# Reticulation of a quantale, pure elements and new transfer properties

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#### Abstract

We know from a previous paper that the reticulation of a coherent quantale A is a bounded distributive lattice L(A) whose prime spectrum is homeomorphic to m- prime spectrum of A. In this paper we shall prove several results on the pure elements of the quantale A by means of the reticulation L(A). We shall investigate how the properties of  $\sigma$ - ideals of L(A) can be transferred to pure elements of A. Then the pure elements of A are used to obtain new properties and characterization theorems for some important classes of quantales: normal quantales, mp- quantales, PF- quantales, purified quantales and PP- quantales.

**Keywords**: coherent quantale, reticulation, pure and w -pure elements, normal quantales, mp - quantales, PF - quantales, PP - quantales .

## 1 Introduction

The quantales [42],[21] and the frames [30] are structures that generalise the lattices of ideals, filters and congruences for various classes of algebras. Several algebraic and topological properties of rings, distributive lattices, l-groups and l-rings, MV-algebras,BL-algebras, residuated lattices, etc. can be extended to quantales and frames. To work in this abstract setting is not only a unification of some existing particular results, but also an efficient way to prove new properties for many types of algebras.

The pure elements in a quantale were introduced in [38]. They constitute an abstractization of pure ideals of rings [11] (named virginal ideals in [12] and neat ideals in [30]),  $\sigma$  - ideals of bounded distributive lattices [17],[18],[26], pure ideals in MV-algebras [14], pure filters in BL-algebras [32] and residuated lattices [36], etc. Similar notions of pure elements can be found in other kind of multiplicative lattices: multiplicative ideal structures [27] and 2-side carriers [44].

On the other hand, the reticulation of a coherent quantale A is a bounded distributive lattice L(A) whose m - prime spectrum Spec(A) is homeomorphic to the prime spectrum  $Spec_{Id}(L(A))$  of L(A) (see [23]). Then L(A) unifies various notions of reticulations defined for commutative rings [43],[30], l-rings [30], MV-algebras [9], BL -algebras [32], residuated lattices [36], etc.

The reticulation is a functorial construction: from a category of algebras to the category of bounded distributive lattices. The reticulation functor allows us a transfer of properties from bounded distributive lattices to algebras and vice-versa.

This paper studies the pure elements of a coherent quantale A by means of the reticulation L(A). We investigate how some results on the  $\sigma$ - ideals of the lattice L(A) can be transferred to the pure elements of A. We use the properties of pure elements in order to obtain new characterization theorems for some important classes of quantales: normal quantales, mp- quantales, PF- quantales, purified quantales and PP- quantales.

Now we shall describe the content of the paper. In Section 2 we recall from [42],[21] some basic notions and results in quantale theory: m - prime elements, radical elements, m - prime and maximal spectra with spectral and flat topologies, regular and max - regular elements. Section 3 contains some fundamental facts on the reticulation L(A) of a coherent quantale A: axiomatic definition, arithmetic, algebraic and topological constructions, the isomorphism between the Boolean centers B(A) and B(L(A)) of A and L(A), the preservation theorems for annihilators, etc.

Section 4 concerns the pure elements in a coherent quantale A. We know from [27] that the set Vir(A) of pure elements of A is a spatial frame. In [27], a lot of properties of Vir(A) were established in the more large framework of multiplicative ideals structures (= mi - structures). We continue the line of [27] and obtain new results on the pure elements. We define the weakly pure elements (= w - pure elements), a notion that enlarges the class of pure elements. Mainly we study the relationship between the operators  $Vir(\cdot)$ ,  $Ker(\cdot)$  and  $O(\cdot)$  (introduced in [27]) and the pure and w - pure elements of A. The main results of the section establish how the reticulation commutes with  $Vir(\cdot)$ ,  $Ker(\cdot)$  and  $O(\cdot)$ . In this way one obtains the relationship between the pure and w - pure elements of A and the  $\sigma$  - ideals of the lattice L(A).

In Section 5 we continue the study of pure elements. Firstly, we prove that there exists a surjective continuos function from the prime spectrum of the frame Vir(A) to the Pierce spectrum Sp(A) of A. Secondly, in the continuation of [27] we investigate the pure elements in a normal quantale. Various properties that characterize the normal coherent quantales point out the role of pure elements and operators  $Vir(\cdot)$ ,  $Ker(\cdot)$  and  $O(\cdot)$  in studying this class of quantales (see Proposition 5.6).

Section 6 contains a lot of results on pure elements in PF - quantales, a quantale abstractization of PF - rings. If A is PF - quantale, then we characterize its pure elements as intersections  $\bigwedge(Min(A)\bigcap E)$ , where E is a closed subset of the minimal m - prime spectrum Min(A) of A (endowed with the restriction of spectral topology on Spec(A))). We prove the equality of Min(A)

with the maximal spectrum Max(Vir(A)) of the frame Vir(A). Another theorem of the section shows that for any PF - quantale A, the frame Vir(A) is hyperarchimedean.

In Section 7 we define the purified quantales as a generalization of purified rings, introduced in [1]. The main result of the section is a characterization theorem of the purified quantales.

The PP - quantales, introduced in Section 8, generalize the PP - rings ( = Baer rings). We prove that a semiprime quantale A is a PP - quantale if and only if the reticulation L(A) is a Stone lattice, extending a theorem of Simmons from [43]. By using this result one obtains some characterization theorems of PP - quantales.

We mention that most of the results of this paper extend some theorems that appear in the case of commutative rings ([1],[3],[5],[6],[11],[12],[30],[35],[43],[48],[49]), bounded distributive lattices ([4],[17],[18],[26],[30],[40],[43],[45]), MV-algebras ([9],[14]), BL-algebras [32], residuated lattices ([36],[41]),etc.

## 2 Preliminaries on quantales

This section contains some basic notions and results in quantale theory [42], [21]. Recall from [42], [21] that a quantale is an algebraic structure  $(A, \vee, \wedge, \cdot, 0, 1)$  such that  $(A, \vee, \wedge, 0, 1)$  is a complete lattice and  $(A, \cdot)$  is a semi-group with the property that the multiplication  $\cdot$  satisfies the infinite distributive laws: for all  $a \in A$  and  $X \subseteq A$ , we have  $a \cdot \bigvee X = \bigvee \{a \cdot x | x \in X\}$  and  $(\bigvee X) \cdot a = \bigvee \{x \cdot a | x \in X\}$ .

Let  $(A, \vee, \wedge, \cdot, 0, 1)$  be a quantale and K(A) the set of its compact elements. A is said to be *integral* if  $(A, \cdot, 1)$  is a monoid and *commutative*, if the multiplication  $\cdot$  is commutative. A *frame* is a quantale in which the multiplication coincides with the meet [30]. The quantale A is algebraic if any  $a \in A$  has the form  $a = \bigvee X$  for some subset X of K(A). An algebraic quantale A is coherent if  $1 \in K(A)$  and K(A) is closed under the multiplication. Throughout this paper, the quantales are assumed to be integral and commutative. Often we shall write ab instead of  $a \cdot b$ . We fix a quantale A.

**Lemma 2.1** [10] For all elements a, b, c of the quantale A the following hold:

- (1) If  $a \lor b = 1$  then  $a \cdot b = a \land b$ ;
- (2) If  $a \lor b = 1$  then  $a^n \lor b^n = 1$  for all integer number  $n \ge 1$ ;
- (3) If  $a \lor b = a \lor c = 1$  then  $a \lor (b \cdot c) = a \lor (b \land c) = 1$ ;

On each quantale A one can define a residuation operation  $a \to b = \bigvee \{x | ax \le b\}$  and a negation operation  $a^{\perp} = a \to 0 = \bigvee \{x | ax = 0\}$ . Thus for all  $a, b, c \in A$  the following equivalence holds:  $a \le b \to c$  if and only if  $ab \le c$ , so  $(A, \vee, \wedge, \cdot, \to, 0, 1)$  becomes a (commutative) residuated lattice [22].

In this paper we shall use without mention the basic arithmetical properties of a residuated lattice [22].

An element p < 1 of A is m-prime if for all  $a, b \in A$ ,  $ab \leq p$  implies  $a \leq b$  or  $b \leq p$ . If A is an algebraic quantale, then p < 1 is m-prime if and only if for all  $c, d \in K(A)$ ,  $cd \leq p$  implies  $c \leq p$  or  $d \leq p$ . Let us introduce the following notations: Spec(A) is the set of m-prime elements and Max(A) is the set of maximal elements of A. If  $1 \in K(A)$  then for any a < 1 there exists  $m \in Mar(A)$  such that  $a \leq m$ . The same hypothesis  $1 \in K(A)$  implies that  $Max(A) \subseteq Spec(A)$ .

The main example of quantale is the set Id(R) of ideals of a (unital) commutative ring R and the main example of frame is the set Id(L) of ideals of a bounded distributive lattice L. Thus the set Spec(R) of prime ideals in R is the prime spectrum of the quantale Id(R) and the set of prime ideals in L is the prime spectrum of the frame Id(L).

Following [42], the radical  $\rho(a) = \rho_A(a)$  of an element  $a \in A$  is defined by  $\rho_A(a) = \bigwedge \{ p \in Spec(A) | a \leq p \}$ ; if  $a = \rho(a)$  then a is a radical element. We shall denote by R(A) the set of radical elements of A. The quantale A is said to be semiprime if  $\rho(0) = 0$ .

**Lemma 2.2** [42] For all elements  $a, b \in A$  the following hold:

- (1)  $a \leq \rho(a)$ ;
- (2)  $\rho(a \wedge b) = \rho(ab) = \rho(a) \wedge \rho(b);$
- (3)  $\rho(a) = 1$  iff a = 1;
- (4)  $\rho(a \vee b) = \rho(\rho(a) \vee \rho(b));$
- (5)  $\rho(\rho(a)) = \rho(a)$ ;
- (6)  $\rho(a) \vee \rho(b) = 1 \text{ iff } a \vee b = 1$ :
- (7)  $\rho(a^n) = \rho(a)$ , for all integer  $n \ge 1$ .

For an arbitrary family  $(a_i)_{i\in I}\subseteq A$ , the following equality holds:  $\rho(\bigvee_{i\in I}a_i)=$ 

$$\rho(\bigvee_{i\in I}\rho(a_i))$$
. If  $(a_i)_{i\in I}\subseteq R(A)$  then we denote  $\bigvee_{i\in I}^{\cdot}a_i=\rho(\bigvee_{i\in I}a_i)$ . Then it is easy

to prove that  $(R(A), \bigvee, \wedge, \rho(a), 1)$  is a frame [42].

**Lemma 2.3** [15] If  $1 \in K(A)$  then Spec(A) = Spec(R(A)) and Max(A) = Max(R(A)).

**Lemma 2.4** [33] Let A be a coherent quantale and  $a \in A$ . Then

- (1)  $\rho(a) = \bigvee \{c \in K(A) | c^k \le a \text{ for some integer } k \ge 1\};$
- (2) For any  $c \in K(A)$ ,  $c \le \rho(a)$  iff  $c^k \le a$  for some  $k \ge 1$ .

**Lemma 2.5** [15] If A is a coherent quantale then  $K(R(A)) = \rho(K(A))$  and R(A) is a coherent frame.

For any element a of a coherent quantale A let us consider the interval  $[a)_A = \{x \in A | a \leq x\}$  and for all  $x, y \in [a)_A$  denote  $x \cdot_a y = xy \vee a$ . Thus  $[a)_A$  is closed under the multiplication  $\cdot_a$  and  $([a)_A, \vee, \wedge, \cdot_a, 0, 1)$  is a coherent quantale.

**Lemma 2.6** [15] The quantale  $([\rho(a))_A, \vee, \wedge, \cdot_a, 0, 1)$  is semiprime and  $Spec(A) = Spec([\rho(a))_A), Max(A) = Max([\rho(a))_A).$ 

Let A, B be two quantales. A function  $f: A \to B$  is a morphism of quantales if it preserves the arbitrary joins and the multiplication; f is an integral morphism if f(1) = 1.

**Lemma 2.7** [15] Let A be a coherent quantale and  $a \in A$ .

- (1) The function  $u_a^A: A \to [a)_A$ , defined by  $u_a^A(x) = x \vee a$ , for all  $x \in A$ , is an integral quantale morphism;
- (2) If  $c \in K(A)$  then  $u_a^A(c) \in K([a))$ .

Let A be a quantale such that  $1 \in K(A)$ . For any  $a \in A$ , denote  $D(a) = \{p \in Spec(A) | a \not\leq p\}$  and  $V(a) = \{p \in Spec(A) | a \leq p\}$ . Then Spec(A) is endowed with a topology whose closed sets are  $(V(a))_{a \in A}$ . If the quantale A is algebraic then the family  $(D(c))_{c \in K(A)}$  is a basis of open sets for this topology. The topology introduced here generalizes the Zariski topology (defined on the prim spectrum Spec(R) of a commutation ring R [2]) and the Stone topology (defined on the prime spectrum  $Spec_{Id}(L)$  of a bounded distributive lattice L [7]).

Thus we denote by  $Spec_Z(A)$  the prime spectrum Spec(A) endowed with the topology above defined;  $Max_Z(A)$  will denote the maximal spectrum Max(A) considered as a subspace of  $Spec_Z(A)$ . According to [24],  $Spec_Z(A)$  is a  $spectral\ space$  in the sense of [28]. The  $flat\ topology$  associated to this spectral space has as basis the family of the completents of compact open subsets of  $Spec_Z(A)$  (cf.[19], [30]). Recall from [24] that the family  $\{V(c)|c\in K(A)\}$  is a basis of open sets for the flat topology on Spec(A). We shall denote by  $Spec_F(A)$  this topological space.

For any  $p \in Spec(A)$ , let us denote  $\Lambda(p) = \{q \in Spec(A) | q \leq p\}$ . According to Propositions 5.6 and 5.7 of [24], if  $p \in Spec(A)$  then the flat closure of the set  $\{p\}$  is  $cl_F\{p\} = \Lambda(p)$  and, if  $S \subseteq Spec(A)$  is compact in  $Spec_Z(A)$  then its flat closure is  $cl_F(S) = \bigcup \Lambda(p)$ .

An element  $a \in A$  is regular if it is a join of complemented elements. A maximal element in the set of proper regular elements is called max-regular. The set Sp(A) of max-regular elements of A is called the Pierce spectrum of the quantale A. For any proper regular element a there exists  $p \in Sp(A)$  such that  $a \leq p$ . If  $e \in B(A)$  then we denote  $U(e) = \{p \in Sp(A) | e \nleq a\}$ . Thus it

easy to prove that the family  $(U(e))_{e \in B(A)}$  is a basis of open sets for a topology on Sp(A).

