Discrete IV d_G-Choquet integrals with respect to admissible orders

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Abstract

In this work, we introduce the notion of d_G-Choquet integral, which generalizes the discrete Choquet integral replacing, in the first place, the difference between inputs represented by closed subintervals of the unit interval $[0, 1]$ by a dissimilarity function; and we also replace the sum by more general appropriate functions. We show that particular cases of d_G-Choquet integral are both the discrete Choquet integral and the d-Choquet integral. We define interval-valued fuzzy measures and we show how they can be used with d_G-Choquet integrals to define an interval-valued discrete Choquet integral which is monotone with respect to admissible orders. We finally study the validity of this interval-valued Choquet integral by means of an illustrative example in a classification problem.

Keywords: Choquet integral; Interval-valued dissimilarity function; Interval-valued fuzzy measure; d-Choquet integral

1. Introduction

In recent years there exists a growing interest on fuzzy integrals [1, 2] for fusing information. This interest has been specially focused on discrete Choquet integrals [3], which have shown themselves very useful in a wide variety of problems in machine learning [4, 5, 6, 7, 8, 9]. Due to this usefulness, several generalizations of Choquet integrals in the framework of real numbers have been proposed in the literature, leading to notions such as Choquet-like integrals [10]; concave integrals [11]; universal integrals [12]; C_T -integrals [13]; C_F-integrals [14]; CC -integrals [15] or C_{F1F2}-integrals [16, 17]; see [18] for an overview of many of these extensions.

Besides, in some problems the use of intervals can provide a means to represent uncertainty linked to the data [19, 20]. In order to compare and fuse this type of data, different interval-valued extensions of notions such as those of aggregation function [21] or similarity functions [22] can be found in the literature.

However, the extension of the Choquet integral in the interval-valued framework is not a trivial or straightforward task. Note the the standard discrete Choquet makes use of the difference between real numbers, and the difference between intervals is not well defined, in general. On the other hand, the difference in the standard Choquet integral is used to determine how dissimilar two inputs are, so it makes sense to replace it by another function, such as a dissimilarity for instance. This, in the case of real numbers, was the approach followed in [23], which led to the notion of d-Choquet integral.

Moreover, one key property of discrete Choquet integrals in the real setting is monotonicity with respect to the usual order between real numbers, which is linear. In the interval-valued setting, on the contrary, it does not exist a "standard" linear order, although several different linear orders, called admissible orders [24],

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can be defined and used to consider monotonicity. In this sense, this problem was considered in [25], where lower and upper bounds of the involved intervals were considered separately, and also in [26, 27], but it was restricted to the case or real-valued fuzzy measures, due to the problem of appropriately defining operations for intervals. In [28] and [29] two different generalizations of the Choquet integral to the setting of fuzzy numbers were proposed. In the former, the restriction to our setting of closed subintervals in $[0, 1]$ with respect to a fuzzy measure only gives decomposable interval functions, i.e., the left (or right) endpoint of the output interval can be obtained separately in terms of the left (or right) endpoints of the input intervals, whereas our goal is to also obtain non-decomposable generalizations of the Choquet integral to intervals. In the later, the generalization was based on the representation of the Choquet integral by means of the Mobius transform which was a way to overcome the difficulty of the ordering of fuzzy numbers (or ordering intervals in the restriction to intervals), however, our goal was to propose a tool for intervals fusion/aggregation which takes into account the specific features/structure of intervals, mainly the linear order, hence our approach is based on the definiton of the discrete Choquet integrals in terms of the ordered inputs.

Taking into account these facts, our objective in this paper is to extend the notion of discrete Choquet integral to the interval-valued setting in such a way that the difference between two intervals is given by the dissimilarity between them and, moreover, monotonicity with respect to admissible orders is preserved and interval-valued fuzzy measures are used.

In order to reach this objective, we are going to define the notion of discrete d_G-Choquet integral, which generalizes the usual Choquet integral by replacing, in the standard definition of the discrete Choquet integral, the difference of inputs by a binary function d (e.g. distance, dissimilarity function etc.), the product by a binary function P and the sum by an n-ary function S, all of them with possibly different domains and ranges. We also replace the fuzzy measure by a set function m which needs not be real-valued. In this way, we show that d_G-Choquet integrals generalize both standard Choquet integrals and d-Choquet integrals. Furthermore, by an appropriate choice of the functions d, P and S, as well as of the set function m, we show that, from d_G-Choquet integrals, we can define interval-valued Choquet integrals which are monotone with respect to admissible orders. It is worth to mention that such generalization is obtained, in particular, by considering that the values of m are intervals. In other words, m can be seen, for the specific case of interval-valued Choquet integrals, as an interval-valued fuzzy measure.

For displaying the usefulness of our theoretical proposal, we discuss an illustrative example where we use the proposed interval-valued Choquet integral for combining the predictions of an ensemble of IVTURS classifiers [30], whose predictions are interval-valued. The accuracy of the proposed methodology will be compared with existing alternatives of the interval-valued Choquet integral proposed in [26].

The structure of this paper is as follows. In Section 2 we present some preliminary definitions and results which are necessary for the rest of the work. In Section 3 we introduce the new notion of discrete d_G-Choquet integral. In Section 4 we study interval-valued fuzzy measures and interval-valued dissimilarity functions that we use in Section 5 to propose our notion of discrete interval-valued Choquet integral with respect to admissible orders. Section 6 is devoted to the illustrative example and we finish with some conclusions and references.

2. Preliminaries

In this section, we recall some basic notions and terminology that are necessary for our subsequent developments.

Based on Fodor's equivalence function [31], a dissimilarity function is defined as follows.

Definition 2.1. A function $\delta : [0, 1]^2 \to [0, 1]$ is called a dissimilarity function on $[0, 1]$ if it satisfies, for all $x, y, z \in [0, 1]$, the following conditions:

1. $\delta(x, y) = \delta(y, x)$;
2. $\delta(0, 1) = 1$;
3. $\delta(x, x) = 0$;
4. if $x \leq y \leq z$, then $\delta(x, y) \leq \delta(x, z)$ and $\delta(y, z) \leq \delta(x, z)$.

We denote the set $\{1,\dots,n\}$ by $[n]$.

Definition 2.2. A function $\mu : 2^{[n]} \to [0,1]$ is called a fuzzy measure on $[n]$ if $\mu(\emptyset) = 0$, $\mu([n]) = 1$ and $\mu(A) \leq \mu(B)$ for all $A \subseteq B \subseteq [n]$.

Definition 2.3 ([32]). Let n be a positive integer. A function $F : [0,1]^n \to [0,1]$ is called an n-ary aggregation function if $F(0) = 0$, $F(1) = 1$ and F is non-decreasing in each variable. A function $F : \bigcup_{k=1}^{n} [0,1]^k \to [0,1]$ is called an extended aggregation function if its restriction to $[0,1]^k$, for any $k \in [n]$, is an aggregation function in $[0,1]^k$.

