)* in [19]. Glöckner states that DFSs "capture all important aspects of systematic and coherent interpretations". As we will see, this claim is problematic, since it neglects some features that might well be considered highly desirable, in particular from the point of view of linguistic adequateness.

We have already pointed out that Preservation of Monotonicity for non-decreasing quantifiers follows from Preservation of Monotonicity for non-increasing quantifiers if Dualization is assumed. But, as indicated above, Dualization might be considered problematic. Therefore it is better to explicitly consider both versions of Preservation of Monotonicity.

Another form of preserving monotonicity is called 'Monotonicity in the Quantifier' by Glöckner. It is not concerned with the truth degrees of the respective argument formulas, but rather with the relative degrees of truth resulting from different quantifiers applied to the same arguments. We suggest an alternative name for the relevant property and the corresponding principle.

Definition 11. *A quantifier* Q_1 *is called* at least as strong *as a quantifier* Q_2 *— in signs:* $\mathsf{Q}_1 \geq \mathsf{Q}_2$ *— if for every formula* F $v_{\mathcal{I}}(\mathsf{Q}_1xF(x)) \geq v_{\mathcal{I}}(\mathsf{Q}_2xF(x))$*.*

Again, the corresponding postulate already follows from others, when one is willing to follow Glöckner's (problematic) detour via non-monadic quantifiers. We prefer to state it explicitly.

Preservation of Quantifier Strength: If $\mathsf{Q}_1 \geq \mathsf{Q}_2$ then $\mathcal{F}(\mathsf{Q}_1) \geq \mathcal{F}(\mathsf{Q}_2)$.

A further rather natural principle calls for a certain 'robustness' in evaluating quantified fuzzy statements. It seeks to capture the intuition that small variations in the truth values of the (instantiated) argument formula should only lead to small changes of the truth value of the quantified formula.

Continuity in the Argument: For any semi-fuzzy Q the truth function of the corresponding fuzzy version $\mathcal{F}(\mathsf{Q})$ is *continuous*. More precisely, the following holds for all formulas F and F': for every $\epsilon > 0$ there exists $\delta > 0$ such that $\sup_{c \in D} |v_{\mathcal{I}}(F(c)) - v_{\mathcal{I}}(F'(c))| < \delta$ implies $|v_{\mathcal{I}}(\mathcal{F}(\mathsf{Q})xF(x)) - v_{\mathcal{I}}(\mathcal{F}(\mathsf{Q})F'(x))| < \epsilon$.

Glöckner states that this condition on $\mathcal{F}$ (called *arg-continuity* in [19]) "is crucial to the utility" and "must be possessed by every practical model". Nevertheless he does not include it in his list of axioms, although it is not derivable from his axioms, explaining that he aims for a level of generality that encompasses discontinuous cases.

A further desideratum, not considered explicitly by Glöckner, is called 'Coherence with Logic' in [8]. This postulate is specific to universal and existential quantification. For monadic quantification, it just states that the truth functions for $\forall$ and for $\exists$ are given by the infimum and supremum of the truth values of the argument formulas, respectively. Note that, while this is obvious for crisp arguments (i.e. ordinary classical logic), it amounts to an explicit desideratum for fuzzy (Type II) versions of $\forall$ and $\exists$. The name 'Coherence with Logic' for this simple principle becomes understandable only if one considers binary quantifiers and additionally requires that binary universal and existential quantification is reduced to the unary case by strict analogy to classical logic, using implication and conjunction, respectively.

A similar kind of principle is introduced in [9] under the name 'Average Property for the Identity Quantifier'. The identity quantifier in [9] is basically our Π and the property in our setting amounts to $v_{\mathcal{I}}(\mathcal{F}(\Pi)xF(x)) = Prop_{\mathcal{I}}F$. We combine 'Coherence with Logic' with the 'Average Property for the Identity Quantifier' and speak of the 'Infimum/Supremum/Average principle´.

Infimum/Supremum/Average principle: The following hold

- (Infimum) $v_{\mathcal{I}}(\mathcal{F}(\forall)xF(x)) = \inf_{c \in D} v_{\mathcal{I}}(F(c))$
- (Supremum) $v_{\mathcal{I}}(\mathcal{F}(\exists)xF(x)) = \sup_{c \in D} v_{\mathcal{I}}(F(c))$
- (Average) $v_{\mathcal{I}}(\mathcal{F}(\Pi)F(x)) = Prop_{\mathcal{I}}F$

In the rest of the paper, following what is customary in Mathematical Fuzzy Logic, the symbols $\forall$ and $\exists$ will denote both the semi-fuzzy and the fully-fuzzy quantifiers interpreted by the infimum and the supremum function, respectively. We also use the symbol Π for both semi-fuzzy and the fully-fuzzy quantifiers interpreted by the $Prop_{\mathcal{I}}$ function (see above). Hence the Supremum/Infimum/Average principle can be equivalently formulated as $v_{\mathcal{I}}(\mathcal{F}(\forall)xF(x)) = v_{\mathcal{I}}(\forall xF(x))$, $v_{\mathcal{I}}(\mathcal{F}(\exists)xF(x)) = v_{\mathcal{I}}(\exists xF(x))$, $v_{\mathcal{I}}(\mathcal{F}(\Pi)xF(x)) = v_{\mathcal{I}}(\Pi xF(x))$.

Finally, for the last desideratum, we consider sets of semi-fuzzy quantifiers which can be seen as *fuzzy partitions* [33] of a given domain. Recall that the concept of fuzzy partition arise naturally in approaches which interpret intermediate degrees of truth as probabilities that certain given labels are adequate, see e.g. [23]. A fuzzy partition determined by quantifiers is called a *quantified partition* and is just a collection of quantifiers which exhaust all possible adequate labels for a set.

As an informal example, one might consider the quantified partition *clearly less than half / about half / clearly more than half.* A precise rendering of the idea in our logical setting is given by the following.

Definition 12. *A set of semi-fuzzy quantifiers* $\{\mathsf{Q}_1, \ldots, \mathsf{Q}_n\}$ *is a* quantified partition *iff for every interpretation* $\mathcal{I}$*, classical formula* F*, we have* $v_{\mathcal{I}}(\mathsf{Q}_1 x F(x)) + \cdots + v_{\mathcal{I}}(\mathsf{Q}_n x F(x)) = 1$

It is straightforward to extend the definition above to fuzzy quantifiers and fuzzy sets. The following desideratum requests then that the fuzzification of a quantified partition is a quantified partition as well.

Preservation of quantified partitions: If $\{\mathsf{Q}_1, \ldots, \mathsf{Q}_n\}$ is a quantified partition, then $\{\mathcal{F}(\mathsf{Q}_1), \ldots, \mathcal{F}(\mathsf{Q}_n)\}$ is a quantified partition, too.

We emphasize that the above list of desiderata for fuzzification mechanisms is not exhaustive. Remember that we restrict attention to unary quantification, here. This renders postulates like Glöckner's axioms for 'internal joins' and 'functional application', but also 'argument insertion' irrelevant to our context. But even for unary quantifiers, further principles might be relevant, at least for particular application scenarios. In fact, we suggest that certain more general methodological principles for the design of fuzzy quantifier models should be respected as well. Most importantly, we think that ideally such models should be embeddable into certain full-fledged (deductive) fuzzy logics.

Definition 13. *We call a QFM* $\mathcal{F}$ expressible *in a given logic over a language* $\mathcal{L}$*, if for every semi-fuzzy quantifier* Q *there is a formula in the language* $\mathcal{L}$ *extended by* Q *that specifies the truth function of* $\mathcal{F}(\mathsf{Q})$*.*

In particular, we argue that it is desirable for a QFM to be expressible in Łukasiewicz logic Ł or at least in some conservative extension of Ł, like the one arising by adding the $\triangle$-operator or the basic random choice quantifier $\boldsymbol{\Pi}$, introduced in [13]. Note that Ł is distinguished among all the t-norm based fuzzy logic as the only one, where the truth functions for *all* connectives are continuous. However, Glöckner (in a footnote on page 156 of [19]) explicitly dismisses the choice of the residuum of the Łukasiewicz t-norm for modeling implication, citing the desirability of 'preserving Aristotelian squares', which is incompatible with Łukasiewicz logic under the suggested generalization of Aristotelian concepts to the fuzzy setting. (We refer to [25,26,28] for an alternative generalization based on fuzzy type theory, which extends Łukasiewicz logic to a higher order setting.)

That properly extended Łukasiewicz logic should indeed be considered as a distinguished basis for modeling reasoning with vague notions, including quantifier expressions has been argued, e.g., by Novak [27]. Another important reason why compatibility with Ł is highly desirable has been discussed at length in [14]: Giles [18] introduced a game based model of approximate reasoning that allows one to *derive* the truth functions of Ł from first principles, rather than to just stipulate them. In [14] and, more recently, in [1], this game based interpretation of Ł has been extended to derive (rather than just impose) truth functions for certain families of semi-fuzzy quantifiers, based on randomized choices of witness elements. The quantifier models for fully-fuzzy quantification introduced below do not directly depend on (extensions of) Giles's game; however they are partly inspired by this semantic framework, as we will point out at some places.

4 Łukasiewicz logic with Delta: Ł$_\triangle$

For reasons explained in the previous section, we take Ł$_\triangle$, i.e. first order Łukasiewicz logic extended with the Delta operator ($\triangle$) as our base logic (cf. [6]). The syntax of Ł$_\triangle$ formulas can be specified as follows:

$$\gamma ::= \bot \mid R(\boldsymbol{t}) \mid \neg\gamma \mid \triangle\gamma \mid \gamma \wedge \gamma \mid \gamma \vee \gamma \mid \gamma \odot \gamma \mid \gamma \oplus \gamma \mid \gamma \to \gamma \mid \gamma \leftrightarrow \gamma \mid \exists v\gamma \mid \forall v\gamma,$$

where R is our meta variable for predicate symbols and $\boldsymbol{t}$ denotes a sequence of terms (either constant symbols or object variables) matching the arity of the preceding predicate symbol; v is our meta variable for object variables, which we usually name $x, y, \ldots$.

As usual, an interpretation $\mathcal{I}$ assigns truth values in $[0,1]$ to atomic formulas. More precisely, $\mathcal{I}$ fixes a *finite* domain D and maps every predicate symbol P of arity n into a fuzzy relation $\widetilde{\mathrm{P}}_\mathcal{I}$ over D^n, i.e., a function $\widetilde{\mathrm{P}} : D^n \to [0,1]$. We identify the elements of D with constant symbols. $\mathcal{I}$ also contains a variable assignment $\xi_\mathcal{I}$ that sends every object variable into an element of D. We define

$$v_\mathcal{I}(P(t_1, \ldots, t_n)) = \widetilde{\mathrm{P}}_\mathcal{I}(c_1, \ldots, c_n),$$

where $c_i = \xi_\mathcal{I}(t_i)$, if t_i is a variable and $c_i = t_i$ if t_i is constant symbol, i.e., an element of D. The special atomic formula $\bot$ is interpreted as *(absolute) falsity*, hence we set $v_\mathcal{I}(\bot) = 0$. The evaluation function is extended from atomic to compound formulas as follows:

$$\begin{array}{ll}
v_\mathcal{I}(\triangle F) = 1 \text{ if } v_\mathcal{I}(F) = 1, \text{ else } v_\mathcal{I}(F) = 0 & v_\mathcal{I}(\neg F) = 1 - v_\mathcal{I}(F) \\
v_\mathcal{I}(F \to G) = \min(1, 1 - v_\mathcal{I}(F) + v_\mathcal{I}(G)) & v_\mathcal{I}(F \leftrightarrow G) = 1 - |v_\mathcal{I}(F) - v_\mathcal{I}(G)| \\
v_\mathcal{I}(F \wedge G) = \min(v_\mathcal{I}(F), v_\mathcal{I}(G)) & v_\mathcal{I}(F \vee G) = \max(v_\mathcal{I}(F), v_\mathcal{I}(G)) \\
v_\mathcal{I}(F \odot G) = \max(0, v_\mathcal{I}(F) + v_\mathcal{I}(G) - 1) & v_\mathcal{I}(F \oplus G) = \min(1, v_\mathcal{I}(F) + v_\mathcal{I}(G)) \\
v_\mathcal{I}(\forall x F(x)) = \min_{c \in D} v_\mathcal{I}(F(c)) & v_\mathcal{I}(\exists x F(x)) = \max_{c \in D} v_\mathcal{I}(F(c))
\end{array}$$

Actually, as is well known, there is a lot of redundancy in the definition of $v_\mathcal{I}(\cdot)$. For example, negation ($\neg$), weak and strong conjunction ($\wedge$ and $\odot$), weak and strong disjunction ($\vee$ and $\oplus$), as well as the biconditional ($\leftrightarrow$) can all be defined in terms of implication ($\to$) and falsity ($\bot$). Moreover, the universal quantifier ($\forall$) can be defined using negation and the existential quantifier ($\exists$), or vice versa, just like in classical logic.

In the remainder of this paper, we use the notation Ł$_\triangle(\mathsf{Q})$ to denote the expansion of the logic Ł$_\triangle$ with the quantifier Q. Such expansions will be handy when discussing various QFMs. As we will see below, for certain QFMs $\mathcal{F}$, Ł$_\triangle(\mathsf{Q})$ is not sufficiently expressive to define formulas of the form $\mathcal{F}(\mathsf{Q})xF(x)$ in the given language. We will therefore introduce further extensions of Łukasiewicz logic as needed. Throughout the rest of the paper, we will simply speak of 'formulas', when we refer to either a Ł$_\triangle$ formula or to a formula of some appropriate extension of Ł$_\triangle$, depending on the given context.

5 Expressibility of Quantifier Fuzzification Mechanisms

In this section we will recall the most well-behaved Quantifier Fuzzification Mechanisms (QFMs) introduced in the literature, and we discuss how to express them in suitable expansions of Łukasiewicz logic, and whether they fulfill the additional desideratum of Preservation of Monotonicity under conservative extensions, that we introduced in the previous section. We will focus our attention on three particularly well-behaved QFMs, which are introduced by Glöckner and qualify as Determiner Fuzzification Schemes (DFS). Such QFMs are originally denoted by Glöckner by $\mathcal{M}$, $\mathcal{M}_{CX}$, $\mathcal{F}^{OWA}$, but here we will use $\mathcal{F}^{\mathcal{M}}, \mathcal{F}^{CX}, \mathcal{F}^{OWA}$ instead, for coherence.

Let us consider first some expansions of the language of $\text{L}_\triangle$, in preparation for the discussion of expressibility of the QFMs. First, we consider extensions of the language with (countably many) propositional variables, which we usually denote by $p, q, \ldots$. This enables us to refer to all possible truth values, avoiding the use of an explicit, uncountable, additional set of truth constants.

Definition 14. *For any atomic formula A we define:*

$$v_{\mathcal{I}[p\to\alpha]}(A) = \begin{cases} \alpha & if\, A = p \\ v_{\mathcal{I}}(A) & otherwise \end{cases}$$

For arbitrary formulas F, we obtain $v_{\mathcal{I}[p\to\alpha]}(F)$ by ordinary truth-functional extension.

We can then define the following propositional version of the object quantifiers $\exists, \forall, \mathit{\Pi}$.

Definition 15. *For any interpretation $\mathcal{I}$, formula F we define:*

$$\begin{aligned} v_{\mathcal{I}}(\exists p F(p)) &= \sup_{\alpha\in[0,1]} v_{\mathcal{I}[p\to\alpha]}(F(p)) \\ v_{\mathcal{I}}(\forall p F(p)) &= \inf_{\alpha\in[0,1]} v_{\mathcal{I}[p\to\alpha]}(F(p)) \\ v_{\mathcal{I}}(\boldsymbol{\Pi} p F(p)) &= \int_0^1 v_{\mathcal{I}[p\to\alpha]}(F(p)) d\alpha. \end{aligned}$$

Let us also introduce some compact notation which will result useful in the following.