For any  $p \in Spec(A)$  we define  $s_A(p) = \bigvee \{e \in B(A) | e \leq p\}$ ;  $s_A(p)$  is regular and  $s_A(p) \leq p < 1$ . According to Lemma 5.8 of [24], for each  $p \in Spec(A)$ ,  $s_A(p)$  is a max - regular element of A, so one obtains a function  $s_A : Spec(A) \rightarrow Sp(A)$ . We know from Proposition 5.9 of [24] that Sp(A) is a Boolean space and  $s_A : Spec(A) \rightarrow Sp(A)$  is surjective and continuous w.r.t. both flat and spectral topologies on Spec(A). If R is a commutative ring then Sp(Id(R)) is exactly the Pierce spectrum of R (see [30], p.181).

Let L be a bounded distributive lattice. For any  $x \in L$ , denote  $D_{Id}(x) = \{P \in Spec_{Id}(L) | x \notin P\}$  and  $V_{Id}(x) = \{P \in Spec_{Id,Z}(L) | x \in P\}$ . The family  $(D_{Id}(x))_{x \in L}$  is a basis of open sets for the Stone topology on  $Spec_{Id}(L)$ ; this topological space will be denoted by  $Spec_{Id,Z}(L)$ . Let  $Max_{Id}(L)$  be the set of maximal ideals of L. Thus  $Max_{Id}(L) \subseteq Spec_{Id}(L)$  and  $Max_{Id}(L)$  becomes a subspace of  $Spec_{Id}(L)$ , denoted  $Max_{Id,Z}(L)$ .

## 3 Reticulation of a coherent quantale

The reticulation L(A) of a quantale A was introduced in [23] as a generalization of the reticulation of a commutative ring, given in [43]. In [23], the reticulation L(A) was characterized as a bounded distributive lattice whose prime spectrum  $Spec_{Id}(L(A))$  is homeomorphic to the prime spectrum Spec(A) of the quantale A. In this section we shall recall from [15],[23] the axiomatic definition of the reticulation of the coherent quantale and some of its basic properties. Let A be a coherent quantale and K(A) the set of its compact elements.

**Definition 3.1** [15] A reticulation of the quantale A is a bounded distributive lattice L together a surjective function  $\lambda: K(A) \to L$  such that for all  $a, b \in K(A)$  the following properties hold

- (1)  $\lambda(a \vee b) \leq \lambda(a) \vee \lambda(b)$ ;
- (2)  $\lambda(ab) = \lambda(a) \wedge \lambda(b)$ ;
- (3)  $\lambda(a) \leq \lambda(b)$  iff  $a^n \leq b$ , for some integer  $n \geq 1$ .

In [15],[23] there were proven the existence and the unicity of the reticulation for each coherent quantale A; this unique reticulation will be denoted by  $(L(A), \lambda_A : K(A) \to L(A))$  or shortly L(A). The reticulation L(R) of a commutative ring R was introduced by many authors, but the main references on this topic remain [43], [30]. We remark that L(R) is isomorphic to the reticulation L(Id(R)) of the quantale Id(R).

**Lemma 3.2** [15] For all elements  $a, b \in K(A)$  the following properties hold:

- (1)  $a < b \text{ implies } \lambda_A(a) < \lambda_A(b)$ ;
- (2)  $\lambda_A(a \vee b) = \lambda_A(a) \vee \lambda_A(b)$ ;

- (3)  $\lambda_A(a) = 1$  iff a = 1;
- (4)  $\lambda_A(0) = 0$ ;
- (5)  $\lambda_A(a) = 0$  iff  $a^n = 0$ , for some integer  $n \ge 1$ ;
- (6)  $\lambda_A(a^n) = \lambda_A(a)$ , for all integer  $n \ge 1$ ;
- (7)  $\rho(a) = \rho(b)$  iff  $\lambda_A(a) = \lambda_A(b)$ ;
- (8)  $\lambda_A(a) = 0 \text{ iff } a \le \rho(0);$
- (9) If A is semiprime then  $\lambda_A(a) = 0$  implies a = 0.

Often the previous nine properties shall be used in the proofs without mention.

For any  $a \in A$  and  $I \in Id(L(A))$  let us denote  $a^* = \{\lambda_A(c) | c \in K(A), c \leq a\}$  and  $I_* = \bigvee \{c \in K(A) | \lambda_A(c) \in I\}$ .

Lemma 3.3 [15] The following assertions hold

- (1) If  $a \in A$  then  $a^*$  is an ideal of L(A) and  $a \leq (a^*)_*$ ;
- (2) If  $I \in Id(L(A))$  then  $(I_*)^* = I$ ;
- (3) If  $p \in Spec(A)$  then  $(p^*)_* = p$  and  $p^* \in Spec_{Id}(L(A))$ ;
- (4) If  $P \in Spec_{Id}((L(A)) \ then \ P_* \in Spec(A);$
- (5) If  $p \in K(A)$  then  $c^* = (\lambda_A(c)]$ ;
- (6) If  $c \in K(A)$  and  $I \in Id(L(A))$  then  $c \leq I_*$  iff  $\lambda_A(c) \in I$ ;
- (7) If  $a \in A$  and  $I \in Id(L(A))$  then  $\rho(a) = (a^*)_*, a^* = (\rho(a))^*$  and  $\rho(I_*) = I_*;$
- (8) If  $c \in K(A)$  and  $p \in Spec(A)$  then  $c \leq p$  iff  $\lambda_A(c) \in p^*$ .

**Lemma 3.4** Let A be a coherent quantale. The following assertions hold

- (1) If  $a, b \in A$  then  $(ab)^* = (a \wedge b)^* = a^* \cap b^*$ ;
- (2) If  $(a_i)_{i\in I}$  is a family of elements of A then  $(\bigvee_{i\in I} a_i)^* = \bigvee_{i\in I} a_i^*$ .

**Proof.** First we remark that  $(ab)^* \subseteq (a \wedge b)^* \subseteq a^* \cap b^*$ . In order to prove that  $a^* \cap b^* \subseteq (ab)^*$ , let us assume that  $x \in a^* \cap b^*$ . Then  $x = \lambda_A(c) = \lambda_A(d)$ , for some compact elements c,d that verify the properties  $c \leq a$  and  $d \leq b$ . Thus one gets  $\lambda_A(cd) \leq \lambda_A(ab)$ , so there exists a positive integer n such that  $c^n d^n \leq ab$ . Therefore  $x = \lambda_A(c^n d^n) \leq \lambda_A(ab)$ , hence it follows that  $x \in (ab)^*$ . The property (2) follows similarly.

According to Lemma 3.3, one can consider the following order- preserving functions:  $u: Spec(A) \to Spec_{Id}(L(A))$  and  $v: Spec_{Id}(L(A)) \to Spec(A)$ , defined by  $u(p) = p^*$  and  $v(P) = P_*$ , for all  $p \in Spec(A)$  and  $P \in Spec_{Id}(L(A))$ .

**Lemma 3.5** [15] The functions u and v are homeomorphisms, inverse to one another.

Corollary 3.6  $Max_Z(A)$  and  $Max_{Id,Z}(L(A))$  are homeomorphic.

**Proposition 3.7** [15] The functions  $\Phi: R(A) \to Id(L(A))$  and  $\Psi: Id(L(A)) \to R(A)$  defined by  $\Phi(a) = a^*$  and  $\Psi(I) = I_*$ , for all  $a \in R(A)$  and  $I \in Id(L(A))$ , are frame isomorphisms, inverse to one another.

The Boolean center of an arbitrary quantale A is the Boolean algebra B(A) of complemented elements of A (cf. [10],[29]). The following lemma collects some elementary properties of the elements of B(A).

**Lemma 3.8** [10],[29] Let A be a quantale and  $a, b \in A$ ,  $e \in B(A)$ . Then the following properties hold:

- (1)  $a \in B(A)$  iff  $a \vee a^{\perp} = 1$ ;
- (2)  $a \wedge b = ae$ ;
- (3)  $e \rightarrow a = e^{\perp} \vee a$ ;
- (4) If  $a \lor b = 1$  and ab = 0, then  $a, b \in B(A)$ ;
- (5)  $(a \wedge b) \vee e = (a \vee e) \wedge (b \wedge e)$ ;
- (6) For any integer  $n \ge 1$ ,  $a \lor b = 1$  and  $a^n b^n = 0$  implies  $a^n, b^n \in B(A)$ .

**Lemma 3.9** [15] If  $1 \in K(A)$  then  $B(A) \subseteq K(A)$ .

For a bounded distributive lattice L we shall denote by B(L) the Boolean algebra of the complemented elements of L. It is well-known that B(L) is isomorphic to the Boolean center B(Id(L)) of the frame Id(L) (see [10], [30], [25]).

Let us fix a coherent quantale A.

**Lemma 3.10** [24] Assume  $c \in K(A)$ . Then  $\lambda_A(c) \in B(L(A))$  if and only if  $c^n \in B(A)$ , for some integer  $n \ge 1$ .

**Corollary 3.11** [15] The function  $\lambda_A|_{B(A)}: B(A) \to B(L(A))$  is a Boolean isomorphism.

If L is bounded distributive lattice and  $I \in Id(L)$  then the annihilator of I is the ideal  $Ann(I) = \{x \in L | x \land y = 0, \text{ for all } y \in L\}.$ 

**Lemma 3.12** If  $c \in K(A)$  and  $p \in Spec(A)$  then  $Ann(\lambda_A(c)) \subseteq p^*$  if and only if  $c \to \rho(0) \le p$ .

The next two propositions concern the behaviour of reticulation w.r.t. the annihilators.

**Proposition 3.13** If a is an element of a coherent quantale then  $Ann(a^*) = (a \to \rho(0))^*$ ; if A is semiprime then  $Ann(a^*) = (a^{\perp})^*$ .

**Proposition 3.14** Assume that A is a coherent quantale. If I is an ideal of L(A) then  $(Ann(I))_* = I_* \to \rho(0)$ ; if A is semiprime then  $(Ann(I))_* = (I_*)^{\perp}$ .

If A is a quantale then we denote by Min(A) the set of minimal m - prime elements of A; Min(A) is called the minimal prime spectrum of A. If  $1 \in K(A)$  then for any  $p \in Spec(A)$  there exists  $q \in Min(A)$  such that  $q \leq p$ .

**Proposition 3.15** Let A be a coherent quantale. If  $p \in Spec(A)$  then the following are equivalent:

- (1)  $p \in Min(A)$ ;
- (2) For all  $c \in K(A)$ ,  $c \le p$  if and only if  $c \to \rho(0) \le p$ .

**Corollary 3.16** If A is semiprime coherent quantale and  $p \in Spec(A)$  then  $p \in Min(A)$  if and only if for all  $c \in K(A)$ ,  $c \leq p$  implies  $c^{\perp} \not\leq p$ .

# 4 Pure and w - pure elements in a quantale

The pure elements in a quantale extend the pure ideals of a ring [31],[44] and the  $\sigma$ -ideals of a bounded distributive lattice [18], [26]. More precisely, an ideal I of bounded distributive lattice L is a  $\sigma$ -ideal if for all  $x \in I$ , we have  $I \vee Ann(x) = L$ . An ideal I of a commutative ring R is pure (or virginal, in terminology of [12]) if for all  $x \in I$ , we have  $I \vee Ann(x) = L$ . The notions of pure ideals and  $\sigma$ -ideals have been generalized to various abstract structures: frames [30], quantales [38], multiplicative - ideal structures [26], two - side carriers [43], etc. In this way appeared the abstract notion of pure element.

An element a of an arbitrary algebraic quantale A is said to be *pure* (or *virginal*, in the terminology of [27]) if for all  $c \in K(A)$ ,  $c \le a$  implies  $a \lor c^{\perp} = 1$ . Then an ideal I of a commutative ring R is pure if and only if I is a pure element of the quantale Id(R). An ideal I of a bounded distributive lattice L is a  $\sigma$ -ideal if and only if I is a pure element of the frame Id(L). The set of pure elements of the quantale A will be denoted by Vir(A).

The quantales are *multiplicative - ideals structures* in sense of [27], so all the results of this paper remain true for the pure elements in a quantale.

**Lemma 4.1** [27] If A is an algebraic quantale then the following hold:

(1) If 
$$a \in A$$
 is pure then  $a = \bigvee \{c \in K(A) | a \lor c^{\perp} = 1\};$ 

- (2) If  $a, b \in A$  are pure the  $ab = a \wedge b$ ;
- (3) If  $(a_i)_{i\in I}$  is a family of pure elements then  $\bigvee_{i\in I} a_i$  is pure;
- (4) The structure  $(Vir(A), \bigvee, \wedge, 0, 1)$  is a frame.

Keeping the notations from [12], [27], for any  $a \in A$  we define the following elements of A:

- $O(a) = \bigvee \{u \in A | uv = 0, for some v \in A, v \nleq a\},\$
- $Ker(a) = \bigvee \{c \in K(A) | c \le a, a \lor c^{\perp} = 1\};$
- $Vir(a) = \bigvee \{b \in Vir(A) | b \le a\}.$

It is easy to show that  $Ker(a) = \bigvee \{c \in K(A) | a \vee c^{\perp} = 1\}$ , because  $a \vee c^{\perp} = 1$  implies that  $c \leq a$ . In general we have  $Vir(a) \leq Ker(a) \leq a$  and a is pure if and only if a = Vir(a). If A is an algebraic quantale then

- $O(a) = \bigvee \{c \in K(A) | cd = 0, forsomed \in K(A), d \not\leq a\}.$
- For any  $p \in Spec(A)$  we have  $O(p) = \bigvee \{c \in K(A) | c^{\perp} \not\leq p\}$ , hence  $O(p) \leq p$ .

**Lemma 4.2** [27] Let A be is algebraic quantale such that  $1 \in K(A)$ . Then the following hold:

- (1) If  $m \in Max(A)$  then O(m) = Ker(m);
- (2) If  $a \in Vir(A)$  then

$$a = Vir(\bigwedge\{m \in Max(A)|a \le m\}) = \bigwedge\{Vir(m)|m \in Max(A), a \le m\};$$

- (3) The map  $\rho: Vir(A) \to R(A)$  is an injective frame morphism, left adjoint to  $Vir: R(A) \to Vir(A)$ ;
- (4) Vir(A) is a spatial frame and  $Vir: Spec(A) \rightarrow Spec(Vir(A))$  is a continuous map.