Definition 2.4 ([32]). Let A, B be non-empty sets, n be a positive integer and $F : \bigcup_{k=1}^{n} A^k \to B$ be a function. An element $e \in A$ is called an extended neutral element of F if, for any $k \in [n]$ and any $x_1,\dots,x_k \in A$ such that $x_j = e$ for some $j \in [k]$ it holds

$$F(x_1,\dots,x_k) = F(x_1,\dots,x_{j-1},x_{j+1},\dots,x_k).$$

We consider closed subintervals of the unit interval $[0,1]$, we denote:

$$L([0,1]) = \{X = [\underline{X}, \overline{X}] \mid 0 \leq \underline{X} \leq \overline{X} \leq 1\}.$$

The width of the interval $X \in L([0,1])$ will be denoted by $w(X)$, clearly $w(X) = \overline{X} - \underline{X}$. We will denote by $\precsim_L$ the partial order relation on $L([0,1])$ induced by the usual partial order on $\mathbb{R}^2$, that is:

$$[\underline{X}, \overline{X}] \precsim_L [\underline{Y}, \overline{Y}] \quad \text{if} \quad \underline{X} \leq \underline{Y} \text{ and } \overline{X} \leq \overline{Y}. \tag{1}$$

This is the order relation most widely used in the literature [33]. Regarding total orders on $L([0,1])$, we are going to consider the so-called admissible orders.

Definition 2.5 ([24]). An admissible order on $L([0,1])$ is a total order $\leq_{\mathrm{A}}$ in $L([0,1])$ such that it refines the partial order $\precsim_L$, that is, for every $X, Y \in L([0,1])$, if $X \precsim_L Y$ then $X \leq_A Y$.

We denote by $\leq_L$ any order on $L([0,1])$ (which can be partial or total) with $0_L = [0,0]$ as its minimal element and $1_L = [1,1]$ as its maximal element. To denote an admissible order on $L([0,1])$ we use the notation $\leq_{\mathrm{A}}$. An interesting feature of admissible orders is that they can be built using aggregation functions, as stated in the following proposition.

Proposition 2.6 ([24])**.** *Let $M_1, M_2 : [0,1]^2 \to [0,1]$ be two aggregation functions such that for all $X, Y \in L([0,1])$, the equalities $M_1(\underline{X}, \overline{X}) = M_1(\underline{Y}, \overline{Y})$ and $M_2(\underline{X}, \overline{X}) = M_2(\underline{Y}, \overline{Y})$ can only hold simultaneously if $X = Y$. The order $\leq_{M_1,M_2}$ on $L([0,1])$ given by*

$$X \leq_{M_1,M_2} Y \quad \textit{if} \quad \begin{cases} M_1(\underline{X}, \overline{X}) < M_1(\underline{Y}, \overline{Y}) \textit{ or} \\ M_1(\underline{X}, \overline{X}) = M_1(\underline{Y}, \overline{Y}) \textit{ and } M_2(\underline{X}, \overline{X}) \leq M_2(\underline{Y}, \overline{Y}) \end{cases}$$

is an admissible order on $L([0,1])$.

Example 2.7. (i) Xu and Yager's order (see [34]):

$$[\underline{X}, \overline{X}] \leq_{\mathrm{XY}} [\underline{Y}, \overline{Y}] \;\Leftrightarrow\; \begin{cases} \underline{X} + \overline{X} < \underline{Y} + \overline{Y} \text{ or} \\ \underline{X} + \overline{X} = \underline{Y} + \overline{Y} \text{ and } \overline{X} - \underline{X} \leq \overline{Y} - \underline{Y}, \end{cases}$$

is an example of admissible order with $M_1(x,y) = \frac{x+y}{2}$ and $M_2(x,y) = y$.

(ii) The lexicographical orders $\leq_{Lex1}$ and $\leq_{Lex2}$:

$$[\underline{X}, \overline{X}] \leq_{Lex_1} [\underline{Y}, \overline{Y}] \;\Leftrightarrow\; \begin{cases} \underline{X} < \underline{Y} \text{ or} \\ \underline{X} = \underline{Y} \text{ and } \overline{X} \leq \overline{Y} \end{cases}$$

and

$$[\underline{X}, \overline{X}] \leq_{Lex_2} [\underline{Y}, \overline{Y}] \;\Leftrightarrow\; \begin{cases} \overline{X} < \overline{Y} \text{ or} \\ \overline{X} = \overline{Y} \text{ and } \underline{X} \leq \underline{Y}, \end{cases}$$

are also examples of admissible orders with $M_1(x, y) = x, M_2(x, y) = y$ for $\leq_{Lex1}$ and $M_1(x, y) = y, M_2(x, y) = x$ for $\leq_{Lex2}$.

(iii) More generally, if, for $\alpha \in [0, 1]$, we define the aggregation function

$$K_\alpha(x, y) = \alpha x + (1 - \alpha)y$$

then, for $\alpha, \beta \in [0, 1]$ with $\alpha \neq \beta$, we can obtain an admissible order $\leq_{\alpha,\beta}$ just taking $M_1(x, y) = K_\alpha(x, y)$ and $M_2(x, y) = K_\beta(x, y)$. See [24] for more details.

Definition 2.8. Let $n \geq 2$. An (n-dimensional) interval-valued (IV) aggregation function on $L([0, 1])$ with respect to $\leq_L$ is a mapping $M : (L([0, 1]))^n \to L([0, 1])$ which verifies:

(i) $M(0_L, \cdots, 0_L) = 0_L$.
(ii) $M(1_L, \cdots, 1_L) = 1_L$.
(iii) M is a non-decreasing function in each variable with respect to $\leq_L$.

We say that $M : (L([0, 1]))^n \to L([0, 1])$ is a decomposable n-dimensional IV aggregation function associated with M_L and M_U, if there exist n-dimensional aggregation functions $M_L, M_U : [0, 1]^n \to [0, 1]$ such that $M_L \leq M_U$ and

$$M(X_1, \ldots, X_n) = \left[M_L\left(\underline{X_1}, \ldots, \underline{X_n}\right), M_U\left(\overline{X_1}, \ldots, \overline{X_n}\right)\right] \tag{2}$$

for all $X_1, \ldots, X_n \in L([0, 1])$.

Definition 2.9. Let $\leq_L$ be an order relation on $L([0, 1])$. A function $N \colon L([0, 1]) \to L([0, 1])$ is an interval-valued negation function (IV negation) if it is a non-increasing function with respect to the order $\leq_L$ such that $N(0_L) = 1_L$ and $N(1_L) = 0_L$.

Definition 2.10. Let $\leq_L$ be an order on $L([0, 1])$. An interval-valued (IV) implication function in $L([0, 1])$ with respect to $\leq_L$ is a function $I \colon (L([0, 1]))^2 \to L([0, 1])$ which verifies the following properties:

1. I is a non-increasing function in the first component and an non-decreasing function in the second component with respect to the order $\leq_L$.
2. $I(0_L, 0_L) = I(0_L, 1_L) = I(1_L, 1_L) = 1_L$.
3. $I(1_L, 0_L) = 0_L$.