Definition 16. *Let F and G be formulas. We denote by $F^{\geq G}$ the formula $\triangle(G \to F)$, and by $F^{\leq G}$ the formula $\triangle(F \to G)$. We let also $F^{>G} := \neg F^{\leq G}$, $F^{<G} := \neg F^{\geq G}$ and $F^{=G} := F^{\leq G} \wedge F^{\geq G}$. Let $F(x)$ be a formula, with x variable occurring in F. To simplify the notation, we will denote the formulas of kind $F(x)^{\geq F(y)}$, $F(x)^{\geq F(c)}$ by $F^{\geq y}(x)$ and $F^{\geq c}(x)$, respectively.*

Before going into the technical details of expressibility, let us stress that we will think of fuzzification mechanisms from an angle slightly different than the usual one in the literature: we take a QFM $\mathcal{F}$ to be a way of reducing the evaluation of a formula $\mathcal{F}(\mathsf{Q})xF(x)$ over a fuzzy interpretation $\mathcal{I}$ to the evaluation of semi-fuzzy quantified sentences of the kind $\mathsf{Q}x\hat{F}(x)$ over a set of classical interpretations. Following the terminology used in supervaluationist accounts of vagueness [12,17], we call such a set of interpretations the *admissible precisifications* of $\mathcal{I}$. Note that, despite the fact that a precisification evaluates the argument formulas classically, the valuation under a precisification of a formula involving a semi-fuzzy quantifier might be an intermediate value in $[0, 1]$. Informally, admissible precisifications are the 'reasonable' ways of turning $\mathcal{I}$ into a precise (i.e. classical) interpretation of $F(x)$. As an example, consider a natural language predicate such as young. This is a inherently vague notion, but for some purposes (e.g. in legal contexts), one might need to deal with an artificially precise counterpart of it. We might then consider admissible precisifications of young, such as the sharply defined *young*$_1$:=*set of people of age* < 18, *young*$_2$:=*set of people of age* < 16, and so on. In our framework, since we model vague predicate as fuzzy sets, a natural candidate for the set of admissible precisifications is obtained by considering various

α-cuts: for any fuzzy set $\widetilde{\mathrm{F}}$ over domain D and any $\alpha \in [0,1]$, the α-cut $\widetilde{\mathrm{F}}_{\geq\alpha}$ is defined as the crisp set $\{c \mid \widetilde{\mathrm{F}}(c) \geq \alpha\}$.

Glöckner criticizes approaches based on α-cuts [19] and suggests instead a more complex route: first, he fixes a "cautiousness parameter" $\gamma \in [0,1]$, which determines two extreme α-cuts, a lower and an upper bound, and selects then all the precisifications included within the two bounds. The results obtained for each choice of the parameter $\gamma \in [0,1]$ are then aggregated by way of different operations. More formally, we can render Glöckner's approach in our setting as follows. Given a formula $F(x)$ a quantifier Q and a fuzzy interpretation $\mathcal{I}$ we let:

$$C_\gamma(F^{\mathcal{I}}) = \{\mathcal{I}' \mid F^{\mathcal{I}^{\geq 0.5+0.5\gamma}} \subseteq F^{\mathcal{I}'} \subseteq F^{\mathcal{I}^{>0.5-0.5\gamma}}\} \text{ for } \gamma \neq 0$$
$$C_0(F^{\mathcal{I}}) = \{\mathcal{I}^{\geq 0.5}, \mathcal{I}^{>0.5}\}$$

where γ is the cautiousness parameter and all the $\mathcal{I}'$ are precisifications of $\mathcal{I}$, i.e. they have the same domain of $\mathcal{I}$ and evaluate the predicate F classically.

We introduce then the following notation, standing for the maximum and a minimum value of semi-fuzzy quantified expressions over C_γ, respectively:

$$\bot_{\mathsf{Q}}(C_\gamma(F^{\mathcal{I}})) = \inf_{\mathcal{I}' \in C_\gamma(F^{\mathcal{I}})} v_{\mathcal{I}'}(\mathsf{Q}xF(x))$$
$$\top_{\mathsf{Q}}(C_\gamma(F^{\mathcal{I}})) = \sup_{\mathcal{I}' \in C_\gamma(F^{\mathcal{I}})} v_{\mathcal{I}'}(\mathsf{Q}xF(x)).$$

Different QFMs can be then defined, by aggregating the values $\bot_{\mathsf{Q}}(C_\gamma(F^{\mathcal{I}}))$ and $\top_{\mathsf{Q}}(C_\gamma(F^{\mathcal{I}}))$ over the different choices of $\gamma \in [0,1]$. Let us start from the QFM $\mathcal{F}^{\mathcal{M}}$ (see [19], Def. 7.22). In such mechanism, one computes first the *fuzzy* median $med(\top_{\mathsf{Q}}(C_\gamma(F^{\mathcal{I}})), \bot_{\mathsf{Q}}(C_\gamma(F^{\mathcal{I}})))$, for each γ. Recall that the fuzzy median is a function $med\colon [0,1]^2 \to [0,1]$, defined by:

$$med(x,y) = \begin{cases} min(x,y) & \text{if } min(x,y) > 0.5 \\ max(x,y) & \text{if } max(x,y) < 0.5 \\ 0.5 & \text{otherwise} \end{cases} \tag{1}$$

The evaluation of the QFM is then obtained by integrating over all the values of γ in $[0,1]$. We get thus the following:

$$v_{\mathcal{I}}(\mathcal{F}^{\mathcal{M}}(\mathsf{Q})xF(x)) = \int_0^1 med(\bot_{\mathsf{Q}}(C_\gamma(F^{\mathcal{I}}))\ ,\ \top_{\mathsf{Q}}(C_\gamma(F^{\mathcal{I}})))d\gamma$$

Let us now investigate how to express $\mathcal{F}^{\mathcal{M}}$ in expansions of $Ł_\triangle$. First, we note that the fuzzy median is expressible in $Ł_\triangle$ expanded with a truth constant $\overline{0.5}$ such that $v_{\mathcal{I}}(\overline{0.5}) = 0.5$, for every interpretation $\mathcal{I}$. We then let, for any formulas α, β

$$\alpha * \beta := ((\alpha \wedge \beta) \wedge (\alpha \wedge \beta)^{\leq \overline{0.5}}) \vee ((\alpha \vee \beta) \wedge (\alpha \vee \beta)^{\leq \overline{0.5}}) \vee \overline{0.5}$$

It is then straightforward to check that $v_{\mathcal{I}}(\alpha * \beta) = v_{\mathcal{I}}(\alpha)\ med\ v_{\mathcal{I}}(\beta)$.

Unfortunately, we are already stuck when trying to express $\bot_{\mathsf{Q}}(C_\gamma(F^{\mathcal{I}}))$ and $\bot_{\mathsf{Q}}(C_\gamma(F^{\mathcal{I}}))$. Indeed, we are not able to quantify over all precisifications in $C_\gamma(F^{\mathcal{I}})$, unless we are willing to move to the setting of second-order logic. Let us thus consider only the special case of monotone quantifiers. There, recalling the definition of $C_\gamma(F^{\mathcal{I}})$ and of $\bot_{\mathsf{Q}}$ and $\top_{\mathsf{Q}}$, we get, for non-decreasing quantifiers:

$$\bot_{\mathsf{Q}}(C_\gamma(F^{\mathcal{I}})) = v_{\mathcal{I}'}(\mathsf{Q}xF(x)) \ \text{ where } F^{\mathcal{I}'} = F^{\mathcal{I}^{\geq 0.5+0.5\gamma}}$$
$$\top_{\mathsf{Q}}(C_\gamma(F^{\mathcal{I}})) = v_{\mathcal{I}'}(\mathsf{Q}xF(x)) \ \text{ where } F^{\mathcal{I}'} = F^{\mathcal{I}^{>0.5-0.5\gamma}}$$

while for non-increasing quantifiers the values $\bot_{\mathsf{Q}}(C_\gamma(F^{\mathcal{I}}))$ and $\top_{\mathsf{Q}}(C_\gamma(F^{\mathcal{I}}))$ are just the other way round. In both cases the two values reduce then to α-cuts based precisifications.

In order to express such α-cuts in $\text{Ł}_\triangle$ we need to be able to express the truth value 0.5, and the product of the truth value of a formula by 0.5. Both are not expressible in Łukasiewicz logic, while this is possible in well-known expansions of it, such as Rational Łukasiewicz logic [4] and the expansion with the the propositional random choice connective π investigated in [11]. Here, however, we prefer to use the more expressive device of propositional quantifiers, since we will need them anyway in the following. Letting then $\overline{0.5} := \boldsymbol{\Pi}p\ p$, we get:

$$v_{\mathcal{I}}(\overline{0.5}) = v_{\mathcal{I}}(\boldsymbol{\Pi}p\ p) = \int_0^1 v_{\mathcal{I}_{[p\to\alpha]}}(p)d\alpha = 0.5$$

Similarly, letting

$$\overline{0.5}\alpha := \exists p(\alpha^{=p\oplus p} \wedge p)$$

we get that $v_{\mathcal{I}}(\overline{0.5}\alpha) = 0.5 \cdot v_{\mathcal{I}}(\alpha)$. Let now:

$$\mathsf{Q}^{(\vee,p)}xF(x) := \ (\mathsf{Q}xF^{\geq\overline{0.5}\oplus\overline{0.5}p}(x) \vee \mathsf{Q}xF^{>\overline{0.5}\ominus\overline{0.5}p}(x) \ \wedge \neg p^{=\overline{0.5}}) \vee (\mathsf{Q}xF^{\geq\overline{0.5}}(x) \vee \mathsf{Q}xF^{>\overline{0.5}}(x) \wedge p^{=\overline{0.5}})$$
$$\mathsf{Q}^{(\wedge,p)}xF(x) := \ (\mathsf{Q}xF^{\geq\overline{0.5}\oplus\overline{0.5}p}(x) \wedge \mathsf{Q}xF^{>\overline{0.5}\ominus\overline{0.5}p}(x) \ \wedge \neg p^{=\overline{0.5}}) \vee (\mathsf{Q}xF^{\geq\overline{0.5}}(x) \wedge \mathsf{Q}xF^{>\overline{0.5}}(x) \wedge p^{=\overline{0.5}}).$$

By straightforward checks, one gets:

$$v_{\mathcal{I}_{[p\to\gamma]}}(\mathsf{Q}^{(\vee,p)}xF(x)) = \top_{\mathsf{Q}}(C_\gamma(F^{\mathcal{I}}))$$
$$v_{\mathcal{I}_{[p\to\gamma]}}(\mathsf{Q}^{(\wedge,p)}xF(x)) = \bot_{\mathsf{Q}}(C_\gamma(F^{\mathcal{I}})).$$

In the light of the previous discussions, we finally obtain the following.

Theorem 1. *The QFM $\mathcal{F}^{\mathcal{M}}$ is expressible in $L_\triangle(\boldsymbol{\Pi})$ for monotone quantifiers. In particular, for each monotone semi-fuzzy quantifier Q, formula F, interpretation $\mathcal{I}$ we have :*

$$v_{\mathcal{I}}(\mathcal{F}^{\mathcal{M}}(\mathsf{Q})xF(x)) = v_{\mathcal{I}}(\boldsymbol{\Pi}p(\mathsf{Q}^{(\vee,p)}xF(x) * \mathsf{Q}^{(\wedge,p)}xF(x)))$$

Let us move now to the next DFS, $\mathcal{F}^{OWA}$. Translating within our terminology the Definition 8.13 in [19], we get

$$v_{\mathcal{I}}(\mathcal{F}^{OWA}(\mathsf{Q})xF(x)) = 0.5\int_0^1 \bot_{\mathsf{Q}}(C_\gamma(F^{\mathcal{I}}))d\gamma + 0.5\int_0^1 \top_{\mathsf{Q}}(C_\gamma(F^{\mathcal{I}}))d\gamma$$

In the light of what we have done for $\mathcal{F}^{\mathcal{M}}$, we obtain then the following for $\mathcal{F}^{OWA}$.

Theorem 2. *The QFM $\mathcal{F}^{OWA}$ is expressible in $Ł_\triangle(\boldsymbol{\Pi})$. In particular, for each monotone semi-fuzzy quantifier* Q*, formula* F*, interpretation* $\mathcal{I}$ *we have :*

$$v_{\mathcal{I}}(\mathcal{F}^{OWA}(\mathsf{Q})xF(x)) = v_{\mathcal{I}}(\overline{0.5}\Pi p\mathsf{Q}^{(\wedge,p)}xF(x) \oplus \overline{0.5}\Pi p\mathsf{Q}^{(\vee,p)}xF(x))$$

We note in particular that, for non-decreasing quantifiers, it is shown in [19] that $\mathcal{F}^{OWA}$ coincides with the methods based on the Choquet integral [7,19], i.e.

$$\mathcal{F}^{OWA}(\mathsf{Q})xF(x) = \int_0^1 v_{\mathcal{I}[p\to\alpha]}(\mathsf{Q}xF^{\geq p}(x))d\alpha$$

The function obtained above for $\mathcal{F}^{OWA}$ in the case of non-decreasing quantifiers can also be taken as a QFM on its own, which can be applied to non-increasing quantifiers as well. For further reference we denote it by $\mathcal{F}^R$ and let

$$v_{\mathcal{I}}(\mathcal{F}^R(\mathsf{Q})xF(x)) = \int_0^1 v_{\mathcal{I}[p\to\alpha]}(\mathsf{Q}xF^{\geq p}(x))d\alpha)) \tag{2}$$

We can express $\mathcal{F}^R$ (or, equivalently $\mathcal{F}^{OWA}$ over non-decreasing quantifier) with a simpler formula than that in Theorem 2, although we still need to resort to the quite expressive language of $Ł_\triangle(\boldsymbol{\Pi})$. Indeed, we obtain the following.

Theorem 3. *The QFM $\mathcal{F}^R$ is expressible in the logic $Ł_\triangle(\boldsymbol{\Pi})$. In particular, for any semi-fuzzy quantifier* Q*, and formula* F*, we have:*

$$v_{\mathcal{I}}(\mathcal{F}^R(\mathsf{Q})xF(x)) = v_{\mathcal{I}}(\boldsymbol{\Pi}p\mathsf{Q}xF^{\geq p}(x)).$$

Proof. Straightforward computation. □

The QFM $\mathcal{F}^R$ has also a game-semantic reading, as we show in the following.

Example 1. Let D be a domain such that $|D| = 4$, and assume that the four objects from the domain represent balls. Two such balls are fully black , i.e., for a fuzzy predicate B, standing for the property of being black, we have $v_{\mathcal{I}}(B(c_1)) = v_{\mathcal{I}}(B(c_2)) = 1$, while the remaining two are grey, or putting it differently, black to a certain degree. We label those two grey balls c_3 and c_4, with $v_{\mathcal{I}}(B(c_3)) = 0.9$ and $v_{\mathcal{I}}(B(c_4)) = 0.7$. Let us consider now the Type III quantifier Π.

Game semantically, the evaluation of $\mathcal{F}^R(\Pi)xB(x)$, based on a randomly sampled threshold value, can be interpreted as follows. *Nature* samples an assignment $v_{\mathcal{I}[p\to\alpha]}()$ to the propositional variable p, and the proponent player has to accept the payoff associated to asserting $\Pi xB^{\geq p}(x)$, with respect to this assignment to p. This payoff corresponds to the (possibly) intermediate truth value, as Π is a semi-fuzzy quantifier. Then, the overall truth value is the average of the individual results. The result is straightforward and yields the truth value 0.9.

We conclude the discussion of the QFMs $\mathcal{F}^{\mathcal{M}}$,$\mathcal{F}^{OWA}$ and $\mathcal{F}^R$, by showing that they comply with our additional desideratum of Preservation of Monotonicity under Conservative Extension. For the other desiderata, we refer the reader to [7,9].

Theorem 4. *The QFMs $\mathcal{F}^{\mathcal{M}}$, $\mathcal{F}^{OWA}$ and $\mathcal{F}^R$ comply with* Preservation of Monotonicity under Conservative Extension*.*

Proof. We show the non-decreasing case of Preservation of Monotonicity under Conservative Extension.

Let Q be a semi-fuzzy quantifier and assume $\mathcal{I}'$ conservatively extends $\mathcal{I}$, and that for all crisp formulas $\hat{F}$ we have that $v_{\mathcal{I}}(\mathsf{Q}x\hat{F}(x)) \leq v_{\mathcal{I}'}(\mathsf{Q}x\hat{F}(x))$. It then holds that $v_{\mathcal{I}[p\to\alpha]}(\mathsf{Q}xF^{\geq p}(x)) \leq v_{\mathcal{I}'[p\to\alpha]}(\mathsf{Q}xF^{\geq p}(x))$, under any assignment to p. As a consequence, $v_{\mathcal{I}[p\to\alpha]}(\mathsf{Q}^{(\vee,p)}xF^{\geq p}(x)) \leq v_{\mathcal{I}'[p\to\alpha]}(\mathsf{Q}^{(\vee,p)}xF^{\geq p}(x))$ and $v_{\mathcal{I}[p\to\alpha]}(\mathsf{Q}^{(\wedge,p)}xF^{\geq p}(x)) \leq v_{\mathcal{I}'[p\to\alpha]}(\mathsf{Q}^{(\wedge,p)}xF^{\geq p}(x))$. Hence, we have that $v_{\mathcal{I}}(\mathcal{F}^{\mathcal{M}}(\mathsf{Q})xF(x)) \leq v_{\mathcal{I}'}(\mathcal{F}^{\mathcal{M}}(\mathsf{Q})xF(x))$, $v_{\mathcal{I}}(\mathcal{F}^{OWA}(\mathsf{Q})xF(x)) \leq v_{\mathcal{I}'}(\mathcal{F}^{OWA}(\mathsf{Q})xF(x))$, and also $v_{\mathcal{I}}(\mathcal{F}^{R}(\mathsf{Q})xF(x)) \leq v_{\mathcal{I}'}(\mathcal{F}^{R}(\mathsf{Q})xF(x))$. The non-increasing case is similar.

□

Before proceeding to the last of Glöckner's main QFMs, we note that our logical framework immediately suggest a variant of $\mathcal{F}^R$, which take α-cuts at the level of atomic formulas, rather than of the whole argument. Let us first introduce the following.

Definition 17. *For any $\alpha \in [0,1]$, let $\mathcal{I}^{at}_{\geq\alpha}$ be the interpretation, such that for any atomic formula A, $v_{\mathcal{I}^{at}_{\geq\alpha}}(A) = 1$ if $v_{\mathcal{I}}(A) \geq \alpha$ and $v_{\mathcal{I}^{at}_{\geq\alpha}}(A) = 0$ otherwise.*

The precisifications such as those in the Definition above can also be represented syntactically via the following formulas, in analogy with Definition 16.

Definition 18. *Let F, G be formulas. We denote by $F^{\geq G}_{at}$ the formula obtained replacing any atomic formula A in F by $\triangle(G \to A)$. Similarly, we define $F^{>G}_{at}$,$F^{\leq G}_{at}$, $F^{<G}_{at}$,$F^{=G}_{at}$.*

We now introduce another QFM, based on atomic-level α-cuts.

Definition 19 (QFM $\mathcal{F}^L$). *Let Q be any semi-fuzzy quantifier, F formula. We let:*

$$v_{\mathcal{I}}(\mathcal{F}^L(\mathsf{Q})xF(x)) = \int_0^1 v_{\mathcal{I}^{at}_{\geq\alpha}}(\mathsf{Q}xF(x))d\alpha \qquad (3)$$

The following requires then just an easy check.

Theorem 5. *The QFM $\mathcal{F}^L$ is expressible in $\text{Ł}_\triangle(\boldsymbol{\Pi})$, as follows:*

$$v_{\mathcal{I}}(\mathcal{F}^L(\mathsf{Q})xF(x)) = v_{\mathcal{I}}(\boldsymbol{\Pi}p\mathsf{Q}xF^{\geq p}_{at}(x))$$

Since the QFM $\mathcal{F}^L$ takes α-cuts already at the atomic level, all the connectives of the formula in the argument of the quantifier behave as the classical ones: in this sense the mechanism can be seen as a special case of the level-based generalized quantification mechanism [32], which is in turn inspired by the notion of gradual element [10]. Note that, however, contrary to [32], in our formulation the QFM fails to comply with the Desideratum of Correct Generalization, as the following theorem shows. This is due to our different framework, which includes in the language also the connective $\triangle$.

Example 2. Assume $D = \{c_1, c_2\}$ with $v_{\mathcal{I}}(A(c_1)) = 0.2$ and $v_{\mathcal{I}}(A(c_2)) = 1$ and let $\hat{F}(x) = \triangle A(x)$. Clearly, $v_{\mathcal{I}}(\forall x\hat{F}(x)) = 0$. On the other hand, for any $\epsilon > 0$, we have $v_{\mathcal{I}}(\mathcal{F}^R(\forall)x\hat{F}(x)) = 0.2 \cdot v_{\mathcal{I}_{\geq 0.2}}(\forall x\hat{F}(x)) + 0.8 \cdot v_{\mathcal{I}_{\geq 0.2+\epsilon}}(\forall x\hat{F}(x)) = 0.2 \cdot 1 + 0.8 \cdot 0 = 0.2 \neq 0$.

This already tells us that $\mathcal{F}^L$ and $\mathcal{F}^R$ cannot coincide, since $\mathcal{F}^R$ complies instead with the desideratum of correct generalization. We provide an explicit counterexample that shows that they are different, thus correcting what has been claimed in [1].

Example 3. Assume that $D = \{c\}$, and Q is either $\forall, \exists$ or Π. $A(x)$ and $B(x)$ are fuzzy atoms, with $0.1 = a = v_{\mathcal{I}}(A(c)) < v_{\mathcal{I}}(B(c)) = b = 0.2$. Then we look at $F(x)$ defined as $A(x) \oplus B(x)$, and plug it into the QFMs:

$$v_{\mathcal{I}}(\mathcal{F}^L(\mathsf{Q})xF(x)) = v_{\mathcal{I}}(\boldsymbol{\Pi} p(A(c)^{\geq p} \oplus B(c)^{\geq p}))$$
$$= \int_0^1 v_{\mathcal{I}[p\to\alpha]}(A(c)^{\geq p} \oplus B(c)^{\geq p})d\alpha = \int_0^a 1d\alpha + \int_a^b 1d\alpha + \int_b^1 0d\alpha = b$$

$$v_{\mathcal{I}}(\mathcal{F}^R(\mathsf{Q})xF(x)) = v_{\mathcal{I}}(\boldsymbol{\Pi} p(A(c) \oplus B(c))^{\geq p})$$
$$= \int_0^1 v_{\mathcal{I}[p\to\alpha]}((A(c) \oplus B(c))^{\geq p})d\alpha = \int_0^{a+b} 1d\alpha + \int_{a+b}^1 0d\alpha = \min(1, a+b)$$

Since $b = 0.2$ and $min(1, a+b) = 0.3$, we get that $v_{\mathcal{I}}(\mathcal{F}^R(\mathsf{Q})xF(x)) \neq v_{\mathcal{I}}(\mathcal{F}^L(\mathsf{Q})xF(x))$.

Let us now conclude this section, discussing Glöckner's DFS $\mathcal{F}^{CX}$ [19]. In our terminology, which slightly departs from Glöckner's, we define it as :

$$v_{\mathcal{I}}(\mathcal{F}^{CX}(\mathsf{Q})xF(x)) = \sup\{\min(\Xi_{\mathcal{I}_1,\mathcal{I}_2}(\mathsf{Q}xF(x)), v_{\mathcal{I}_1,\mathcal{I}_2}(\mathsf{Q}xF(x))) \mid \mathcal{I}_1, \mathcal{I}_2 \text{ precisification}, F^{\mathcal{I}_1} \subseteq F^{\mathcal{I}_2}\}$$

where

$$\begin{aligned}\Xi_{\mathcal{I}_1,\mathcal{I}_2}(\mathsf{Q}xF(x)) &= \min(\inf\{v_{\mathcal{I}}(F(c)) \mid c \in F^{\mathcal{I}_1}\}, \inf\{1 - v_{\mathcal{I}}(F(c)) \mid c \notin F^{\mathcal{I}_2}\}) \\ &= \min(\inf\{v_{\mathcal{I}}(F(c)) \mid c \in F^{\mathcal{I}_1}\}, 1 - \sup\{v_{\mathcal{I}}(F(c)) \mid c \notin F^{\mathcal{I}_2}\})\end{aligned}$$

and

$$v_{\mathcal{I}_1,\mathcal{I}_2}(\mathsf{Q}xF(x)) = \inf\{v_{\mathcal{I}'}(\mathsf{Q}xF(x)) \mid \mathcal{I}' \text{ precisification }, F^{\mathcal{I}_1} \subseteq F^{\mathcal{I}'} \subseteq F^{\mathcal{I}_2}\}.$$

As for $\mathcal{F}^{\mathcal{M}}$ and $\mathcal{F}^{OWA}$ we investigate expressibility, focusing our attention on monotone quantifiers. We refer the reader to [19,9] for the remaining desiderata.

Before proceeding, let us fix some notation. Given a formula F and interpretation $\mathcal{I}$, we let $\alpha_0 = 0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_n \leq \alpha_{n+1} = 1$ where $\alpha_1, \ldots, \alpha_n$ are the n distinct truth values taken by $v_{\mathcal{I}}(F(x))$. Let c_i be constants (domain elements) such that $\alpha_i = v_{\mathcal{I}}(F(c_i))$, hence in particular $v_{\mathcal{I}}(F(c_i) \leq v_{\mathcal{I}}(F(c_{i+1}))$. Slightly abusing notation, for simplicity, we let also $F(c_0)$ stand for $\perp$ and $F(c_{n+1})$ for $\top$. We can prove the following.

Lemma 3. *Let Q be a monotone quantifier, $\mathcal{I}$ an interpretation, F a formula. If Q is non-decreasing, we have :*

$$v_{\mathcal{I}}(\mathcal{F}^{CX}(\mathsf{Q})F(x)) = \max_{0\leq i\leq n+1} \min(v_{\mathcal{I}}(F(c_i)), v_{\mathcal{I}}(\mathsf{Q}xF^{\geq c_i}(x)))$$

while, if Q is nonincreasing

$$v_{\mathcal{I}}(\mathcal{F}^{CX}(\mathsf{Q})xF(x)) = \min_{0\leq i\leq n+1} \max(1 - v_{\mathcal{I}}(F(c_i)), v_{\mathcal{I}}(\mathsf{Q}xF^{\geq c_i}(x)))$$

Proof. We can reduce the choice of the precisifications $\mathcal{I}_1, \mathcal{I}_2$ to the choice of two elements c_i, c_j such that $i \geq j, v_{\mathcal{I}_1}(F(c)) = 1$ iff $v_{\mathcal{I}}(F(c)) \geq v_{\mathcal{I}}(F(c_i))$ and $v_{\mathcal{I}_2}(F(c)) = 1$ iff $v_{\mathcal{I}}(F(c)) \geq v_{\mathcal{I}}(F(c_j))$. Hence, we get:

$$\begin{aligned}\Xi_{\mathcal{I}_1,\mathcal{I}_2}(\mathsf{Q}xF(x)) &= \min(\inf\{v_{\mathcal{I}}(F(c)) \mid c \in F^{\mathcal{I}_1}\}, 1 - \sup\{v_{\mathcal{I}}(F(c)) \mid c \notin F^{\mathcal{I}_2}\}) \\ &= \min(v_{\mathcal{I}}(F(c_i)), 1 - v_{\mathcal{I}}(F(c_{j-1}))).\end{aligned}$$

Now, if Q is a non-decreasing semi-fuzzy quantifier, we get

$$v_{\mathcal{I}_1,\mathcal{I}_2}(\mathsf{Q}xF(x)) = v_{\mathcal{I}_1}(\mathsf{Q}xF(x))) = v_{\mathcal{I}}(\mathsf{Q}xF^{\geq c_i}(x))$$

hence:

$$v_{\mathcal{I}}(\mathcal{F}^{CX}(\mathsf{Q})xF(x)) = \max_{0 \leq j \leq i \leq n+1} \min(v_{\mathcal{I}}(F(c_i)), 1 - v_{\mathcal{I}}(F(c_{j-1})), v_{\mathcal{I}}(\mathsf{Q}xF^{\geq c_i}(x)))$$

Note that, for every choice of c_i the highest value of $1 - v_{\mathcal{I}}(F(c_{j-1}))$ is obtained for $j = 1$, which implies $\min(v_{\mathcal{I}}(F(c_i)), 1 - v_{\mathcal{I}}(F(c_0))) = \min(v_{\mathcal{I}}(F(c_i)), 1) = v_{\mathcal{I}}(F(c_i))$. We obtain thus :

$$v_{\mathcal{I}}(\mathcal{F}^{CX}(\mathsf{Q})xF(x)) = \max_{0 \leq i \leq n+1} \min(v_{\mathcal{I}}(F(c_i)), v_{\mathcal{I}}(\mathsf{Q}xF^{\geq c_i}(x)))$$

If Q is non-increasing, we have that the quantifier $\neg\mathsf{Q}$ is non-decreasing, and since $\mathcal{F}^{CX}$ complies with preservation of external negation, we obtain:

$$\begin{aligned}v_{\mathcal{I}}(\mathcal{F}^{CX}(\mathsf{Q})xF(x)) &= 1 - v_{\mathcal{I}}(\neg\mathcal{F}^{CX}(\mathsf{Q})xF(x)) = 1 - v_{\mathcal{I}}(\mathcal{F}^{CX}(\neg\mathsf{Q})xF(x)) \\ &= 1 - \max_{0 \leq i \leq n+1} \min(v_{\mathcal{I}}(F(c_i)), 1 - v_{\mathcal{I}}(\mathsf{Q}xF^{\geq c_i}(x))) \\ &= \min_{0 \leq i \leq n+1} \max(1 - v_{\mathcal{I}}(F(c_i)), v_{\mathcal{I}}(\mathsf{Q}xF^{\geq c_i}(x)))\end{aligned}$$

□

The representation of $\mathcal{F}^{CX}$ that we obtained can also be reformulated in terms of α-cut based precisifications, as we show in the following.

Lemma 4. *Let $\mathcal{I}$ be an interpretation, F a formula, Q be a monotone quantifier. If Q is non-decreasing, we have:*

$$(1) \quad v_{\mathcal{I}}(\mathcal{F}^{CX}(\mathsf{Q})xF(x)) = \sup_{\alpha \in [0,1]} \min(\alpha, v_{\mathcal{I}[p \to \alpha]}(\mathsf{Q}xF^{\geq p}(x))$$

If Q is non-increasing, we have:

$$(2) \quad v_{\mathcal{I}}(\mathcal{F}^{CX}(\mathsf{Q})xF(x)) = \inf_{\alpha \in [0,1]} \max(1 - \alpha, v_{\mathcal{I}[p \to \alpha]}(\mathsf{Q}xF^{\geq p}(x)))$$

Proof. Let us prove (1). By the definition of the α_i, it follows that, for any α in $(\alpha_{i-1}, \alpha_i]$, we get $v_{\mathcal{I}[p \to \alpha]}(\mathsf{Q}xF^{\geq p}(x)) = v_{\mathcal{I}[p \to \alpha_i]}(\mathsf{Q}xF^{\geq p}(x))$. We have then:

$$\sup_{\alpha \in (\alpha_{i-1}, a_i]} \min(\alpha, v_{\mathcal{I}[p \to \alpha]}(\mathsf{Q}xF^{\geq p}(x)) =$$

$$\min(\sup_{\alpha \in (\alpha_{i-1}, a_i]} \alpha, v_{\mathcal{I}[p \to \alpha_i]}(\mathsf{Q}xF^{\geq p}(x)) \ ) =$$

$$\min(\alpha_i, v_{\mathcal{I}[p \to \alpha_i]}(\mathsf{Q}xF^{\geq p}(x))).$$

From this we get that $\sup_{\alpha\in[0,1]}\min(\alpha, v_{\mathcal{I}[p\to\alpha]}(\mathsf{Q}xF^{\geq p}(x)))$ is equal to:

$$\begin{aligned}
&\max(\min(\alpha_0, v_{\mathcal{I}[p\to\alpha_0]}(\mathsf{Q}xF^{\geq p}(x))), \max_{1\leq i\leq n+1}(\sup_{\alpha\in(a_{i-1},\alpha_i]} \min(\alpha, v_{\mathcal{I}[p\to\alpha]}(\mathsf{Q}xF^{\geq p}(x))))\\
&= \max_{0\leq i\leq n+1} \min(\alpha_i, v_{\mathcal{I}[p\to\alpha_i]}(\mathsf{Q}xF^{\geq p}(x)))\\
&= \max_{0\leq i\leq n+1} \min(v_{\mathcal{I}}(F(c_i)), v_{\mathcal{I}}(\mathsf{Q}xF^{\geq c_i}(x))) = v_{\mathcal{I}}(\mathcal{F}^{CX}(\mathsf{Q})F(x)).
\end{aligned}$$

where the last equation is given by Lemma 3. The statement in (2) is shown in a similar fashion, applying Lemma 3 to the case of non-increasing quantifiers.

□

We can now easily obtain the expressibility of $\mathcal{F}^{CX}$, using either propositional quantifiers or only Ł$_\triangle$, at the price of a somewhat more complex representation.

Theorem 6. *The QFM $\mathcal{F}^{CX}$ is expressible in the logics $Ł_\triangle(\exists)$ and $Ł_\triangle$ for monotone quantifiers. In particular, for any non-decreasing semi-fuzzy quantifier Q, and formula F, we have:*

$$\begin{aligned}
v_{\mathcal{I}}(\mathcal{F}^{CX}(\mathsf{Q})xF(x)) &= v_{\mathcal{I}}(\exists p(p \wedge \mathsf{Q}xF^{\geq p}(x)))\\
v_{\mathcal{I}}(\mathcal{F}^{CX}(\mathsf{Q})xF(x)) &= v_{\mathcal{I}}(\exists y(F(y) \wedge \mathsf{Q}xF^{\geq y}(x)) \vee \mathsf{Q}x\triangle F(x)).
\end{aligned}$$

On the other hand, for non-increasing semi-fuzzy quantifiers we have:

$$\begin{aligned}
v_{\mathcal{I}}(\mathcal{F}^{CX}(\mathsf{Q})xF(x)) &= v_{\mathcal{I}}(\forall p(\neg p \vee \mathsf{Q}xF^{\geq p}(x)))\\
v_{\mathcal{I}}(\mathcal{F}^{CX}(\mathsf{Q})xF(x)) &= v_{\mathcal{I}}(\forall y(\neg F(y) \vee \mathsf{Q}xF^{\geq y}(x)) \wedge \mathsf{Q}x\triangle F(x)).
\end{aligned}$$

Proof. Both in the case of non-decreasing and non-increasing quantifiers, the first claim follows from Lemma 4, while the second is a direct consequence of the Lemma 3.

□

Let us show now the *Preservation of Monotonicity under Conservative Extension.*

Theorem 7. *The QFM $\mathcal{F}^{CX}$ complies with the desideratum of* Preservation of Monotonicity under Conservative Extension*.*

Proof. We show the the case of Preservation of Monotonicity under Conservative Extension for non-decreasing quantifier. Other cases are similar.

Let Q be a semi-fuzzy quantifier and assume $\mathcal{I}'$ conservatively extends $\mathcal{I}$, as well as for all crisp formulas $\hat{F}$ we have that $v_{\mathcal{I}}(\mathsf{Q}x\hat{F}(x)) \leq v_{\mathcal{I}'}(\mathsf{Q}x\hat{F}(x))$. It then holds that $v_{\mathcal{I}[p\to\alpha]}(\mathsf{Q}xF^{\geq p}(x)) \leq v_{\mathcal{I}'[p\to\alpha]}(\mathsf{Q}xF^{\geq p}(x))$, under any assignment to p. Hence, it also holds:

$$v_{\mathcal{I}[p\to\alpha]}(p \wedge \mathsf{Q}xF^{\geq p}(x)) \leq v_{\mathcal{I}'[p\to\alpha]}(p \wedge \mathsf{Q}xF^{\geq p}(x)),$$

under any assignment to p. Therefore:

$$v_{\mathcal{I}}(\exists p(p \wedge \mathsf{Q}xF^{\geq p}(x))) \leq v_{\mathcal{I}'}(\exists p(p \wedge \mathsf{Q}xF^{\geq p}(x))).$$

□

Note that the function we obtained above for $\mathcal{F}^{CX}$ in case of non-decreasing quantifiers has a quite transparent reading: the fuzzy argument F, of a semi-fuzzy quantifier Q, is cut at various level $\alpha \in [0,1]$, determining suitable precisifications, where truth values of at least α are rendered true while all others are projected to false. The idea is then to pick an optimal threshold α which maximizes the truth value of the semi-fuzzy quantifier over the corresponding precisification, in conjunction with the threshold value α itself. This, as shown also in [19], is a reformulation of the Sugeno integral, which falls under the broader family of "possibilistic method" for QFMs [7]. Let us discuss some examples of $\mathcal{F}^{CX}$ over non-decreasing quantifiers.

Example 4. Let us get back to the setting of Example 1. From a game semantic perspective, we can understand the functioning of $\mathcal{F}^{CX}$ over a non-decreasing quantifier as follows. If we evaluate $\mathcal{F}^{CX}(\Pi)xB(x)$, the proponent player has to decide how many of the grey balls are accepted as black balls. If the threshold value is chosen as 0.9, 3 of the 4 balls are accepted as black. If the proponent also accepts the last ball, c_4, as black, he would have all 4 balls to qualify as black, but the conjunction in $\mathcal{F}^{CX}$ with the threshold value 0.7 would make that a non-rational move. Hence, $v_{\mathcal{I}}(\mathcal{F}^{CX}(\Pi)xB(x)) = 0.75$.

Example 5. Let us fix the domain D such that $\mid D \mid = 4$. Regarding Q, we use the quantifier "at least 50%", modeled as Type III quantifier. It is non-decreasing and denoted by $\mathsf{Q}_{[\geq \frac{1}{2}]}$, with: $v_{\mathcal{I}}(\mathsf{Q}_{[\geq \frac{1}{2}]}x\hat{F}(x)) = \min(1, 2 \cdot \mathit{Prop}_{\mathcal{I}}\hat{F})$, for a crisp formula $\hat{F}$. Also, we assume that there are two fuzzy predicates A and B, with the following truth value distribution: $v_{\mathcal{I}}(A(c_1)) = v_{\mathcal{I}}(A(c_2)) = 1, v_{\mathcal{I}}(A(c_3)) = v_{\mathcal{I}}(A(c_4)) = 0$, and $v_{\mathcal{I}}(B(c_1)) = v_{\mathcal{I}}(B(c_2)) = v_{\mathcal{I}}(B(c_3)) = v_{\mathcal{I}}(B(c_4)) = 0.7$. Hence, $\mathit{Prop}_{\mathcal{I}}A = 0.5 \leq 0.7 = \mathit{Prop}_{\mathcal{I}}B$. This means that, although there are more fully true witnesses for A than for B, the proportion of objects fulfilling A is lower than the one of B. Still, $v_{\mathcal{I}}(\mathcal{F}^{CX}\mathsf{Q}_{[\geq \frac{1}{2}]}xA(x))$, which is 1, is greater than $v_{\mathcal{I}}(\mathcal{F}^{CX}\mathsf{Q}_{[\geq \frac{1}{2}]}xB(x))$, which is 0.7.

This shows also that it is not immediate to extend Lemma 2 to fully fuzzy quantifiers. Indeed, although the quantifier $\mathcal{F}^{CX}(\mathsf{Q}_{[\geq \frac{1}{2}]})$ is non-decreasing, since $\mathcal{F}^{CX}$ preserves monotonicity, it is not non-decreasing *in proportion*. The role of proportions for fully fuzzy quantifier is indeed less central than for semi-fuzzy ones: even in principle, we do not expect that the fuzzified version of a proportional semi-fuzzy quantifier is proportional, see Definition 1 and the discussion thereafter. This also explains why we confine ourselves to "preservation of monotonicity" as a desideratum, and we would rather not consider "preservation of monotonicity in proportion" as such.

Given its rather transparent reading, one might wonder whether the function we derived by restricting $\mathcal{F}^{CX}$ to the special case of non-decreasing quantifier could be treated as a QFM on its own right, to be applied to any semi-fuzzy quantifier. However, if we apply the function (a) (equivalently (b)) in Lemma 3 rather than (c) (equivalently (d)) to the case of non-increasing quantifiers, we would obtain the following, degenerate, case:

$$v_{\mathcal{I}}(\exists p(p \wedge \mathsf{Q}xF^{\geq p}(x))) = v_{\mathcal{I}}(\mathsf{Q}xF^{\geq 1}(x)).$$

As an immediate consequence of this fact, we also see that preservation of internal and external negation cannot be fulfilled, even when the function is applied to a non-decreasing quantifier Q. Indeed both $\mathsf{Q}^{\neg}$ and $\neg\mathsf{Q}$ are then non-increasing quantifiers, and we would obtain, for the internal negation condition:

$$v_{\mathcal{I}}(\exists p(p \wedge \mathsf{Q}^{\neg}xF^{\geq p}(x))) = v_{\mathcal{I}}(\mathsf{Q}^{\neg}xF^{\geq 1}(x)) \neq v_{\mathcal{I}}(\exists p(p \wedge \mathsf{Q}x(\neg F)^{\geq p}(x)))$$

and, similarly, for external negation:

$$v_{\mathcal{I}}(\exists p(p \wedge {}^{\neg}\mathsf{Q}xF^{\geq p}(x))) = v_{\mathcal{I}}({}^{\neg}\mathsf{Q}xF^{\geq 1}(x)) \neq v_{\mathcal{I}}(\neg\exists p(p \wedge \mathsf{Q}xF^{\geq p}(x))).$$

Note that, since the dual of a nondecreasing quantifier is itself non-decreasing, it follows from Glöckner results on $\mathcal{F}^{CX}$, that the desideratum of dualization is instead satisfied by $\exists p(p \wedge \mathsf{Q}xF^{\geq p}(x))$.

To conclude this section, we consider two further issues. First, let us consider what happens when we further restrict the QFMs considered so far to non-decreasing Type I quantifiers. We can show that $\mathcal{F}^{CX}$ there coincides with $\mathcal{F}^{R}$, and hence with $\mathcal{F}^{OWA}$.

Theorem 8. *For any formula F, and non-decreasing Type I quantifier Q, we have:*

$$v_{\mathcal{I}}(\mathcal{F}^{CX}(\mathsf{Q})xF(x)) = v_{\mathcal{I}}(\mathcal{F}^{OWA}(\mathsf{Q})xF(x)) = v_{\mathcal{I}}(\mathcal{F}^{R}(\mathsf{Q})xF(x)).$$

Proof. First, note that the quantifiers we consider are either quantitative or of the form Δ_c for some $c \in D$. For the latter, the claim holds, since, by Theorem 4 and Theorem 7, we have $v_{\mathcal{I}}(\mathcal{F}^{CX}(\Delta_c)xF(x)) = v_{\mathcal{I}}(F(c)) = v_{\mathcal{I}}(\mathcal{F}^{R}(\Delta_c)xF(x))$.

Let us assume now that Q is a quantitative quantifier. Then, by Lemma 2, Q is non-decreasing in proportion. If $v_{\mathcal{I}}(\mathsf{Q}x\hat{F}(x)) = 0$ for any interpretation $\mathcal{I}$, then the claim is obvious. Let us assume that under some interpretation $\mathcal{I}$, $v_{\mathcal{I}}(\mathsf{Q}x\hat{F}(x)) = 1$. Then, since Q is non-decreasing in proportion there is a value $k \in \{0, 1, \ldots, n\}$ such that, for any crisp formula $\hat{F}$, if $Prop_{\mathcal{I}}\hat{F} \geq \frac{k}{n}$ then we also have $v_{\mathcal{I}}(\mathsf{Q}x\hat{F}(x)) = 1$. Let $\alpha_0, \ldots, \alpha_{n+1}$ be as in Lemma 3. We define the set $I_{\mathsf{Q}} = \{k, \ldots, n\}$ and the indicator function $\mathbb{I}_{\{x \in A\}}$, which is 1, if $x \in A$ and 0 otherwise. We have:

(1) $v_{\mathcal{I}}(\mathcal{F}^{R}(\mathsf{Q})xF(x)) = \sum_{i=0}^{n}(\alpha_{i+1} - \alpha_i) \cdot \mathbb{I}_{\{n-j \in I_{\mathsf{Q}}\}}$, and
(2) $v_{\mathcal{I}}(\mathcal{F}^{CX}(\mathsf{Q})xF(x)) = \max_{(i \in \{1,\ldots,n+1\})}(\min(\alpha_i, \mathbb{I}_{\{n-(i-1) \in I_{\mathsf{Q}}\}})))$.

For (1), the indicator function is 1, if $n - i \geq k$, which is equivalent to $k \leq n - i$, and for (2) the indicator function is 1, if $n - (i - 1) \geq k$, which is equivalent to $j \leq n - k + 1$. Hence:

(1) $v_{\mathcal{I}}(\mathcal{F}^{R}(\mathsf{Q})xF(x)) = \sum_{i=0}^{n-k}(\alpha_{i+1} - \alpha_i) = \alpha_{n-k+1}$, and
(2) $v_{\mathcal{I}}(\mathcal{F}^{CX}(\mathsf{Q})xF(x)) = \alpha_{n-k+1}$.

□

Finally, let us stress that no major technical obstacle arises when extending the above expressibility results to the case of n-ary semi-fuzzy quantifiers, provided that they satisfy suitable form of monotonicity with the respect to their arguments. In particular, our results on $\mathcal{F}^{\mathcal{M}}$ and $\mathcal{F}^{OWA}$ can be straightforwardly adapted to semi-fuzzy quantifiers which are monotone in all their arguments. The same holds for $\mathcal{F}^{CX}$, but only for the representation in terms of propositional quantifiers, in Theorem 6. On the other hand, possible contextual dependencies between the arguments of semi-fuzzy quantifiers might make such a straightforward extension problematic from a linguistic point of view (see the discussion in Section 2). A possible way to address this problem has been presented, e.g., in the maximum dependence $\mathcal{F}^{MD}$ and the maximal independence model $\mathcal{F}^{I}$ in [8], which collapse to $\mathcal{F}^{R}$ and $\mathcal{F}^{OWA}$ in the unary case. These models do not pose additional challenges for expressibility over $\text{Ł}_{\triangle}(\Pi)$.

6 Fuzzification mechanisms based on closeness measures: $\mathcal{F}^C, \mathcal{F}^{C'}$

We now introduce a new family of QFMs, the *closeness-based* ones. The core idea is to evaluate a Type IV sentence by picking a precisification which maximizes the evaluation of the corresponding semi-fuzzy quantifier and the "closeness" of the precisification to the original fuzzy interpretation. Let us give the following schematic definition of a closeness-based QFM.

Definition 20 (Scheme for closeness-based QFMs). *Let $\mathcal{I}$ be a fuzzy interpretation, $F(x)$ a formula in the language of $L_\triangle$, Q a semi-fuzzy quantifier. A closeness-based QFM $\mathcal{F}$ is determined by the following parameters:*

(i) the set of admissible precisifications $\mathcal{C}_\mathcal{I}(F(x))$ associated to the fuzzy interpretation $\mathcal{I}$,
(ii) a measure evaluating, for any precisification $\mathcal{I}' \in \mathcal{C}_\mathcal{I}(F(x))$, how close the (classical) set $F^{\mathcal{I}'}$ interpreting F in $\mathcal{I}'$ is to the fuzzy set $F^\mathcal{I}$, interpreting F in $\mathcal{I}$
(iii) a t-norm operation combining the measure of closeness and the evaluation of a semi-fuzzy quantified sentence.

Once the parameters are settled, the evaluation of the fully-fuzzy sentence $\mathcal{F}(\mathsf{Q})xF(x)$ is obtained by picking in $\mathcal{C}_\mathcal{I}(F(x))$ (parameter (i)) the precisification $\mathcal{I}'$ which maximizes the value of closeness (parameter (ii)) "and"(parameter (iii)) the evaluation of $v_{\mathcal{I}'}(\mathsf{Q}xF(x))$.

As a first concrete study case, we consider the fuzzification mechanism $\mathcal{F}^C$ introduced in [1]. The set of admissible precisification (i) is reduced to the set of threshold values in $[0, 1]$ or, equivalently, the elements of the domain, acting as thresholds, as we did in the previous section. Using the notation of Definition 16, in this setting precisifications are identified with crisp formulas of the form $F^{\geq c}(x)$.

Since precisifications are expressed as formulas $F^{\geq c}(x)$, the closeness measure (item (ii)) reduces to the closeness between the formulas $F^{\geq c}(x)$ and $F(x)$. Recall that we have an obvious way to measure the closeness of (the evaluation of) two propositional formulas A and B in Łukasiewicz logic, that is by evaluating the formula $A \leftrightarrow B$. What about two first-order formulas such as $F^{\geq c}(x)$ and $F(x)$? Given an interpretation $\mathcal{I}$, the QFM $\mathcal{F}^C$ takes the closeness of $F(x)$ and $F^{\geq c}(x)$ to be their average closeness i.e. $Prop_\mathcal{I}(F \leftrightarrow F^{\geq c})$. Note that other legitimate options might be the minimum or the maximum closeness between the two formulas, i.e. to let $v_\mathcal{I}(\forall x(F(x) \leftrightarrow F^{\geq c}(x))$ or $v_\mathcal{I}(\exists x(F(x) \leftrightarrow F^{\geq c}(x))$ stand for the measure of closeness.

Finally, for item (iii), the evaluation of the semi-fuzzy quantified sentence and the evaluation of closeness are combined using the Łukasiewicz conjunction $\odot$ (another reasonable option would be to use the weak conjunction $\wedge$). To summarize, $\mathcal{F}^C$ instantiates the parameters in Definition 20 with:

(i) $F^{\geq c}(x)$ for any $c \in D$, with the addition of $F^{\geq \top}(x)$ for the set of precisifications.
(ii) The average $Prop_\mathcal{I}(F \leftrightarrow F^{\geq c})$ for the measure of closeness.
(iii) The Łukasiewicz conjunction $\odot$.

We have then the following definition.

Definition 21 (QFM $\mathcal{F}^C$). *Let* Q *be a semi-fuzzy quantifier,* F *a formula. The QFM* $\mathcal{F}^C$ *is defined as follows*[9]*:*

$$v_{\mathcal{I}}(\mathcal{F}^C(\mathsf{Q})xF(x)) = \max_{c \in D \cup \{\top\}} \big(v_{\mathcal{I}}(\mathsf{Q}xF^{\geq c}(x)) \odot Prop_{\mathcal{I}}(F^{\geq c} \leftrightarrow F)\big).$$

As shown in [1], $\mathcal{F}^C$ has the important advantage of providing quantifiers expressible in the logic Ł$_\triangle(\Pi)$.

Theorem 9. *[1] The QFM* $\mathcal{F}^C(\mathsf{Q})$ *is expressible in the logic Ł*$_\triangle$*. In particular, for any semi-fuzzy quantifier* Q *and formula* F *, we have that* $v_{\mathcal{I}}(\mathcal{F}^C(\mathsf{Q})xF(x))$ *is equal to:*

$$v_{\mathcal{I}}(\exists z(\mathsf{Q}xF^{\geq z}(x) \odot \Pi y(F^{\geq z}(y) \leftrightarrow F(y))) \vee (\mathsf{Q}x(\triangle F(x)) \odot \Pi y(F(y) \leftrightarrow \triangle F(y))))$$

Note that the second disjunct in the formula above corresponds to the choice of $\top$ as a threshold value for a precisification.

While intuitively plausible, $\mathcal{F}^C$ does not square very well with most of the desiderata for QFMs discussed in Section 3, see [29]. Let us consider, for instance, the desiderata of preservation of monotonicity, testing it on the semi-fuzzy quantifier $\mathsf{Q}_{[\geq \frac{1}{2}]}$ standing for At least half, whose truth function is given by $v_{\mathcal{I}}(\mathsf{Q}_{[\geq \frac{1}{2}]}x\hat{F}(x))) = \min(1, 2Prop_{\mathcal{I}}\hat{F})$. $\mathsf{Q}_{[\geq \frac{1}{2}]}$ is clearly a non-decreasing quantifier, hence one would expect the corresponding fuzzy quantifier to be also non-decreasing. This is, however, not the case, as we show in the next example.

Example 6. Let $D = \{c_1, \dots, c_4\}$, and consider two unary predicates $F(x)$ and $G(x)$, such that $v_{\mathcal{I}}(F(c_1)) = v_{\mathcal{I}}(F(c_2)) = 0.1$, $v_{\mathcal{I}}(F(c_3)) = v_{\mathcal{I}}(F(c_4)) = 0.9$, and $v_{\mathcal{I}}(G(c_1)) = v_{\mathcal{I}}(G(c_2)) = 0.4$, $v_{\mathcal{I}}(G(c_3)) = v_{\mathcal{I}}(G(c_4)) = 0.9$. Note that $v_{\mathcal{I}}(F(c)) \leq v_{\mathcal{I}}(G(c))$ for any $c \in D$.

The supremum in $v_{\mathcal{I}}(\mathcal{F}^C(\mathsf{Q}_{[\geq \frac{1}{2}]})xF(x))$ and in $v_{\mathcal{I}}(\mathcal{F}^C(\mathsf{Q}_{[\geq \frac{1}{2}]})xG(x))$ is obtained by choosing the precisification determined by c_3, or equivalently by c_4. We have $v_{\mathcal{I}}(\mathsf{Q}_{[\geq \frac{1}{2}]}xF^{\geq c_4}(x)) = v_{\mathcal{I}}(\mathsf{Q}_{[\geq \frac{1}{2}]}xG^{\geq c_4}(x)) = 1$. We obtain instead different results on the closeness measure, which lead to different values for $v_{\mathcal{I}}(\mathcal{F}^C(\mathsf{Q}_{[\geq \frac{1}{2}]})xF(x))$ and $v_{\mathcal{I}}(\mathcal{F}^C(\mathsf{Q}_{[\geq \frac{1}{2}]})xG(x))$, i.e.

$$v_{\mathcal{I}}(\mathcal{F}^C(\mathsf{Q}_{[\geq \frac{1}{2}]})xF(x)) = 1 \odot v_{\mathcal{I}}(\Pi yF^{\geq c_4}(y) \leftrightarrow F(y)) = \frac{0.9 + 0.9 + 0.9 + 0.9}{4} = 0.9$$

$$v_{\mathcal{I}}(\mathcal{F}^C(\mathsf{Q}_{[\geq \frac{1}{2}]})xG(x)) = 1 \odot v_{\mathcal{I}}(\Pi y(G^{\geq c_4}(y) \leftrightarrow G(y)) = \frac{0.6 + 0.6 + 0.9 + 0.9}{4} = 0.75.$$

The problem in the example above is that, when evaluating how close $G(x)$ is to the precisification $G^{\geq c_4}(x)$, we take into account also those elements (c_1 and c_2) for which $v_{\mathcal{I}}(G^{\geq c_4}(x)) = 0$. These values should be indifferent for the evaluation of At least $\frac{1}{2}$. A possible way to address this problem with monotonicity, also presented in [1], is to measure closeness by $Prop_{\mathcal{I}}(F \to F^{\geq c})$ for non-decreasing quantifier, by $Prop_{\mathcal{I}}(F^{\geq c} \to F)$ for non-increasing quantifier, and use $Prop_{\mathcal{I}}(F \leftrightarrow F^{\geq c})$ for the remaining quantifiers. As shown in [1], in this way the desiderata of preservation of monotonicity is satisfied, and at the same time the expressibility in Ł$_\triangle(\Pi)$ is retained. The solution is, however, in some respects, unsatisfactory. It is instructive to see why, considering also what happens with the projection quantifier, in the following example.

[9] Note that here, and elsewhere in this section, with a slight abuse of notation, we use the same symbol $\odot$ for the connective in $\alpha \odot \beta$ and for the truth-function interpreting it, i.e. $v_{\mathcal{I}}(\alpha) \odot v_{\mathcal{I}}(\beta) = \max(v_{\mathcal{I}}(\alpha) + v_{\mathcal{I}}(\beta) - 1, 0)$.

Example 7. Let $\mathcal{I}$ be an interpretation with domain $D = \{c_1, c_2\}$ such that $v_{\mathcal{I}}(F(c_1)) = 0.1$, $v_{\mathcal{I}}(F(c_2)) = 0.9$. A maximal value in $v_{\mathcal{I}}(\mathcal{F}^C(\triangle_{c_1})xF(x))$ is obtained by considering the precisification determined by $F^{\geq c_1}(x)$. Indeed we have $v_{\mathcal{I}}(\triangle_{c_1}xF^{\geq c_1}(x)) = 1$ and $Prop_{\mathcal{I}}(F^{\geq c_1} \leftrightarrow F) = (0.1 + 0.9)/2 = 0.5$, hence $v_{\mathcal{I}}(\mathcal{F}^C(\triangle_{c_1})xF(x)) = 0.5 \neq v_{\mathcal{I}}(F(c_1))$.

Here the problem is that we would expect that only the truth value of $F(c_1)$ should be relevant in evaluating both $\triangle_{c_1}$ and $\mathcal{F}(\triangle_{c_1})$, but the closeness measure considered takes into account the truth value of $F(c_2)$ as well. Using $Prop_{\mathcal{I}}(F \to F^{\geq c})$ as a measure of closeness would return exactly the same value.

Therefore, we do not investigate further the properties of $\mathcal{F}^C$ and its variants, as we aim at a more systematic and principled approach. Note that, in both examples above, the core of the problem is that the measure of closeness should be restricted to a suitable subset of the domain.

Let us introduce then a second instance of closeness-based fuzzification mechanism, $\mathcal{F}^{C'}$, which deals with this issue and meets more of the desiderata discussed in Section 3. The price to pay w.r.t. $\mathcal{F}^C$ is that $\mathcal{F}^{C'}$ does not, at least in the general case, provide quantifiers expressible in expansions of Ł$_\triangle$.

The closeness mechanism $\mathcal{F}^{C'}$ is obtained by instantiating the parameters of Definition 20 as follows. Let us start from item (i).

Definition 22. *Let $\mathcal{I}$ be an interpretation, F a formula, $\mathcal{I}'$ a classical interpretation over the same domain of $\mathcal{I}$. If $\mathcal{I}'$ satisfies the conditions:*

1. *For any $c \in D$, if $v_{\mathcal{I}}(F(c)) \in \{0, 1\}$ then $v_{\mathcal{I}'}(F(c)) = v_{\mathcal{I}}(F(c))$ ($\mathcal{I}'$ respects the classical truth values).*
2. *For any $c_1, c_2 \in D$, if $v_{\mathcal{I}}(F(c_1)) < v_{\mathcal{I}}(F(c_2))$ then $v_{\mathcal{I}'}(F(c_1)) \leq v_{\mathcal{I}'}(F(c_2))$ ($\mathcal{I}'$ respects the strict ordering of truth values).*

then $\mathcal{I}'$ is said to be a coherent precisification[10] *of $\mathcal{I}$ over $F(x)$. We denote by $C_{\mathcal{I}}(F(x))$ the set of coherent precisifications of $\mathcal{I}$ over $F(x)$.*

For the measure of closeness, i.e. item (ii) in Definition 20, the idea is to evaluate closeness only on some subsets of the domain, which are determined by the semi-fuzzy quantifier and the precisification at hand. We formalize the idea by the following definition.

Definition 23. *Let $\mathcal{I}$ be a fuzzy interpretation, Q a semi-fuzzy quantifier, $F(x)$ a first-order formula in the language of Ł$_\triangle$, and $\mathcal{I}'$ be a precisification in $\mathcal{C}_{\mathcal{I}}(F(x))$. A Q-kernel of $\mathcal{I}'$ is a minimal (w.r.t. to the ordered set $(\mathcal{P}(D), \subseteq)$) nonempty subset D' of the domain D such that any D'-based variant $\mathcal{I}''$ of $\mathcal{I}'$ (see Definition 5) satisfies $v_{\mathcal{I}''}(\mathsf{Q}xF(x)) \geq v_{\mathcal{I}'}(\mathsf{Q}xF(x))$.*

A Q-kernel D' of $\mathcal{I}'$ is said to be positive *if $v_{\mathcal{I}'}(F(c)) = 1$ for any $c \in D'$ and* negative *if $v_{\mathcal{I}'}(F(c)) = 0$ for any $c \in D'$.*

We denote by $K_{\mathcal{I}'}(\mathsf{Q}xF(x))$ the set of Q-kernels of $\mathcal{I}'$. Despite its apparently complicated definition, the kernel can be usually easily read off from the semi-fuzzy quantifier at hand, as we illustrate in the next examples.

[10] Note that not all the threshold-based precisifications determined by formulas $F^{\geq c}(x)$ are coherent. In particular, any formulas $F^{\geq c}(x)$ where $v_{\mathcal{I}}(F(c)) = 0$ does not determine a coherent precisification, since it does not satisfy 1. in the Definition 22. On the other hand, threshold-based precisifications satisfy a property stronger than 2. , namely that, for any $c_1, c_2 \in D$, if $v_{\mathcal{I}}(F(c_1)) \leq v_{\mathcal{I}}(F(c_2))$, then $v_{\mathcal{I}'}(F(c_1)) \leq v_{\mathcal{I}'}(F(c_2))$. This means that the threshold-based precisifications $F^{\geq c}(x)$, differently from the coherent precisifications, need to preserve also the equality of truth values in $\mathcal{I}$.

Example 8. Consider the projection quantifier $\triangle_c$ and a precisification $\mathcal{I}'$ of $\mathcal{I}$ such that $v_{\mathcal{I}'}(\triangle_c x F(x)) = 1$. The only $\triangle_c$-kernel of $\mathcal{I}'$ is the subset $\{c\}$ of the domain.

Let us now consider the semi-fuzzy quantifiers $\mathsf{Q}_{[\geq j]}$, standing for $\mathsf{At\ least}\ j$ and their proportional counterpart $\mathsf{Q}_{[\geq \frac{k}{r}]}$, standing for $\mathsf{At\ least}\ \frac{k}{r}$, whose truth functions are:

$$v_{\mathcal{I}}(\mathsf{Q}_{[\geq j]} x \hat{F}(x)) = \min(1, \tfrac{|D|}{j} Prop_{\mathcal{I}}(\hat{F}))$$
$$v_{\mathcal{I}}(\mathsf{Q}_{[\geq \frac{k}{r}]} x \hat{F}(x))) = \min(1, \tfrac{r}{k} Prop_{\mathcal{I}}(\hat{F}))$$

For $\mathsf{Q}_{[\geq j]} x F(x)$ and any coherent precisification $\mathcal{I}'$ such that $v_{\mathcal{I}'}(\mathsf{Q}_{[\geq j]} x F(x)) = 1$, the kernels of $\mathcal{I}'$ are those subsets D' such that $|D'| = j$ and $v_{\mathcal{I}'}(F(c)) = 1$ for any $c \in D'$. Similarly, for $\mathsf{Q}_{[\geq \frac{k}{n}]} x F(x)$, if we pick a coherent precisification $\mathcal{I}'$ such that $v_{\mathcal{I}'}(\mathsf{Q}_{[\geq \frac{k}{n}]} x F(x)) = 1$, the kernels are subsets D' such that $|D'| = \frac{k}{n} * |D|$ and $v_{\mathcal{I}'}(F(c)) = 1$, for any $c \in D'$.

We now define the measure of closeness (item (ii) in Definition 20) for the fuzzification mechanism $\mathcal{F}^{C'}$. The idea is to restrict it only to elements in a kernel. For this we introduce another piece of notation: given two interpretations $\mathcal{I}, \mathcal{I}'$ over the same domain D, a formula $F(x)$, and $D' \subseteq D$, we let:

$$CL_{D'}(F^{\mathcal{I}}, F^{\mathcal{I}'}) = \frac{\sum_{c \in D'} 1 - |v_{\mathcal{I}}(F(c)) - v_{\mathcal{I}'}(F(c))|}{|D'|}.$$

We will keep using the conjunction $\odot$ (item (iii) in Definition 20) for combining the measure of closeness and the evaluation of the semi-fuzzy quantifier.

In summary, given a fuzzy interpretation $\mathcal{I}$, $v_{\mathcal{I}}(\mathcal{F}^{C'}(\mathsf{Q}) x F(x))$ is obtained by picking a coherent precisification $\mathcal{I}'$ in $\mathcal{C}_{\mathcal{I}}(F(x))$ and a kernel D' in $K_{\mathcal{I}'}(\mathsf{Q} x F(x))$, such that the closeness measure $CL_{D'}(F^{\mathcal{I}}, F^{\mathcal{I}'})$ and the evaluation $v_{\mathcal{I}'}(\mathsf{Q} x F(x))$ are maximized. Formally, we have the following definition.

Definition 24 (QFM $\mathcal{F}^{C'}$). *Let Q be a semi-fuzzy quantifier, $\mathcal{I}$ a fuzzy interpretation, $F(x)$ a formula. The QFM $\mathcal{F}^{C'}$ is defined by:*

$$v_{\mathcal{I}}(\mathcal{F}^{C'}(\mathsf{Q}) x F(x)) = \max_{\substack{\mathcal{I}' \in \mathcal{C}_{\mathcal{I}}(F(x)) \\ D' \in K_{\mathcal{I}'}(\mathsf{Q} x F(x))}} v_{\mathcal{I}'}(\mathsf{Q} x F(x)) \odot CL_{D'}(F^{\mathcal{I}}, F^{\mathcal{I}'}).$$

Before proceeding, let us show an important lemma on kernels for monotone quantifiers.

Lemma 5. *If Q is a non-decreasing quantifier, then any Q-kernel D' is positive. Similarly, if Q is a non-increasing quantifier, any Q-kernel D' is negative.*

Proof. Let $\mathcal{I}$ be a fuzzy interpretation, Q be a non-decreasing semi-fuzzy quantifier, $F(x)$ a first-order formula in the language of Ł$_\triangle$, and $\mathcal{I}' \in \mathcal{C}_{\mathcal{I}}(F(x))$. Assume that D' is a Q-kernel of $\mathcal{I}'$ which is not positive, i.e. $v_{\mathcal{I}'}(F(c)) = 0$ for some $c' \in D'$. We show that, given any $D' \backslash \{c'\}$-based variant $\mathcal{I}''$ of $\mathcal{I}'$, it holds that $v_{\mathcal{I}''}(\mathsf{Q} x F(x)) \geq v_{\mathcal{I}'}(\mathsf{Q} x F(x))$, thus contradicting the minimality of the kernel D'. Assume first that $v_{\mathcal{I}''}(F(c)) = v_{\mathcal{I}'}(F(c)) = 0$. Then $\mathcal{I}''$ is already a D'-based variant of $\mathcal{I}'$, and $v_{\mathcal{I}''}(\mathsf{Q} x F(x)) \geq v_{\mathcal{I}'}(\mathsf{Q} x F(x))$. In case $v_{\mathcal{I}''}(F(c)) = 1$, since Q non-decreasing, we also have $v_{\mathcal{I}''}(\mathsf{Q} x F(x)) \geq v_{\mathcal{I}'}(\mathsf{Q} x F(x))$.

By similar reasoning we can show that the kernel of a non-decreasing quantifier need to be negative. $\square$

Using the previous lemma, we can greatly simplify the evaluation of $\mathcal{F}^{C'}$ for monotone quantifiers. Let us introduce first some additional notation.

Let Q be a non-decreasing quantifier, $F(x)$ a formula, $\mathcal{I}$ a fuzzy interpretation over a domain D. Henceforth we assume that $D = \{c_1, \dots, c_n\}$ where, w.l.o.g. $v_{\mathcal{I}}(F(c_1)) \leq v_{\mathcal{I}}(F(c_2)) \leq \cdots \leq v_{\mathcal{I}}(F(c_n))$. Let us denote by $\mathcal{I}_{F\uparrow m}$ a coherent precisification such that $v_{\mathcal{I}_{F\uparrow m}}(F(c_n) = v_{\mathcal{I}_{F\uparrow m}}(F(c_{n-1})) = \cdots = v_{\mathcal{I}_{F\uparrow m}}(F(c_{n-m+1})) = 1$ and $v_{\mathcal{I}_{F\uparrow m}}(F(c_1)) = \cdots = v_{\mathcal{I}_{F\uparrow m}}(F(c_{n-m})) = 0$. In other words, $\mathcal{I}_{F\uparrow m}$ evaluates $F(c)$ as true for m elements of the domain for which $v_{\mathcal{I}}(F(c))$ is highest.

We let $\kappa^+_{\mathcal{I}}(\mathsf{Q}xF(x))$ be the set of the integers m such that the coherent precisification $\mathcal{I}_{F\uparrow m}$ has $\{c_{n-m+1}, \dots, c_n\}$ as its unique, positive Q kernel.

Similarly, if Q is a non-increasing quantifier, we denote by $\mathcal{I}_{F\downarrow m}$ a coherent precisification such that $v_{\mathcal{I}_{F\downarrow m}}(F(c_1)) = \cdots = v_{\mathcal{I}_{F\downarrow m}}(F(c_m)) = 0$ and $v_{\mathcal{I}_{F\downarrow m}}(F(c_{m+1})) = \cdots = v_{\mathcal{I}_{F\downarrow m}}(F(c_n)) = 1$, and we let $\kappa^-_{\mathcal{I}}(\mathsf{Q}xF(x))$ be the set of the integers m such that the coherent precisification $\mathcal{I}_{\downarrow n}$ has $\{c_1, \dots, c_m\}$ as its unique, negative Q-kernel.

Lemma 6. *Let $\mathcal{I}$ be a fuzzy interpretation, Q a semi-fuzzy quantifier, $F(x)$ a first-order formula in the language of $L_\triangle$. If Q is a non-decreasing quantifier, we have:*

$$v_{\mathcal{I}}(\mathcal{F}^{C'}(\mathsf{Q})xF(x)) = \max_{m \in \kappa^+_{\mathcal{I}}(\mathsf{Q}xF(x))} v_{\mathcal{I}_{F\uparrow m}}(\mathsf{Q}xF(x)) \odot \frac{\sum\limits_{i=n-m+1}^{n} v_{\mathcal{I}}(F(c_i))}{|m|}.$$

and, if Q is a non-increasing quantifier:

$$v_{\mathcal{I}}(\mathcal{F}^{C'}(\mathsf{Q})xF(x)) = \max_{m \in \kappa^-_{\mathcal{I}}(\mathsf{Q}xF(x))} v_{\mathcal{I}_{F\downarrow m}}(\mathsf{Q}xF(x)) \odot \frac{\sum\limits_{i=1}^{m} 1 - v_{\mathcal{I}}(F(c_i))}{|m|}.$$

Proof. Throughout the proof, we assume that the quantifier Q is quantitative. We refer to the proof of Theorem 10 for the case of the quantifier $\triangle_c$. Let us show the case for a non-decreasing (quantitative) quantifier. It is easy to see that:

$$\begin{aligned} v_{\mathcal{I}}(\mathcal{F}^{C'}(\mathsf{Q})xF(x)) &= \max_{\substack{\mathcal{I}' \in \mathcal{C}_{\mathcal{I}}(F(x)) \\ D' \in K_{\mathcal{I}'}(\mathsf{Q}xF(x))}} v_{\mathcal{I}'}(\mathsf{Q}xF(x)) \odot CL_{D'}(F^{\mathcal{I}}, F^{\mathcal{I}'}) \\ &\geq \max_{m \in \kappa^+_{\mathcal{I}}(\mathsf{Q}xF(x))} v_{\mathcal{I}_{F\uparrow m}}(\mathsf{Q}xF(x)) \odot \frac{\sum\limits_{i=n-m+1}^{n} v_{\mathcal{I}}(F(c_i))}{|m|} \end{aligned}$$

This holds, since any element in the range of the maximum in the last expression is also in the range of the maximum in the second expression. Indeed, a few straightforward computations reveal, in particular:

$$CL_{\{c_{n-m+1}, \dots, c_n\}}(F^{\mathcal{I}}, F^{\mathcal{I}_{F\uparrow m}}) = \frac{\sum\limits_{i=n-m+1}^{n} v_{\mathcal{I}}(F(c_i))}{|m|}.$$

We now show the other inequality. Let $\mathcal{I}'$ be a precisification in $\mathcal{C}_{\mathcal{I}}(F(x))$ and D' a kernel such that:

$$v_{\mathcal{I}}(\mathcal{F}^{C'}(\mathsf{Q})xF(x)) = v_{\mathcal{I}'}(\mathsf{Q}xF(x)) \odot CL_{D'}(F^{\mathcal{I}}, F^{\mathcal{I}'})$$

Assume $|D'| = m$ and $|F^{\mathcal{I}'}| = m + k$. From Lemma 5 and some straightforward computations, it follows:

$$CL_{D'}(F^{\mathcal{I}}, F^{\mathcal{I}'}) = \frac{\sum_{c \in D'} v_{\mathcal{I}}(F(c))}{m}.$$

Since Q is quantitative, we have that if D' is a positive kernel for $\mathcal{I}'$, then the set $\{c_{n-m+1}, \ldots, c_n\} \subseteq D$ (which has the same cardinality as D') is a kernel as well, for the precisification $\mathcal{I}_{F\uparrow m}$. Therefore $m \in \kappa^+_{\mathcal{I}}(\mathsf{Q}xF(x))$ and, by assumption, for any $c_j \in D', c_i \in \{c_{n-m+1}, \ldots, c_n\}$, we have $v_{\mathcal{I}}(F(c_j)) \leq v_{\mathcal{I}}(F(c_i))$. Hence we obtain

$$CL_{D'}(F^{\mathcal{I}}, F^{\mathcal{I}'}) = \frac{\sum_{c_j \in D'} v_{\mathcal{I}}(F(c_j))}{m} \leq \frac{\sum\limits_{i=n-m+1}^{n} v_{\mathcal{I}}(F(c_i))}{m}.$$

Now, consider the precisification $\mathcal{I}_{F\uparrow m+k}$. Since $\{c_{n-m-k+1}, \ldots, c_n\}$ has the same cardinality as $F^{\mathcal{I}'}$ and Q is quantitative, we have $v_{\mathcal{I}_{F\uparrow m+k}}(\mathsf{Q}xF(x)) = v_{\mathcal{I}'}(\mathsf{Q}xF(x))$. Moreover $\mathcal{I}_{F\uparrow m}$ is a $\{c_{n-m+1}, \ldots, c_n\}$-based variant of $\mathcal{I}_{F\uparrow m+k}$. Since $\{c_{n-m+1}, \ldots, c_n\}$ is a kernel, by Definition 5 we have $v_{\mathcal{I}_{F\uparrow m+k}}(\mathsf{Q}xF(x)) \leq v_{\mathcal{I}_{F\uparrow m}}(\mathsf{Q}xF(x))$. On the other hand, we have $v_{\mathcal{I}'}(\mathsf{Q}xF(x)) = v_{\mathcal{I}_{F\uparrow m+k}}(\mathsf{Q}xF(x))$, hence $v_{\mathcal{I}'}(\mathsf{Q}xF(x)) \leq v_{\mathcal{I}_{F\uparrow m}}(\mathsf{Q}xF(x))$. Finally, we obtain:

$$\begin{aligned} v_{\mathcal{I}}(\mathcal{F}^{C'}(\mathsf{Q})xF(x)) &= v_{\mathcal{I}'}(\mathsf{Q}xF(x)) \odot CL_{D'}(F^{\mathcal{I}}, F^{\mathcal{I}'}) \\ &\leq v_{\mathcal{I}_{F\uparrow m}}(\mathsf{Q}xF(x)) \odot \frac{\sum\limits_{i=n-m+1}^{n} v_{\mathcal{I}}(F(c_i))}{m} \\ &\leq \max_{m \in \kappa^+_{\mathcal{I}}} v_{\mathcal{I}_{F\uparrow m}}(\mathsf{Q}xF(x)) \odot \frac{\sum\limits_{i=n-m+1}^{n} v_{\mathcal{I}}(F(c_i))}{|m|}. \end{aligned}$$

The case for non-increasing quantifiers is similar. □

We are now ready to discuss the desiderata satisfied by the QFM $\mathcal{F}^{C'}$.

Theorem 10. *The fuzzification mechanism $\mathcal{F}^{C'}$ satisfies the desiderata of* preservation of quantitativity, correct generalization, projection quantifier, internal negation, preservation of monotonicity in the argument, continuity*. Moreover, it preserves* quantifier strength *for monotone quantifiers.*

Proof. Preservation of quantitativity is obvious. For correct generalization, note that, if $\mathcal{I}$ is already a classical interpretation, then the only coherent precisification is $\mathcal{I}$ itself. Hence, we obtain:

$$\begin{aligned} v_{\mathcal{I}}(\mathcal{F}^{C'}(\mathsf{Q})xF(x)) &= \max_{\substack{\mathcal{I}' \in \mathcal{C}_{\mathcal{I}}(F(x)) \\ D' \in K_{\mathcal{I}'}(\mathsf{Q}xF(x)}} (v_{\mathcal{I}'}(\mathsf{Q}x\hat{F}(x)) \odot CL_{D'}(F^{\mathcal{I}}, F^{\mathcal{I}'}) \\ &= v_{\mathcal{I}}(\mathsf{Q}x\hat{F}(x)) \odot CL_{D'}(F^{\mathcal{I}}, F^{\mathcal{I}}) = v_{\mathcal{I}}(\mathsf{Q}x\hat{F}(x)). \end{aligned}$$

(note that,whatever kernel D' is considered, $CL_{D'}(F^{\mathcal{I}}, F^{\mathcal{I}}) = 1$).

For the projection quantifier, we show that $v_{\mathcal{I}}(\mathcal{F}^{C'}(\triangle_c)xF(x)) = v_{\mathcal{I}}(F(c))$. Recall the definition of $v_{\mathcal{I}}(\mathcal{F}^{C'}(\triangle_c)xF(x))$ above. We distinguish two cases. If $v_{\mathcal{I}}(F(c)) = 0$, then, for all coherent precisifications $\mathcal{I}'$ in $\mathcal{C}_{\mathcal{I}}(F(x))$ we have $v_{\mathcal{I}'}(F(c)) = 0$. Hence, in particular $v_{\mathcal{I}'}(\triangle_c F(x)) = 0$, and $v_{\mathcal{I}}(\mathcal{F}^{C'}(\triangle_c)xF(x)) = 0 = v_{\mathcal{I}}(F(c))$. Let us consider now the case $v_{\mathcal{I}}(F(c)) \neq 0$. As $\triangle_c$ is essentially a Type I quantifier, we obtain a maximum value in the evaluation of $v_{\mathcal{I}}(\mathcal{F}^{C'}(\triangle_c)xF(x))$ for the coherent precisification $\mathcal{I}' \in \mathcal{C}_{\mathcal{I}}(F(x))$ such that $v_{\mathcal{I}'}(F(c)) = 1$, with kernel $\{c\}$. Hence, we obtain

$$v_{\mathcal{I}}(\mathcal{F}^{C'}(\triangle_c)xF(x)) = v_{\mathcal{I}'}(\triangle_c xF(x)) \odot CL_{\{c\}}(F^{\mathcal{I}}, F^{\mathcal{I}'}) = 1 \odot \frac{v_{\mathcal{I}}(F(c))}{|\{c\}|} = v_{\mathcal{I}}(F(c)).$$

For internal negation, first note that any coherent precisification $\mathcal{I}'$ in $\mathcal{C}_{\mathcal{I}}(F(x))$, is also a coherent precisification in $(\mathcal{C}_{\mathcal{I}}(\neg F(x))$, and vice versa. Moreover, we have that $CL_{D'}(F^{\mathcal{I}}, F^{\mathcal{I}'}) = CL_{D'}((\neg F)^{\mathcal{I}}, (\neg F)^{\mathcal{I}'})$. Hence we obtain:

$$\begin{aligned} v_{\mathcal{I}}(\mathcal{F}^{C'}(\mathsf{Q}^{\neg})xF(x)) &= \max_{\substack{\mathcal{I}' \in \mathcal{C}_{\mathcal{I}}(F(x)) \\ D' \in K_{\mathcal{I}'}(\mathsf{Q}^{\neg}xF(x))}} v_{\mathcal{I}'}(\mathsf{Q}^{\neg}xF(x)) \odot CL_{D'}(F^{\mathcal{I}}, F^{\mathcal{I}'}) \\ &= \max_{\substack{\mathcal{I}' \in \mathcal{C}_{\mathcal{I}}(\neg F(x)) \\ D' \in K_{\mathcal{I}'}(\mathsf{Q}x\neg F(x))}} v_{\mathcal{I}'}(\mathsf{Q}x\neg F(x)) \odot CL_{D'}(\neg F^{\mathcal{I}}, \neg F^{\mathcal{I}'}) \\ &= v_{\mathcal{I}}(\mathcal{F}^{C'}(\mathsf{Q})x\neg F(x)) = v_{\mathcal{I}}((\mathcal{F}^{C'}(\mathsf{Q}))^{\neg}xF(x)). \end{aligned}$$

For preservation of monotonicity, let Q be a non-decreasing quantifier and assume that $v_{\mathcal{I}}(F(c)) \leq v_{\mathcal{I}}(G(c))$ for any $c \in D$. Recall that we assumed that $v_{\mathcal{I}}(F(c_1)) \leq \cdots \leq v_{\mathcal{I}}(F(c_n))$. To avoid further complications in the notation, let us further assume that, for a permutation σ, $v_{\mathcal{I}}(G(c_{\sigma(1)})) \leq \cdots \leq v_{\mathcal{I}}(G(c_{\sigma(n)}))$. Recalling Lemma 6, let m be the integer in $\kappa^{+}_{\mathcal{I}}(\mathsf{Q}xF(x))$ such that:

$$v_{\mathcal{I}}(\mathcal{F}^{C'}(\mathsf{Q})xF(x)) = v_{\mathcal{I}_{F\uparrow m}}(\mathsf{Q}xF(x)) \odot \frac{\sum\limits_{i=n-m+1}^{n} v_{\mathcal{I}}(F(c_i))}{m}.$$

Since Q is non-decreasing, we have $v_{\mathcal{I}_{F\uparrow m}}(\mathsf{Q}xF(x)) \leq v_{\mathcal{I}_{G\uparrow m}}(\mathsf{Q}xG(x))$. Moreover, we have $v_{\mathcal{I}}(F(c_i)) \leq v_{\mathcal{I}}(G(c_i))$, hence we obtain:

$$\begin{aligned} v_{\mathcal{I}}(\mathcal{F}^{C'}(\mathsf{Q})xF(x)) &= v_{\mathcal{I}_{F\uparrow m}}(\mathsf{Q}xF(x)) \odot \frac{\sum\limits_{i=n-m+1}^{n} v_{\mathcal{I}}(F(c_i))}{m} \\ &\leq v_{\mathcal{I}_{G\uparrow m}}(\mathsf{Q}xG(x)) \odot \frac{\sum\limits_{i=n-m+1}^{n} v_{\mathcal{I}}(G(c_i))}{m} \\ &\leq v_{\mathcal{I}_{G\uparrow m}}(\mathsf{Q}xG(x)) \odot \frac{\sum\limits_{i=n-m+1}^{n} v_{\mathcal{I}}(G(c_{\sigma(i)}))}{m} \\ &\leq v_{\mathcal{I}}(\mathcal{F}^{C'}(\mathsf{Q})xG(x)). \end{aligned}$$

For continuity, note that for any $\mathcal{I}' \in C_{\mathcal{I}}(\mathsf{Q}xF(x))$, $D' \in K^{\mathsf{Q}}(\mathcal{I}')$ the function $v_{\mathcal{I}'}(\mathsf{Q}xF(x)) \odot CL_{D'}(F^{\mathcal{I}}, F^{\mathcal{I}'})$ is continuous, in the sense explained above for continuity in the argument. Hence $v_{\mathcal{I}}(\mathcal{F}^{C'}(\mathsf{Q})xF(x))$ is continuous as well.

Finally, assume that Q_1 and Q_2 are semi-fuzzy non-decreasing quantifier, with $\mathsf{Q}_1 \leq \mathsf{Q}_2$. Assume

$$v_{\mathcal{I}}(\mathcal{F}^{C'}(\mathsf{Q}_1)xF(x)) = v_{\mathcal{I}_{F\uparrow m_1}}(\mathsf{Q}_1 xF(x)) \odot \frac{\sum\limits_{i=n-m_1+1}^{n} v_{\mathcal{I}}(F(c_i))}{m_1}.$$

Now, $\mathcal{I}_{F\uparrow m_1}$ is also a coherent precisification of $\mathsf{Q}_2 xF(x)$, but $\{c_{n-m_1+1}, \ldots, c_n\}$ need not be a Q_2-kernel. Since Q_2 is non-decreasing, a kernel of $\mathcal{I}_{F\uparrow m_1}$ w.r.t. to Q_2 can only be positive, hence it is a subset of $\{c_{n-m_1+1}, \ldots, c_n\}$, say of cardinality $m_2 \leq m_1$. Consider now $\mathcal{I}_{F\uparrow m_2}$. We have

$$\frac{\sum\limits_{i=n-m_1+1}^{n} v_{\mathcal{I}}(F(c_i))}{m_1} \leq \frac{\sum\limits_{i=n-m_2+1}^{n} v_{\mathcal{I}}(F(c_i))}{m_2}.$$

Finally, recalling that $\mathsf{Q}_1 \leq \mathsf{Q}_2$, we get:

$$\begin{aligned} v_{\mathcal{I}}(\mathcal{F}^{C'}(\mathsf{Q}_1)xF(x)) &= v_{\mathcal{I}'}(\mathsf{Q}_1 xF(x)) \odot \frac{\sum\limits_{i=n-m_1+1}^{n} v_{\mathcal{I}}(F(c_i))}{m_1} \\ &\leq v_{\mathcal{I}'}(\mathsf{Q}_2 xF(x)) \odot \frac{\sum\limits_{i=n-m_2+1}^{n} v_{\mathcal{I}}(F(c_i))}{m_2} \\ &\leq v_{\mathcal{I}}(\mathcal{F}^{C'}(\mathsf{Q}_2)xF(x)). \end{aligned}$$

The latter shows that $\mathcal{F}^{C'}$ preserves quantifier strength, when Q_1 and Q_2 are non-decreasing quantifiers. The case for non-increasing quantifiers is similar. $\square$

As we show below, $\mathcal{F}^{C'}$ also satisfies preservation of monotonicity under conservative extension, provided that additional conditions on the kernels of the involved quantifiers are imposed.

Definition 25. *Let Q be a quantifier, $\mathcal{I}$ an interpretation and $\mathcal{I}_1$ a conservative extension of $\mathcal{I}$. We say that Q has* extensive *kernels if $K^{\mathsf{Q}}(\mathcal{I}_1) \supseteq K^{\mathsf{Q}}(\mathcal{I})$ and that Q has* restrictive *kernels if, instead, $K^{\mathsf{Q}}(\mathcal{I}_1) \subseteq K^{\mathsf{Q}}(\mathcal{I})$.*

Theorem 11. *Let Q be a semi-fuzzy quantifier non-decreasing (non-increasing) in extension . Then $\mathcal{F}^{C'}(\mathsf{Q})$ is non-decreasing (non-increasing), provided that Q has extensive(restrictive) kernels.*

Proof. let Q be a quantifier non-decreasing in extension, $F(x)$ any formula, $\mathcal{I}$ a fuzzy interpretation with domain D and $\mathcal{I}_1$ with domain D_1 a conservative extension of $\mathcal{I}$.

Let $\mathcal{I}'$ be the precisification of $\mathcal{I}$ and $D' \in K^{\mathsf{Q}}(\mathcal{I}')$ the kernel such that $v_{\mathcal{I}}(\mathcal{F}^{C'}\mathsf{Q}xF(x)) = v_{\mathcal{I}'}(\mathsf{Q}xF(x)) \odot CL_{D'}(F^{\mathcal{I}'}, F^{\mathcal{I}})$

Consider a coherent precisification $\mathcal{I}_1'$ of $\mathcal{I}_1$, which is also a conservative extension of $\mathcal{I}'$. Hence in particular $CL_{D'}(F^{\mathcal{I}'}, F^{\mathcal{I}}) = CL_{D'}(F^{\mathcal{I}_1'}, F^{\mathcal{I}_1})$. Note that since Q has extensive kernel, D' is a Q-kernel for $\mathcal{I}_1'$ as well. Moreover, as Q is non-decreasing in extension, we get $v_{\mathcal{I}_1'}(\mathsf{Q}xF(x)) \geq v_{\mathcal{I}'}(\mathsf{Q}xF(x))$. Hence, putting the pieces together we obtain

$$\begin{aligned} v_{\mathcal{I}}(\mathcal{F}^{C'}\mathsf{Q}xF(x)) &= v_{\mathcal{I}'}(\mathsf{Q}xF(x)) \odot CL_{D'}(F^{\mathcal{I}'}, F^{\mathcal{I}}) \\ &\leq v_{\mathcal{I}_1'}(\mathsf{Q}xF(x)) \odot CL_{D'}(F^{\mathcal{I}_1'}, F^{\mathcal{I}_1}) \\ &\leq v_{\mathcal{I}_1}(\mathcal{F}^{C'}\mathsf{Q}xF(x)). \end{aligned}$$

$\square$

Let us now focus on the desiderata which are not satisfied by $\mathcal{F}^{C'}$.

Theorem 12. *The supremum/infimum principle is not fully satisfied by $\mathcal{F}^{C'}$. In particular, the supremum part of the principle holds, i.e.*

$$v_{\mathcal{I}}(\mathcal{F}^{C'}(\exists)xF(x)) = \sup_{c\in D} v_{\mathcal{I}}(F(c))$$

while the infimum and average part does not, i.e.

$$v_{\mathcal{I}}(\mathcal{F}^{C'}(\forall)xF(x)) \neq \inf_{c\in D} v_{\mathcal{I}}(F(c))$$

and

$$v_{\mathcal{I}}(\mathcal{F}^{C'}(\Pi)xF(x)) \neq Prop_{\mathcal{I}} v_{\mathcal{I}}(F(c))$$

The desiderata of Dualization, external negation and quantified partition are also not satisfied by $\mathcal{F}^{C'}$.

Proof. Let us first prove that the supremum principle holds. For any precisification $\mathcal{I}'$, we have either $v_{\mathcal{I}'}(\exists xF(x)) = 0$ or $v_{\mathcal{I}'}(\exists xF(x)) = 1$. Hence, recalling that we assumed $v_{\mathcal{I}}(F(c_1) \leq \cdots \leq v_{\mathcal{I}}(F(c_n))$, we obtain a maximum in $v_{\mathcal{I}}(\mathcal{F}^{C'}(\exists)xF(x))$, for the precisification $\mathcal{I}_{F\uparrow 1}$, with kernel $\{c_n\}$. Thus, $v_{\mathcal{I}}(\mathcal{F}^{C'}(\exists)xF(x))$ boils down to $\max_{c\in D} v_{\mathcal{I}}(F(c))$. Now let us show a counterexample to the infimum principle.

For $\forall$, again any precisification behaves as a Type I quantifier. Now, we have $\mathcal{I}_{F\uparrow n}$ as a coherent precisification of $\mathcal{I}$ we would obtain $v_{\mathcal{I}_{F\uparrow n}}(\forall xF(x)) = 1$, since $v_{\mathcal{I}_{F\uparrow n}}(F(c_1)) = \cdots = v_{\mathcal{I}_{F\uparrow n}}(F(c_n)) = 1$. Recalling that the kernel associated with $\mathcal{I}_{F\uparrow n}$ is D and $|D| = n$ we get :

$$v_{\mathcal{I}}(\mathcal{F}^{C'}(\forall)xF(x)) = v_{\mathcal{I}_{F\uparrow n}}(\forall xF(x)) \odot \frac{\sum_{c\in D} v_{\mathcal{I}}(F(c))}{n} = Prop_{\mathcal{I}}(F) \neq \inf_{c\in D} v_{\mathcal{I}}(F(c)).$$

Let us show a counterexample to the average principle. Assume $|D| = 10$ and let F be a predicate, $\mathcal{I}$ an interpretation such that $v_{\mathcal{I}}(F(c_{10})) = 1, v_{\mathcal{I}}(F(c_9)) = 0.1$ and $v_{\mathcal{I}}(F(c_i) = 0$ for each $i = 1, \ldots, 8$. We get $Prop_{\mathcal{I}}F = 1.1/10 = 0.11$. On the other hand, the only two coherent precisifications are $\mathcal{I}_{F\uparrow 1}$ and $\mathcal{I}_{F\uparrow 2}$, for which we get $v_{\mathcal{I}_{F\uparrow 1}}(\Pi xF(x)) = 0.1$ and $v_{\mathcal{I}_{F\uparrow 2}}(\Pi xF(x)) = 0.2$. Recalling that Π is a nondecreasing, we obtain:

$$\begin{aligned}
v_{\mathcal{I}}(\mathcal{F}^{C'}(\Pi)xF(x)) &= \max_{m\in\kappa^{+}_{\mathcal{I}}(\Pi xF(x))} v_{\mathcal{I}_{F\uparrow m}}(\Pi xF(x)) \odot \frac{\sum\limits_{i=n-m+1}^{n} v_{\mathcal{I}}(F(c_i))}{m} \\
&= \max\left(0.1 \odot v_{\mathcal{I}}(F(c_1)), 0.2 \odot \frac{v_{\mathcal{I}}(F(c_1)) + v_{\mathcal{I}}(F(c_2))}{2}\right) \\
&= \max(0.1 \odot 1, 0.2 \odot \frac{1.1}{2}) = 0.1
\end{aligned}$$

Hence $v_{\mathcal{I}}(\mathcal{F}^{C'}(\Pi)xF(x)) \neq Prop_{\mathcal{I}}$.

A counterexample to the Dualization desideratum can be obtained from what we showed for the Infimum principle. Indeed, we have

$$v_{\mathcal{I}}((\mathcal{F}^{C'}(\exists^d))xF(x)) = v_{\mathcal{I}}(\mathcal{F}^{C'}(\forall)xF(x)) = Prop_{\mathcal{I}}(F),$$
$$\text{while } v_{\mathcal{I}}(\mathcal{F}^{C'}(\exists)^d xF(x)) = 1 - \sup_{c\in D} 1 - v_{\mathcal{I}}(F(c)) = \inf_{c\in D} v_{\mathcal{I}}(F(c)).$$

Let us now assume that $\mathcal{F}^{C'}$ satisfies external negation. Since, by Theorem 10, $\mathcal{F}^{C'}$ satisfies internal negation, this would imply that $\mathcal{F}^{C'}$ satisfies dualization as well, but this is in contradiction with what we just showed above. Finally, quantified partition cannot hold, since external negation would follow as a consequence (just take the quantifiers Q and $\neg Q$ as a quantified partition). □

Remark 1. The failure of the average principle is due to the use of the Łukasiewicz t-norm in item (iii) of Definition 20. It is easy to show that, replacing it by the product t-norm, the principle would be satisfied.

Remark 2. The result on the supremum principle might struck one as a rather implausible consequence of the fuzzification mechanism $\mathcal{F}^{C'}$. Is there a sense in which $Prop_{\mathcal{I}}(F)$ can be seen as a reasonable truth function for the natural language quantifier all in a fuzzy setting?

From the logical point of view this move seems costly, as $Prop_{\mathcal{I}}(F)$ does not enjoy the logical properties we would expect from a quantifier standing for all: there is no connection with a conjunction connective, and all usual axioms relating the quantifier all with the implication connective turn out to be invalid.

On the other hand, it is disputable whether the usual truth function inf can provide a good model for the quantifier all in the fuzzy setting. Consider a domain $D = \{c_1, \ldots, c_{100}\}$, predicate $F(x)$ and an interpretation $\mathcal{I}$. If, say $v_{\mathcal{I}}(F(c_1)) = \cdots = v_{\mathcal{I}}(F(c_{99})) = 1$ and $v_{\mathcal{I}}(F(c_{100})) = 0$, we would obtain $v_{\mathcal{I}}(\forall xF(x)) = 0$, while on other hand $Prop_{\mathcal{I}}(F) = 0.99$. We contend that, in this setting, the latter is at least as plausible as a choice for the truth value of the sentence All (the elements of the domain) are F. Indeed, reframing the issue in terms of hedging, we imagine that a competent speaker would accept the sentence It is almost true that all (the elements of the domain) are F. This is consistent with the reading of the quantifier as $Prop_{\mathcal{I}}(F)$, but not as the usual infimum.

Moreover, in case $v_{\mathcal{I}}(F(c)) = 1$ for every element c of the domain, we have anyway $v_{\mathcal{I}}(\forall xF(x)) = Prop_{\mathcal{I}}(F) = 1$. Note that, in any case, this result of the fuzzification mechanism can be easily put aside: both infimum and $Prop_{\mathcal{I}}(F)$ are anyway expressible in the language of $\text{Ł}_{\triangle}(\Pi)$ and can be both used, according to the situation to be modeled.

To conclude the section, we note that the QFM $\mathcal{F}^{CX}$ can also be seen as an instance of the closeness based fuzzification mechanism, for monotone quantifiers. It can be obtained by instantiating the schema on Definition 20 as follows:

(i) The set of precisifications is identified with the crisp predicates $F^{\geq c}(x)$, for any $c \in D$.
(ii) The closeness is measured as $\inf\,(F(x) \leftrightarrow F^{\geq c}(x))$, where inf ranges over the kernels of the quantifier at hand.
(iii) The conjunction $\wedge$.

This determines the value

$$\max_{\substack{c \in D \cup \{\top\} \\ D' \in K_{\geq c}(\mathsf{Q}xF(x))}} v_{\mathcal{I}}(\mathsf{Q}xF^{\geq c}(x)) \wedge \inf_{d \in D'} (F^{\geq c}(d) \leftrightarrow F(d))$$

where the $K_{\geq c}(\mathsf{Q}xF(x))$ is a shorthand for $K_{\mathcal{I}'}(\mathsf{Q}xF(x))$, for $\mathcal{I}'$ a precisification such that $v_{\mathcal{I}'}(F(d)) = 1$ if $v_{\mathcal{I}}(F(d)) \geq v_{\mathcal{I}}(F(c))$, $v_{\mathcal{I}'}(F(d)) = 0$ otherwise.

Theorem 13. *Let* Q *be a monotone semi-fuzzy quantifier,* $F(x)$ *a formula,* $\mathcal{I}$ *a fuzzy interpretation. We have :*

$$v_{\mathcal{I}}(\mathcal{F}^{CX}(\mathsf{Q})xF(x)) = \max_{\substack{c \in D \cup \{\top\} \\ D' \in K_{\geq c}(\mathsf{Q}xF(x))}} v_{\mathcal{I}}(\mathsf{Q}xF^{\geq c}(x)) \wedge (\inf_{d \in D'} F^{\geq c}(d) \leftrightarrow F(d)).$$

Proof. Let us show the case for non-decreasing quantifiers. By Lemma 6, any kernel D' is positive, i.e. $v_{\mathcal{I}}(F^{\geq c}(d)) = 1$ for any $d \in D'$.

Hence by straightforward computations we obtain:

$$\inf_{d \in D'} (v_{\mathcal{I}}(F^{\geq c}(d) \leftrightarrow F(d))) = \inf_{d \in D'} (v_{\mathcal{I}}(F^{\geq c}(d) \rightarrow F(d))) = \min_{d \in D'} v_{\mathcal{I}}(F(d))$$

Assume that the latter equals $v_{\mathcal{I}}(F(c'))$ for a certain $c' \in D'$ Note that, since D' is a positive kernel in $K_{\geq c}(\mathsf{Q}xF(x))$, we have that $F^{\geq c'}(d)$ determines a D'-based variant of $F^{\geq c}(d)$. Hence we have $v_{\mathcal{I}}(\mathsf{Q}xF^{\geq c'}(x)) \geq v_{\mathcal{I}}(\mathsf{Q}xF^{\geq c}(x))$. We thus get

$$v_{\mathcal{I}}(\mathsf{Q}xF^{\geq c}(x)) \wedge \inf_{d \in D'} (v_{\mathcal{I}}(F^{\geq c}(d) \leftrightarrow F(d))) = v_{\mathcal{I}}(\mathsf{Q}xF^{\geq c}(x) \wedge F(c')) \leq v_{\mathcal{I}}(\mathsf{Q}xF^{\geq c'}(x) \wedge F(c')).$$

Hence, we finally obtain:

$$\begin{aligned} \max_{\substack{c \in D \cup \{\top\} \\ D' \in K_{\geq c}(\mathsf{Q}xF(x))}} v_{\mathcal{I}}(\mathsf{Q}xF^{\geq c}(x)) \wedge (\inf_{d \in D'} F^{\geq c}(d) \leftrightarrow F(d)) &= \max_{c' \in D \cup \{\top\}} v_{\mathcal{I}}(\mathsf{Q}xF^{\geq c'}(x) \wedge F(c')) \\ &= v_{\mathcal{I}}(\mathcal{F}^{CX}(\mathsf{Q})xF(x)). \end{aligned}$$

□

To conclude, let us note that, regarding closeness-based QFMs, only the closeness measure $CL_{D'}$ is strictly calibrated for the case of unary quantifiers. In order to extend our results to n-ary quantifiers, one has to consider suitable extensions of such a closeness measure to the n-ary case. As for previous cases, however, the biggest challenges would actually be assessing the adequacy of the n-ary fuzzy quantifiers thus obtained for linguistic modeling.

6.1 Applying Closeness-based QFMs to Type I Quantifiers

So far, following Glöckner, we assumed that functions interpreting semi-fuzzy quantifiers were given and we showed possible ways of lifting such functions to the fully fuzzy case. However, modeling semi-fuzzy quantifiers is, in itself, a nontrivial task, difficult to ground on purely linguistic considerations. As mentioned in the introduction, the problem has usually been ignored, with the notable exception of [14], where functions for semi-fuzzy quantifiers are systematically introduced, on the basis of games semantics principles. In this section we show an alternative approach to such issues.

We consider a variant of our closeness-based fuzzification mechanism, to be applied to quantifiers of Type I, rather than to semi-fuzzy quantifiers. This way, we obtain models of semi-fuzzy quantifiers, which can be also directly applied to the fully-fuzzy case.

In particular, we show that some of the models of semi-fuzzy quantifiers introduced in [14] can be rediscovered in this way.

Clearly, QFMs satisfying correct generalization will not work for our purpose: no proper semi-fuzzy quantifier can be obtained if we insist that over a crisp input, the quantifier should output the same truth value as in the original Type I case.

For this reason, we need to consider a variant of our last QFM $\mathcal{F}^{C'}$, dropping the condition that the precisifications should be coherent. Given a Type I quantifier Q, a formula $F(x)$ and an interpretation $\mathcal{I}$, we consider the set $Id(\mathsf{Q}x(F(x))$ containing every classical interpretation $\mathcal{I}'$ over the same domain of $\mathcal{I}$, such that $v_{\mathcal{I}'}(\mathsf{Q}xF(x)) = 1$, no matter what the original fuzzy interpretation $\mathcal{I}$ was. Here, rather than precisifications of $\mathcal{I}$, we should speak of ideal interpretations, hence the name Id. We have the following.

Definition 26. *Let Q be a Type I quantifier, $\mathcal{I}$, a fuzzy interpretation. We define:*

$$v_{\mathcal{I}}(\mathcal{F}^{Id}(\mathsf{Q})xF(x)) = \max_{\substack{\mathcal{I}' \in Id(\mathsf{Q}xF(x)) \\ D' \in K_{\mathcal{I}'}(\mathsf{Q}xF(x)}} CL_{D'}(F^{\mathcal{I}}, F^{\mathcal{I}'}).$$

In the definition above, $K_{\mathcal{I}'}(\mathsf{Q}xF(x))$ and $CL_{D'}(F^{\mathcal{I}}, F^{\mathcal{I}'})$ are as in the previous section. Comparing $\mathcal{F}^{Id}$ with $\mathcal{F}^{C'}$, note that, in the former, the conjunction with $v_{\mathcal{I}'}(\mathsf{Q}xF(x))$ is missing: this is because, differently than for $C_{\mathcal{I}}$, by definition $v_{\mathcal{I}'}(\mathsf{Q}xF(x)) = 1$ for every $\mathcal{I}' \in Id(\mathsf{Q}xF(x))$.

Let us consider now what happens with the quantifiers $\mathsf{Q}_{[\geq j]}$ and $\mathsf{Q}_{[\geq \frac{k}{n}]}$. If we treat them as Type I quantifiers, their truth function is uniquely determined, in the obvious way, in terms of cardinality of their arguments. Let $\mathcal{I}$ be an interpretation over a domain D, $F(x)$ a first-order formula in the language of $\text{Ł}_{\triangle}$.

Recall, from Example 8, that the kernel of a precisification $\mathcal{I}'$, such that $v_{\mathcal{I}'}(\mathsf{Q}_{[\geq j]}xF(x)) = 1$, is a positive kernel D' such that $|D'| = j$. Let us assume, as in the previous section that $D = \{c_1, \ldots, c_n\}$ with $v_{\mathcal{I}}(F(c_1)) \leq \cdots \leq v_{\mathcal{I}}(F(c_n))$. Hence, recalling Lemma 6, the maximum in $v_{\mathcal{I}}(\mathcal{F}^{Id}(\mathsf{Q}_{[\geq j]})xF(x))$ is obtained for the precisification $\mathcal{I}_{F\uparrow j}$, i.e. we have

$$v_{\mathcal{I}}(\mathcal{F}^{Id}(\mathsf{Q}_{[\geq j]})xF(x)) = \frac{\sum_{i=n-j+1}^{n} v_{\mathcal{I}}(F(c_i))}{j}.$$

That is, $\mathcal{F}^{Id}(\mathsf{Q}_{[\geq j]})xF(x)$ is the average truth value of the j elements of the domain for which $v_{\mathcal{I}}(F(c))$ is highest. We then obtain:

$$v_{\mathcal{I}}(\mathcal{F}^{Id}(\mathsf{Q}_{[\geq \frac{k}{m}]})xF(x)) = v_{\mathcal{I}}(\mathcal{F}^{Id}(\mathsf{Q}_{[\geq j]})xF(x)) \quad \text{for} \quad j = \lceil (k/m) \cdot |D| \rceil.$$

Let us now recall the models of semi-fuzzy quantifiers proposed in [14]. Consider the family of quantifiers G^k_m introduced in [14] on the basis of game theoretical principles, which are meant to provide models for linguistic expressions such as $\mathsf{at\ least\ k/m}$. Their truth function is given by

$$v_{\mathcal{I}}(\mathsf{G}^k_m x\hat{F}(x)) = \min\{1, \max\{0, (k+m)Prop(\hat{F}) - k + 1\}\}.$$

For the particular case $k = 1$, we get

$$v_{\mathcal{I}}(\mathsf{G}^1_m x \hat{F}(x)) = \min\{1, (1+m)Prop(\hat{F})\}\}.$$

On the other hand, if $|D| = n$ is divisible by $(m+1)$, we have:

$$\begin{aligned} v_{\mathcal{I}}(\mathcal{F}^{Id}(\mathsf{Q}_{[\geq \frac{1}{m+1}]})x\hat{F}(x)) &= \frac{\sum\limits_{i=n-n/(m+1)+1}^{n} v_{\mathcal{I}}(\hat{F}(c_i))}{n/(m+1)} \\ &= \begin{cases} 1 & \text{if } Prop(F) \geq 1/(m+1) \\ (m+1)\cdot Prop(\hat{F}) & \text{otherwise.} \end{cases} \end{aligned}$$

That is, G^1_m coincides with $\mathcal{F}^{C'}(\mathsf{Q}_{[\geq \frac{1}{1+m}]})$. In the general case, we will not have that G^k_m coincides with $\mathcal{F}^{Id}(\mathsf{Q}_{[\geq \frac{k}{k+m}]})$, as we get:

$$\begin{aligned} v_{\mathcal{I}}(\mathcal{F}^{Id}(\mathsf{Q}_{[\geq \frac{k}{k+m}]})x\hat{F}(x)) &= \frac{\sum\limits_{i=n-nk/(k+m)+1}^{n} v_{\mathcal{I}}(\hat{F}(c_i))}{nk/(k+m)} \\ &= \begin{cases} 1 & \text{if } Prop(F) \geq \frac{k}{k+m} \\ Prop(F)\cdot \frac{m+k}{k} & \text{otherwise.} \end{cases} \end{aligned}$$

To obtain the truth function of G^k_m in the general case, we can introduce a *strictness* parameter in the evaluation of closeness for $\mathcal{F}^{C'}(\mathsf{Q}_{[\geq \frac{k}{k+m}]})$. Given a quantifier Q, let us denote by $\mathcal{F}^{Id^s}$ the fuzzification mechanism obtained by letting:

$$v_{\mathcal{I}}(\mathcal{F}^{Id^s}(\mathsf{Q})xF(x)) = \max_{\substack{\mathcal{I}'\in Id(\mathsf{Q}xF(x)) \\ D'\in K_{\mathcal{I}'}(\mathsf{Q}xF(x)}} (CL^s_{D'}(F^{\mathcal{I}}, F^{\mathcal{I}'}))$$

where the power s is taken w.r.t. to the Łukasiewicz conjunction $\odot$. We have that:

$$v_{\mathcal{I}}(\mathcal{F}^{Id^s}(\mathsf{Q}_{[\geq \frac{k}{n}]})xF(x)) = \max(0, s\cdot v_{\mathcal{I}}(\mathcal{F}^{Id}(\mathsf{Q}_{[\geq \frac{k}{n}]})xF(x)) - (s-1))$$

and in particular

$$\begin{aligned} v_{\mathcal{I}}(\mathcal{F}^{Id^k}(\mathsf{Q}_{[\geq \frac{k}{k+m}]})xF(x)) &= \max(0, k\cdot v_{\mathcal{I}}(\mathcal{F}^{Id}(\mathsf{Q}_{[\geq \frac{k}{k+m}]})xF(x)) - k + 1) \\ &= \min(1, \max(0, k\cdot Prop(F)\cdot \frac{m+k}{k} - k + 1)) = v_{\mathcal{I}}(\mathsf{G}^k_m xF(x)). \end{aligned}$$

7 Summary and Conclusion

In this paper we have revisited and analyzed Glöckner's desiderata for quantifiers fuzzification mechanisms [19]. As discussed in the introduction, we focused on unary quantification. This move allowed us to set aside broader concerns with truth functionality, which are pressing issues already in the case of binary quantifiers, where vagueness related dependencies between range and scope predicates should be respected in linguistically adequate models. Moreover, the restriction to unary quantification allowed a more focused analysis of Glöckner's desiderata for QFMs. We also emphasized that Glöckner's approach is incompatible with a central paradigm of contemporary Mathematical Fuzzy Logic: t-norm based truth functions, in particular including (full) Łukasiewicz logic.

Desiderata	$\mathcal{F}^{C'}$
Correct Generalization	✓
Projection Quantifiers	✓
Quantitativity	✓
Internal Negation	✓
External Negation	×
Dualization	×
Monotonicity (argument)	✓
Quantifier strength	(✓)[a]
Continuity	✓
Supremum/Infimum/Average	(✓)[b]
Quantified partitions	×

Table 1. Desiderata satisfied by $\mathcal{F}^{C'}$

[a] For monotone quantifiers, see Theorem 10.
[b] Only the supremum part, see Theorem 12.

	Expressibility
$\mathcal{F}^{\mathcal{M}}$	$Ł_\triangle(\boldsymbol{\Pi})$[a]
$\mathcal{F}^{OWA}$	$Ł_\triangle(\boldsymbol{\Pi})$[b]
$\mathcal{F}^{R}$	$Ł_\triangle(\boldsymbol{\Pi})$
$\mathcal{F}^{L}$	$Ł_\triangle(\boldsymbol{\Pi})$
$\mathcal{F}^{CX}$	$Ł_\triangle$
$\mathcal{F}^{C}$	$Ł_\triangle$
$\mathcal{F}^{C'}$	×

	Monotonicity (cons. extension)
$\mathcal{F}^{\mathcal{M}}$	✓
$\mathcal{F}^{OWA}$	✓
$\mathcal{F}^{R}$	✓
$\mathcal{F}^{L}$	✓
$\mathcal{F}^{CX}$	✓
$\mathcal{F}^{C'}$	(✓)[c].

Table 2. Our two new desiderata: expressibility and monotonicity under conservative extension

[a] For monotone quantifiers.
[b] For monotone quantifiers.
[c] For quantifiers with extensive kernels, see Theorem 11

We have investigated the main DFS introduced by Glockner and showed how they fared with the respect to expressibility. We focused in particular on the case of monotone quantifiers, where the DFS $\mathcal{F}^{OWA}$ and $\mathcal{F}^{CX}$ coincide with previously known QFMs in the literature, i.e. those based on the Choquet and the Sugeno integral, respectively. While $\mathcal{F}^{\mathcal{M}}$ and $\mathcal{F}^{OWA}$ required the use of propositional quantifiers, $\mathcal{F}^{CX}$ turned out to be the only DFS already expressible in $Ł_\triangle$.

In Section 6 we then introduced a new family of QFM, which we called "closeness-based". The core principle for these QFMs is to evaluate a fully fuzzy sentence by picking a precisification which maximizes the evaluation of a corresponding semi-fuzzy quantified sentence and the "closeness" of the precisification to the original fuzzy interpretation. Various QFMs can in principle be introduced by concrete instantiations of this idea, in particular by suitable choices of the measure of closeness.

In this setting, we recalled the QFM $\mathcal{F}^C$ from [1] and saw its shortcomings: essential desiderata such as correct generalization and monotonicity are not satisfied. Investigating the reason for such failures, we ended up with our last QFM, $\mathcal{F}^{C'}$, which, to the best of our knowledge, was not previously discussed in the literature. We contend that the models it provides are intuitively appealing, despite not fulfilling all of the given desiderata. From our perspective, a particularly important issue is that this QFM may not be expressible in suitable expansions of Łukasiewicz logic, at least in the general case. This is a topic for future research.

In Table 1 and Table 2 we provide a summary of our investigations.

Concerning future work, we believe that the concept of closeness-based models establishes a different research direction for modeling quantifiers. In particular, as we hinted in Section 6.1, one could follow a path different from that of Glöckner, and model fully-fuzzy quantifiers directly from generalized classical quantifiers, avoiding the intermediate step of modeling semi-fuzzy quantifiers. This results in a reduced arbitrariness in the modeling choice: in contrast to the case of semi-fuzzy quantifiers, truth functions for generalized classical quantifiers can be read off more transparently from their linguistic specification. We believe that fruitful insights for this kind of investigation can come from modeling truth values as distances from prototypes and counterexamples [36].

We also plan to further explore the advantages of embedding fuzzy quantifiers models into logical calculi, in particular for t-norm based logics. An axiomatization and a proof-theoretic study of semi-fuzzy and fuzzy quantifiers is still lacking, even for the "basic" logic $\text{Ł}(\mathit{\Pi})$. Promising steps in this direction consider modal counterparts of quantifiers, e.g. along the lines suggested in Chapter 8 of Hajek's monograph [20]. Finally, another task for future research arises from the linguistic phenomenon of vagueness-induced dependencies between different predicates involved in higher-arity quantification, discussed in Section 2. Models that take corresponding contextual shifts in memberships degrees into full account will no longer be truth-functional, but rather call for extending the machinery of quantifier fuzzification mechanisms by intensional components, akin to modal logic.

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