The following lemma improves the property (2) from Lemma 4.1.

**Lemma 4.3** Let A be is an algebraic quantale such that  $1 \in K(A)$ . If  $a \in A$  and  $b \in VA$  then  $a \wedge b = ab$ .

**Proof.** Assume that c is a compact element of A such that  $c \le a \land b$ . From  $c \le b$  one gets  $b \lor c^{\perp} = 1$ , hence  $c = c(b \lor c^{\perp}) = cb$ . Thus  $c \le ab$ , hence  $a \land b \le ab$ . The converse inequality is obvious.

**Lemma 4.4** Let A be is an algebraic quantale such that  $1 \in K(A)$ . If p is an m - of prime element A then Vir(p) = Vir(O(p)).

**Proof.** Let p be an element of Spec(A). Since  $O(p) \leq p$  then the inequality  $Vir(O(p)) \leq Vir(p)$  holds. In order to obtain the inverse inequality  $Vir(p) \leq Vir(O(p))$ , we must prove that for any compact element c of A,  $c \leq Vir(p)$  implies  $c \leq Vir(O(p))$ . If  $c \leq Vir(p)$  then  $Vir(p) \vee c^{\perp} = 1$ , so  $p \vee c^{\perp} = 1$ . Thus  $p \vee d = 1$  for some  $d \in K(A)$  such that  $d \leq c^{\perp}$ , hence  $d \not\leq p$  and cd = 0. It follows that  $c \leq O(p)$ , therefore  $Vir(p) \leq O(p)$ . The last inequality implies  $Vir(p) \leq Vir(O(p))$ .

**Proposition 4.5** Let A a semiprime algebraic quantale such that  $1 \in K(A)$ . If  $a \in Vir(A)$  then  $\rho(a) = a$ .

**Proof.** Assume that  $c \in K(A)$  and n is a positive integer such that  $c^n \leq a$ . Then  $a \vee (c^n)^{\perp} = 1$ , so there exists  $d \in K(A)$  such that  $d \leq (c^n)^{\perp}$  and  $a \vee d = 1$ . It follows that  $c^n d = 0$ , hence  $\lambda_A(cd) = \lambda_A(c) \wedge \lambda_A(d) = \lambda_A(c^n d) = 0$ . Since A is semiprime one gets cd = 0, therefore  $c = c(a \vee d) = ca$ . By using Lemma 4.3 one obtains  $c = c \wedge a$ , hence  $c \leq a$ . According to Lemma 2.4 it results that  $\rho(a) = a$ .

An element a of a quantale A is said to be weakly - pure (= w - pure) if for all  $c \in K(A)$ ,  $c \le a$  implies  $a \lor (c \to \rho(0)) = 1$ .

**Lemma 4.6** Any pure element a of A is w - pure.

**Proof.** For any compact element  $c \le a$  we have  $c^{\perp} = c \to 0 \le c \to \rho(0)$ , so  $1 = a \lor c^{\perp} \le a \lor (c \to \rho(0))$ , hence a is w - pure.

If A is semiprime then  $a \in A$  is pure if and only if it is w - pure. We denote by  $Vir_w(A)$  the set of w - pure elements of A. By the previous lemma we have  $Vir(A) \subseteq Vir_w(A)$ .

**Lemma 4.7** Let A be an algebraic quantale such that  $1 \in K(A)$ .

- (1)  $Vir_w(A)$  is closed under  $\cdot$  and  $\wedge$ ;
- (2) For any family  $(a_i)_{i \in I} \subseteq Vir_w(A)$  we have  $\bigvee_{i \in I} a_i \in Vir_w(A)$ .

**Proof.** (1) Assume that a,b are two w - pure elements of A. If c is a compact element of A such that  $c \le ab$  then  $c \le a$  and  $c \le b$ , so we have  $a \lor (c \to \rho(0) = b \lor (c \to \rho(0) = 1)$ . By Lemma 2.1,(3) we get  $(ab) \lor (c \to \rho(0)) = 1$ , so ab is w - pure. Similarly,  $c,d \in K(A)$  and  $c \le a \land b$  implies  $(a \land b) \lor (c \to \rho(0)) = 1$ , hence  $a \land b$  is w - pure.

(2) Let us denote  $a = \bigvee_{i \in I} a_i$ . Assume that c is a compact element of A such that  $c \leq a$ , hence  $c \leq \bigvee_{i \in J} a_i$ , for some finite subset J of I. Thus there exist the compact elements  $d_i, i \in J$  such that  $c \leq \bigvee_{i \in J} d_i$  and  $d_i \leq a_i$ , for all  $i \in J$ . For any  $i \in J$  we have  $a \vee (d_i \to \rho(0)) \geq a_i \vee (d_i \to \rho(0)) = 1$ , so, by using Lemma 2.1,(3) it follows that  $a \vee \bigwedge_{i \in J} (d_i \to \rho(0)) = 1$ . Observing that  $c \to \rho(0) \geq (\bigvee_{i \in J} d_i) \to \rho(0) = \bigwedge_{i \in J} (d_i \to \rho(0))$ , one obtains the inequality  $a \vee (\to \rho(0)) \geq a \vee \bigwedge_{i \in J} (d_i \to \rho(0)) = 1$ . Thus  $a \vee (c \to \rho(0)) = 1$ , hence a is wpure.

**Lemma 4.8** If an element a of coherent quantale A is w - pure then  $a^*$  is a  $\sigma$ -ideal of the reticulation L(A). Particularly, if a is pure then  $a^*$  is a  $\sigma$ -ideal.

**Proof.** Assume that  $x \in a^*$ , hence  $x = \lambda_A(c)$  for some compact element c of A, such that  $c \leq a$ . Then  $a \vee (c \to \rho(0)) = 1$ , so there exist  $c, d \in K(A)$  such that  $d \leq a$ ,  $e \leq c \to \rho(0)$  and  $d \vee e = 1$ . It follows that  $\lambda_A(d) \in a^*$ ,  $ec \leq \rho(0)$  and  $\lambda_A(d) \vee \lambda_A(e) = \lambda_A(d \vee e) = 1$ . On the other hand we have  $\lambda_A(d) \wedge \lambda_A(e) = \lambda_A(de) \leq \lambda_A(\rho(0)) = 0$ , so  $\lambda_A(e) \in Ann(\lambda_A(c))$ . Thus  $a^* \vee Ann(x) = a^* \vee Ann(\lambda_A(c)) = L(A)$ , so  $a^*$  is a  $\sigma$ - ideal of L(A). The second part of proposition follows by Lemma 4.5.

**Lemma 4.9** Let A be a coherent quantale and J a  $\sigma$  - ideal of L(A). Then  $J_*$  is a w - pure element of A.

**Proof.** Let c be a compact element of A such that  $c \leq J_*$ . By Lemma 3.3,(6) we have  $\lambda_A(c) \in J$ , hence  $J \vee Ann(\lambda_A(c)) = L(A)$ . Then there exist  $d, e \in K(A)$  such that  $\lambda_A(c) \in J$ ,  $\lambda_A(e) \in Ann(\lambda_A(c))$  and  $\lambda_A(d \vee e) = \lambda_A(d) \vee \lambda_A(e) = 1$ . By Lemmas 3.3,(6) and 3.2,(3) we obtain  $d \leq J_*$  and  $d \vee e = 1$ . From  $\lambda_A(e) \in Ann(\lambda_A(c))$  we get  $\lambda_A(ce) = \lambda_A(c) \wedge \lambda_A(e) = 0$ , hence  $ce \leq \rho(0)$  (cf. Lemma 3.2,(8)). It follows that  $e \leq c \to \rho(0)$ , so  $1 = c \vee d \leq J_* \vee (c \to \rho(0))$ . Thus  $J_* \vee (c \to \rho(0)) = 1$ , hence  $J_*$  is w - pure.

Corollary 4.10 Let A be a semiprime quantale. If J is a  $\sigma$  - ideal of L(A) then  $J_*$  is a pure element of A.

Corollary 4.11 Let A be a coherent quantale.

- (1) If  $a \in A$  is w pure then  $\rho(a)$  is w pure;
- (2) An ideal J of L(A) is a  $\sigma$  ideal if and only if  $J_*$  is a w pure element of A.

**Proof.** (1) If  $a \in A$  is w - pure then  $a^*$  is a  $\sigma$  - ideal of L(A) (cf. Lemma 4.8). By applying Lemma 4.9 it follows that  $\rho(a) = (a^*)_*$  is w - pure.

(2) We apply Lemmas 4.8, 4.9 and 3.3,(2).

Following [18], [26], if I is an ideal of a bounded distributive lattice then we denote  $\sigma(I) = \{x \in L | I \vee Ann(x) = L\}$ . We remark that in the frame Id(L) we have  $\sigma(I) = Ker(I)$ .

**Lemma 4.12** Let A be a coherent quantale. Then for all  $a \in A$  and  $c \in K(A)$ , the following equivalence holds:  $c \leq Ker(a)$  if and only if  $a \vee c^{\perp} = 1$ .

**Proof.** If  $c \leq Ker(a)$  then there exist  $d_1, ..., d_n \in K(A)$  such that  $c \leq d_1 \vee ... \vee d_n$  and  $d_i \leq a, a \vee d_i^{\perp} = 1$ , for all i = 1, ..., n. Denoting  $d = \bigvee_{i=1}^n d_i$  we have  $d \in K(A)$  and, by using Lemma 2.1,(3), the following equalities hold:

$$a \vee d^{\perp} = a \vee (\bigvee_{i=1}^{n} d_i)^{\perp} = a \vee \bigwedge_{i=1}^{n} d_i^{\perp} = 1.$$

Since  $c \leq d$  it follows that  $d^{\perp} \leq c^{\perp}$ , so  $a \vee c^{\perp} = 1$ . The proof of converse implication is obvious.

The following two theorems emphasize the way in wich the reticulation preserves the operator  $Ker(\cdot)$ .

**Theorem 4.13** Let A be a coherent quantale.

- (1) For any  $a \in A$  we have  $(Ker(a))^* = \sigma(a^*)$ ;
- (2) For any ideal I of L(A) we have  $(\sigma(I))_* = Ker(a_*)$ .

**Proof.** (1) Assume that  $x \in \sigma(a^*)$ , so  $a^* \vee Ann(x) = L(A)$ . Then there exists  $c \in K(A)$  such that  $x = \lambda_A(c)$ , so  $a^* \vee Ann(\lambda_A(c)) = L(A)$ . Then one can find  $d, e \in K(A)$  such that  $\lambda_A(d) \in a^*$ ,  $\lambda_A(e) \in Ann(\lambda_A(c))$  and  $\lambda_A(d \vee e) = \lambda_A(d) \vee \lambda_A(e) = 1$ . Thus  $\lambda_A(ec) = \lambda_A(e) \wedge \lambda_A(c) = 0$ , so there exists a positive integer n such that  $e^n c^n = 0$ , hence  $e^n \leq (c^n)^{\perp}$ . According to Lemma 3.2,(3) have  $d \vee e = 1$ , so  $d^n \vee e^n = 1$  (by Lemma 2.2,(2)). On can take  $d \leq a$ , hence  $a \vee (c^n)^{\perp} = 1$ . By Lemma 4.12 we get  $c^n \leq Ker(a)$ , so  $x = \lambda_A(c) = \lambda_A(c^n) \in (Ker(a))^*$ . We have proven that  $\sigma(a^*) \subseteq (Ker(a))^*$ .

In order to prove that  $(Ker(a))^* \subseteq \sigma(a^*)$ , let us assume that  $x \in (Ker(a))^*$ , so  $x = \lambda_A(c)$ , for some compact element c having the property  $c \leq Ker(a)$ . By

Lemma 4.12 we have  $a \vee c^{\perp} = 1$ , hence, by using Lemma 3.4,(2), one obtains  $a^* \vee (c^{\perp})^* = L(A)$ . According to Proposition 3.13,  $Ann(x) = Ann(\lambda_A(c)) = (c \to \rho(0))^*$ , therefore the inequality  $c^{\perp} = c \to 0 \le c \to \rho(0)$  implies  $(c^{\perp})^* \subseteq (c \to \rho(0))^* = Ann(x)$ . It follows that  $a^* \vee Ann(x) = L(A)$ , i.e  $x \in \sigma(a^*)$ 

(2) Assume that I is an ideal of L(I) and c is a compact element of A. If  $I_* \vee c^\perp = 1$  then, by using Lemmas 3.3,(2) and 3.4,(2), one obtains  $I \vee (c^\perp)^* = (I_*)^* \vee (c^\perp)^* = (I_* \vee c^\perp)^* = L(A)$ . Conversely, assuming that  $I \vee (c^\perp)^* = L(A)$ , one infers that there exist  $d, e \in K(A)$  such that  $\lambda_A(d) \in I$ ,  $\lambda_A(e) \in (c^\perp)^*$  and  $\lambda_A(d \vee e) = \lambda_A(d) \vee \lambda_A(e) = 1$ . We remark that  $\lambda_A(d) \in I$  implies  $d \leq I_*$  and  $\lambda_A(d \vee e) = 1$  implies  $d \vee e = 1$ . It is obvious that one can assume that  $e \leq c^\perp$ , so  $I_* \vee c^\perp = 1$ . Thus we get the following equivalence:  $I_* \vee c^\perp = 1$  if and only if  $I \vee (c^\perp)^* = L(A)$ . By using this equivalence and Lemma 4.12, the following hold:  $c \leq Ker(I_*)$  iff  $I \vee c^\perp = 1$  iff  $I \vee (c^\perp)^* = L(A)$  iff  $I \vee Ann(\lambda_A(c)) = L(A)$  iff  $\lambda_A(c) \in \sigma(I)$  iff  $c \leq (\sigma(I))_*$ . The previous equivalences hold for all  $c \in K(A)$ , therefore one gets the equality  $(\sigma(I))_* = Ker(a_*)$ .

**Theorem 4.14** If a is a w - pure radical element of a coherent quantale A then  $a = \rho(Ker(a))$  and Ker(a) = Vir(a).

**Proof.** First we shall show that  $a = \rho(Ker(a))$ . From  $Ker(a) \leq a$  we get  $\rho(Ker(a)) \leq \rho(a) \leq a$ . In order to prove that  $a \leq \rho(Ker(a))$ , assume that c is a compact element of A such that  $c \leq a$ . Since a is w - pure, we get  $a \vee (c \to \rho(0)) = 1$ . By the compactness of 1, there exists  $d \in K(A)$  such that  $d \leq c \to \rho(0)$  and  $a \vee d = 1$ . According to Lemma 2.4,(2), from  $dc \leq \rho(0)$  one obtains  $d^nc^n = 0$  for some positive integer n, hence  $d^n \leq (c^n)^{\perp}$ . Applying Lemma 2.1,(3), from  $a \vee d = 1$  we obtain  $a \vee d^n = 1$ , hence  $a \vee (c^n)^{\perp} = 1$ . Thus the compact element  $c^n$  verifies the inequality  $c^n \leq Ker(a)$  (cf. Lemma 4.1), hence, by using Lemma 2.4,(2) one obtains  $c \leq \rho(Ker(a))$ . It follows that  $a \leq \rho(Ker(a))$ , hence  $a = \rho(Ker(a))$ .

In order to show that Ker(a) = Vir(a), we recall that  $Ker(a) \leq Vir(a)$ , hence it suffices to check that Ker(a) is pure. Assume that c is a compact element such that  $c \leq Ker(a)$ , so  $a \vee c^{\perp} = 1$  (cf. Lemma 4.1). We have proven that  $a = \rho(Ker(a))$ , so  $\rho(Ker(a)) \vee c^{\perp} = 1$ . By the compactness of 1 there exists  $d \in K(A)$  such that  $d \leq \rho(Ker(a))$  and  $d \vee c^{\perp} = 1$ . According to Lemma 2.4,(1), there exists a positive integer n such that  $d^n \leq Ker(a)$  and, by Lemma 4.1, on gets  $d^n \vee c^{\perp} = 1$ . It follows that  $Ker(a) \vee c^{\perp} = 1$ , so Ker(a) is pure.

**Lemma 4.15** If a is a pure element of a coherent quantale A then  $a = Vir(\rho(a))$ .

**Proof.** We remark that  $a \leq \rho(a) \leq \bigwedge \{m \in Max(A) | a \leq m\}$ . In accordance with Lemma 4.2,(3) it follows that the following hold:  $a = Vir(a) \leq Vir(\rho(a)) \leq Vir(\bigwedge \{m \in Max(A) | a \leq m\}) = a$ , so  $a = Vir(\rho(a))$ .

According to Lemma 4.7, one can consider the function  $w: Vir(A) \to Vir(Id(L(A)))$ , defined by  $w(a) = a^*$ , for all  $a \in Vir(A)$ . By using Lemma 3.4,(2) it follows that w is a frame morphism.

By using the previous results one can obtain a new proof of a theorem given in [23].

**Theorem 4.16** The map w is a frame isomorphism.

**Proof.** According to Lemma 4.8 one can consider the composition  $Vir \circ (\cdot)_*$  of the following functions:

$$Vir(Id(L(A))) \xrightarrow{(\cdot)_*} Vir_w(A) \xrightarrow{Vir} Vir(A).$$

We shall prove that  $Vir \circ (\cdot)_*$  is the inverse function of w. Let a be an arbitrary pure element of A. By using Lemmas 4.11 and 3.3(7), the following equalities hold:  $a = Vir(\rho(a)) = Vir((a^*)_*) = (Vir \circ (\cdot)_*)(w(a))$ .

If  $J \in V(Id(L(A)))$  then  $a = J_*$  is a w - pure radical element of A (cf. Lemma 4.8). Then by using Theorem 4.3 and Lemma 3.3, the following equalities hold:  $w(Vir(J_*)) = (Vir(a))^* = (Ker(a))^* = (\rho(Ker(a)))^* = a^* = (J_*)^* = J$ .

Let A be a semiprime coherent quantale. By Proposition 4.5 we have  $Vir(A) \subseteq R(A)$ . Therefore, by using the proof of the previous theorem it follows that the functions

 $(\cdot)^*: Vir(A) \to Vir(Id(L(A))), \ (\cdot)_*: Vir(Id(L(A))) \to Vir(A)$  are the inverse frame morphisms that give that the frames Vir(A) and

Lemma 4.17 Let A be a coherent quantale.

Vir(Id(L(A))) are isomorphic.

- (1) For any  $p \in Spec(A)$  we have  $O(p) = \bigvee \{c \in K(A) | c \leq p, c^{\perp} \not < p\}$ :
- (2) For any  $p \in Spec(A)$ , c < O(p) if and only if  $c^{\perp} \not < p$ .

For any  $p \in Spec(A)$ , define  $\tilde{O}(p) = \bigvee \{c \in K(A) | c \leq p, c \rightarrow \rho(0) \not\leq p\}$ . It is easy to prove that  $\tilde{O}(p) \leq O(p)$ .

**Lemma 4.18** For all  $p \in Spec(A)$  and  $c \in K(A)$ ,  $c \leq \tilde{O}(p)$  if and only if  $c \leq p$  and  $c \to \rho(0) \not\leq p$ .

#### Proof.

Assume that  $c \leq \tilde{O}(p)$ , so there exist  $d_1, ..., d_n \in K(A)$  such that  $c \leq \bigvee_{i=1}^n d_i$ ,  $d_i \leq p$  and  $d_i \to \rho(0) \not\leq p$ , for each i = 1, ..., n. Then  $c \leq p$  and  $(\bigvee_{i=1}^n d_i) \to \rho(0)$ 

 $=\bigwedge_{i=1}^n (d_i \to \rho(0)) \not\leq p$  (because p is m - prime). The converse implication is obvious.

**Theorem 4.19** Let A be a coherent quantale.

- (1) For any  $p \in Spec(A)$  we have  $O(p^*) = (\tilde{O}(p))^*$ ;
- (2) For any  $P \in Spec_{Id}(L(A))$  we have  $(O(P))_* = \tilde{O}(P_*)$ .

**Proof.** (1) Assume that  $x \in O(p^*)$ , hence  $Ann(x) \not\subseteq p^*$  and  $x \in p^*$ . Let us take a compact elemenent c such that  $c \leq p$  and  $x = \lambda_A(c)$ , so  $Ann(\lambda_A(c)) \not\subseteq p^*$ . By Lemma 3.12 we have  $c \to \rho(0) \not\leq p$ , so  $c \leq \tilde{O}(p)$ . It follows that  $x = \lambda_A(c) \in (\tilde{O}(p))^*$ , so one obtains the inclusion  $O(p^*) \subseteq (\tilde{O}(p))^*$ .

In order to prove that  $(\tilde{O}(p))^* \subseteq p^*$ , let us assume that  $x \in (\tilde{O}(p))^*$ , hence there exists  $c \in K(A)$  such that  $c \leq \tilde{Q}(p)$  and  $x = \lambda_A(c)$ . By Lemma 4.18 we have  $c \leq p$  and  $c \to \rho(0) \not\leq p$ . By using Lemma 3.12 one gets  $Ann(\lambda_A(c)) \not\subseteq p^*$ , hence  $x = \lambda_A(c) \in O(p^*)$ .

(2) Let c be a compact element of A such that  $c \leq (O(P))_*$ , hence  $\lambda_A(c) \in O(P)$  (cf. Lemma 3.3,(6)). It follows that  $\lambda_A(c) \in P$  and  $Ann(\lambda_A(c)) \not\subseteq P = (P_*)^*$ . By using Lemma 3.12 we get  $c \to \rho(0) \not\leq P_*$ , therefore by applying Lemma 4.18 we have  $c \leq \tilde{O}(P_*)$ . We conclude that  $(O(P))_* \leq \tilde{O}(P_*)$ .

In order to prove the converse inequality  $O(P_*) \leq (O(P))_*$ , let us assume that  $c \in K(A)$  and  $c \leq \tilde{O}(P_*)$ . By Lemma 4.18 we have  $c \leq P_*$  and  $c \to \rho(0) \not\leq P_*$ . Applying Lemma 3.12 we obtain  $Ann(\lambda_A(c)) \not\leq (P_*)^* = P$ . From  $c \leq P_*$  we get  $\lambda_A(c) \in P$  (according to Lemma 3.3,(6)). Therefore  $\lambda_A(c) \in O(P)$ , so we conclude that  $c \leq (O(P))_*$ .

Corollary 4.20 Let A be a semiprime coherent quantale.

- (1) For any  $p \in Spec(A)$  we have  $(O(p^*) = (O(p))^*$ ;
- (2) For any  $P \in Spec_{Id}(L(A))$  we have  $(O(P))_* = O(P_*)$ .

Theorem 4.19 and Corollary 4.20 show us how the reticulation preserves the operator  $O(\cdot)$ . They will be used many-times in the proofs of the theorems in the next sections (see e.g. the proof of Theorem 5.10).

# 5 Further properties of pure elements

Lemma 5.1 Let A be an algebraic quantale. Any regular element of A is pure.

**Proof.** Let e be a complemented element of A. If c is a compact element of A such that  $c \le e$  then  $e^{\perp} \le d^{\perp}$ , therefore  $1 = e \lor e^{\perp} \le e \lor c^{\perp}$ . Then e is pure.

Let now q be a regular element of A and  $e \in B(A)$  such that  $e \le q$ . We have proven that any complemented element is pure, so  $e = Vir(e) \le Vir(p)$ . It follows that  $q = \bigvee \{e \in B(A) | e \le q\} \le Vir(q)$ , so q = Vir(q). Therefore q is pure.

For any  $p \in Spec(Vir(A))$  let us define  $t_A(p) = \bigvee \{e \in B(A) | e \leq q\}$ . Thus  $t_A(p)$  is regular and  $t_A(p) \leq p < 1$ .

**Lemma 5.2** If  $p \in Spec(Vir(A))$  then  $t_A(p)$  is a max - regular element.

- **Proof.** It suffices to prove that  $e \in B(A)$  and  $e \not\leq p$  imply  $t_A(p) \lor e = 1$ . Assume by absurdum that there exists  $e \in B(A)$  such that  $e \not\leq p$  and  $t_A(p) \lor e < 1$ . The element  $t_A(p) \lor e$  is regular so there exists a max - regular element q such that  $t_A(p) \lor e < q$ . Since the complemented elements e and  $\neg e$  are pure, it follows that  $e \not\leq p$  and  $e \land \neg e = 0$  implies  $\neg e \leq p$ , hence  $\neg e = t_A(\neg e) \leq t_A(p)$ . Thus one gets  $1 \leq e \lor \neg e \leq e \lor t_A(p)$ , contradicting  $q \in Sp(A)$ . We conclude that  $t_A(p) \lor e = 1$ .

In accordance with the previous lemma, one can consider the function  $t_A$ :  $Spec(Vir(A)) \to Sp(A)$  defined by the assignment  $p \mapsto t_A(p)$ .

**Proposition 5.3** The function  $t_A$  is surjective and continuous.

**Proof.** If  $q \in Sp(A)$  then there exists  $m \in Max(A)$  with  $q \leq m$ . By Lemma 5.1 q is pure, so  $q \leq Vir(m)$ . Since q is regular, we have  $q = t_A(q) \leq t_A(Vir(m))$ . Thus  $q = t_A(Vir(m))$ , because both q and  $t_A(Vir(m))$  are max - regular. In a straightforward manner one can show that  $t_A$  is continous.

Following [38], a quantale A is said to be normal if for all  $a,b \in A$  such that  $a \vee b = 1$  there exist  $e, f \in A$  such that  $a \vee e = b \vee f = 1$  and ef = 0. If  $1 \in K(A)$  then A is normal if and only if for all  $c,d \in K(A)$  such that  $c \vee d = 1$  there exist  $e, f \in K(A)$  such that  $c \vee e = d \vee f = 1$  and ef = 0 (cf. Lemma 20 of [15]). One observes that a commutative ring R is a Gelfand ring iff Id(R) is a normal quantale and a bounded distributive lattice L is normal iff Id(L) is a normal frame.

The normal quantales offer an abstract framework in order to unify some algebraic and topological properties of commutative Gelfand rings [30], [8], [35], normal lattices [30], [26], [40], [43], commutative unital l - groups [10], F - rings [10], [30], MV - algebras and BL - algebras [22], [32], Gelfand residuated lattices [25], etc.

Let us fix a coherent quantale A.

**Lemma 5.4** [27] Let A be a normal coherent quantale and  $a \in A, m \in Max(A)$ . Then the following hold:

- (1) m is the unique maximal element of A such that  $O(m) \leq m$ ;
- (2)  $Ker(m) \le a$  if and only if a = m or a = 1;
- (3)  $Ker(a) \le m \text{ implies } a \le m;$
- (4) Vir(a) = Ker(a);
- (5)  $Vir(a) \leq m$  if and only if  $a \leq m$ .

By Lemmas 4.2,(1) and 5.4,(4), for each maximal element m of A we have O(m) = Ker(m) = Vir(m), so O(m) is pure.

**Proposition 5.5** [15] The quantale A is normal if and only if the reticulation L(A) is a normal lattice (in the sense of [43],[30]).

For any element  $a \in A$  we denote  $r(a) = \bigwedge (Max(A) \cap V(a))$ . In particular, r(0) is exactly the radical r(A) of the quantale A (cf.[24]). We observe that r(a) = r(0) is an abstractization of the Jacobson radical of a ring.

The literature of ring theory contains several properties that characterize the (commutative) Gelfand rings (see[43],[30],[35],[1],[49]). The following result extends the main characterization theorems of Gelfand rings. It collects various conditions that characterize normal quantales. In particular, the below properties (1) - (7) correspond to some conditions from Theorem 4.3 of [1] and the properties (8) - (14) generalize the conditions contained in Theorem 4.6 of [49].

**Proposition 5.6** [38],[27],[44] If A is a coherent quantale then the following are equivalent:

- (1) A is a normal quantale;
- (2) For all distinct  $m, n \in Max(A)$  there exist  $c_1, c_2 \in K(A)$  such that  $c_1 \not\leq m, c_2 \not\leq n$  and  $c_1c_2 = 0$ ;
- (3) The inclusion  $Max(A) \subseteq Spec(A)$  is Hausdorff embedding (i.e. any pair of distinct points in Max(A) have disjoint neighbourhoods in  $Spec_Z(A)$ );
- (4) For any  $p \in Spec(A)$  there exists a unique  $m \in Max(A)$  such that  $p \leq m$ ;
- (5)  $Spec_Z(A)$  is a normal space;
- (6) The inclusion  $Max_Z(A) \subseteq Spec_Z(A)$  has a continuous retraction  $\gamma : Spec_Z(A) \to Max_Z(A)$ ;
- (7) If  $m \in Max(A)$  then  $\Lambda(m)$  is a closed subset of  $Spec_Z(A)$ .
- (8) If m, n are two distinct maximal elements of A then  $Vir(m) \vee Vir(n) = 1$ ;

- (9) For all  $a, b \in A$ ,  $a \lor b = 1$  implies  $Vir(a) \lor Vir(b) = 1$ ;
- (10)  $Vir: A \rightarrow Vir(A)$  preserves arbitrary joins;
- (11) For all  $a \in A$  and  $m \in Max(A)$ ,  $Vir(a) \leq m$  implies  $a \leq m$ ;
- (12) For all  $a \in A$ , the following equality holds:  $Max(A) \cap V(a) = Max(A) \cap V(Vir(a))$ ;
- (13) For all  $a \in A$ , the following equality holds: r(a) = r(Vir(a));
- (14) The function  $\eta: Max_Z(A) \to Spec_Z(Vir(A))$  defined by  $m \mapsto Vir(a)$  is a homeomorphism.

#### Proof.

- $(1) \Leftrightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4)$  By Proposition 3.2 of [27].
- $(1) \Leftrightarrow (8) \Leftrightarrow (9) \Leftrightarrow (10)$  By Proposition 3.4 of [27] or Theorem 3.5 of [44].
- $(1) \Leftrightarrow (11)$  By Theorem 3.5 of [44].
- $(11) \Leftrightarrow (12) \Leftrightarrow (13)$  These equivalences follow in a straightforward manner.
- $(1) \Leftrightarrow (4) \Leftrightarrow (5)$  These equivalences follow from [38],[46] or by using Proposition 5.5 and [30], (p.68, Proposition 3.7).
  - $(1) \Rightarrow (14)$  By Theorem 3.5 of [27].
- $(14) \Rightarrow (6)$  According to Lemma 4.2,(4),  $Vir : Spec_Z(A) \rightarrow Spec_Z(Vir(A))$  is a continuous map and, by the hypothesis (14), the function  $\eta : Max_Z(A) \rightarrow Spec_Z(Vir(A))$  defined by  $m \mapsto Vir(a)$  is a homeomorphism. Then it is easy to see that  $\gamma = \eta^{-1} \circ Vir : Max_Z(A) \rightarrow Spec_Z(Vir(A))$  is a continuous retraction of the inclusion  $Max_Z(A) \subseteq Spec_Z(A)$ .
- $(1) \Leftrightarrow (7)$  According to [28], for each bounded distributive lattice L there exists a commutative ring R such that the reticulation L(R) of R is isomorphic to L. Then for any coherent quantale A there exists a commutative ring R such that the reticulations L(A) and L(R) are isomorphic. By Proposition 5.5, A is a normal quantale iff L(R) is a normal lattice iff L(R) is a normal quantale iff L(R) is a Gelfand ring. Applying twice Lemma 3.5 we get a homeomorphism  $H(R) = \frac{1}{2} \int_{R}^{R} dR dR$  is an order isomorphism. Thus the following assertions are equivalent:
  - for all  $m \in Max(A)$ ,  $\Lambda(m) = \{ p \in Spec(A) | p \leq m \}$  is closed in  $Spec_Z(A)$ ;
- for all  $M \in Spec(R)$ ,  $\Lambda(M) = \{P \in Spec(R) | P \subseteq M\}$  is closed in  $Spec_Z(R)$ .

In accordance with the equivalence  $(i) \Leftrightarrow (viii)$  from Theorem 4.3 of [1], the following equivalences hold: A is a normal quantale iff R is a Gelfand ring iff for all  $M \in Max(R)$ ,  $\Lambda(M)$  is closed in  $Spec_Z(R)$  iff for all  $m \in Max(A)$ ,  $\Lambda(m)$  is closed in  $Spec_Z(A)$ .

Let L be a bounded distributive lattice and  $P \in Spec_{Id}(L)$ , hence L - P is a filter of L. Following [13], the quotient bounded distributive lattice  $L_P = L/L - P$  is called the *lattice of fractions* of L associated with the prime ideal P. The congruence  $\equiv_P$  of L modulo the filter L - P has the following form:  $x \equiv_P y$ 

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iff  $x \wedge t = y \wedge t$ , for some  $t \notin P$ . We denote by  $x_P$  the congruence class of the element  $x \in L$ . Let us consider the lattice morphism  $\pi_P : L \to L_P$ , defined by  $\pi_P(x) = x_P$ , for all  $x \in L$ .

**Lemma 5.7** For all  $x \in L$ , we have  $x \in O(P)$  if and only if  $\pi_P(x) = 0_P$ .

**Proof.** For all  $x \in L$ , the following equivalences hold:  $\pi_P(x) = 0_P$  iff  $x \equiv_P 0$  iff there exists  $t \notin P$  such that  $x \wedge t = 0$  iff  $x \in O(P)$ .

**Remark 5.8** [13] For any prime ideal P of L, we shall denote  $\Lambda_{Id}(P) = \{Q \in Spec_{Id}(L)|Q \subseteq P\}$ . Let us consider the function  $\pi_P^* : \Lambda_{Id}(L) \to Spec_{Id}(L)$ , defined by the assignment  $Q \mapsto \pi_P^*(Q) = \{x_P | x \in Q\}$ . It is straightforward to prove that  $\pi_P^*$  is an order - isomorphism.

**Proposition 5.9** For all prime ideal P of the lattice L, we have  $O(P) = \bigcap \Lambda_{Id}(P)$ .

**Proof.** For any element  $x \in O(P)$  there exists  $y \notin P$  such that  $x \wedge y = 0$ . Then for any  $Q \in V_{Id}(L)$  we have  $y \notin Q$ , hence  $x \in Q$ . It follows that the inclusion  $O(P) \subseteq \bigcap V_{Id}(P)$  holds.

In order to establish the converse inclusion  $\bigcap \Lambda_{Id}(P) \subseteq O(P)$ , assume that  $x \in \bigcap \Lambda_{Id}(P)$ . Then  $\pi_P(x) \in \pi_P^*(Q)$  for all prime ideal Q such that  $Q \subseteq P$ , hence by using Remark 5.8, one gets

 $\pi_P(x) \in \bigcap \{\pi_P^*(Q) | Q \in \Lambda_{Id}(P)\} = \bigcap Spec_{id}(L) = \{0_P\}.$ In accordance with Lemma 5.7, one obtains  $x \in O(P)$ .

**Theorem 5.10** Let A be a semiprime coherent quantale. For any m - prime element p of A, the equality  $O(p) = \bigwedge \Lambda(p)$  holds.

**Proof.** Let  $c \in K(A)$  such that  $c \leq O(p)$ , hence  $c \leq p$  and  $c^{\perp} \not\leq p$  (cf. Lemma 4.17,(2)). For all  $q \in \Lambda(p)$  we have  $c^{\perp} \not\leq q$ , so  $c \leq q$ . It follows that  $c \leq \bigwedge \Lambda(p)$ , hence  $O(p) \leq \bigwedge \Lambda(p)$ .

In order to prove that  $\bigwedge \Lambda(p) \leq O(p)$ , let us consider an element  $c \in K(A)$  such that  $c \leq \bigwedge \Lambda(p)$ . Then for all m - prime elements q such that  $q \leq p$  we have  $c \leq q$ . According to Lemma 3.5, it follows that for all prime ideals of L(A) such that  $Q \subseteq p^*$  we have  $\lambda_A(c) \in Q$ . In accordance with Proposition 5.9 it follows that  $\lambda_A(c) \in O(p^*)$ . Since the quantale A is semiprime, by using Corollary 4.20 one gets  $\lambda_A(c) \in (O(p))^*$ . Therefore, by applying Lemma 3.3 and Corollary 4.20, the following hold:  $c \leq ((O(p))^*)_* = (O(p^*))_* = O((p^*)_*) = O(p)$ . We conclude that  $\bigwedge \Lambda(p) \leq O(p)$ .

We observe that the previous theorem is obtained by transferring Proposition 5.9 from lattices to quantales by using the reticulation.

**Corollary 5.11** Let A be a semiprime coherent quantale. For any minimal m - prime element p of A, the equality O(p) = p holds.

**Corollary 5.12** If A is a semiprime coherent quantale then  $\bigwedge \{O(m)|m \in Max(A)\} = 0$ .

**Proof.** By using Theorem 5.10, for each  $m \in Max(A)$  we have  $O(m) = \Lambda(m)$ . By observing that  $Spec(A) = \bigcup \{\Lambda(m) | m \in Max(A), \text{ the following equality holds:}$ 

 $\bigwedge \{O(m) | m \in Max(A)\} = \bigwedge Spec(A) = 0.$ 

**Lemma 5.13** If a is a pure element of a normal coherent quantale A then  $a = \bigwedge \{O(m) | m \in Max(A) \cap V(a)\}.$ 

**Proof.** Let a be a pure element of A. Recall from Lemma 4.2,(ii) the equality  $a = \bigwedge \{Vir(m) | m \in Max(A) \cap V(a)\}$ . By Lemmas 4.2,(1) and 5.4,(4) one gets O(m) = Ker(m) = Vir(m), hence  $a = \bigwedge \{O(m) | m \in Max(A) \cap V(a)\}$ .

**Lemma 5.14** Let A be a normal coherent quantale.

- (1) If  $m \in Max(A)$  then O(m) is pure;
- (2) If p is a pure m prime element of A then O(p) = p.

**Proof.** (1) By Lemmas 4.2,(1) and 5.4,(4) we have O(m) = Ker(m) = Vir(m); (2) The inequality  $O(p) \le p$  is always true. In order to prove the converse inequality  $p \le O(p)$  let us consider a compact element c such that  $c \le p$ . Since p is pure we have  $p \lor c^{\perp} = 1$ , so  $c^{\perp} \not\le p$ . From  $cc^{\perp} = 0$  and  $c^{\perp} \not\le p$  it follows that  $c \le p$ , so  $p \le O(p)$ .

**Corollary 5.15** If A be a normal coherent quantale then a maximal element of A is pure if and only if O(m) = m.

Following [15],[24], a quantale A is said to be hyperarchimedean if for all  $c \in K(A)$  there exists a positive integer n such that  $c^n \in B(A)$ . In [15] we proved that a coherent quantale is hyperarchimedean iff L(A) is a Boolean algebra iff Max(A) = Spec(A).

**Proposition 5.16** Let A be a semiprime coherent quantale. Then rhe following are equivalent:

(1) A is hyperarchimedean;

(2) Any maximal element of A is pure.

#### Proof.

 $(1) \Rightarrow (2)$  Let m be a maximal element of A. According to Theorem 1 of [15], we have Max(A) = Spec(A), so  $m \in Min(A)$ . By Corollary 5.11 we get O(m) = m, so m is pure.

 $(2) \Rightarrow (1)$  Assume that  $m \in Max(A)$ , so m is pure, hence O(m) = m (cf. Lemma 5.14). By Proposition 5.10, it follows that  $\bigwedge \Lambda(m) = O(m) = m$ , hence m is a minimal m - prime element. Therefore Min(A) = Max(A), so, by Theorem 1 of [15], we conclude that A is hyperarchimedean.

**Proposition 5.17** Let A be a normal coherent quantale and  $a \in A$ . Then  $K(a) = \bigwedge (Max(A) \bigcap V(a))$  is a pure element of A.

**Proof.** Let c be a compact element of A such that  $c \leq K(a)$ . We have to prove that  $K(a) \vee c^{\perp} = 1$ . For all  $m \in Max(A) \cap V(a)$  we have  $c \leq O(m)$ , hence, by Lemma 4.17,(2), we get  $c^{\perp} \not\leq m$ , so  $d_m c = 0$  and  $d_m \not\leq m$  for some  $d_m \in Max(A)$ .

For all maximal element m such that  $m \notin V(a)$  we take a compact element  $e_m$  such that  $e_m \leq a$  and  $e_m \not\leq m$ . Assume that

 $\bigvee \{d_m | m \in Max(A), a \leq m\} \vee \bigvee \{e_m | m \in Max(A), a \not\leq m\} < 1,$  so there exists  $n \in Max(A)$  such that

 $\bigvee \{d_m | m \in Max(A), a \leq m\} \vee \bigvee \{e_m | m \in Max(A), a \not\leq m\} \leq n.$ 

Thus  $a \leq n$  implies  $d_n \leq n$  and  $a \not\leq n$  implies  $e_n \leq n$ . In the both cases we have obtained a contradiction, therefore

 $\bigvee \{d_m | m \in Max(A), a \le m\} \vee \bigvee \{e_m | m \in Max(A), a \not\le m\} = 1.$ 

Since  $1 \in K(A)$  there exist the maximal elements  $m_1,...,m_k,n_1,...,n_l$  such that  $d_{m_1} \vee ... \vee d_{m_k} \vee e_{n_1} \vee ... \vee e_{n_l} = 1, \ a \leq m_i$ , for i=1,...,k and  $a \not\leq n_j$ , for j=1,...,l. Let us denote  $d_i=d_{m_i}$  for i=1,...,k and  $e_j=e_{n_j}$  for j=1,...,l. Thus  $cd_i=0,\ d_i \leq m_i$ , for i=1,...,k and  $e_j \leq a,\ e_j \not\leq n_j$ , for j=1,...,l.

If  $d=d_1\vee\ldots\vee d_k$  and  $e=e_1\vee\ldots\vee e_l$  then  $d,e\in K(A)$  and  $d\vee e=1$ . It follows that cd=0 and  $e\leq a$ . Since A is a normal quantale, from  $d\vee e=1$  we infer that there exist  $x,y\in K(A)$  such that  $d\vee x=e\vee y=1$  and xy=0. Then  $x=x(e\vee y)=xe\vee xy=xe$ , hence  $x\leq e\leq a$ . In a similar way we get  $y=yd\leq d$ .

We shall prove that  $x \leq K(a)$ . Let m be a maximal element of A such that  $a \leq m$ , hence  $e \leq a \leq m$ . Since  $e \vee y = 1$  and  $e \leq m$  it follows that  $y \not\leq m$ . On the other hand, xy = 0 implies  $y \leq x^{\perp}$ , hence  $x^{\perp} \not\leq m$ . According to Lemma 4.17,(2) we have  $x \leq O(m)$ . We have proven that for all maximal element m,  $a \leq m$  implies  $x \leq O(m)$ , hence  $x \leq K(a)$ .

Recall that cd = 0, so  $d \le c^{\perp}$ . Thus  $1 = x \lor d \le K(a) \lor c^{\perp}$ , so  $K(a) \lor c^{\perp} = 1$ . We conclude that K(a) is pure.

**Theorem 5.18** Let A be a normal coherent quantale. Then an element  $a \in A$  is pure if and only if  $a = \bigwedge \{O(m) | m \in Max(A) \cap E\}$ , for some closed subset E of  $Spec_Z(A)$ .

**Proof.** By Lemma 5.13, any pure element  $a \in A$  has the form  $a = \bigwedge \{O(m) | m \in Max(A) \cap V(a)\}$ . Conversely, assume that  $a = \bigwedge \{O(m) | m \in Max(A) \cap E\}$ , for some closed subset E of  $Spec_Z(A)$ . Then E = V(x) for some  $x \in A$ , hence a = K(x). In accordance with Proposition 5.17, a = K(x) is a pure element of A.

For any m - prime element p of a coherent quantale denote  $\Omega(p) = \bigwedge \Lambda(p)$ . According to Theorem 5.10, if A is semiprime and  $p \in Spec(A)$  then  $O(p) = \Omega(p)$ .

**Corollary 5.19** Let A be a normal and semiprime coherent quantale. Then an element  $a \in A$  is pure if and only if  $a = \bigwedge \{\Omega(m) | m \in Max(A) \cap E\}$ , for some closed subset E of  $Spec_Z(A)$ .

# 6 Pure elements in PF - quantales

Following [24], a quantale A is said to be an mp-quantale if for any  $p \in Spec(A)$  there exists a unique  $q \in Min(A)$  such that  $q \leq p$ . An mp-frame is an mp-quantale wich is a frame. We remark that a ring R is an mp- ring in the sense of [1] if and only if the quantale Id(R) of ideals of R is an mp- quantale. The mp- quantales can be related to the conormal lattices, introduced by Cornish in [17] under the name of "normal lattices". According to [43],[30], a conormal lattice is a bounded distributive lattice L such that for all  $x, y \in L$  with  $x \wedge y = 0$  there exist  $u, v \in L$  having the properties  $x \wedge u = y \wedge v = 0$  and  $u \vee v = 1$ . Cornish obtained in [17] several characterizations of the conormal lattices. For example, a bounded distributive lattice L is conormal if and only if any prime ideal of L contains a unique minimal prime ideal.

**Lemma 6.1** [24] A coherent quantale A is an mp - quantale if and only if the reticulation L(A) is a conormal lattice.

Let us denote by  $Min_Z(A)$  (resp.  $Min_F(A)$ ) the topological space obtained by restricting the topology of  $Spec_Z(A)$  (resp.  $Spec_F(A)$ ) to Min(A). By using Lemma 3.5,  $Min_Z(A)$  is homeomorphic to the space  $Min_{Id,Z}(A)$  of minimal prime ideals in L(A) with the Stone topology and  $Min_F(A)$  is homeomorphic to the space  $Min_{Id,F}(A)$  of minimal prime ideals in L(A) with the flat topology. Then  $Min_Z(A)$  is a zero - dimensional Hausdorff space and  $Min_F(A)$  is a compact T1 space [24].

**Theorem 6.2** [24] If A is a semiprime quantale then the following are equivalent:

- (1)  $Min_Z(A) = Min_F(A);$
- (2)  $Min_Z(A)$  is a compact space;
- (3)  $Min_Z(A)$  is a Boolean space;
- (4) For any  $c \in K(A)$  there exists  $d \in K(A)$  such that cd = 0 and  $(c \vee d)^{\perp} = 0$ .

**Theorem 6.3** [24] If A is a coherent quantale then the following are equivalent:

- (1) A is an mp quantale;
- (2) For any distinct elements  $p, q \in Min(A)$  we have  $p \vee q = 1$ ;
- (3) R(A) is an mp frame;
- (4)  $[\rho(0)]_A$  is an mp quantale;
- (5) The inclusion  $Min_F(A) \subseteq Spec_F(A)$  has a flat continuous retraction;
- (6)  $Spec_F(A)$  is a normal space;
- (7) If  $p \in Min(A)$  then V(p) is a closed subset of  $Spec_F(A)$ .

Recall from [5] that a commutative ring R is said to be PF - ring if the annihilator of each element of R is a pure ideal. Following [24], a quantale A is a PF - quantale if for each  $c \in K(A)$ ,  $c^{\perp}$  is a pure element. For any commutative ring R, Id(R) is a PF - quantale if and only if R is a PF - ring.

### Lemma 6.4 [24]

- (1) Any PF quantale A is semiprime;
- (2) If A is a PF quantale then the reticulation L(A) is a conormal lattice.

Now we remind from [24] some characterization theorems of PF - quantales. In what follows they will be intensely used in proving some algebraic and topological results on pure elements in a PF - algebra.

**Theorem 6.5** [24] Let A be a coherent quantale. Then A is a PF - quantale if and only if A is a semiprime mp - quantale.

**Theorem 6.6** [24] For a coherent quantale A consider the following conditions:

- (1) Any minimal m prime element of A is pure;
- (2) A is an mp quantale.

Then (1) implies (2). If the quantale A is semiprime then the converse implication holds.

**Corollary 6.7** [24] Let A be a semiprime quantale. Then A is a PF - quantale if and only if any minimal m - prime element of A is pure.

**Theorem 6.8** [24] For a coherent quantale A the following are equivalent:

- (1) A is a PF quantale;
- (2) A is a semiprime mp quantale;
- (3) If  $c, d \in K(A)$  then cd = 0 implies  $c^{\perp} \vee d^{\perp} = 1$ ;
- (4) If  $c, d \in K(A)$  then  $(cd)^{\perp} = c^{\perp} \vee d^{\perp}$ ;
- (5) For each  $c \in K(A)$ ,  $c^{\perp}$  is a pure element.

**Proposition 6.9** Let A be a coherent PF - quantale. If  $p \in Spec(A)$  then  $O(p) \in Min(A)$ .

**Proof.** By taking into account Theorem 5.10, it suffices to show that O(p) is an m - prime element of A. Let us consider two compact elements c,d of A such that  $cd \leq O(p)$ . By Lemma 4.17,(2) it follows that  $(cd)^{\perp} \not\leq p$ . Since  $(cd)^{\perp} = c^{\perp} \vee d^{\perp}$  (by Theorem 6.8,(3)), we have  $c^{\perp} \vee d^{\perp} \not\leq p$ , hence  $c^{\perp} \not\leq p$  or  $d^{\perp} \not\leq p$ . According to Lemma 4.17,(2), we get  $c \leq O(p)$  or  $d \leq O(p)$ , hence O(p) is m -prime.

**Theorem 6.10** Let A be a PF - quantale. The pure elements of A have the form  $\bigwedge(Min(A) \cap E)$ , where E is a closed subset of  $Spec_F(A)$ .

**Proof.** Let a be a pure element of A. According to Theorem 6.6, the pure element a is minimal m - prime, hence  $a = \bigwedge (Min(A) \bigcap V(a))$ . By Theorem 6.3,(7),  $Min(A) \bigcap V(a)$  is a closed subset of  $Min_F(A)$ .

Conversely, assume that E is a closed subset of  $Spec_F(A)$  and  $a = \bigwedge (Min(A) \bigcap E)$ . Let c be a compact element of A such that  $c \leq a$ . We have to prove that  $a \vee c^{\perp} = 1$ . Assume by absurdum that  $a \vee c^{\perp} < 1$ , so  $a \vee c^{\perp} \leq m$ , for some  $m \in Max(A)$ . Let us consider a minimal m - prime element q of A such that  $q \leq m$ .

Assume that  $q \in E$ , so we have  $c \leq q$ . By Theorem 6.6, the minimal m-prime element q is pure, hence  $q \vee c^{\perp} = 1$ . From  $q \leq m$  and  $c^{\perp} \leq m$  we obtain  $1 = q \vee c^{\perp} \leq m$ , contradicting that m is a maximal element. We conclude that  $q \notin E$ , so  $q \neq p$  for all  $p \in Min(A) \cap E$ .

Since A is an mp - quantale, we can apply Theorem 6.3,(2), hence for all  $p \in Min(A) \cap E$  there exist  $s_p, y_p \in K(A)$  such that  $x_p \leq p, y_p \leq q$  and  $x_p \vee y_p = 1$ . Assume  $r \in E$  and take  $p \in Min(A)$  such that  $p \leq r$ . It follows that  $x_p \leq p \leq r$ , so  $r \in V(x_p)$ . Thus we obtain the following inclusion:

 $E \subseteq \bigcup \{V(x_p) | p \in Min(A) \cap E\}.$ 

We remind that E is a closed subset of the compact space  $Spec_F(A)$ , hence E is itself compact. Then there exist a positive integer n and  $p_1, ..., p_n \in Min(A) \cap E$  such that  $E \subseteq V(x_{p_1}) \bigcup ... \bigcup V(x_{p_n})$ .

Let us denote  $x_i = x_{p_i}$ ,  $y_i = y_{p_i}$  for all i = 1, ..., n and  $x = x_1 \cdot ... \cdot x_n$ ,  $y = y_1 \vee ... \vee y_n$ . Thus  $x, y \in K(A)$  and  $y \vee x_i = 1$  for all i = 1, ..., n, hence by applying Lemma 2.1,(3) we obtain  $y \vee x = y \vee x_1 \cdot ... \cdot x_n = 1$ .

Since  $y_i \leq q$  for all i = 1, ..., n, we get  $y \leq q$ . For all  $r \in Min(A) \cap E$  we have  $r \in V(x_1) \cup ... \cup V(x_n)$ , so there exists  $i \in \{1, ..., n\}$  such that  $x_i \leq r$ . It follows that  $x \leq x_i \leq r$ . Thus for all  $r \in Min(A) \cap E$  we have  $x \leq r$ , hence  $x \leq \bigwedge (Min(A) \cap E) = a$ .

From  $x \le a \le m$  and  $y \le q \le m$  we obtain  $1 = x \lor y \le m$ , contradicting  $m \in Max(A)$ . Therefore  $a \lor c^{\perp} = 1$ , so a is a pure element of A.

Let R be a PF - ring. If we apply Theorem 6.10 for the PF - quantale A = Id(R) then we obtain Theorem 7.3 of [1].

**Lemma 6.11** Let A be a coherent quantale. For any  $m \in Max(Vir(A))$  there exists  $n \in Max(A)$  such that Vir(n) = m.

**Proof.** Let m be an element of Max(Vir(A)) and  $n \in Max(A)$  such that  $m \leq n$ . By Lemma 3.2,(4) one can consider the function  $Vir: Spec(A) \rightarrow Spec(Vir(A))$ . Then  $m = Vir(m) \leq Vir(n)$ , so we get m = Vir(n), because  $m \in Max(Vir(A))$  and  $Vir(n) \in Spec(Vir(A))$ .

**Theorem 6.12** If A is a coherent PF - quantale then Min(A) = Max(Vir(A)).

**Proof.** Firstly we shall establish the inclusion  $Min(A) \subseteq Max(Vir(A))$ . Assume that p is a minimal m - prime element of A. By Corollary 6.7, p is a pure element of A.

Since p < 1 there exists  $m \in Max(Vir(A))$  such that  $p \le m$ . Let us consider a compact element x of A such that  $x \le m$ . Since m is pure we have  $m \lor x^{\perp} = 1$ , hence  $x^{\perp} \not\le m$ . Since  $p \le m$ , we get  $x^{\perp} \not\le m$ . It follows that  $x \le m$ , hence p = m. We conclude that  $p \in Max(Vir(A))$ .

Conversely, assume that  $m \in Max(Vir(A))$ . By the previous lemma there exists  $n \in Max(A)$  such that Vir(n) = m. Let us take a minimal m - prime element p such that  $p \leq n$ . According to the first part of the proof, p is a maximal element of the frame Vir(A). Thus  $p = Vir(p) \leq Vir(n) = m$ , hence p = m, because  $m, p \in Max(Vir(A))$ . It results that  $m \in Min(A)$ , hence the inclusion  $Max(Vir(A)) \subseteq Min(A)$  is established.

**Proposition 6.13** Let A be a coherent PF - algebra and  $a \in A$ . Then  $K(a) = \bigwedge(Max(A) \bigcap V(a))$  is a pure element of A.

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**Proof.** Let c be a compact element of A such that  $c \leq K(a)$ . We shall prove that  $K(a) \vee c^{\perp} = 1$ . By using the same argument as in the proof of Theorem one can find two compact element d and e such that  $e \leq a$ , cd = 0 and  $d \vee e = 1$ . Since A is a PF - quantale, from cd = 0 we get  $c^{\perp} \vee d^{\perp} = 1$  (cf. Theorem 6.8,(4)). Then there exist the compact elements x and y such that  $x \leq c^{\perp}$ ,  $y \leq d^{\perp}$  and  $x \vee y = 1$ .

We shall prove that  $y \leq K(a)$ . Let m be a maximal element of A such that  $a \leq m$ , hence  $e \leq a \leq m$ . If  $d \leq m$  then  $1 = e \vee d \leq m$ , contradicting  $m \in Max(A)$ . Then we obtain  $d \not\leq m$ . From  $y \leq d^{\perp}$  we get  $d \leq y^{\perp}$ , hence  $y^{\perp} \not\leq m$ . According to Lemma 4.17,(2) we have  $y \leq O(m)$ . We have proven that for all maximal element m,  $a \leq m$  implies  $y \leq O(m)$ , hence  $y \leq K(a)$ .

We remark that  $1 = y \vee x \leq K(a) \vee c^{\perp}$ , so  $K(a) \vee c^{\perp} = 1$ . We conclude that K(a) is pure.

**Lemma 6.14** Let A be a coherent PF - quantale,  $a \in Var(A)$  and  $m \in Max(A)$  such that  $a \leq m$ . Assume that the following condition holds:

(\*) For all  $c \in K(A)$ ,  $c \le O(m)$  implies  $K(c) \le a$ . Then a = O(m).

**Proof.** Since  $a \leq m$  and a is pure we get  $a = Vir(a) \leq Vir(m) = O(m)$ . In order to show that  $O(m) \leq a$ , assume that c is a compact element such that  $c \leq O(m)$ . Since O(m) = Vir(m) is pure we have  $O(m) \vee c^{\perp} = 1$ , so there exist the compact elements d, e such that  $d \leq O(m)$ ,  $e \leq c^{\perp}$  and  $d \vee e = 1$ .

Now we shall prove that  $c \leq K(d)$ . Let us consider a maximal element n of A such that  $d \leq n$ . Assume that  $c^{\perp} \leq n$ , so  $e \leq c^{\perp} \leq n$ , therefore  $1 = d \vee e \leq n$ . This contradicts  $n \in Max(A)$ , hence  $c^{\perp} \not\leq n$ . By Lemma 4.17,(2) we obtain  $c \leq O(n)$ . It follows that  $c \leq \bigwedge\{O(n)|n \in Max(A), d \leq n\} = K(d)$ . In accordance with (\*),  $d \leq O(m)$  implies  $K(d) \leq a$ , hence  $c \leq K(d) \leq a$ . It follows that  $O(m) = \bigvee\{c \in K(A)|c \leq O(m)\} \leq a$ .

**Theorem 6.15** If A is a coherent PF - quantale then Spec(Vir(A)) = Max(Vir(A)).

**Proof.** It suffices to prove that  $Spec(Vir(A)) \subseteq Max(Vir(A))$ . Assume that  $p \in Spec(Vir(A))$ . Let q be a maximal element of the frame Vir(A) such that  $p \leq q$ .

Let us consider a compact element c of A such that  $c \leq q$ . For each  $n \in Max(A)$  there exist two possible casses:

Case 1:  $c \le n$ . By the definition of K(c) we have  $K(c) \le O(n)$ .

 $Case2: c \leq n$ . Since n is m - prime it follows that  $c^{\perp} \leq n$ . In accordance with Theorem 6.8,(5),  $c^{\perp}$  is a pure element of A, so  $c^{\perp} \leq Vir(n) = O(n)$ .

In virtue of these two cases one obtains  $K(c)c^{\perp} \leq \bigwedge \{O(n)|n \in Max(A), \text{ hence } K(c)c^{\perp} = 0 \text{ (by Corrolary 5.12).}$ 

Since  $c \leq q = O(q)$  we have  $c^{\perp} \not\leq q$  (by Lemma 4.17,(2)). According to Theorem 6.8,(5) and Proposition 6.13,  $c^{\perp}$  and K(c) are pure elements, hence  $K(c) \leq p$ .

We have proven that for each compact element  $c \leq q$  we have  $K(c) \leq p$ . We remark that q < 1 (because  $q \in Max(Vir(A))$ ). Let us consider  $m \in Max(A)$  such that  $q \leq m$ . Thus  $q = Vir(q) \leq Vir(m)$ , hence q = Vir(m) = O(m). Therefore for each compact element  $c, c \leq O(m)$  implies  $K(c) \leq p$ , i.e. the condition (\*) is satisfied (with p instead of a). By applying Lemma 6.14 we conclude that p = O(m) = q, so  $p \in Max(Vir(A))$ .

**Remark 6.16** Let A be a PF - quantale. By applying the previous result and Proposition 6.2 of [24] it follows that Vir(A) is a hyperarchimedean frame. Thus, by using Theorem 6.8,(8) of [24], we have the following identity of topological spaces:  $Spec_Z(Vir(A)) = Spec_F(Vir(A))$ . Taking into account Theorem 6.12, it follows that  $Min_F(A)$  and  $Spec_Z(Vir(A))$  are identical as topological spaces.

**Remark 6.17** Let R be a PF - ring. By applying the previous Theorems 6.12 and 6.16 to the PF - quantale A = Id(R) we obtain Theorems 5.1 and 5.3 of [49]. Remark 6.17 can be view as an abstract version of Theorem 5.4 of [49].

## 7 Purified quantales

Let us recall from [1] that a commutative ring R is a purified ring if for all distinct minimal ideals P and Q of R there exists an idempotent element e such that  $e \in P$  and  $1 - e \in Q$ . A quantale A is said to be a purified quantale if for all minimal m - prime elements p and q of A there exists  $e \in B(A)$  such that  $e \leq p$  and  $\neg e \leq q$ . Then a commutative ring R is a purified ring if and only in Id(R) is a purified quantale.

The results of this section generalize to purified quantales some theorems proved in [1] for purified rings.

Lemma 7.1 Any purified quantale A is an mp - quantale.

**Proof.** Let p,q be two distinct elements of Min(A). Thus there exists a complemented element e of A such that  $e \leq p$  and  $\neg e \leq q$ , hence  $p \vee q = 1$ . In accordance to Theorem 6.3,(2) it follows that A is a mp - quantale.

Let A be a coherent quantale and  $a \in A$ . Let us consider the quantale  $[a)_A$  (defined in Section 2) and the unital quantale morphism  $u_a^A: A \to [a)_A$ , defined by  $u_a^A(x) = x \vee a$ , for all  $x \in A$ . It is easy to see that the negation operation  $\neg^a$  of  $[a)_A$  is defined by  $\neg^a(x) = \neg x \vee a$ , for any  $x \in [a)_A$ . By Lemma 14 of [15], one can consider the Boolean morphism  $B(u_a^A): B(A) \to B([a)_A)$  defined by  $B(u_a^A)(e) = u_a^A(e)$ , for all  $e \in B(A)$ . According to [15], an element  $a \in A$  has the lifting property (LP) if the Boolean morphism  $B(u_a^A): B(A) \to B([a)_A)$  is surjective. The quantale A has LP if each element of A has LP.

Lemma 7.2  $\rho(0)$  has LP.

**Proof.** Let us denote  $a=\rho(0)$ . We have to prove that  $B(u_a^A)$  is surjective. Let x be a complemented element of  $[a)_A$ , so there exists  $y\in [a)_A$  such that  $x\vee y=1$  and  $xy\vee a=x\cdot_a y=a$ . Since 1 is compact, there exist  $c,d\in K(A)$  such that  $c\leq x,d\leq y$  and  $c\vee d=1$ . On the other hand,  $xy\leq a=\rho(0)$  implies  $cd\leq \rho(0)$ , so by Lemma 2.4,(2), there exists a positive integer n such that  $c^nd^n=0$ . In accordance with Lemma 2.1,(2) we have  $c^n\vee d^n=1$ . By using Lemma 3.8(6), it follows that  $c^n,d^n\in B(A)$  and  $c^n=\neg d^n$ . One remarks that  $u_a^A(c^n)\leq x$  and  $u_a^A(d^n)\leq y$ . The second inequality implies  $x=\neg^a y\leq \neg^a(u_a^A(d^n))=u_a^A(\neg d^n)=u_a^A(c^n)$ . We have proven that  $u_a^A(c^n)=x$  and  $u_a^A(c^n)=x$  and  $u_a^A(c^n)=x$  and  $u_a^A(c^n)=x$ .

**Theorem 7.3** A coherent quantale A is a purified quantale if and only if  $[\rho(0)]_A$  is a purified quantale.

**Proof.** Let us denote  $a=\rho(0)$ . Assume that A is a purified quantale and consider  $p,q\in Min([a)_A)$  such that  $p\neq q$ . Observing that  $Min(A)=Min([a)_A)$ , it results that there exists  $e\in B(A)$  such that  $e\leq p$  and  $\neg e\leq q$ . It follows that  $u_a^A(e)\in B([a)_A),\ u_a^A(e)\leq u_a^A(p)=p$  and  $\neg^a u_a^A(e)=u_a^A(\neg e)\leq u_a^A(q)=q$ , therefore  $[a)_A$  is a purified quantale.

Conversely, suppose that  $[a)_A$  is a purified quantale and consider two distinct minimal m - prime elements p,q of A. Then  $p,q \in Min([a)_A)$ , hence, by taking into account the hypothesis that  $[a)_A$  is a purified quantale there exists  $f \in B([a)_A)$  such that  $f \leq p$  and  $\neg^a f \leq q$ . In accordance with Lemma 7.2,  $a = \rho(0)$  has the lifting property, so there exists  $e \in B([a)_A)$  such that  $a \vee e = u_a^A(e) = f$ . Sine  $B(u_a^A)$  is a Boolean morphism, we have  $a \vee \neg e = u_a^A(\neg e) = \neg^a u_a^A(e) = \neg^a f$ , so  $\neg^a e \leq q$ . Thus A is a purified quantale.

**Lemma 7.4** Let U be a subset of  $Spec_F(A)$ . Then U is a clopen subset of  $Spec_F(A)$  if and only if U = V(e), for some  $e \in B(A)$ .

**Proof.** Assume that U is a clopen subset of  $Spec_F(A)$ . Then  $V = Spec_F(A) - U$  is a clopen subset of  $Spec_F(A)$  and  $U \cup V = Spec_F(A)$ ,  $U \cap V = \emptyset$ . Since  $Max_F(A)$  is compact, there exist  $e, f \in K(A)$  such that  $V(e) \subseteq U$ ,  $V(f) \subseteq V$  and  $V(ef) = V(e) \cup V(f) = Spec_F(A)$ , hence  $ef \leq \rho(0)$ . We remark that  $V(e \vee f) = V(e) \cap V(f) \subseteq U \cap V = \emptyset$ , so  $e \vee f = 1$ . From  $ef \leq \rho(0)$  we have  $e^n f^n = 0$ , for some positive integer n. By Lemma 2.1,(2) we have  $e^n \vee f^n = 1$ , therefore by using Lemma 3.8,(6), we obtain  $e^n, f^n \in B(A)$ . Now it is easy to prove that  $U = V(e^n)$  and  $V = V(f^n)$ . The converse implication is obvious.

Recall from [30], p.69 that a topological space X is said to be

ullet totally disconnected, if the only connected subsets of X are single points;

• totally separated, if for all distinct points  $x, y \in X$ , there exists a clopen subset of X containing x but not y.

**Theorem 7.5** For a semiprime coherent quantale A the following are equivalent:

- (1) A is a purified quantale;
- (2) A is an mp quantale and  $Min_F(A)$  is totally separated;
- (3) A is an mp quantale and  $Min_F(A)$  is totally disconnected;
- (4) A is an mp quantale and  $Min_F(A)$  is a Boolean space;
- (5) The family  $(V(e) \cap Min(A))_{e \in B(A)}$  is a basis of open sets for  $Min_F(A)$ .
- (6) Any minimal m prime element p of A is regular;
- (7) Min(A) = Sp(A);
- (8) A is an mp quantale and any pure element of A is regular.

**Proof.** Recall from Corollary 8.6 of [24] that  $Min_F(A)$  is a compact T1 - space. According to Theorem 4.2 of [30], the properties (2),(3) and 4 are equivalent.

- $(1)\Rightarrow (2)$  In accordance with Lemma 7.1, A is an mp- quantale. Then for a distinct points p,q of  $Min_F(A)$  there exists  $e\in B(A)$  such that  $e\leq p$ ,  $\neg e\leq q$ , so  $p\in V(e)$ ,  $q\in V(\neg e)$  and  $V(e)\bigcap V(\neg e)=\emptyset$ . Then  $Min_F(A)$  is totally separated.
  - $(4) \Rightarrow (5)$  We apply Lemma 7.4.
- $(5) \Rightarrow (1)$  Let p, q be two distinct points of Min(A). Then  $U = Spec(A) \{q\}$ =  $Spec(A) - \Lambda(q)$  is an open subset of  $Spec_F(A)$  and  $p \in U$ , hence there exists  $e \in B(A)$  such that  $p \in V(e) \cap Min_F(A) \subseteq U \cap Min_F(A)$ . It follows that  $e \leq p$  and  $\neg e \leq q$ , so A is a purified quantale.
- $(1) \Rightarrow (6)$  Assume that  $p \in Min(A)$  and  $c \in K(A)$  such that  $c \leq p$ . The property (1) is equivalent to (4), hence  $Min_F(A)$  is a Boolean space. Since  $p \in V(c)$  and  $V(c) \cap Min(A)$  is an open subset of  $Min_F(A)$  one can find an element  $e \in B(A)$  such that  $p \in V(e) \cap Min(A) \subseteq V(c) \cap Min(A)$ . Therefore  $c \leq \bigwedge (V(c) \cap Min(A) \leq \bigwedge (V(e) \cap Min(A) \leq \rho(e))$ ,
- so  $c^n \leq e$  for some positive integer n. According to Theorem 6.8,(4) we have  $(c^n)^{\perp} = c^{\perp}$ , hence  $c^n \leq e$  implies  $\neg e = e^{\perp} \leq (c^n)^{\perp} = c^{\perp}$ . It follows that  $c \leq (c^{\perp})^{\perp} \leq \neg \neg e = e$ . We have proven that  $p \leq \bigvee \{e \in B(A) | e \leq p\}$ , hence  $p = \bigvee \{e \in B(A) | e \leq p\}$ . We conclude that the minimal m prime element p is regular.
- $(6) \Rightarrow (1)$  Let p,q be two distinct minimal m prime elements of A. By taking into account the hypothesis, we have  $p = \bigvee \{e \in B(A) | e \leq p\}$  and  $q = \bigvee \{e \in B(A) | e \leq q\}$ . Since p,q are distinct we can find an element  $e \in B(A)$  such that  $e \leq p$  and  $e \not\leq q$ . Then  $\neg e \leq q$ , hence A is a purified quantale.
- $(6) \Rightarrow (7)$  We shall prove that  $Min(A) \subseteq Sp(A)$ . Consider an element  $p \in Min(A)$ , so by the condition (6), p is regular. In order to show that p is

max - regular, let us consider  $e \in B(A)$  such that  $e \not\leq p$ . Thus  $\neg e \leq p$ , hence  $1 = e \vee \neg e \leq p \vee e$ , so  $p \vee e = 1$ . It follows that  $p \in Sp(A)$ .

Now we shall prove that  $Sp(A) \subseteq Min(A)$ . Let us consider  $p \in Sp(A)$ and take an element  $q \in Spec(A)$  such that  $s_A(q) = p$  (because the function  $s_A: Spec(A) \to Sp(A)$  is surjective). If r is a minimal m - prime element of A such that  $r \leq q$  then we have  $s_A(r) \leq s_A(q) = p$ , so  $s_A(r) = p$  (because p and  $s_A(r)$  are max - regular). The minimal m - prime element r is regular, hence  $s_A(r) = r$ . Thus r = p, hence we get  $p \in Min(A)$ .

 $(7) \Rightarrow (1)$  Obviously.

 $(5) \Rightarrow (8)$  We have proven that (5) is equivalent to (2), so A is a semiprime mp - quantale (i.e. a PF - quantale). Let a be a pure element of A. By Theorem 6.6, a is a minimal m - prime element, hence V(a) is a closed subset of  $Spec_F(A)$ . Then U = Spec(A) - V(a) is an open subset of  $Spec_F(A)$ . Applying the hypothesis (5) we find a family  $(e_i)_{i \in I}$  of complement elements of A such

that  $U \cap Min(A) = \bigcup_{i \in I} (V(e_i) \cap Min(A))$ . We shall prove that  $U = \bigcup_{i \in I} V(e_i)$ . Let  $p \in \bigcup_{i \in I} V(e_i)$ , so  $e_j \leq p$  for some  $j \in I$ . Assume by absurdum that  $a \leq p$ , so  $a \notin U \cap Min(A)$ . By taking into account the equality  $U \cap Min(A) = \bigcup_{i \in I} (V(e_i) \cap Min(A))$ , it follows that  $a \notin V(e_j)$ , i.e.  $e_j \not\leq a$ . Therefore  $\neg e_j \leq a \leq p$ , so we obtain  $1 = e_j \vee \neg e_j \leq p$ , contradicting that  $p \in Spec(A)$ . It follows that  $a \not\leq p$ , so  $p \in U$ .

In order to prove the converse inclusion  $U \subseteq \bigcup_{i \in I} V(e_i)$ , let us assume that

 $p \notin \bigcup_{i \in I} V(e_i)$ , hence for all  $i \in I$  we have  $e_i \not \leq p$ . Consider a minimal m prime element n such that  $n \leq p$ , hence  $e_i \not\leq n$  for all  $i \in I$ . This implies  $n \notin V(e_i) \cap Min(A)$  for all  $i \in I$ , hence  $n \notin U \cap Min(A)$ . Since  $n \notin U$  implies  $a \leq n$ , we obtain  $a \leq n \leq p$ , hence  $p \in U$ . We have proven that  $U = \bigcup_{i \in I} V(e_i)$ ,

hence the following equalities hold:

$$V(a) = Spec(A) - \bigcup_{i \in I} V(e_i) = \bigcap_{i \in I} D(e_i) = \bigcap_{i \in I} V(\neg e_i) = V(\bigvee_{i \in I} \neg e_i).$$
 Let us consider the regular element  $b = \bigvee_{i \in I} \neg e_i$ . Then  $\rho(a) = \bigwedge V(a) = \bigcup_{i \in I} V(a$ 

 $\bigwedge V(b) = \rho(b)$ . By Lemma 5.1, the regular element b is pure. In accordance with Proposition 4.5, for the pure elements a and b we have  $a = \rho(a) = \rho(b) = b$ . Therefore a is regular element.

 $(8) \Rightarrow (6)$  According to Theorem 6.6, any minimal m - prime element p is pure, so p is regular.

Corollary 7.6 Any hyperarchimedean coherent quantale A is purified.

**Proof.** In accordance with the characterization theorems of hyperarchimedean quantales given in [15],[24] it follows that Max(A) = Spec(A) = Min(A) and  $Spec_F(A)$  is a Boolean space. Thus A is an mp- quantale (by Theorem 6.3,(6)) and  $Min_F(A)$  is a Boolean space. By applying Theorem 7.5,(4) we conclude that A is purified.

Corollary 7.7 Let A be a coherent PF - quantale. If  $Min_Z(A)$  is compact then A is purified.

**Proof.** Since  $Min_Z(A)$  is compact it follows that  $Min_Z(A) = Min_F(A)$  (cf.Theorem 6.2,(1)). We know that  $Min_Z(A)$  is a zero - dimensional Hausdorff space and  $Min_F(A)$  is a compact space, hence in our case,  $Min_F(A)$  is a Boolean space. By applying Theorem 7.5,(4) we conclude that A is a purified quantale.

## 8 PP - quantales

In this section we shall define the PP - quantales as an abstraction of PP - rings (= Baer rings) [1],[31],[43], Stone lattices [7],[43], Stone MV - algebras [9], Baer BL - algebras [32], Stone residuated lattices [36],[41],etc.

Let A be an algebraic quantale such that  $1 \in K(A)$ . Then A will be called a PP - quantale if for any  $c \in K(A)$  we have  $c^{\perp} \in B(A)$ . A PP - frame is a PP - quantale which is a frame.

Let L be a bounded distributive lattice. Following [7],[43], L is said to be a *Stone lattice* if for any  $x \in L$  there exists  $e \in B(L)$  such that Ann(x) is the ideal [e) of L generated by the point set  $\{x\}$ . Then L is a Stone lattice if and only if Id(L) is a PP - frame.

Let R be a commutative ring. Then R is said to be a PP - ring if the annihilator of any element of R is generated by an idempotent element. Then R is a PP - ring if and only if Id(R) is a PP - quantale.

Throughout this section we shall assume that A is coherent quantale.

**Lemma 8.1** Any PP - quantale A is semiprime.

**Proof.** Firstly we remark that for any  $a \in A$  such that  $a \leq a^{\perp}$  and  $a^{\perp} \in B(A)$  we have  $a = a \wedge a^{\perp} = aa^{\perp} = 0$ .

In order to prove that A is semiprime let c be a compact element such that  $c^n = 0$  for some positive integer n. Thus  $c^{n-1} \leq (c^{n-1})^{\perp}$  and  $(c^{n-1})^{\perp} \in B(A)$ , hence  $c^{n-1} = 0$ . By using many times this argument one gets c = 0. By using Lemma 2.4 it follows that A is semiprime.

**Lemma 8.2** If A is semiprime then the following hold:

- (1) If  $e \in B(A)$  then  $\rho(e) = e$ ;
- (2) If  $a \in A$  and  $\rho(a) \in B(A)$  then  $a = \rho(a)$ .
- **Proof.** (1) Assume that  $c \in K(A)$  and  $c^2 \le e$ . By Lemma 3.8,(1) we have  $c^{\perp} \in B(A)$ , hence  $\lambda_A(ce^{\perp}) = \lambda_A(c) \wedge \lambda_A(e^{\perp}) = \lambda_A(c^2) \wedge \lambda_A(e^{\perp}) = \lambda_A(c^2e^{\perp}) = \lambda_A(0) = 0$ . Since A is semiprime one gets  $ce^{\perp} = 0$ , so  $c \le e^{\perp \perp} = e$ . By using the same argument, one can prove by induction that for any  $c \in K(A)$  and for any positive integer  $n, c^n \le e$ . In accordance with Lemma 2.4 one obtains  $\rho(e) \le e$ , so  $\rho(e) = e$ .
- (2) Assume that  $a \in A$  and  $\rho(a) \in B(A)$ . By Lemma 3.9,  $c = \rho(a)$  is a compact element of A, hence there exists a positive integer n such that  $c^n \leq a$ . Since  $c \in B(A)$  we have  $c^n = c$ , hence  $c \leq a$ . Thus  $\rho(a) \leq a$ , so we obtain  $\rho(a) = a$ .

The following theorem is a generalization of a result proved by Simmons for the case of PP - rings (see [43]).

**Theorem 8.3** For a quantale A let us consider the following properties:

- (1) A is a PP quantale;
- (2) R(A) is a PP frame;
- (3) The reticulation L(A) is a Stone lattice.

Then  $(1) \Rightarrow (3)$  and  $(1) \Leftrightarrow (2)$  hold. If A is semiprime then the implication  $(3) \Rightarrow (1)$  is valid.

#### Proof.

- (1)  $\Rightarrow$  (3) Assume that A is PP quantale. By Lemma 8.1, A is semiprime and, by Corollary 3.11, the function  $\lambda_A|_{B(A)}:B(A)\to B(L(A))$  is a Boolean isomorphism. Let  $x\in L(A)$ , hence there exists  $c\in K(A)$  such that  $x=\lambda_A(c)$ . Since A is a PP quantale we have  $c^\perp\in B(A)$ , hence  $\lambda_A(c^\perp)\in B(L(A))$ . We shall prove that  $Ann(\lambda_A(c))=(\lambda_A(c^\perp)]$ . By Proposition 3.13 we have  $Ann(\lambda_A(c))=Ann(c^*)=(c^\perp)^*$ . Let us consider an element  $y\in Ann(\lambda_A(c))=(c^\perp)^*$ , so there exists  $d\in K(A)$  such that  $d\leq c^\perp$  and  $y=\lambda_A(d)$ . Thus  $y=\lambda_A(d)\leq \lambda_A(c^\perp)$ , so  $y\in (\lambda_A(c^\perp)]$ . We have proven that  $Ann(\lambda_A(c))\subseteq (\lambda_A(c^\perp)]$ . On the other hand, from  $\lambda_A(c^\perp)\wedge\lambda_A(c)=\lambda_A(cc^\perp)=\lambda_A(0)=0$  we obtain  $\lambda_A(c^\perp)\in Ann(\lambda_A(c))$ , hence  $(\lambda_A(c^\perp)]\subseteq Ann(\lambda_A(c))$ .
- $(1) \Leftrightarrow (2)$  In accordance with Proposition 3.7, the frames R(A) and Id(L(A)) are isomorphic. Then L(A) is a Stone lattice iff Id(L(A)) is a PP frame iff R(A) is a PP frame.
- $(3) \Rightarrow (1)$  Assume now that A is semiprime and L(A) is a Stone lattice. Let c be a compact element of A, so there exists  $f \in B(L(A))$  such that  $Ann(\lambda_A(c)) = (f]$ . By Corollary 3.11 there exists  $e \in B(A)$  such that  $f = \lambda_A(e)$ . According to Proposition 3.13 we have  $(c^{\perp})^* = Ann(\lambda_A(c)) = (\lambda_A(e)] = e^*$ , hence, by

using Proposition 3.3,(7) one gets  $\rho(c^{\perp}) = ((c^{\perp})^*)_* = (e^*)_* = \rho(e)$ . By Lemma 8.2,(1) we have  $\rho(0) = 0$ . Since  $\rho(c^{\perp}) = e \in B(A)$ , by applying Lemma 8.2,(2) one obtains  $c^{\perp} = e$ , hence  $c^{\perp} \in B(A)$ , so A is a PP - quantale.

By using the previous result one can obtain characterization theorems for PP - quantales by transferring from lattices to rings the properties that describe the Stone lattices.

**Lemma 8.4** [17] Let L be a bounded distributive lattice. Then L is a Stone lattice if and only if L is conormal and  $Min_{Id,Z}(L)$  is compact.

**Theorem 8.5** For a semiprime quantale A the following properties are equivalent:

- (1) A is a PP quantale;
- (2) The reticulation L(A) is a Stone lattice;
- (3) L(A) is a conormal lattice and  $Min_{Id,Z}(L(A))$  is compact;
- (4) A is a PF quantale and  $Min_Z(A)$  is compact;
- (5) A is a PF quantale and  $Min_Z(A)$  is a Boolean space;
- (6) A is an mp quantale and  $Min_Z(A)$  is a Boolean space;
- (7)  $Spec_F(A)$  is a normal space and  $Min_Z(A)$  is a Boolean space;

#### Proof.

- $(1) \Leftrightarrow (2)$  By Theorem 8.3.
- $(2) \Leftrightarrow (3)$  By Lemma 8.4.
- $(3) \Leftrightarrow (4)$  Since A is semiprime, the reticulation L(A) is a conormal lattice if and only if A is a PF quantale (cf. Lemma 6.1 and Theorem 6.5). On the other hand,  $Min_Z(A)$  and  $Min_{Id,Z}(L(A))$  are homeomorpic topological spaces. Then the equivalence of (3) and (4) follows.
  - $(4) \Leftrightarrow (5)$  By Theorem 6.2.
  - $(5) \Leftrightarrow (6)$  By Theorem 6.5.
  - $(6) \Leftrightarrow (7)$  By Theorem 6.3.

**Lemma 8.6** [18] Let L be a conormal lattice. Then L is a Stone lattice if and only if the inclusion  $Min_Z(L) \subseteq Spec_Z(A)$  has a continuous retraction.

**Theorem 8.7** For a PF - quantale A the following properties are equivalent:

- (1) A is a PP quantale;
- (2) The reticulation L(A) is a Stone lattice;

- (3) The inclusion  $Min_{Id,Z}(L(A)) \subseteq Spec_{Id,Z}(L(A))$  has a continuous retraction;
- (4) The inclusion  $Min_Z(A) \subseteq Spec_Z(A)$  has a continuous retraction;
- (5) For any  $c \in K(A)$ ,  $Min(A) \cap D(c)$  is an open subset of  $Spec_Z(Vir(A))$ ;
- (6)  $Min_Z(A)$  is a compact space.

#### Proof.

- $(1) \Leftrightarrow (2)$  By Theorem 8.3.
- $(2) \Leftrightarrow (3)$  By Lemma 8.6.
- (3)  $\Leftrightarrow$  (4) This equivalence follows because  $Spec_Z(A)$  (resp.  $Min_Z(A)$ ) is homeomorphic to  $Spec_{Id,Z}(L(A))$  (resp.  $Min_{Id,Z}(L(A))$ ).
- $(1) \Leftrightarrow (5)$  Let c be a compact element of A, hence  $c^{\perp} \in B(A)$ . By Theorems 6.12 and 6.16 we have Min(A) = Spec(Vir(A)). According to Corolarry 3.16, for each  $p \in Min(A)$  the following equivalence holds:  $c \leq p$  iff  $c^{\perp} \nleq p$ . Therefore one obtain the equality  $Min(A) \cap D(c) = \{p \in Spec(Vir(A)) | c^{\perp} \nleq p\}$ , hence  $Min(A) \cap D(c)$  is open in Spec(Vir(A)).
- $(5)\Leftrightarrow (6)$  According to Lemma 4.2,(4) one can consider the composition  $Vir\circ i$  of the following two continuous maps: the inclusion  $i:Min_Z(A)\to Spec_Z(A)$  and  $Vir:Spec_Z(A)\to Spec_Z(Vir(A))$ . By Theorems 6.12 and 6.16,  $Vir\circ i$  is a continuous bijection. The hypothesis (5) implies that  $Vir\circ i$  is an open map, so it is a homeomorphism. Since  $Spec_Z(Vir(A))$  is a compact space, it follows that  $Min_Z(A)$  is also compact.
  - $(6) \Leftrightarrow (1)$  By Theorem 8.5,(4).

Corollary 8.8 Any PP - quantale is a purified quantale.

**Proof.** Let A be a PP - quantale. By Theorem 8.5,(5),  $Min_Z(A)$  is a Boolean space, hence  $Min_Z(A) = Min_F(A)$  (cf. Theorem 6.2,(1)). Then  $Min_F(A)$  is a Boolean space, therefore, according to Theorem 7.5,(4), it follows that A is a purified quantale.

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