3. Discrete d_G-Choquet integrals

The discrete Choquet integral on $[0, 1]$ with respect to μ is defined as a mapping $C_\mu : [0, 1]^n \to [0, 1]$ such that

$$C_\mu(x_1, \ldots, x_n) = \sum_{i=1}^{n} (x_{\sigma(i)} - x_{\sigma(i-1)})\mu\left(A_{\sigma(i)}\right) \tag{3}$$

where $\mu : 2^{[n]} \to [0, 1]$ is a fuzzy measure on $[n]$, σ is a permutation on $[n]$ with $x_{\sigma(1)} \leq \ldots \leq x_{\sigma(n)}$, with the convention $x_{\sigma(0)} = 0$ and $A_{\sigma(i)} := \{\sigma(i), \ldots, \sigma(n)\}$.

In order to generalize the discrete Choquet integral to various different settings, we replace the difference $x_{\sigma(i)} - x_{\sigma(i-1)}$ by a binary function d (e.g. distance, dissimilarity function etc.), product by a binary function P, sum by an n-ary function S and fuzzy measure by a set function m, all of them with possibly different domains and ranges.

Definition 3.1. Let n be a positive integer. Let

- $S : \bigcup\limits_{i=1}^{n} D^i \to E$ be a function with $S(d) = d$ for all $d \in D$, where D, E are sets such that $D \subseteq E$ and there exists an extended neutral element $0_D \in D$ of the function S;
- $P : B \times C \to D$ be a function, where B, C are sets such that there exists $0_B \in B$ satisfying, for all $c \in C$, $P(0_B, c) = 0_D$;
- $d : A^2 \to B$ be a function such that $d(a, a) = 0_B$ for all $a \in A$ where A is a set with a total order $\leq$ and the minimum element 0_A;
- $m : 2^{[n]} \to C$ be a set function.

Then an n-ary discrete d_G-Choquet integral on A with respect to S, P, m, d is defined as a mapping $C_{S,P,m,d} : A^n \to E$ such that

$$C_{S,P,m,d}(x_1, \ldots, x_n) = S_{i=1}^{n}\Big(P\Big(d\big(x_{\sigma(i)}, x_{\sigma(i-1)}\big), m\left(A_{\sigma(i)}\right)\Big)\Big) \tag{4}$$

for all $x_1, \ldots, x_n \in A$, where σ is a permutation on $[n]$ with $x_{\sigma(1)} \leq \ldots \leq x_{\sigma(n)}$, with the convention $x_{\sigma(0)} = 0_A$ and $A_{\sigma(i)} := \{\sigma(i), \ldots, \sigma(n)\}$.

Remark 3.2. (i) The existence of elements 0_B and 0_D with the described property and the property $d(a, a) = 0_B$ for all $a \in A$ ensures that if there exist several possible permutations such that $x_{\sigma(1)} \leq \ldots \leq x_{\sigma(n)}$, the result for any of them when applying (4) is the same. Hence, $C_{S,P,m,d}$ is well-defined.

(ii) Clearly, taking $A = B = C = D = E = [0, 1]$, the sum as S, the product as P, a fuzzy measure μ as m and $d(x, y) = |x - y|$ for all $x, y \in [0, 1]$, we obtain the 'standard' discrete Choquet integral with respect to μ as defined in Equation (3).

(iii) In [23] we have introduced the so-called discrete d-Choquet integral which is a particular case of $C_{S,P,m,d}$ such that:

- $A = B = C = D = [0, 1]$ and $E = [0, n]$;
- as a set function m we considered a fuzzy measure $\mu : 2^{[n]} \to [0, 1]$;
- as a binary function d we considered a restricted dissimilarity function $\delta : [0, 1]^2 \to [0, 1]$;
- as an n-ary function $S : D^n \to E$ we considered the sum;
- as a binary function $P : B \times C \to D$ we considered the product.

Hence, we have studied the function $C_{\mu,\delta} : [0, 1]^n \to [0, n]$ defined as:

$$C_{\mu,\delta}(x_1, \ldots, x_n) = \sum_{i=1}^{n} \delta(x_{\sigma(i)}, x_{\sigma(i-1)})\mu\left(A_{\sigma(i)}\right) \tag{5}$$

where σ is a permutation on $[n]$ with $x_{\sigma(1)} \leq \ldots \leq x_{\sigma(n)}$, with the convention $x_{\sigma(0)} = 0$ and $A_{\sigma(i)} := \{\sigma(i), \ldots, \sigma(n)\}$.

As a next step, we are going to generalize the discrete Choquet integral to intervals, so we obtain particular cases of discrete d_G-Choquet integrals considering the functions:

- as the function d we use an interval-valued dissimilarity function $\delta : (L([0, 1]))^2 \to L([0, 1])$;
- as the set function m we use
  - a fuzzy measure $\mu : 2^{[n]} \to [0, 1]$, or

  - an interval-valued fuzzy measure $m : 2^{[n]} \to L([0,1])$;

- as the function P we use
  - $P : L([0,1]) \times [0,1] \to L([0,1])$ defined by $P([a,b],c) = [ac, bc]$ in the case of fuzzy measure, and
  - $P : (L([0,1]))^2 \to L([0,1])$ defined by $P([a,b],[c,d]) = [ac, bd]$ in the case of interval-valued fuzzy measure;

- as the function S we use
  - $S : \bigcup_{i=1}^{n} (L([0,1]))^i \to L([0,n])$ defined by $P([a_1,b_1],\dots,[a_n,b_n]) = [a_1+\dots+a_n, b_1+\dots+b_n]$, or
  - $\bigoplus : \bigcup_{i=1}^{n} (L([0,1]))^i \to L([0,n])$ defined as a new operation on intervals based on K_α operators introduced in Theorem 5.11.

Recall that, according to Definition 3.1, the obtained Choquet integrals have to be with respect to a total (not only partial) order. Hence, also the considered IV dissimilarity functions, IV fuzzy measures as well as functions P and S have to be defined with respect to the total order.

4. Interval-valued fuzzy measures and Interval-valued dissimilarity functions

4.1. Interval-valued fuzzy measures

Definition 4.1. Let n be a positive integer and $\leq_{\mathrm{A}}$ be an admissible order on $L([0,1])$. A function $m : 2^{[n]} \to L([0,1])$ is called an interval-valued fuzzy measure on $[n]$ w.r.t. $\leq_{\mathrm{A}}$ if $m(\emptyset) = 0_L$, $m([n]) = 1_L$ and $m(A) \leq_{\mathrm{A}} m(B)$ for all $A \subseteq B \subseteq [n]$.

The IV fuzzy measure $m : 2^{[n]} \to L([0,1])$ is a generalization of fuzzy measure $\mu : 2^{[n]} \to [0,1]$ as defined in Definition 2.2. According to the following proposition, the fuzzy measure is a special case of the IV fuzzy measure. Recall that an interval $[x,x]$ is called degenerate interval and we can write x instead of $[x,x]$.

Proposition 4.2. *Let n be a positive integer, $\leq_{\mathrm{A}}$ be an admissible order on $L([0,1])$ and $m : 2^{[n]} \to L([0,1])$ be an interval-valued fuzzy measure on $[n]$ w.r.t. $\leq_{\mathrm{A}}$. If, for all $A \subseteq [n]$, it holds $m(A) = [x,x]$ for some $x \in [0,1]$, then $\mu : 2^{[n]} \to [0,1]$ given by $\mu(A) = x$ if and only if $m(A) = [x,x]$, is a fuzzy measure on $[n]$.*

Proof. It is immediate that $\mu(\emptyset) = 0$ and $\mu([n]) = 1$. Since any admissible order satisfies $[a,b] \leq_{\mathrm{A}} [c,d]$ whenever $a \leq c$ and $b \leq d$, the monotonicity of μ follows from the monotonicity of m. □

Theorem 4.3. *Let n be a positive integer and $\leq_{\mathrm{A}}$ be an admissible order on $L([0,1])$. Then $m : 2^{[n]} \to L([0,1])$ is an IV fuzzy measure on $[n]$ w.r.t. $\leq_{\mathrm{A}}$ if and only if there exists an IV aggregation function $M_m : L([0,1])^n \to L([0,1])$ w.r.t. $\leq_{\mathrm{A}}$ such that for every $A \in 2^{[n]}$*

$$m(A) = M_m(1_A)$$

where

$$1_A = (X_1,\dots,X_n) \qquad \textit{with} \qquad X_i = \begin{cases} 1_L, & \textit{if } i \in A \\ 0_L, & \textit{otherwise.} \end{cases}$$

Example 4.4. (i) Let M_m be the IV aggregation function defined by $M_m(X_1,\dots,X_n) = [1/n \sum_{i=1}^{n} \underline{X_i}, 1/n \sum_{i=1}^{n} \overline{X_i}]$. Then the corresponding IV fuzzy measure is given by:

$$m(A) = \left[\frac{|A|}{n}, \frac{|A|}{n}\right].$$

(ii) For $M_m(X_1, \dots, X_n) = \left[\sqrt[n]{\prod_{i=1}^n \underline{X_i}}, \sqrt[n]{\prod_{i=1}^n \overline{X_i}} \right]$ we have

$$m(A) = \begin{cases} 1_L, & \text{if } A = [n], \\ 0_L, & \text{otherwise.} \end{cases}$$

Note that in (i) and (ii) we obtained fuzzy measures on $[n]$, i.e. all the outputs are degenerate intervals.
(iii) For $M_m(X_1, \dots, X_n) = \left[\sqrt[n]{\prod_{i=1}^n \underline{X_i}}, 1/n \sum_{i=1}^n \overline{X_i} \right]$ we have

$$m(A) = \begin{cases} 1_L, & \text{if } A = [n], \\ \left[0, \frac{|A|}{n}\right], & \text{otherwise.} \end{cases}$$

(iv) If $M_m(X_1, \dots, X_n) = \left[\min\{\underline{X_1}, \dots, \underline{X_n}\}, \max\{\overline{X_1}, \dots, \overline{X_n}\}\right]$ we have

$$m(A) = \begin{cases} 0_L, & \text{if } A = \emptyset, \\ 1_L, & \text{if } A = [n], \\ [0,1], & \text{otherwise.} \end{cases}$$

A construction of IV fuzzy measure in terms of two fuzzy measures is given next.

Proposition 4.5. *Let n be a positive integer and $\mu_L, \mu_U : 2^{[n]} \to [0,1]$ be fuzzy measures such that $\mu_L \leq \mu_U$. Then $m : 2^{[n]} \to L([0,1])$ given by*

$$m(A) = [\mu_L(A), \mu_U(A)]$$

for all $A \subseteq [n]$, is an IV fuzzy measure on $[n]$ w.r.t. any admissible order $\leq_{\mathrm{A}}$.

Proof. Clearly, m is well-defined and $m(\emptyset) = 0_L$, $m([n]) = 1_L$. From the monotonicity of μ_L, μ_U and the fact that each admissible order extends standard partial order of intervals we have the monotonicity of m. □

We say that $m : 2^{[n]} \to L([0,1])$ introduced in Proposition 4.5 is a decomposable IV fuzzy measure associated with μ_L and μ_U. A construction of decomposable IV fuzzy measure is given in the following theorem.

Theorem 4.6. *If an IV aggregation function $M_m : (L([0,1]))^n \to L([0,1])$ is decomposable and associated with $M_L, M_U : [0,1]^n \to [0,1]$, then the corresponding IV fuzzy measure $m : 2^{[n]} \to L([0,1])$ induced from M_m as in Theorem 4.3 is decomposable associated with $\mu_L, \mu_U : 2^{[n]} \to [0,1]$ where $\mu_L(A) = M_L(1_A)$ and $\mu_U(A) = M_U(1_A)$.*

Note that if $m : 2^{[n]} \to L([0,1])$ is an IV fuzzy measure w.r.t. some admissible order, the set functions $\mu_1, \mu_2 : 2^{[n]} \to [0,1]$ such that $\mu_1(A) = \underline{m(A)}$ and $\mu_2(A) = \overline{m(A)}$ may not be fuzzy measures, since the monotonicity of μ_1, μ_2 is not secured.

Example 4.7. Let $\leq_{\mathrm{XY}}$ be Xu and Yager's order and let m be an IV fuzzy measure w.r.t. $\leq_{\mathrm{XY}}$ such that $m(B) = [0.2, 0.6]$ and $m(C) = [0.1, 0.7]$ for some $B \subset C$. Then $\mu_1(B) = \underline{m(B)} = 0.2 > 0.1 = \underline{m(C)} = \mu_1(C)$, hence the monotonicity is not satisfied and consequently μ_1 is not a fuzzy measure.

We will use the following notation and results from [35]. Let $c \in [0,1]$ and $\alpha \in]0,1[$. We denote by $d_\alpha(c)$ the maximal possible width of an interval $Z \in L([0,1])$ such that $K_\alpha(Z) = c$. Moreover, for any $X \in L([0,1])$ let

$$\lambda_\alpha(X) = \frac{w(X)}{d_\alpha(K_\alpha(X))} \tag{6}$$

where we set $\frac{0}{0} = 0$.

Proposition 4.8 ([35]). *For all $\alpha \in]0,1[$ and $X \in L([0,1])$ it holds that*

$$d_\alpha(K_\alpha(X)) = \min\left\{\frac{K_\alpha(X)}{\alpha}, \frac{1-K_\alpha(X)}{1-\alpha}\right\}. \tag{7}$$

Now, considering a fuzzy measure μ on $[n]$, we can construct IV fuzzy measures on $[n]$ with respect to orders $\leq_{\alpha,\beta}$.

Proposition 4.9. *Let n be a positive integer and $\alpha, \beta \in]0,1[$, $\beta \neq \alpha$. Let $\mu : 2^{[n]} \to [0,1]$ be a fuzzy measure on $[n]$. Then $m : 2^{[n]} \to L([0,1])$ defined by:*

$$m(A) = Y, \qquad \textit{where} \qquad \begin{cases} K_\alpha(Y) = \mu(A), \\ \lambda_\alpha(Y) = \begin{cases} \frac{\alpha-\mu(A)}{\alpha}, & \textit{if } \mu(A) \leq \alpha, \\ \frac{1-\mu(A)}{1-\alpha}, & \textit{otherwise.} \end{cases} \end{cases} \tag{8}$$

for all $A \in 2^{[n]}$, is an IV fuzzy measure on $[n]$ with respect to $\leq_{\alpha,\beta}$.

Proof. First observe that $K_\alpha(m(A))$ as well as $\lambda_\alpha(m(A))$ is a number from $[0,1]$ for any $A \in 2^{[n]}$, hence the function m is well-defined. Since $K_\alpha(m(\emptyset)) = \mu(\emptyset) = 0$ and $K_\alpha(m([n])) = \mu([n]) = 1$, we have $m(\emptyset) = 0_L$ and $m([n]) = 1$. Now let $A \subseteq B \subseteq [n]$, then there are two possibilities:

1. $\mu(A) = \mu(B)$, which implies that $K_\alpha(m(A)) = K_\alpha(m(B))$ and $\lambda_\alpha(m(A)) = \lambda_\alpha(m(B))$, i.e. $m(A) = m(B)$;
2. $\mu(A) < \mu(B)$, which implies that $K_\alpha(m(A)) < K_\alpha(m(B))$, i.e. $m(A) <_{\alpha,\beta} m(B)$,

which proves the monotonicity. □

Remark 4.10. The explicit form for computing $\lambda_\alpha(Y)$ in Equation (8) is:

$$\lambda_\alpha(Y) = \frac{\max\{0, \alpha-\mu(A)\}}{\alpha} + \frac{\max\{0, \mu(A)-\alpha\}(1-\mu(A))}{(1-\alpha)(\mu(A)-\alpha)}. \tag{9}$$

Example 4.11. Consider the IV fuzzy measure given in Proposition 4.9 where $n = 5$, $\alpha = \frac{1}{3}$ and $\mu(A) = \frac{|A|}{n}$.

- If $|A| = 1$, then $K_\alpha(m(A)) = \frac{1}{5}$, $\lambda_\alpha(m(A)) = \frac{2}{5}$, $d_\alpha(K_\alpha)) = \frac{3}{5}$, $w(m(A)) = \frac{6}{25}$, hence $\mu(A) = \left[\frac{3}{25}, \frac{9}{25}\right]$.
- If $|A| = 2$, then $K_\alpha(m(A)) = \frac{2}{5}$, $\lambda_\alpha(m(A)) = \frac{9}{10}$, $d_\alpha(K_\alpha)) = \frac{9}{10}$, $w(m(A)) = \frac{81}{100}$, hence $\mu(A) = \left[\frac{13}{100}, \frac{94}{100}\right]$.
- If $|A| = 3$, then $K_\alpha(m(A)) = \frac{3}{5}$, $\lambda_\alpha(m(A)) = \frac{3}{5}$, $d_\alpha(K_\alpha)) = \frac{3}{5}$, $w(m(A)) = \frac{9}{25}$, hence $\mu(A) = \left[\frac{12}{25}, \frac{21}{25}\right]$.
- If $|A| = 4$, then $K_\alpha(m(A)) = \frac{4}{5}$, $\lambda_\alpha(m(A)) = \frac{3}{10}$, $d_\alpha(K_\alpha)) = \frac{3}{10}$, $w(m(A)) = \frac{9}{100}$, hence $\mu(A) = \left[\frac{77}{100}, \frac{86}{100}\right]$.

By the following proposition we obtain a construction method for indecomposable IV aggregation functions w.r.t. $\leq_{\alpha,\beta}$.

Proposition 4.12 ([36]). *Let $\alpha, \beta \in]0,1[$, $\beta \neq \alpha$. Let $M_{\mathrm{M1}}, M_{\mathrm{M2}} : [0,1]^n \to [0,1]$ be aggregation functions where M_{M1} is strictly increasing. Then $M : (L([0,1]))^n \to L([0,1])$ defined by:*

$$M(X_1, \ldots, X_n) = Y, \qquad \textit{where} \qquad \begin{cases} K_\alpha(Y) = M_{\mathrm{M1}}(K_\alpha(X_1), \ldots, K_\alpha(X_n)), \\ \lambda_\alpha(Y) = M_{\mathrm{M2}}(\lambda_\alpha(X_1), \ldots, \lambda_\alpha(X_n)), \end{cases}$$

for all $X_1, \ldots, X_n \in L([0,1])$, is an IV aggregation function with respect to $\leq_{\alpha,\beta}$.

Note that applying the indecomposable IV aggregation functions from Proposition 4.12 in Theorem 4.3 we receive an IV fuzzy measure such that, for all $A \subseteq [n]$, it holds $\mu(A) = [x, x]$ for some $x \in [0,1]$, i.e., in fact the fuzzy measure with the range in $[0,1]$.

Remark 4.13. It is worth pointing out that the explicit form of the IV aggregation function given in Proposition 4.12 can be expressed as follows. To shorten the notation we set $\mathbf{K} = M_{M1}(K_\alpha(X_1), \ldots, K_\alpha(X_n))$ and $\mathbf{L} = M_{M2}(\lambda_\alpha(X_1), \ldots, \lambda_\alpha(X_n))$. Then,

$$M(X_1, \ldots, X_n) = Y,$$

where

$$\underline{Y} = \mathbf{K} - \alpha \mathbf{L} \min\left\{\frac{\mathbf{K}}{\alpha}, \frac{1-\mathbf{K}}{1-\alpha}\right\} \qquad \text{and} \qquad \overline{Y} = \frac{\mathbf{K}}{\alpha} - \frac{1-\alpha}{\alpha}\underline{Y}.$$

4.2. Interval-valued dissimilarity functions

In order to define discrete IV Choquet integral, we deal with IV dissimilarity functions in this part.

Definition 4.14. Let $\leq_{\mathrm{A}}$ be an admissible order on $L([0,1])$. A function $d : (L([0,1]))^2 \to L([0,1])$ is called an interval-valued dissimilarity function on $L([0,1])$ w.r.t. $\leq_{\mathrm{A}}$ if it satisfies, for all $X, Y, Z \in L([0,1])$, the following conditions:

1. $d(X,Y) = d(Y,X)$;
2. $d(0_L, 1_L) = 1_L$;
3. $d(X,X) = 0_L$;
4. if $X \leq_{\mathrm{A}} Y \leq_{\mathrm{A}} Z$, then $d(X,Y) \leq_{\mathrm{A}} d(X,Z)$ and $d(Y,Z) \leq_{\mathrm{A}} d(X,Z)$.

The construction of IV dissimilarity function using an IV negation and IV implication is given in the following theorem.

Theorem 4.15 ([36])**.** *Let $\leq_{\mathrm{A}}$ be an admissible order on $L([0,1])$, N be an IV negation with respect to $\leq_{\mathrm{A}}$ and I be an IV implication function such that, for all $X, Y \in L([0,1])$, $I(X,Y) = 1_L$ whenever $X \leq_{\mathrm{A}} Y$. Let $M : L([0,1])^2 \to L([0,1])$ be an IV aggregation function with respect to $\leq_{\mathrm{A}}$ such that $M(1_L, 0_L) = 0_L$ and $M(X,Y) = M(Y,X)$ for all $X, Y \in L([0,1])$. Then the function d: $L([0,1])^2 \to L([0,1])$ defined by*

$$d(X,Y) = N\big(M(I(X,Y), I(Y,X))\big) \tag{10}$$

is an IV dissimilarity function on $L([0,1])$ w.r.t. $\leq_{\mathrm{A}}$.

Proof. 1. Symmetry of d immediately follows from the symmetry of M.

2.
$$d(0_L, 1_L) = N(M(I(0_L, 1_L), I(1_L, 0_L))) = N(M(1_L, 0_L)) = N(0_L) = 1_L$$

3.
$$d(X,X) = N(M(I(X,X), I(X,X))) = N(M(1_L, 1_L)) = N(1_L) = 0_L$$

4. If $X \leq_{\mathrm{A}} Y \leq_{\mathrm{A}} Z$, then $I(X,Y) = I(X,Z) = 1_L$ and $I(Z,X) \leq_{\mathrm{A}} I(Y,X)$, hence, by the monotonicity of M and N, we have $d(X,Y) \leq_{\mathrm{A}} d(X,Z)$. Similarly, $d(Y,Z) \leq_{\mathrm{A}} d(X,Z)$. □

Example 4.16. (i) Let us consider Xu and Yager's order. The function $M : (L([0,1]))^2 \to L([0,1])$ given by

$$M(X,Y) = \left[\frac{\underline{X}\overline{Y} + \overline{X}\underline{Y}}{2}, \frac{\underline{X}\underline{Y} + \overline{X}\overline{Y}}{2}\right] \tag{11}$$

is an IV aggregation function w.r.t. Xu and Yager's order satisfying the assumptions of Theorem 4.15.

The function $I\colon L([0,1])^2 \to L([0,1])$ defined by

$$I(X,Y) = \begin{cases} 1_L, & \text{if } X \leq_{\mathrm{A}} Y, \\ M_I(N(X),Y), & \text{if } X >_{\mathrm{A}} Y, \end{cases} \tag{12}$$

is an IV implication w.r.t. Xu and Yager's order satisfying the assumptions of Theorem 4.15. Note that, in Equation (12), M_I can be any IV aggregation function w.r.t. Xu and Yager's order (for instance M from Equation (11)) and N can be any IV negation w.r.t. Xu and Yager's order, for instance

$$N(X) = [c' - r', c' + r'] \text{ with } \begin{cases} c' = 1 - c, \\ r' = a - r. \end{cases} \tag{13}$$

where $c = \frac{\underline{X}+\overline{X}}{2}$, $r = \frac{\overline{X}-\underline{X}}{2}$ and $a = \min\{c, 1-c\}$, see [35, Theorem 2].

Hence, applying Equations (11), (12) and (13) into Equation (10), we obtain an IV dissimilarity measure w.r.t. Xu and Yager's order.

(ii) Now consider $K_{\alpha,\beta}$ order for some $\alpha, \beta \in]0,1[$, $\beta \neq \alpha$. To build an IV dissimilarity measure w.r.t. $\leq_{\alpha,\beta}$ we can, for instance, use:

- The IV implication function given by Equation (12).
- Some IV negation w.r.t. $\leq_{\alpha,\beta}$ defined in [35].
- The IV aggregation function $M : (L([0,1]))^n \to L([0,1])$ given in Proposition 4.12.

Remark 4.17. The IV dissimilarity functions defined in Example 4.16 are not suitable for our intention to build a discrete IV Choquet integral with range in $L([0,1])$, since it is possible that, for some $0_L \leq_{\mathrm{XY}} X_1 \leq_{\mathrm{XY}} \ldots \leq_{\mathrm{XY}} X_n \leq 1_L$, it holds $d(X_1,X_2) + d(X_2,X_3) + \ldots + d(X_{n-1},X_n) >_{\mathrm{XY}} 1_L$ (or w.r.t. $\leq_{\alpha,\beta}$ in case (ii)). Hence, the output of $C_{m,d}$ could be greater than 1_L.

In the following proposition, in line with the previous remark, an IV dissimilarity function that can be used to construct a discrete IV Choquet integral with the range in $L([0,1])$ is introduced.

Proposition 4.18. *Let $\alpha, \beta \in]0,1[$ with $\alpha \neq \beta$. Let $M_d : [0,1]^2 \to [0,1]$ be a symmetric aggregation function, $\delta_d : [0,1]^2 \to [0,1]$ be a strictly monotone dissimilarity function and λ_α be given as in (6). Then the function $d\colon L([0,1])^2 \to L([0,1])$ defined by*

$$d(X,Y) = Z, \qquad \text{where} \qquad \begin{cases} K_\alpha(Z) = \delta_d(K_\alpha(X), K_\alpha(Y)); \\ \lambda_\alpha(Z) = M_d(\lambda_\alpha(X), \lambda_\alpha(Y)), \end{cases} \tag{14}$$

is an IV dissimilarity function w.r.t. $\leq_{\alpha,\beta}$.

Proof. 1. Symmetry of d immediately follows from the symmetry of δ_d and M_d.

2. Let $d(0_L, 1_L) = Z$. Then $K_\alpha(Z) = \delta(0,1) = 1$ and $\lambda_\alpha(Z) = M_d(0,0) = 0$, hence $Z = 1_L$.
3. Let $d(X,X) = Z$. Then $K_\alpha(Z) = \delta(K_\alpha(X), K_\alpha(X)) = 0$, hence $d_\alpha(K_\alpha(Z)) = 0$ and consequently $Z = 0_L$.
4. Let $X \leq_{\alpha,\beta} Y \leq_{\alpha,\beta} Z$ and $d(X,Y) = U, d(X,Z) = V$. We consider two cases:
   (i) If $K_\alpha(Y) < K_\alpha(Z)$, then $K_\alpha(U) < K_\alpha(V)$, thus $d(X,Y) \leq_{\alpha,\beta} d(X,Z)$.
   (ii) If $K_\alpha(Y) = K_\alpha(Z)$ and $K_\beta(Y) \leq K_\beta(Z)$, then $K_\alpha(U) = K_\alpha(V)$ and there are two cases:
      (a) Let $\alpha < \beta$. Then $w(Y) \leq w(Z)$, thus $\lambda_\alpha(Y) \leq \lambda_\alpha(Z)$ and by the monotonicity of M we have $\lambda_\alpha(U) \leq \lambda_\alpha(V)$, hence $w(U) \leq w(V)$ and consequently $U \leq_{\alpha,\beta} V$.
      (b) Let $\alpha > \beta$. Then $w(Y) \geq w(Z)$, thus $\lambda_\alpha(Y) \geq \lambda_\alpha(Z)$, hence $\lambda_\alpha(U) \geq \lambda_\alpha(V)$, from which it follows $w(U) \geq w(V)$ and finally $U \leq_{\alpha,\beta} V$.

□

Remark 4.19. Similarly to Remark 4.13 we can express the IV dissimilarity function given in Proposition 4.18 by the explicit form as follows. To shorten the notation we set $\mathbf{K} = \delta_d(K_\alpha(X), K_\alpha(Y))$ and $\mathbf{L} = M_d(\lambda_\alpha(X), \lambda_\alpha(Y))$. Then,

$$d(X, Y) = Z,$$

where

$$\underline{Z} = \mathbf{K} - \alpha \mathbf{L} \min\left\{\frac{\mathbf{K}}{\alpha}, \frac{1-\mathbf{K}}{1-\alpha}\right\} \qquad \text{and} \qquad \overline{Z} = \frac{\mathbf{K}}{\alpha} - \frac{1-\alpha}{\alpha}\underline{Z}.$$

It is worth pointing out that considering degenerate intervals in Equation (14) we recover dissimilarity measure for numbers, in particular the measure δ.

Proposition 4.20. *Under the assumptions of Proposition 4.18,*

$$d(X, Y) = \delta_d(x, y)$$

for all $x, y \in [0, 1]$ where $X = [x, x], Y = [y, y]$.

Proof. Let $X = [x, x], Y = [y, y]$ for some $x, y \in [0, 1]$ and $d(X, Y) = Z$. Then $K_\alpha(Z) = \delta_d(x, x)$ and $\lambda_\alpha(Z) = M(0, 0) = 0$, hence $w(Z) = 0$. □

5. Discrete interval-valued Choquet integral

Similarly to the case with numbers [23], we define discrete interval-valued Choquet integral in such a way that we consider interval-valued dissimilarity function instead of difference of input intervals. The discrete interval-valued Choquet integral is a special case of discrete d_G-Choquet integral introduced in Definition 3.1.

Definition 5.1. A discrete d_G-Choquet integral given by Definition 3.1 is called a discrete interval-valued Choquet integral if $A = L([0, 1])$ and $E = L([0, \infty[)$.

5.1. Discrete interval-valued Choquet integral with respect to the sum of intervals

First, we consider the standard sum of intervals. Hence, in this part, we are going to study the discrete IV Choquet integral (as a special case of d_G-Choquet integral) w.r.t. $\leq_L$ in the form:

$$C_{\sum, m, d}(X_1, \ldots, X_n) = \sum_{i=1}^{n} d(X_{\sigma(i)}, X_{\sigma(i-1)}) m\left(A_{\sigma(i)}\right) \tag{15}$$

where $d\colon L([0, 1])^2 \to L([0, 1])$ is an IV dissimilarity function, $m : 2^{[n]} \to L([0, 1])$ is an IV fuzzy measure, σ is a permutation on $[n]$ with $X_{\sigma(1)} \leq_L \ldots \leq_L X_{\sigma(n)}$, with the convention $X_{\sigma(0)} = 0_L$ and $A_{\sigma(i)} := \{\sigma(i), \ldots, \sigma(n)\}$.

Clearly, the domain of this function is $(L([0, 1]))^n$ and the range is a subset of $L([0, n])$. For numbers, a sufficient condition $(P1)$ to keep the range of $C_{\mu,\delta}$ in $[0, 1]$ was given in [23]:

$(P1)$ $\quad \delta(0, x_1) + \delta(x_1, x_2) + \ldots + \delta(x_{n-1}, x_n) \leq 1$ for all $x_1, \ldots, x_n \in [0, 1]$ where $x_1 \leq \ldots \leq x_n$.

However, as we show in the following example, taking δ_d satisfying $(P1)$ does not ensure that the IV dissimilarity function $d\colon L([0, 1])^2 \to L([0, 1])$ w.r.t. $\leq_{\alpha,\beta}$ given as in Proposition 4.18 satisfies:

$(IP1)$ $\quad \delta(0_L, X_1) + \delta(X_1, X_2) + \ldots + \delta(X_{n-1}, X_n) \leq_{\alpha,\beta} 1_L$ for all $X_1, \ldots, X_n \in L([0, 1])$ where $X_1 \leq_{\alpha,\beta} \ldots \leq_{\alpha,\beta} X_n$.

Example 5.2. Consider the IV dissimilarity function $d\colon L([0,1])^2 \to L([0,1])$ w.r.t. $\leq_{\alpha,\beta}$ given as in Proposition 4.18 where $\alpha = 0.5$, $\delta_d(x,y) = |x-y|$ and $M_d(x,y) = \min\{x,y\}$ for all $x, y \in [0,1]$. It is easy to see that δ_d satisfies $(P1)$, however,

$$d\big([0,0.1],[0.1,0.2],\ldots,[0.9,1]\big) =$$

$$= \left[\frac{1}{30},\frac{5}{30}\right] + \left[\frac{3}{50},\frac{7}{50}\right] + \left[\frac{5}{70},\frac{9}{70}\right] + \left[\frac{7}{90},\frac{11}{90}\right] + \left[\frac{7}{90},\frac{11}{90}\right] + \left[\frac{7}{90},\frac{11}{90}\right] + \left[\frac{5}{70},\frac{9}{70}\right] + \left[\frac{3}{50},\frac{7}{50}\right] + \left[\frac{1}{30},\frac{5}{30}\right] =$$

$$= \left[\frac{197}{350},\frac{433}{350}\right] \notin L([0,1]).$$

In spite of the fact that the range of $C_{\sum,m,d}$ can be outside of $L([0,1])$, it is easy to see that the 'boundary' conditions hold.

Proposition 5.3. *Let n be a positive integer. Let $C_{\sum,m,d} : (L([0,1]))^n \to L([0,n[)$ be an n-ary discrete IV Choquet integral w.r.t. $\leq_L$ given by Equation (15). Then $C_{\sum,m,d}(0_L,\ldots,0_L) = 0_L$ and $C_{\sum,m,d}(1_L,\ldots,1_L) = 1_L$.*

Proof. It is enough to observe that $d(0_L,0_L) = d(1_L,1_L) = 0_L$ and $d(0_L,1_L) = 1_L$. □

A discrete IV Choquet integral is idempotent only if d has neutral element 0_L.

Theorem 5.4. *Let n be a positive integer. Let $C_{\sum,m,d} : (L([0,1]))^n \to L([0,n[)$ be an n-ary discrete IV Choquet integral given by Equation (15). Then $C_{\sum,m,d}$ is idempotent for any IV fuzzy measure m on $[n]$ if and only if the IV dissimilarity function d satisfies $d(X,0_L) = X$ for all $X \in L([0,1])$.*

Proof. Let $X \in L([0,1])$. The proof follows from the observation:

$$C_{\sum,m,d}(X,\ldots,X) = d(X,0_L) + \sum_{i=2}^{n} d(X,X)m\left(A_{\sigma(i)}\right) = d(X,0_L).$$

□

Now, a necessary and sufficient condition for monotonicity (with respect to $\leq_{\alpha,\beta}$ order) of $C_{\sum,m,d}$ with respect to a fuzzy measure $m : 2^{[n]} \to [0,1]$ is given.

Theorem 5.5. *Let n be a positive integer and $\alpha,\beta \in [0,1]$ with $\alpha \neq \beta$. Let $C_{\sum,m,d} : (L([0,1]))^n \to L([0,n[)$ be an n-ary discrete IV Choquet integral w.r.t. $\leq_{\alpha,\beta}$ given by Equation (15). Then the following assertions are equivalent:*

(i) For any fuzzy measure $m : 2^{[n]} \to [0,1]$ it holds

$$C_{\sum,m,d}\left(X_1,\ldots,X_n\right) \leq_{\alpha,\beta} C_{\sum,m,d}\left(Y_1,\ldots,Y_n\right)$$

for all $X_1,\ldots,X_n,Y_1,\ldots,Y_n \in L([0,1])$ such that $X_1 \leq_{\alpha,\beta} Y_1,\ldots,X_n \leq_{\alpha,\beta} Y_n$.

(ii) For all $k \in [n]$ it holds

$$d(0_L,X_1) + d(X_1,X_2) + \ldots + d(X_{k-1},X_k) \leq_{\alpha,\beta} d(0_L,Y_1) + d(Y_1,Y_2) + \ldots + d(Y_{k-1},Y_k)$$

for all $X_1,\ldots,X_k,Y_1,\ldots,Y_k \in L([0,1])$ such that $X_1 \leq_{\alpha,\beta} \ldots \leq_{\alpha,\beta} X_k$, $Y_1 \leq_{\alpha,\beta} \ldots \leq_{\alpha,\beta} Y_k$ and $X_1 \leq_{\alpha,\beta} Y_1,\ldots,X_k \leq_{\alpha,\beta} Y_k$.

Proof. First observe that $X \leq_{\alpha,\beta} Y$ implies $c\,X \leq_{\alpha,\beta} c\,Y$ for all $X, Y \in L([0,1])$ and $c \in [0,1]$.

$(ii) \Rightarrow (i)$ Let $X_1, \ldots, X_n, Y_1, \ldots, Y_n \in L([0,1])$ be such that $X_1 \leq_{\alpha,\beta} Y_1, \ldots, X_n \leq_{\alpha,\beta} Y_n$. Then there exist permutations σ_x, σ_y of $[n]$ such that $X_{\sigma_x(1)} \leq_{\alpha,\beta} \ldots \leq_{\alpha,\beta} X_{\sigma_x(n)}$, $Y_{\sigma_y(1)} \leq_{\alpha,\beta} \ldots \leq_{\alpha,\beta} Y_{\sigma_y(n)}$ and $X_{\sigma_x(1)} \leq_{\alpha,\beta} Y_{\sigma_y(1)}, \ldots, X_{\sigma_x(n)} \leq_{\alpha,\beta} Y_{\sigma_y(n)}$. Let us denote, for any $r \in [n]$, $m(\{r, r+1, \ldots, n\})$ by m_r. From (ii) we have

$$(m_1 - m_2) d\left(0_L, X_{\sigma_x(1)}\right) \leq_{\alpha,\beta} (m_1 - m_2) d\left(0_L, Y_{\sigma_y(1)}\right)$$

and

$$(m_2 - m_3)\left(d\left(0_L, X_{\sigma_x(1)}\right) + d\left(X_{\sigma_x(1)}, X_{\sigma_x(2)}\right)\right) \leq_{\alpha,\beta} (m_2 - m_3)\left(d\left(0_L, Y_{\sigma_y(1)}\right) + d\left(Y_{\sigma_y(1)}, Y_{\sigma_y(2)}\right)\right),$$

thus

$$(m_1 - m_3) d\left(0_L, X_{\sigma_x(1)}\right) + (m_2 - m_3) d\left(X_{\sigma_x(1)}, X_{\sigma_x(2)}\right) \leq_{\alpha,\beta} (m_1 - m_3) d\left(0_L, Y_{\sigma_y(1)}\right) + (m_2 - m_3) d\left(Y_{\sigma_y(1)}, Y_{\sigma_y(2)}\right).$$

Moreover, from (ii) it follows

$$(m_3 - m_4)\left(d\left(0_L, X_{\sigma_x(1)}\right) + d\left(X_{\sigma_x(1)}, X_{\sigma_x(2)}\right) + d\left(X_{\sigma_x(2)}, X_{\sigma_x(3)}\right)\right) \leq_{\alpha,\beta}$$
$$\leq_{\alpha,\beta} (m_3 - m_4)\left(d\left(0_L, Y_{\sigma_y(1)}\right) + d\left(Y_{\sigma_y(1)}, Y_{\sigma_y(2)}\right) + d\left(Y_{\sigma_y(2)}, Y_{\sigma_y(3)}\right)\right),$$

hence we obtain

$$(m_1 - m_4) d\left(0_L, X_{\sigma_x(1)}\right) + (m_2 - m_4) d\left(X_{\sigma_x(1)}, X_{\sigma_x(2)}\right) + (m_3 - m_4) d\left(X_{\sigma_x(2)}, X_{\sigma_x(3)}\right) \leq_{\alpha,\beta}$$
$$\leq_{\alpha,\beta} (m_1 - m_4) d\left(0_L, Y_{\sigma_y(1)}\right) + (m_2 - m_4) d\left(Y_{\sigma_y(1)}, Y_{\sigma_y(2)}\right) + (m_3 - m_4) d\left(Y_{\sigma_y(2)}, Y_{\sigma_y(3)}\right).$$

Repeating a similar procedure we have

$$(m_1 - m_n) d\left(0_L, X_{\sigma_x(1)}\right) + (m_2 - m_n) d\left(X_{\sigma_x(1)}, X_{\sigma_x(2)}\right) + \ldots + (m_{n-1} - m_n) d\left(X_{\sigma_x(n-2)}, X_{\sigma_x(n-1)}\right) \leq_{\alpha,\beta}$$
$$\leq_{\alpha,\beta} (m_1 - m_n) d\left(0_L, Y_{\sigma_y(1)}\right) + (m_2 - m_n) d\left(Y_{\sigma_y(1)}, Y_{\sigma_y(2)}\right) + \ldots + (m_{n-1} - m_n) d\left(Y_{\sigma_y(n-2)}, Y_{\sigma_y(n-1)}\right).$$

Finally, since

$$m_n\left(d\left(0_L, X_{\sigma_x(1)}\right) + d\left(X_{\sigma_x(1)}, X_{\sigma_x(2)}\right) + \ldots + d\left(X_{\sigma_x(n-1)}, X_{\sigma_x(n)}\right)\right) \leq_{\alpha,\beta}$$
$$\leq_{\alpha,\beta} m_n\left(d\left(0_L, Y_{\sigma_y(1)}\right) + d\left(Y_{\sigma_y(1)}, Y_{\sigma_y(2)}\right) + \ldots + d\left(Y_{\sigma_y(n-1)}, Y_{\sigma_y(n)}\right)\right),$$

it holds

$$m_1 d\left(0_L, X_{\sigma_x(1)}\right) + m_2 d\left(X_{\sigma_x(1)}, X_{\sigma_x(2)}\right) + \ldots + m_n d\left(X_{\sigma_x(n-1)}, X_{\sigma_x(n)}\right) \leq_{\alpha,\beta} \tag{16}$$
$$\leq_{\alpha,\beta} m_1 d\left(0_L, Y_{\sigma_y(1)}\right) + m_2 d\left(Y_{\sigma_y(1)}, Y_{\sigma_y(2)}\right) + \ldots + m_n d\left(Y_{\sigma_y(n-1)}, Y_{\sigma_y(n)}\right),$$

that is

$$C_{\sum,m,d}(X_1, \ldots, X_n) \leq_{\alpha,\beta} C_{\sum,m,d}(Y_1, \ldots, Y_n).$$

$(i) \Rightarrow (ii)$ Let $X_1, \ldots, X_n, Y_1, \ldots, