

On the construction of fuzzy betweenness relations from metrics

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Abstract

We consider the problem of constructing a fuzzy betweenness relation from a metric. More precisely, given a continuous Archimedean triangular norm, we present two construction methods for a fuzzy betweenness relation from a metric by making use of the pseudo-inverse of either a continuous additive generator or a continuous multiplicative generator of the triangular norm. In case the metric is bounded and given a 1-Lipschitz continuous triangular norm, we present a third construction method for a fuzzy betweenness relation from a metric by making use of the residual im-

plication of the triangular norm. Since the Łukasiewicz and product triangular norms are both continuous Archimedean and 1-Lipschitz continuous, all three construction methods may be used. Interestingly, the construction method based on the residual implication is proved to coincide with that based on a continuous additive generator for the Łukasiewicz triangular norm and with that based on a continuous multiplicative generator for the product triangular norm. We end by noting that all three construction methods result in a fuzzy prebetweenness relation when considering a pseudometric instead of a metric.

Keywords: Fuzzy betweenness relation; Metric; Triangular norm; Fuzzy prebetweenness relation; Pseudometric

1 Introduction

The study of crisp betweenness relations is a classical topic in mathematics that can be traced back as far as the nineteenth century [21]. Through the years, many different axiomatizations of crisp betweenness relations have been presented [1, 15, 16, 23], a key difference lying in the underlying structure with which they are associated (e.g., metric spaces, posets, road systems, etc.). In this work, we adhere to the definition of a crisp betweenness relation in the sense of Pérez-Fernández and De Baets [22], which has been proven to admit a characterization as a family of order relations [29].

In a recent paper [30], we studied the notion of a fuzzy betweenness relation and established a connection between fuzzy betweenness relations and fuzzy order relations, as in the crisp case. More precisely, we followed the two main schools of thought in the study of fuzzy order relations (the one by Zadeh [28] and the one initiated by Höhle and Blanchard [14] and popularized by Bodenhofer [3, 4, 5]) and presented two types of fuzzy betweenness relations. Each of these types admits a characterization as a family of fuzzy order relations in the respective sense. Moreover, we established the connection between these two types of fuzzy betweenness relations.

Although it is admittedly true that crisp betweenness relations have been historically linked to order relations, it is also true that crisp betweenness relations have been linked to metrics [22]. Thus, it is natural at this point to explore the connection between the types of fuzzy betweenness relations introduced in [30] and metrics, especially bearing in mind that (pseudo)metrics have been shown to be closely related to (binary) fuzzy relations. For instance, De Baets and Mesiar studied the correspondence between pseudometrics and fuzzy equivalence relations [7] and between metrics and fuzzy equality relations [8]. Interestingly, this correspondence is based on the use of additive generators of triangular norms and serves as the main source of inspiration for the present paper.

The aim of this paper is then to present some construction methods for a fuzzy betweenness relation from a metric. In particular, any metric d on a universe X naturally induces a crisp betweenness relation B_d on X [22], as follows:

$$B_d = \{(x, y, z) \in X^3 \mid d(x, z) = d(x, y) + d(y, z)\}.$$

It is expected that most of the triplets (x, y, z) do not fulfill the above metric equality. In particular, if d is an ultrametric on X , then $d(x, z) = d(x, y) + d(y, z)$ is equivalent to the fact that $x = y$ or $y = z$. That is to say, if d is an ultrametric on X , then it holds that B_d is exactly the smallest betweenness relation on X , defined as

$$B_0 = \{(x, y, z) \in X^3 \mid x = y \text{ or } y = z\}.$$

At this point, one might wonder, *given a metric d on X , how could the degree to which y is in between x and z be measured for those triplets (x, y, z) that do not fulfill the metric equality?* The answer to this question requires to develop some appropriate construction methods for obtaining a $*$ -betweenness relation from a metric. In the present paper, after recalling some preliminaries in Section 2, we provide such construction methods in case $*$ is either a continuous Archimedean t-norm (Section 3) or a 1-Lipschitz continuous t-norm

(Section 4). More precisely, for continuous Archimedean t-norms, we measure the degree to which the metric equality is (or is not) fulfilled when rewritten as $d(x, y) + d(y, z) - d(x, z) = 0$ (see Subsection 3.2) or as $\frac{d(x, z)}{d(x, y) + d(y, z)} = 1$ (see Subsection 3.3). Similarly, for 1-Lipschitz continuous t-norms, we measure the degree to which the metric equality is fulfilled when written in the classic form $d(x, y) + d(y, z) = d(x, z)$ by making use of the biresidual implication. In the latter case, it is necessary that the considered metric is bounded in order to permit rescaling to the unit interval. Furthermore, we provide similar results for constructing a fuzzy prebetweenness relation from a pseudometric. Section 5 provides some insights into the connection between the presented construction methods in case the given t-norm is both continuous Archimedean and 1-Lipschitz continuous. We end with some conclusions and suggestions for future work in Section 6.

2 Preliminaries

Throughout this paper, X always denotes a nonempty set. In the following, we recall some basic notions and results related to pseudometrics, triangular norms and fuzzy betweenness relations.

2.1 On pseudometrics

A mapping $d : X^2 \rightarrow [0, +\infty[$ is called a pseudometric on X if it satisfies the following conditions [9]:

- (i) *Reflexivity*: $d(x, x) = 0$, for any $x \in X$;
- (ii) *Symmetry*: $d(x, y) = d(y, x)$, for any $x, y \in X$;
- (iii) *Triangle inequality*: $d(x, z) \leq d(x, y) + d(y, z)$, for any $x, y, z \in X$.

A pseudometric d on X is called a metric on X if it additionally satisfies the *identity of indiscernibles* property (i.e., $d(x, y) = 0$ implies $x = y$, for any $x, y \in X$). Additionally, a metric d on X is called an ultrametric on X if it

satisfies the following stronger inequality (usually referred to as the *ultrametric inequality*): $d(x, z) \leq \max(d(x, y), d(y, z))$, for any $x, y, z \in X$.

A pseudometric d on X is called bounded if there exists an upper bound of d , i.e., there exists $N > 0$ such that, for any $x, y \in X$, it holds that $d(x, y) \leq N$.

The following two propositions concerning pseudometrics will be used in the next sections.

Proposition 1. *Let d be a pseudometric on X and $x, y, z \in X$ be such that $d(x, y) = 0$. It holds that $d(x, z) = d(y, z)$.*

Proof. From the triangle inequality and the fact that $d(x, y) = 0$, it follows that $d(x, z) \leq d(x, y) + d(y, z) = d(y, z)$. Similarly, it follows that $d(y, z) \leq d(y, x) + d(x, z) = d(x, z)$. We finally conclude that $d(x, z) = d(y, z)$. \square

Proposition 2. *Let d be a pseudometric on X and $o, x, y, z \in X$ be such that $d(o, x) + d(x, y) > 0$, $d(o, y) + d(y, z) > 0$ and $d(o, x) + d(x, z) > 0$. It holds that*

$$\frac{d(o, y)}{d(o, x) + d(x, y)} \cdot \frac{d(o, z)}{d(o, y) + d(y, z)} \leq \frac{d(o, z)}{d(o, x) + d(x, z)}.$$

Proof. The inequality trivially holds if $d(o, y) = 0$ or $d(o, z) = 0$. Suppose that $d(o, y) > 0$ and $d(o, z) > 0$. By making use of the triangle inequality twice, it follows that

$$\begin{aligned} & \frac{d(o, y)}{d(o, x) + d(x, y)} \cdot \frac{d(o, z)}{d(o, y) + d(y, z)} \\ &= \frac{d(o, y) \cdot d(o, z)}{d(o, x) \cdot d(o, y) + d(x, y) \cdot d(o, y) + (d(o, x) + d(x, y)) \cdot d(y, z)} \\ &\leq \frac{d(o, y) \cdot d(o, z)}{d(o, x) \cdot d(o, y) + d(x, y) \cdot d(o, y) + d(o, y) \cdot d(y, z)} \\ &= \frac{d(o, y) \cdot d(o, z)}{d(o, y) \cdot d(o, x) + d(o, y) \cdot (d(x, y) + d(y, z))} \\ &\leq \frac{d(o, y) \cdot d(o, z)}{d(o, y) \cdot d(o, x) + d(o, y) \cdot d(x, z)} \\ &= \frac{d(o, z)}{d(o, x) + d(x, z)}. \end{aligned}$$

Therefore, the inequality holds. \square

2.2 On triangular norms

A triangular norm (t-norm) $*$ on $[0, 1]$ [19] is a binary operation on $[0, 1]$ that is commutative (i.e., $\alpha * \beta = \beta * \alpha$, for any $\alpha, \beta \in [0, 1]$), associative (i.e., $\alpha * (\beta * \gamma) = (\alpha * \beta) * \gamma$, for any $\alpha, \beta, \gamma \in [0, 1]$), increasing (i.e., $\alpha \leq \beta$ implies $\alpha * \gamma \leq \beta * \gamma$, for any $\alpha, \beta, \gamma \in [0, 1]$) and has neutral element 1 (i.e., $\alpha * 1 = \alpha$, for any $\alpha \in [0, 1]$).

The four basic t-norms $*_M$ (minimum), $*_P$ (product), $*_L$ (Łukasiewicz) and $*_D$ (drastic product) are defined as follows:

- (i) minimum: $x *_M y = x \wedge y$;
- (ii) product: $x *_P y = x \cdot y$;
- (iii) Łukasiewicz: $x *_L y = 0 \vee (x + y - 1)$;
- (iv) drastic product:

$$x *_D y = \begin{cases} x \wedge y, & \text{if } 1 \in \{x, y\}, \\ 0, & \text{otherwise.} \end{cases}$$

For two t-norms $*_1$ and $*_2$, $*_1$ is said to be weaker than $*_2$ (and, equivalently, $*_2$ is said to be stronger than $*_1$), denoted by $*_1 \leq *_2$, if $x *_1 y \leq x *_2 y$, for any $(x, y) \in [0, 1]^2$. We shall write $*_1 < *_2$ meaning that $*_1 \leq *_2$ and there exists $(x_0, y_0) \in [0, 1]^2$ such that $x_0 *_1 y_0 < x_0 *_2 y_0$. It is known that the drastic product is the weakest t-norm and the minimum is the strongest one. Actually, it holds that $*_D < *_L < *_P < *_M$. From this result, it is immediate to see that 0 is an annihilator for any t-norm $*$ (i.e., $\alpha * 0 = 0$ for any $\alpha \in [0, 1]$).

A t-norm $*$ is called continuous if it is continuous as a function in the usual interval topology on $[0, 1]^2$. Similarly, a t-norm $*$ is called left-continuous if it is lower semicontinuous or, equivalently, left-continuous in its first component [10]. We recall that the minimum, the product and the Łukasiewicz t-norms are all continuous (and, thus, left-continuous), but the drastic product t-norm is not left-continuous (and, thus, not continuous). A t-norm $*$ is called 1-Lipschitz

continuous if, for any $x_1, x_2, y_1, y_2 \in [0, 1]$, it holds that $|x_1 * y_1 - x_2 * y_2| \leq |x_1 - x_2| + |y_1 - y_2|$.

A t-norm $*$ is called Archimedean [19] if for any $(x, y) \in]0, 1[^2$, there exists a positive integer n such that $\underbrace{x * x * \dots * x}_n < y$. In particular, a continuous t-norm $*$ is Archimedean if and only if $x * x < x$, for any $x \in]0, 1[$.

The minimum t-norm is 1-Lipschitz continuous but not Archimedean and the t-norm $*_n$ (see [25]) defined as

$$x *_n y = \begin{cases} \sqrt[n]{x^n + y^n - 1}, & \text{if } x^n + y^n > 1, \\ 0, & \text{otherwise,} \end{cases}$$

is continuous Archimedean but not 1-Lipschitz continuous. Hence, there exists no inclusion relation between the class of all continuous Archimedean t-norms and that of all 1-Lipschitz continuous t-norms. However, the intersection of these two classes is not empty since the product and Łukasiewicz t-norms are both continuous Archimedean and 1-Lipschitz continuous.

2.3 On fuzzy betweenness relations

A (binary) fuzzy relation R on X is a mapping $R : X^2 \rightarrow [0, 1]$. Similarly, a ternary fuzzy relation T on X is a mapping $T : X^3 \rightarrow [0, 1]$. In this paper, we are interested in the following properties of a fuzzy relation (see, e.g., [2, 20, 28]):

- (i) reflexivity: $R(x, x) = 1$, for any $x \in X$;
- (ii) symmetry: $R(x, y) = R(y, x)$, for any $x, y \in X$;
- (iii) $*$ -transitivity (with $*$ being a t-norm): $R(x, y) * R(y, z) \leq R(x, z)$, for any $x, y, z \in X$.

A fuzzy relation R on X is said to be a $*$ -equivalence relation¹ if it is reflexive, symmetric and $*$ -transitive.

¹There are several different names for $*$ -equivalence relations in the literature, such as similarity relations [28] and indistinguishability operators [24, 26].

In [30], we presented the definitions of a $*$ - E -betweenness relation and a $*$ -betweenness relation in a lattice-theoretic setting. However, in order to construct fuzzy betweenness relations from a metric, we only consider the lattice to be the real unit interval $[0, 1]$ throughout this paper.

Definition 1. [30] Let $*$ be a t-norm and E be a $*$ -equivalence relation on X . A ternary fuzzy relation B on X is called a $*$ - E -betweenness relation if it satisfies the following four conditions:

- (i) *Symmetry* (w.r.t. the first and the third element): $B(x, y, z) = B(z, y, x)$, for any $x, y, z \in X$;
- (ii) *E-reflexivity* (w.r.t. the second and the third element): $E(y, z) \leq B(x, y, z)$, for any $x, y, z \in X$;
- (iii) *$*$ - E -antisymmetry* (w.r.t. the second and the third element): $B(x, y, z) * B(x, z, y) \leq E(y, z)$, for any $x, y, z \in X$;
- (iv) *$*$ -transitivity* (w.r.t. the second and the third element): $B(o, x, y) * B(o, y, z) \leq B(o, x, z)$, for any $o, x, y, z \in X$.

Definition 2. [30] Let $*$ be a t-norm. A ternary fuzzy relation B on X is called a $*$ -betweenness relation if it satisfies the following four conditions:

- (i) *Symmetry* (w.r.t. the first and the third element): $B(x, y, z) = B(z, y, x)$, for any $x, y, z \in X$;
- (ii) *Reflexivity* (w.r.t. the second and the third element): $B(x, y, y) = 1$, for any $x, y \in X$;
- (iii) *Crisp antisymmetry* (w.r.t. the second and the third element): $B(x, y, z) = B(x, z, y) = 1$ implies $y = z$, for any $x, y, z \in X$;
- (iv) *$*$ -transitivity* (w.r.t. the second and the third element): $B(o, x, y) * B(o, y, z) \leq B(o, x, z)$, for any $o, x, y, z \in X$.

Remark 1. A first notion of fuzzy betweenness relation is due to Jacas and Recasens [17, 18], yet with a stronger transitivity property and a weaker antisymmetry property compared to Definition 2, thus yielding a notion incomparable to that in Definition 2.

A symmetric, reflexive and $*$ -transitive ternary fuzzy relation is called a $*$ -prebetweenness relation.

The following proposition reveals the connection between $*$ - E -betweenness relations and $*$ -prebetweenness relations.

Proposition 3. [30] *Let $*$ be a t-norm, E be a binary fuzzy relation on X and B be a ternary fuzzy relation on X . The following two statements are equivalent:*

- (i) *E is a $*$ -equivalence relation on X and B is a $*$ - E -betweenness relation on X ;*
- (ii) *B is a $*$ -prebetweenness relation on X , $E(x, y) = B(y, x, y)$, for any $x, y \in X$, and*

$$B(x, y, z) * B(x, z, y) \leq B(y, z, y), \text{ for any } x, y, z \in X.$$

3 The case of continuous Archimedean t-norms

In this section, we first recall some basic notions and results related to additive and multiplicative generators of a continuous Archimedean t-norm. Subsequently, given a continuous Archimedean t-norm $*$, we present two construction methods for a $*$ -prebetweenness relation from a pseudometric by making use of the pseudo-inverse of a continuous additive or a continuous multiplicative generator of $*$. Furthermore, it is proved that a $*$ -betweenness relation arises if and only if a metric (rather than just a pseudometric) is considered. Interestingly, both construction methods are shown to result in different fuzzy (pre)betweenness relations.

3.1 On additive and multiplicative generators of a continuous Archimedean t-norm

Let $[a, b]$ and $[c, d]$ be two closed subintervals of the extended real line $[-\infty, +\infty]$ and $f : [a, b] \rightarrow [c, d]$ be a monotone function. The pseudo-inverse $f^{(-1)} : [c, d] \rightarrow [a, b]$ of f is defined by

$$f^{(-1)}(y) = \sup\{x \in [a, b] \mid (f(x) - y)(f(b) - f(a)) < 0\},$$

for any $y \in [c, d]$, where it is assumed that $\sup \emptyset = a$.

In particular, if $f(a) < f(b)$ (thus, f is increasing), then it holds that, for any $y \in [c, d]$,

$$f^{(-1)}(y) = \sup\{x \in [a, b] \mid f(x) < y\}.$$

Similarly, if $f(a) > f(b)$ (thus, f is decreasing), then it holds that, for any $y \in [c, d]$,

$$f^{(-1)}(y) = \sup\{x \in [a, b] \mid f(x) > y\}.$$

Many t-norms can be characterized by an additive generator. Let $\text{Ran}(t)$ denote the range of a function t .

Definition 3. [19] A strictly decreasing function $t : [0, 1] \rightarrow [0, +\infty]$ is said to be an additive generator of a t-norm $*$ if it is right-continuous at 0, $t(1) = 0$, and for any $(x, y) \in [0, 1]^2$, it holds that

$$t(x) + t(y) \in \text{Ran}(t) \cup [t(0), +\infty]$$

and

$$x * y = t^{(-1)}(t(x) + t(y)).$$

Examples of t-norms that have an additive generator are the product, Łukasiewicz and drastic product t-norms.

Example 1. (i) The function $t_P : [0, 1] \rightarrow [0, +\infty]$ defined by $t_P(x) = -\ln x$ is an additive generator of the product t-norm.

(ii) The function $t_L : [0, 1] \rightarrow [0, +\infty]$ defined by $t_L(x) = 1 - x$ is an additive generator of the Łukasiewicz t-norm.

(iii) The function $t_D : [0, 1] \rightarrow [0, +\infty]$ defined by

$$t_D(x) = \begin{cases} 2 - x, & \text{if } x \in [0, 1[, \\ 0, & \text{if } x = 1, \end{cases}$$

is an additive generator of the drastic product t-norm.

Not all t-norms have an additive generator. An example of such a t-norm is the minimum t-norm. For an extensive discussion on additive generators, we refer to [27].

Similarly, many t-norms can be characterized by a multiplicative generator. Interestingly, a function $t : [0, 1] \rightarrow [0, +\infty]$ is an additive generator of a t-norm $*$ if and only if the function $\theta : [0, 1] \rightarrow [0, 1]$ defined by $\theta(x) = e^{-t(x)}$ is a multiplicative generator of $*$.

Definition 4. [19] A strictly increasing function $\theta : [0, 1] \rightarrow [0, 1]$ is said to be a multiplicative generator of a t-norm $*$ if it is right-continuous at 0, $\theta(1) = 1$, and for any $(x, y) \in [0, 1]^2$, it holds that

$$\theta(x) \cdot \theta(y) \in \text{Ran}(\theta) \cup [0, \theta(0)]$$

and

$$x * y = \theta^{(-1)}(\theta(x) \cdot \theta(y)).$$

Remark 2. (1) Examples of t-norms that have a multiplicative generator are the product, Łukasiewicz and drastic product t-norms:

(i) The function $\theta_P : [0, 1] \rightarrow [0, 1]$ defined by $\theta_P(x) = x$ is a multiplicative generator of the product t-norm;

(ii) The function $\theta_L : [0, 1] \rightarrow [0, 1]$ defined by $\theta_L(x) = e^{x-1}$ is a multiplicative generator of the Łukasiewicz t-norm;

(iii) The function $\theta_D : [0, 1] \rightarrow [0, 1]$ defined by

$$\theta_D(x) = \begin{cases} e^{x-2}, & \text{if } x \in [0, 1[, \\ 1, & \text{if } x = 1, \end{cases}$$

is a multiplicative generator of the drastic product t-norm.

(2) Not all t-norms have a multiplicative generator. An example of such a t-norm is the minimum t-norm.

Any t-norm with an additive (or, equivalently, a multiplicative) generator is necessarily Archimedean. The converse is not true, but holds for continuous t-norms.

Proposition 4. [19] *For any t-norm $*$, the following statements are equivalent:*

- (i) *$*$ is continuous Archimedean;*
- (ii) *$*$ has a continuous additive generator, i.e., there exists a continuous and strictly decreasing function $t : [0, 1] \rightarrow [0, +\infty]$ with $t(1) = 0$, which is uniquely determined up to a positive multiplicative constant, such that for any $(x, y) \in [0, 1]^2$, it holds that*

$$x * y = t^{(-1)}(t(x) + t(y)) = t^{-1}((t(x) + t(y)) \wedge t(0));$$

- (iii) *$*$ has a continuous multiplicative generator, i.e., there exists a continuous and strictly increasing function $\theta : [0, 1] \rightarrow [0, 1]$ with $\theta(1) = 1$, which is uniquely determined up to a positive constant exponent, such that for any $(x, y) \in [0, 1]^2$, it holds that*

$$x * y = \theta^{(-1)}(\theta(x) \cdot \theta(y)) = \theta^{-1}((\theta(x) \cdot \theta(y)) \vee \theta(0)).$$

3.2 The additive generator approach

For a given pseudometric d on X , we define the mapping $D_d : X^3 \rightarrow [0, +\infty[$ by

$$D_d(x, y, z) = d(x, y) + d(y, z) - d(x, z).$$

This mapping measures the deviation of $d(x, y) + d(y, z)$ from $d(x, z)$ when rewriting the metric equality in the form $d(x, y) + d(y, z) - d(x, z) = 0$. Note that $D_d(x, y, z) = 0$ if and only if $(x, y, z) \in B_d$.

The following proposition presents some properties of the mapping D_d .

Proposition 5. *Let d be a pseudometric on X . The mapping D_d has the following properties:*

- (i) $D_d(x, y, z) = D_d(z, y, x)$, for any $x, y, z \in X$;
- (ii) $D_d(x, x, y) = D_d(x, y, y) = 0$, for any $x, y \in X$;
- (iii) $D_d(o, x, y) + D_d(o, y, z) \geq D_d(o, x, z)$, for any $o, x, y, z \in X$;
- (iv) $D_d(x, y, z) + D_d(x, z, y) = D_d(y, z, y)$, for any $x, y, z \in X$;
- (v) $D_d(x, y, z) = D_d(x, z, y) = 0$ if and only if $d(y, z) = 0$, for any $x, y, z \in X$.

Proof. The proofs of (i)-(iv) are straightforward. The left-to-right implication of (v) follows from (iv), whereas the right-to-left implication of (v) follows from Proposition 1. \square

The main idea is now to transform the deviation of $d(x, y) + d(y, z)$ from $d(x, z)$ into a degree of betweenness, where triplets fulfilling the triangle equality (i.e. belonging to B_d) should obtain degree of betweenness 1. Inspired by [7, 8] and given the respective domains, we will explore the use of the pseudo-inverse of an additive generator to that end. More explicitly, the following theorem provides a method to construct a $*$ -(pre)betweenness relation from a given (pseudo)metric, with $*$ a continuous Archimedean t-norm, by making use of a continuous additive generator of $*$.

Theorem 1. *Let t be an additive generator of a t-norm $*$ and d be a pseudometric on X . Define the ternary fuzzy relation B_d^t on X as follows:*

$$B_d^t(x, y, z) = t^{(-1)}(D_d(x, y, z)).$$

The following results hold:

- (i) B_d^t is symmetric and reflexive;
- (ii) If B_d^t is crisp antisymmetric, then d is a metric on X . Moreover, if t is continuous, B_d^t is crisp antisymmetric if and only if d is a metric on X ;

- (iii) If t is continuous, then B_d^t is $*$ -transitive;
- (iv) If t is continuous, then B_d^t is a $*$ -prebetweenness relation on X ;
- (v) If t is continuous, then $B_d^t(x, y, z) * B_d^t(x, z, y) = B_d^t(y, z, y)$, for any $(x, y, z) \in X^3$;
- (vi) If t is continuous, then B_d^t is a $*$ -betweenness relation on X if and only if d is a metric on X .

Proof. (i) It follows from Proposition 5(i) and (ii).

(ii) We first prove the left-to-right implication. Suppose that B_d^t is crisp antisymmetric. Let $x, y \in X$ with $d(x, y) = 0$. It follows from (i) and Proposition 5(v) that $B_d^t(x, x, y) = B_d^t(x, y, x) = 1$. Hence, it holds that $x = y$ and d is a metric. We now prove the right-to-left implication. Suppose that d is a metric on X . Let $B_d^t(x, y, z) = B_d^t(x, z, y) = 1$. It follows from the continuity of t that

$$D_d(x, y, z) = D_d(x, z, y) = 0.$$

By Proposition 5(v), it holds that $d(y, z) = 0$, which implies $y = z$.

(iii) Suppose that t is continuous. In this case, it holds that

$$B_d^t(x, y, z) = t^{-1}(D_d(x, y, z) \wedge t(0)).$$

For any $o, x, y, z \in X$, it holds that

$$\begin{aligned} B_d^t(o, x, y) * B_d^t(o, y, z) &= t^{-1}((t(B_d^t(o, x, y)) + t(B_d^t(o, y, z))) \wedge t(0)) \\ &= t^{-1}(((D_d(o, x, y) \wedge t(0)) + (D_d(o, y, z) \wedge t(0))) \wedge t(0)) \\ &= t^{-1}((D_d(o, x, y) + D_d(o, y, z)) \wedge t(0)). \end{aligned}$$

Hence, it follows from Proposition 5(iii) that

$$\begin{aligned} B_d^t(o, x, y) * B_d^t(o, y, z) &\leq t^{-1}(D_d(o, x, z) \wedge t(0)) \\ &= B_d^t(o, x, z). \end{aligned}$$

(iv) It follows from (i) and (iii).

- (v) It follows from the proof of (iii) and Proposition 5(iv).
 (vi) The left-to-right implication follows from (ii). The right-to-left implication follows from (ii) and (iv). \square

Exploiting the results of Theorem 1, the same ternary fuzzy relation B_d^t turns out to be a $*$ - E -betweenness relation for an appropriately defined $*$ -equivalence relation.

Theorem 2. *Let t be a continuous additive generator of a continuous Archimedean t -norm $*$, d be a pseudometric on X . The ternary fuzzy relation B_d^t is a $*$ - E -betweenness relation on X , where $E : X^2 \rightarrow [0, 1]$ is defined by*

$$E(x, y) = B_d^t(y, x, y).$$

Proof. It follows from Theorem 1(iv), (v) and Proposition 3. \square

Remark 3. The $*$ -equivalence relation E in Theorem 2 is actually given by

$$E(x, y) = t^{(-1)}(2d(x, y)),$$

and slightly differs from the following $*$ -equivalence relation (see [7, 8]):

$$E(x, y) = t^{(-1)}(d(x, y)).$$

We give two examples to illustrate Theorems 1 and 2 with two prominent continuous Archimedean t -norms (the product and Łukasiewicz t -norms).

Example 2. For any $k > 0$, the function $t_{P,k} : [0, 1] \rightarrow [0, +\infty]$ defined by $t_{P,k}(x) = -\frac{1}{k} \ln x$ is a continuous additive generator of the product t -norm $*_P$. Obviously, $t_{P,k}^{(-1)} : [0, +\infty] \rightarrow [0, 1]$ is given by $t_{P,k}^{(-1)}(x) = e^{-kx}$. For any pseudometric d on X , the ternary fuzzy relation $B_d^{t_{P,k}}$ on X given by

$$B_d^{t_{P,k}}(x, y, z) = e^{-kD_d(x, y, z)},$$

is a $*_P$ -prebetweenness relation on X as well as a $*_P$ - E -betweenness relation on X , where $E : X^2 \rightarrow [0, 1]$ is given by

$$E(x, y) = e^{-2kd(x, y)}.$$

If d is a metric on X , then $B_d^{t_{P,k}}$ is a $*_P$ -betweenness relation on X .

Example 3. For any $k > 0$, the function $t_{L,k} : [0, 1] \rightarrow [0, +\infty]$ defined by $t_{L,k}(x) = \frac{1}{k}(1 - x)$ is a continuous additive generator of the Łukasiewicz t-norm $*_L$. Obviously, $t_{L,k}^{(-1)} : [0, +\infty] \rightarrow [0, 1]$ is given by

$$t_{L,k}^{(-1)}(x) = \begin{cases} 1 - kx, & \text{if } 0 \leq x < \frac{1}{k}, \\ 0, & \text{if } x \geq \frac{1}{k}. \end{cases}$$

For any pseudometric d on X , the ternary fuzzy relation $B_d^{t_{L,k}}$ on X , given by

$$B_d^{t_{L,k}}(x, y, z) = \begin{cases} 1 - kD_d(x, y, z), & \text{if } D_d(x, y, z) < \frac{1}{k}, \\ 0, & \text{otherwise,} \end{cases}$$

is a $*_L$ -prebetweenness relation on X as well as a $*_L$ - E -betweenness relation on X , where $E : X^2 \rightarrow [0, 1]$ is given by

$$E(x, y) = \begin{cases} 1 - 2kd(x, y), & \text{if } d(x, y) < \frac{1}{2k}, \\ 0, & \text{otherwise.} \end{cases}$$

If d is a metric on X , then $B_d^{t_{L,k}}$ is a $*_L$ -betweenness relation on X .

The first of the following two examples shows that t does not necessarily need to be continuous (as stated in Theorem 1(iii)) for B_d^t to be $*$ -transitive, and, hence, a $*$ -betweenness relation. The second one shows that, if t is not required to be continuous, B_d^t is no longer assured to be $*$ -transitive.

Example 4. Let d be a two-valued metric on X , i.e., there exists $N > 0$ such that

$$d(x, y) = \begin{cases} 0, & \text{if } x = y, \\ N, & \text{if } x \neq y. \end{cases}$$

For any additive generator t of a t-norm $*$, it holds that the ternary fuzzy relation B_d^t is $*$ -transitive.

Note that

$$D_d(x, y, z) = \begin{cases} 0, & \text{if } (x = y) \text{ or } (y = z), \\ 2N, & \text{if } (x \neq y) \text{ and } (y \neq z) \text{ and } (x = z), \\ N, & \text{otherwise.} \end{cases}$$

Hence,

$$B_d^t(x, y, z) = \begin{cases} 1, & \text{if } (x = y) \text{ or } (y = z), \\ t^{(-1)}(2N), & \text{if } (x \neq y) \text{ and } (y \neq z) \text{ and } (x = z), \\ t^{(-1)}(N), & \text{otherwise.} \end{cases}$$

It is not difficult to verify that B_d^t is $*_M$ -transitive and, thus, $*$ -transitive.

Example 5. Let $X = \{o, x, y, z\}$ and d be the metric on X defined as follows:

$$d(u, v) = \begin{cases} 0, & \text{if } u = v, \\ 2, & \text{if } \{u, v\} = \{x, z\}, \\ 1, & \text{otherwise.} \end{cases}$$

For the additive generator $t_D : [0, 1] \rightarrow [0, +\infty]$ of the drastic product t-norm $*_D$ shown in Example 1(iii), the pseudo-inverse $t_D^{(-1)}$ is given by

$$t_D^{(-1)}(x) = \begin{cases} 1, & \text{if } x \in [0, 1], \\ 2 - x, & \text{if } x \in]1, 2[, \\ 0, & \text{if } x \in [2, +\infty]. \end{cases}$$

Hence, it holds that

$$B_d^{t_D}(o, x, y) *_D B_d^{t_D}(o, y, z) = t_D^{(-1)}(1) *_D t_D^{(-1)}(1) = 1 > 0 = t_D^{(-1)}(2) = B_d^{t_D}(o, x, z).$$

This shows that $B_d^{t_D}$ is not $*_D$ -transitive.

3.3 The multiplicative generator approach

For a given pseudometric d on X , we define the mapping $Q_d : X^3 \rightarrow [0, 1]$ by

$$Q_d(x, y, z) = \begin{cases} 1, & \text{if } d(x, y) + d(y, z) = 0, \\ \frac{d(x, z)}{d(x, y) + d(y, z)}, & \text{if } d(x, y) + d(y, z) > 0. \end{cases}$$

This mapping measures the deviation of $d(x, y) + d(y, z)$ from $d(x, z)$ when rewriting the metric equality in the form $\frac{d(x, z)}{d(x, y) + d(y, z)} = 1$. Note that $Q_d(x, y, z) = 1$ if and only if $(x, y, z) \in B_d$. As the range of Q_d is $[0, 1]$, it could be interpreted as a fuzzy relation. However, we will need to further transform it to obtain a proper fuzzy betweenness relation.

The following proposition presents some properties of the fuzzy relation Q_d .

Proposition 6. *Let d be a pseudometric on X . The ternary fuzzy relation Q_d has the following properties:*

- (i) $Q_d(x, y, z) = Q_d(z, y, x)$, for any $x, y, z \in X$;
- (ii) $Q_d(x, x, y) = Q_d(x, y, y) = 1$, for any $x, y \in X$;
- (iii) $Q_d(o, x, y) \cdot Q_d(o, y, z) \leq Q_d(o, x, z)$, for any $o, x, y, z \in X$;
- (iv) $Q_d(x, y, z) = Q_d(x, z, y) = 1$ if and only if $d(y, z) = 0$, for any $x, y, z \in X$.

Proof. The proofs of (i) and (ii) are straightforward.

(iii) For any $o, x, y, z \in X$, we distinguish the following cases:

- (a) If $d(o, x) + d(x, z) = 0$, then the inequality trivially holds.
- (b) If $d(o, x) + d(x, z) > 0$ and $d(o, x) + d(x, y) = 0$, then $d(o, x) = 0$. Hence, it follows from Proposition 1 that $Q_d(o, x, z) = 1$. The inequality holds.
- (c) If $d(o, x) + d(x, z) > 0$, $d(o, x) + d(x, y) > 0$ and $d(o, y) + d(y, z) = 0$, then $d(o, y) = 0$. Hence, $Q_d(o, x, y) = 0$. The inequality holds.
- (d) If $d(o, x) + d(x, z) > 0$, $d(o, x) + d(x, y) > 0$ and $d(o, y) + d(y, z) > 0$, then it follows from Proposition 2 that the inequality holds.

(iv) For the left-to-right implication, suppose that $Q_d(x, y, z) = Q_d(x, z, y) = 1$. We distinguish two cases. If $d(x, y) + d(y, z) = 0$ or $d(x, z) + d(z, y) = 0$, then it holds that $d(y, z) = 0$. Otherwise, it holds that $d(x, z) = d(x, y) + d(y, z)$ and $d(x, y) = d(x, z) + d(z, y)$, and, thus, $d(y, z) = 0$. The right-to-left implication trivially follows from Proposition 1. \square

The following theorem provides a method to construct a \ast -(pre)betweenness relation from a given (pseudo)metric, with \ast a continuous Archimedean t-norm, by making use of a continuous multiplicative generator of \ast .

Theorem 3. *Let θ be a multiplicative generator of a t -norm $*$ and d be a pseudometric on X . Define the ternary fuzzy relation B_d^θ on X as follows:*

$$B_d^\theta(x, y, z) = \theta^{(-1)}(Q_d(x, y, z)).$$

The following results hold:

- (i) B_d^θ is symmetric and reflexive;
- (ii) If B_d^θ is crisp antisymmetric, then d is a metric on X . Moreover, if θ is continuous, B_d^θ is crisp antisymmetric if and only if d is a metric on X ;
- (iii) If θ is continuous, then B_d^θ is $*$ -transitive;
- (iv) If θ is continuous, then B_d^θ is a $*$ -prebetweenness relation on X ;
- (v) If θ is continuous, then B_d^θ is a $*$ -betweenness relation on X if and only if d is a metric on X .

Proof. (i) It follows from Proposition 6(i) and (ii).

(ii) We first prove the left-to-right implication. Suppose that B_d^θ is crisp antisymmetric. Let $x, y \in X$ with $d(x, y) = 0$. It follows from (i) and Proposition 6(iv) that $B_d^\theta(x, x, y) = B_d^\theta(x, y, x) = 1$. Hence, it holds that $x = y$ and d is a metric. We now prove the right-to-left implication. Suppose that d is a metric. Let $B_d^\theta(x, y, z) = B_d^\theta(x, z, y) = 1$. It follows from the continuity of θ that

$$Q_d(x, y, z) = Q_d(x, z, y) = 1.$$

By Proposition 6(iv), it holds that $d(y, z) = 0$, which implies $y = z$.

(iii) Suppose that θ is continuous. In this case, it holds that

$$B_d^\theta(x, y, z) = \theta^{-1}(Q_d(x, y, z) \vee \theta(0)).$$

For any $o, x, y, z \in X$, it holds that

$$\begin{aligned} B_d^\theta(o, x, y) * B_d^\theta(o, y, z) &= \theta^{-1}((\theta(B_d^\theta(o, x, y)) \cdot \theta(B_d^\theta(o, y, z))) \vee \theta(0)) \\ &= \theta^{-1}(((Q_d(o, x, y) \vee \theta(0)) \cdot (Q_d(o, y, z) \vee \theta(0))) \vee \theta(0)) \\ &= \theta^{-1}((Q_d(o, x, y) \cdot Q_d(o, y, z)) \vee \theta(0)). \end{aligned}$$

Hence, it follows from Proposition 6(iii) that

$$\begin{aligned} B_d^\theta(o, x, y) * B_d^\theta(o, y, z) &\leq \theta^{-1}(Q_d(o, x, z) \vee \theta(0)) \\ &= B_d^\theta(o, x, z). \end{aligned}$$

(iv) It follows from (i) and (iii).

(v) The left-to-right implication follows from (ii). The right-to-left implication follows from (ii) and (iv). \square

We give two examples to illustrate Theorem 3 with two prominent continuous Archimedean t-norms (the product and Łukasiewicz t-norms).

Example 6. For any $k > 0$, the function $\theta_{P,k} : [0, 1] \rightarrow [0, 1]$ defined by $\theta_{P,k}(x) = x^{\frac{1}{k}}$ is a continuous multiplicative generator of the product t-norm $*_P$. Obviously, $\theta_{P,k}^{(-1)} : [0, 1] \rightarrow [0, 1]$ is given by $\theta_{P,k}^{(-1)}(x) = x^k$. For any pseudometric d on X , the ternary fuzzy relation $B_d^{\theta_{P,k}}$ given by

$$B_d^{\theta_{P,k}}(x, y, z) = \begin{cases} 1, & \text{if } d(x, y) + d(y, z) = 0, \\ \left(\frac{d(x, z)}{d(x, y) + d(y, z)} \right)^k, & \text{if } d(x, y) + d(y, z) > 0, \end{cases}$$

is a $*_P$ -prebetweenness relation on X . If d is a metric on X , then $B_d^{\theta_{P,k}}$ is a $*_P$ -betweenness relation on X .

Remark 4. If d is a metric on X , then for any pairwise different $x, y, z \in X$, it holds that

$$B_d^{\theta_{P,k}}(x, y, z) *_P B_d^{\theta_{P,k}}(x, z, y) > 0 = B_d^{\theta_{P,k}}(y, z, y).$$

This shows that $B_d^{\theta_{P,k}}$ is not a $*_P$ -E-betweenness relation on X (see Proposition 3). We conclude from Theorem 2 that Theorems 1 and 3 present two different construction methods for a $*$ -(pre)betweenness relation.

Example 7. For any $k > 0$, the function $\theta_{L,k} : [0, 1] \rightarrow [0, 1]$ defined by $\theta_{L,k}(x) = e^{\frac{1}{k}(x-1)}$ is a continuous multiplicative generator of the Łukasiewicz

t-norm $*_L$. Obviously, $\theta_{L,k}^{(-1)} : [0, 1] \rightarrow [0, 1]$ is given by

$$\theta_{L,k}^{(-1)}(x) = \begin{cases} 0, & \text{if } 0 \leq x \leq e^{-\frac{1}{k}}, \\ 1 + k \ln x, & \text{if } e^{-\frac{1}{k}} < x \leq 1. \end{cases}$$

For any pseudometric d on X , the ternary fuzzy relation $B_d^{\theta_{L,k}}$ given by

$$B_d^{\theta_{L,k}}(x, y, z) = \begin{cases} 1, & \text{if } d(x, y) + d(y, z) = 0, \\ 1 + k \ln \frac{d(x, z)}{d(x, y) + d(y, z)}, & \text{if } \frac{d(x, z)}{d(x, y) + d(y, z)} > e^{-\frac{1}{k}}, \\ 0, & \text{otherwise,} \end{cases}$$

is a $*_L$ -prebetweenness relation on X . If d is a metric on X , then $B_d^{\theta_{L,k}}$ is a $*_L$ -betweenness relation on X .

The first of the following two examples shows that θ does not necessarily need to be continuous (as stated in Theorem 3(iii)) for B_d^θ to be $*$ -transitive, and, hence, a $*$ -betweenness relation. The second one shows that, if θ is not required to be continuous, B_d^θ is no longer assured to be $*$ -transitive.

Example 8. Let d be a two-valued metric on X (see Example 4). For any multiplicative generator θ of a t-norm $*$, it holds that B_d^θ is $*$ -transitive.

Note that

$$Q_d(x, y, z) = \begin{cases} 1, & \text{if } (x = y) \text{ or } (y = z), \\ 0, & \text{if } (x \neq y) \text{ and } (y \neq z) \text{ and } (x = z), \\ \frac{1}{2}, & \text{otherwise.} \end{cases}$$

Hence,

$$B_d^\theta(x, y, z) = \begin{cases} 1, & \text{if } (x = y) \text{ or } (y = z), \\ 0, & \text{if } (x \neq y) \text{ and } (y \neq z) \text{ and } (x = z), \\ \theta^{(-1)}\left(\frac{1}{2}\right), & \text{otherwise.} \end{cases}$$

It is not difficult to verify that B_d^θ is $*_M$ -transitive and, thus, $*$ -transitive.

Example 9. Let X and d be defined as in Example 5. For the multiplicative generator $\theta_D : [0, 1] \rightarrow [0, 1]$ of the drastic product t-norm $*_D$ defined in

Remark 2(1), the pseudo-inverse $\theta_D^{(-1)} : [0, 1] \rightarrow [0, 1]$ is given by

$$\theta_D^{(-1)}(x) = \begin{cases} 0, & \text{if } x \in [0, e^{-2}], \\ 2 + \ln x, & \text{if } x \in]e^{-2}, e^{-1}[, \\ 1, & \text{if } x \in [e^{-1}, 1]. \end{cases}$$

Hence, it holds that

$$\begin{aligned} B_d^{\theta_D}(o, x, y) *_{\theta_D} B_d^{\theta_D}(o, y, z) &= \theta_D^{(-1)}\left(\frac{1}{2}\right) *_{\theta_D} \theta_D^{(-1)}\left(\frac{1}{2}\right) = 1 \\ &> 2 - \ln 3 = \theta_D^{(-1)}\left(\frac{1}{3}\right) = B_d^{\theta_D}(o, x, z). \end{aligned}$$

This shows that $B_d^{\theta_D}$ is not $*_{\theta_D}$ -transitive.

4 The case of 1-Lipschitz continuous t-norms for a bounded (pseudo)metric

In this section, we first recall some basic notions and results related to the residual implication of a left-continuous t-norm. Next, we propose a construction method for a $*$ -(pre)betweenness relation from a bounded (pseudo)metric by making use of the residual implication of a 1-Lipschitz continuous t-norm.

4.1 On (bi)residual implications

The residual implication [11, 19] I_* of a left-continuous t-norm $*$ is defined as follows:

$$I_*(a, b) = \sup\{x \in [0, 1] \mid a * x \leq b\} = \max\{x \in [0, 1] \mid a * x \leq b\}.$$

Throughout this section, we will make use of the following properties of I_* [11, 19]:

- (i) $a \leq b \iff I_*(a, b) = 1$, for any $a, b \in [0, 1]$;
- (ii) ($*$ -transitivity) $I_*(a, b) * I_*(b, c) \leq I_*(a, c)$, for any $a, b, c \in [0, 1]$;
- (iii) $I_*(1, a) = a$, for any $a \in [0, 1]$;

(iv) $a * b \leq c \iff a \leq I_*(b, c)$, for any $a, b, c \in [0, 1]$;

(v) I_* is decreasing with respect to the first variable, but increasing with respect to the second one.

The residual implication I_* of a left-continuous t-norm $*$ is called special [12, 25] if, for any $a, b, c \in [0, 1]$ such that $a + c, b + c \in [0, 1]$, it holds that

$$I_*(a, b) \leq I_*(a + c, b + c).$$

Interestingly, the residual implications of the three most prominent (left-)continuous t-norms (i.e., $*_M$, $*_P$ and $*_L$) are all special.

Sainio et al. [25] gave several characterizations of special residual implications. One such characterization states that *the residual implication of a left-continuous t-norm $*$ is special if and only if $*$ is 1-Lipschitz continuous*. Note that a t-norm $*$ is 1-Lipschitz continuous if and only if it is an associative copula [13].

The residual implication I_* of a left-continuous t-norm $*$ is called antispecial [12] if, for any $b \in [0, 1[$ and $a, c \in]0, 1[$ such that $a + c, b + c \in [0, 1]$ and $a > b$, it holds that

$$I_*(a, b) > I_*(a + c, b + c).$$

For any positive integer $n > 1$, the residual implication of the continuous t-norm $*_n$ (see Section 2) is antispecial (see [25]). In particular, the corresponding residual implication I_{*_n} is given by

$$I_{*_n}(a, b) = \begin{cases} 1, & \text{if } a \leq b, \\ \sqrt[n]{1 - a^n + b^n}, & \text{otherwise.} \end{cases}$$

As the terminology suggests, an antispecial residual implication is not special.

The following proposition gives an equivalent characterization of special residual implications, which will be crucial in the proof of the $*$ -transitivity of ternary fuzzy relations induced by a bounded pseudometric in the next subsection.

Proposition 7. *Let $*$ be a left-continuous t-norm. It holds that I_* is special if and only if, for any $a, b, c, d, e \in [0, 1]$ such that $a + c, a + d, b + e \in [0, 1]$ and $c - d \leq e$, it holds that $I_*(a + d, b) \leq I_*(a + c, b + e)$.*

Proof. *Necessity:* Suppose that I_* is special. Consider $a, b, c, d, e \in [0, 1]$ such that $a + c, a + d, b + e \in [0, 1]$ and $c - d \leq e$. We distinguish two cases:

- (i) If $c \leq d$, then $I_*(a + d, b) \leq I_*(a + c, b) \leq I_*(a + c, b + e)$.
- (ii) If $c > d$, then $I_*(a + d, b) \leq I_*(a + d + (c - d), b + (c - d)) \leq I_*(a + c, b + e)$.

Sufficiency: Consider $a, b, c \in [0, 1]$ such that $a + c, b + c \in [0, 1]$, and let $d = 0$ and $e = c$. It holds that $a + d \in [0, 1]$, $b + e \in [0, 1]$ and $c - d \leq e$. From the assumption, we conclude that $I_*(a + d, b) \leq I_*(a + c, b + e)$ and, thus, $I_*(a, b) \leq I_*(a + c, b + c)$. \square

In a similar way as a residual implication measures the degree to which an inequality for two numbers in the unit interval (typically truth values) holds, a biresidual implication measures the degree to which an equality between two numbers in the unit interval (typically truth values) holds [8]. Formally, the biresidual implication \mathcal{E}_* of a left-continuous t-norm $*$ is defined as follows:

$$\mathcal{E}_*(a, b) = \min(I_*(a, b), I_*(b, a)) = I_*(a, b) * I_*(b, a).$$

4.2 The residual implication approach

Let $*$ be a left-continuous t-norm and d be a bounded pseudometric on X with $N > 0$ as upper bound. As in the previous section, we aim at measuring the deviation of $d(x, y) + d(y, z)$ from $d(x, z)$, now when the triangle equality is written in the classic form $d(x, y) + d(y, z) = d(x, z)$ by making use of the biresidual implication. Note that, in order to use the biresidual implication, we need to rescale to the unit interval (thus the reason why the pseudometric now needs to be bounded).

For measuring the degree to which $d(x, y) + d(y, z)$ equals $d(x, z)$, we consider the biresidual implication as follows:

$$\begin{aligned} \mathcal{E}_* \left(\frac{d(x, y)}{2N} + \frac{d(y, z)}{2N}, \frac{d(x, z)}{2N} \right) \\ = \min \left(I_* \left(\frac{d(x, y)}{2N} + \frac{d(y, z)}{2N}, \frac{d(x, z)}{2N} \right), I_* \left(\frac{d(x, z)}{2N}, \frac{d(x, y)}{2N} + \frac{d(y, z)}{2N} \right) \right). \end{aligned}$$

Due to property (i) of a residual implication, the above expression can be reduced to the following one:

$$\mathcal{E}_* \left(\frac{d(x, y)}{2N} + \frac{d(y, z)}{2N}, \frac{d(x, z)}{2N} \right) = I_* \left(\frac{d(x, y)}{2N} + \frac{d(y, z)}{2N}, \frac{d(x, z)}{2N} \right).$$

We thus define the ternary fuzzy relation $B_{d,*}^N : X^3 \rightarrow [0, 1]$ on X by

$$B_{d,*}^N(x, y, z) = I_* \left(\frac{d(x, y)}{2N} + \frac{d(y, z)}{2N}, \frac{d(x, z)}{2N} \right).$$

The following proposition presents some properties of $B_{d,*}^N$.

Proposition 8. *Let $*$ be a left-continuous t -norm and d be a bounded pseudo-metric on X with $N > 0$ as upper bound. The ternary fuzzy relation $B_{d,*}^N$ has the following properties:*

- (i) $B_{d,*}^N(x, y, z) = 1$ if and only if $d(x, z) = d(x, y) + d(y, z)$, for any $x, y, z \in X$;
- (ii) If d is a metric, then $B_{d,*}^N(x, y, z) = 0$ implies that $x = z$ and $x \neq y$, for any $x, y, z \in X$;
- (iii) If d is a metric, then $x \neq z$ and $d(x, z) < d(x, y) + d(y, z)$ imply that $0 < B_{d,*}^N(x, y, z) < 1$, for any $x, y, z \in X$.

Proof. (i) Consider any $x, y, z \in X$. By definition $B_{d,*}^N(x, y, z) = 1$ is equivalent to

$$I_* \left(\frac{d(x, y)}{2N} + \frac{d(y, z)}{2N}, \frac{d(x, z)}{2N} \right) = 1,$$

which, by property (i) of a residual implication, occurs if and only if it holds that

$$\frac{d(x, y)}{2N} + \frac{d(y, z)}{2N} \leq \frac{d(x, z)}{2N},$$

or, equivalently,

$$d(x, y) + d(y, z) \leq d(x, z).$$

Due to the triangle inequality, the above is equivalent to

$$d(x, z) = d(x, y) + d(y, z).$$

(ii) Consider $x, y, z \in X$ such that $B_{d,*}^N(x, y, z) = 0$, i.e.,

$$I_* \left(\frac{d(x, y)}{2N} + \frac{d(y, z)}{2N}, \frac{d(x, z)}{2N} \right) = 0.$$

Since I_* is decreasing with respect to the first variable (see property (v) of a residual implication) and $I_*(1, a) = a$ for any $a \in [0, 1]$ (see property (iii) of a residual implication), it follows that $\frac{d(x, z)}{2N} = 0$ and $\frac{d(x, y)}{2N} + \frac{d(y, z)}{2N} \neq 0$. We conclude that, if d is a metric, then it holds that $x = z$ and $x \neq y$.

(iii) It follows from (i) and (ii). \square

The following theorem provides a method to construct a $*$ -betweenness relation from a bounded metric by making use of the residual implication I_* of a 1-Lipschitz continuous t-norm $*$ (i.e., a special residual implication I_*).

Theorem 4. *Let $*$ be a left-continuous t-norm and d be a bounded pseudometric on X with $N > 0$ as upper bound. The following results hold*

- (i) $B_{d,*}^N$ is symmetric and reflexive;
- (ii) $B_{d,*}^N$ is crisp antisymmetric if and only if d is a metric on X ;
- (iii) If I_* is special, then $B_{d,*}^N$ is $*$ -transitive;
- (iv) If I_* is special, then $B_{d,*}^N$ is a $*$ -prebetweenness relation on X ;
- (v) If I_* is special, then $B_{d,*}^N$ is a $*$ -betweenness relation on X if and only if d is a metric on X .

Proof. (i) It follows from the symmetry of a pseudometric that $B_{d,*}^N$ is symmetric and from Proposition 8(i) that $B_{d,*}^N$ is reflexive.

(ii) *Necessity*: Suppose that $B_{d,*}^N$ is crisp antisymmetric. Consider $x, y \in X$ such that $d(x, y) = 0$. It follows from (i) and Proposition 8(i) that

$$B_{d,*}^N(x, x, y) = B_{d,*}^N(x, y, x) = 1.$$

Hence, from the crisp antisymmetry of $B_{d,*}^N$, we conclude that $x = y$. Thus, d is a metric.

Sufficiency: Suppose that d is a metric on X . Consider $x, y, z \in X$ such that

$$B_{d,*}^N(x, y, z) = B_{d,*}^N(x, z, y) = 1.$$

It follows from Proposition 8(i) that $d(x, z) = d(x, y) + d(y, z)$ and $d(x, y) = d(x, z) + d(z, y)$, which together imply that $d(y, z) = 0$ and, thus, $y = z$. We conclude that $B_{d,*}^N$ is crisp antisymmetric.

(iii) Suppose that I_* is special. For any $o, x, y, z \in X$, let $a = \frac{d(o, x)}{2N}$, $b = \frac{d(o, y)}{2N}$, $c = \frac{d(x, z)}{2N}$, $d = \frac{d(x, y)}{2N}$ and $e = \frac{d(y, z)}{2N}$. It holds that $a, b, c, d, e \in [0, 1]$, $a + c, a + d, b + e \in [0, 1]$ and $c - d \leq e$.

It follows from Proposition 7 that

$$\begin{aligned} B_{d,*}^N(o, x, y) &= I_* \left(\frac{d(o, x)}{2N} + \frac{d(x, y)}{2N}, \frac{d(o, y)}{2N} \right) \\ &\leq I_* \left(\frac{d(o, x)}{2N} + \frac{d(x, z)}{2N}, \frac{d(o, y)}{2N} + \frac{d(y, z)}{2N} \right). \end{aligned}$$

Hence, it follows from this inequality and the $*$ -transitivity of I_* (see property (ii) of a residual implication) that

$$\begin{aligned} &B_{d,*}^N(o, x, y) * B_{d,*}^N(o, y, z) \\ &\leq I_* \left(\frac{d(o, x)}{2N} + \frac{d(x, z)}{2N}, \frac{d(o, y)}{2N} + \frac{d(y, z)}{2N} \right) * I_* \left(\frac{d(o, y)}{2N} + \frac{d(y, z)}{2N}, \frac{d(o, z)}{2N} \right) \\ &\leq I_* \left(\frac{d(o, x)}{2N} + \frac{d(x, z)}{2N}, \frac{d(o, z)}{2N} \right) \\ &= B_{d,*}^N(o, x, z). \end{aligned}$$

Therefore, $B_{d,*}^N$ is $*$ -transitive.

(iv) It follows from (i) and (iii).

(v) The left-to-right implication follows from (ii). The right-to-left implication follows from (ii) and (iv). \square

The following example gives the explicit expressions of the fuzzy betweenness relations $B_{d,*}^N$ induced by the three most prominent special residual implications.

Example 10. Let d be a bounded pseudometric on X with $N > 0$ as upper bound.

(1) If $* = *_M$, then $B_{d,*}^N$ can be computed as follows:

$$B_{d,*_M}^N(x, y, z) = \begin{cases} 1, & \text{if } d(x, z) = d(x, y) + d(y, z), \\ \frac{d(x, z)}{2N}, & \text{if } d(x, z) < d(x, y) + d(y, z). \end{cases}$$

(2) If $* = *_P$, then $B_{d,*}^N$ can be computed as follows:

$$B_{d,*_P}^N(x, y, z) = \begin{cases} 1, & \text{if } d(x, z) = d(x, y) + d(y, z), \\ \frac{d(x, z)}{d(x, y) + d(y, z)}, & \text{if } d(x, z) < d(x, y) + d(y, z). \end{cases}$$

(3) If $* = *_L$, then $B_{d,*}^N$ can be computed as follows:

$$B_{d,*_L}^N(x, y, z) = \begin{cases} 1, & \text{if } d(x, z) = d(x, y) + d(y, z), \\ 1 - \frac{d(x, y) + d(y, z) - d(x, z)}{2N}, & \text{if } d(x, z) < d(x, y) + d(y, z). \end{cases}$$

Remark 5. (1) $B_{d,*_M}^N(x, y, z)$ is decreasing with respect to N .

(2) $B_{d,*_P}^N(x, y, z)$ is constant with respect to N .

(3) $B_{d,*_L}^N(x, y, z)$ is increasing with respect to N .

The first of the following two examples shows that I_* does not necessarily need to be special (as stated in Theorem 4(iii)) for $B_{d,*}^N$ to be $*$ -transitive, and, hence, a $*$ -betweenness relation. The second one shows that, if I_* is not required to be special, $B_{d,*}^N$ is no longer assured to be $*$ -transitive.

Example 11. Let $*$ be a left-continuous t-norm and d be a two-valued metric on X (see Example 4). After some computations, $B_{d,*}^N$ can be expressed as

follows:

$$B_{d,*}^N(x, y, z) = \begin{cases} 1, & \text{if } (x = y) \text{ or } (y = z), \\ 0, & \text{if } (x \neq y) \text{ and } (y \neq z) \text{ and } (x = z), \\ \frac{1}{2}, & \text{otherwise.} \end{cases}$$

It is easy to verify that $B_{d,*}^N$ is $*_M$ -transitive and, thus, $*$ -transitive.

Example 12. Let $X = \{o, x, y, z\}$ and $*$ be a left-continuous t-norm with I_* being antispecial. Consider the metric d on X defined as follows:

$$\begin{aligned} d(o, o) &= d(x, x) = d(y, y) = d(z, z) = 0, \\ d(o, x) &= d(x, o) = d(o, y) = d(y, o) = d(x, y) = d(y, x) = d(y, z) = d(z, y) = \frac{1}{5}, \\ d(o, z) &= d(z, o) = d(x, z) = d(z, x) = \frac{2}{5}. \end{aligned}$$

Obviously, $N = \frac{1}{2}$ is an upper bound of d . After some computations, one can verify that $B_{d,*}^N(o, x, y) = I_*\left(\frac{2}{5}, \frac{1}{5}\right)$, $B_{d,*}^N(o, y, z) = I_*\left(\frac{2}{5}, \frac{2}{5}\right) = 1$ and $B_{d,*}^N(o, x, z) = I_*\left(\frac{3}{5}, \frac{2}{5}\right)$. Hence,

$$B_{d,*}^N(o, x, y) * B_{d,*}^N(o, y, z) = I_*\left(\frac{2}{5}, \frac{1}{5}\right) > I_*\left(\frac{2}{5} + \frac{1}{5}, \frac{1}{5} + \frac{1}{5}\right) = B_{d,*}^N(o, x, z).$$

This implies that $B_{d,*}^N$ is not $*$ -transitive.

5 The case of continuous Archimedean and 1-Lipschitz continuous t-norms

As mentioned in Section 2, there exist t-norms that are both continuous Archimedean and 1-Lipschitz continuous (for instance, the product and Łukasiewicz t-norms). In this section, we explore how the construction methods in the previous two sections relate to each other for such t-norms.

We first prove that when we restrict to the Łukasiewicz t-norm, the construction method presented in Section 4 amounts to the construction method presented in Subsection 3.2 for a carefully-chosen additive generator.

Theorem 5. *Let X be a set with cardinality $|X| \geq 3$ and $*$ be a continuous Archimedean t -norm with $t : [0, 1] \rightarrow [0, +\infty]$ being a continuous additive generator. It holds that $B_{d,*}^N = B_d^t$ for any bounded pseudometric d on X with $N > 0$ as upper bound if and only if t is defined by $t(x) = 2N(1 - x)$ (thus, $*$ is $*_L$).*

Proof. We first prove the right-to-left implication. In case $*$ is continuous Archimedean (with continuous additive generator t), the residual implication I_* can be written as follows [6]:

$$\begin{aligned} I_*(a, b) &= \begin{cases} 1, & \text{if } a \leq b, \\ t^{(-1)}(t(b) - t(a)), & \text{otherwise.} \end{cases} \\ &= \begin{cases} 1, & \text{if } a \leq b, \\ t^{-1}(t(b) - t(a)), & \text{otherwise.} \end{cases} \end{aligned}$$

Thus, the mapping $B_{d,*}^N : X^3 \rightarrow [0, 1]$ can be rewritten as

$$B_{d,*}^N(x, y, z) = \begin{cases} 1, & \text{if } d(x, z) = d(x, y) + d(y, z), \\ t^{-1} \left(t \left(\frac{d(x, z)}{2N} \right) - t \left(\frac{d(x, y)}{2N} + \frac{d(y, z)}{2N} \right) \right), & \text{otherwise.} \end{cases}$$

Recall that, for any $k > 0$, the function $t_{L,k} : [0, 1] \rightarrow [0, +\infty]$ defined by $t_{L,k}(x) = \frac{1}{k}(1 - x)$ is a continuous additive generator of the Łukasiewicz t -norm $*_L$. Consider $k = \frac{1}{2N}$. It is thus easy to verify that, for any $x, y, z \in X$,

$$B_{d,*_L}^N(x, y, z) = B_d^{t_{L,\frac{1}{2N}}}(x, y, z),$$

where $B_{d,*_L}^N(x, y, z)$ is defined as in Theorem 4 and $B_d^{t_{L,\frac{1}{2N}}}(x, y, z)$ is defined as in Theorem 1 (see also Example 3).

Now we prove the left-to-right implication. Let $\{a, b, c\} \subseteq X$ with cardinality 3. For any $x \in]0, 1]$, let d_x be the metric on X defined as follows:

$$d_x(u, v) = \begin{cases} 0, & \text{if } u = v, \\ Nx, & \text{if } \{u, v\} = \{a, c\}, \\ N, & \text{otherwise.} \end{cases}$$

It follows from $B_{d_x,*}^N(a, b, c) = B_{d_x}^t(a, b, c)$ that $t(\frac{x}{2}) - t(1) = (2N - Nx) \wedge t(0)$. Note that $t(1) = 0$ and $t(\frac{x}{2}) < t(0)$. We obtain that $t(\frac{x}{2}) = 2N(1 - \frac{x}{2})$.

For any $x \in]0, 1]$, let m_x be the metric on X defined as follows:

$$m_x(u, v) = \begin{cases} 0, & \text{if } u = v, \\ Nx, & \text{otherwise.} \end{cases}$$

It follows from $B_{m_x, *}^N(a, b, c) = B_{m_x}^t(a, b, c)$ that $t(\frac{x}{2}) - t(x) = (Nx) \wedge t(0)$. Note that $t(\frac{x}{2}) < t(0)$. We have $t(\frac{x}{2}) - t(x) = Nx$. Since $t(\frac{x}{2}) = 2N(1 - \frac{x}{2})$, it follows that $t(x) = t(\frac{x}{2}) - Nx = 2N(1 - \frac{x}{2}) - Nx = 2N(1 - x)$. Note that t is continuous. We thus have that $t(x) = 2N(1 - x)$ for any $x \in [0, 1]$. \square

Interestingly, in case we consider the product t-norm and its continuous additive generator $t_P : [0, 1] \rightarrow [0, +\infty]$ defined by $t_P(x) = -\ln x$, we do not obtain $B_d^{t_P}(x, y, z)$ after computing $B_{d, *P}^N(x, y, z)$. Instead, when we restrict to the product t-norm, the construction method presented in Section 4 amounts to the construction method presented in Subsection 3.3 for a carefully-chosen multiplicative generator.

Theorem 6. *Let X be a set with cardinality $|X| \geq 3$ and $*$ be a continuous Archimedean t-norm with $\theta : [0, 1] \rightarrow [0, 1]$ being a continuous multiplicative generator. It holds that $B_{d, *}^N = B_d^\theta$ for any bounded pseudometric d on X with $N > 0$ as upper bound if and only if θ is defined by $\theta(x) = x$ (thus, $*$ = $*_P$).*

Proof. We first prove the right-to-left implication. In case $*$ is continuous Archimedean (with continuous multiplicative generator θ), the residual implication I_* can be written as follows [6]:

$$\begin{aligned} I_*(a, b) &= \begin{cases} 1, & \text{if } a \leq b, \\ \theta^{(-1)}(\frac{\theta(b)}{\theta(a)}), & \text{otherwise.} \end{cases} \\ &= \begin{cases} 1, & \text{if } a \leq b, \\ \theta^{-1}(\frac{\theta(b)}{\theta(a)}), & \text{otherwise.} \end{cases} \end{aligned}$$

Thus, the mapping $B_{d,*}^N : X^3 \rightarrow [0, 1]$ can be rewritten as

$$B_{d,*}^N(x, y, z) = \begin{cases} 1, & \text{if } d(x, z) = d(x, y) + d(y, z), \\ \theta^{-1} \left(\frac{\theta \left(\frac{d(x, z)}{2N} \right)}{\theta \left(\frac{d(x, y)}{2N} + \frac{d(y, z)}{2N} \right)} \right), & \text{otherwise.} \end{cases}$$

Recall that, for any $k > 0$, the function $\theta_{P,k} : [0, 1] \rightarrow [0, 1]$ defined by $\theta_{P,k}(x) = x^{\frac{1}{k}}$ is a continuous multiplicative generator of the product t-norm $*_P$. Consider $k = 1$. It is thus easy to verify that, for any $x, y, z \in X$,

$$B_{d,*_P}^N(x, y, z) = B_d^{\theta_{P,1}}(x, y, z),$$

where $B_{d,*_P}^N(x, y, z)$ is defined as in Theorem 4 and $B_d^{\theta_{P,1}}(x, y, z)$ is defined as in Theorem 3 (see also Example 6).

Now we prove the left-to-right implication. Let $\{a, b, c\} \subseteq X$ with cardinality 3. For any $x \in]0, 1]$, let d_x be the metric on X defined as follows:

$$d_x(u, v) = \begin{cases} 0, & \text{if } u = v, \\ Nx, & \text{if } \{u, v\} = \{a, c\}, \\ N, & \text{otherwise.} \end{cases}$$

It follows from $B_{d_x,*}^N(a, b, c) = B_{d_x}^\theta(a, b, c)$ that $\frac{\theta(\frac{x}{2})}{\theta(1)} = \frac{xN}{2N} \vee \theta(0)$. Note that $\theta(1) = 1$ and $\theta(\frac{x}{2}) > \theta(0)$. We obtain that $\theta(\frac{x}{2}) = \frac{x}{2}$.

For any $x \in]0, 1]$, let m_x be the metric on X defined as follows:

$$m_x(u, v) = \begin{cases} 0, & \text{if } u = v, \\ Nx, & \text{otherwise.} \end{cases}$$

It follows from $B_{m_x,*}^N(a, b, c) = B_{m_x}^\theta(a, b, c)$ that $\frac{\theta(\frac{x}{2})}{\theta(x)} = \frac{xN}{2xN} \vee \theta(0)$. Note that $\theta(\frac{x}{2}) > \theta(0)$. We have $\frac{\theta(\frac{x}{2})}{\theta(x)} = \frac{1}{2}$. Since $\theta(\frac{x}{2}) = \frac{x}{2}$, it follows that $\theta(x) = x$. Note that θ is continuous. We thus have that $\theta(x) = x$ for any $x \in [0, 1]$. \square

The results above are oddly satisfying. In particular, we see that the construction method presented in Section 4 is not substitutable with any of the two methods presented in Section 3, however, some similarities between the

former and the latter construction methods do exist. For instance, when we restrict to the Łukasiewicz t-norm, the construction method presented in Section 4 amounts to the construction method presented in Subsection 3.2 for a carefully-chosen additive generator and differs from the construction method presented in Subsection 3.3 for any multiplicative generator. Analogously, when we restrict to the product t-norm, the construction method presented in Section 4 amounts to the construction method presented in Subsection 3.3 for a carefully-chosen multiplicative generator and differs from the construction method presented in Subsection 3.2 for any additive generator.

6 Conclusions and future work

In this paper, we have presented different methods for constructing a fuzzy betweenness relation from a metric given a continuous Archimedean t-norm or a 1-Lipschitz continuous t-norm. More precisely, given a continuous Archimedean t-norm, we have developed two construction methods by making use of either the pseudo-inverse of a continuous additive generator of the continuous Archimedean t-norm (see Theorems 1 and 2) or the pseudo-inverse of a continuous multiplicative generator of the continuous Archimedean t-norm (see Theorem 3). Interestingly, these two construction methods yield different fuzzy betweenness relations (see Remark 4). In case the metric is bounded and the given t-norm is 1-Lipschitz continuous, we have developed a third construction method by making use of the residual implication of the 1-Lipschitz continuous t-norm (see Theorem 4).

It remains as future work how to relax the conditions of these theorems. More specifically, conditions (1) the generator of the t-norm is continuous and (2) the residual implication of the t-norm is special, are shown to be not necessary for assuring the construction of a fuzzy betweenness relation from a metric. Furthermore, a construction method for a fuzzy betweenness relation from a metric is still to be developed for t-norms that are neither continuous Archimedean, nor 1-Lipschitz continuous. A further study subject concerns how

to construct fuzzy betweenness relations from fuzzy metrics.

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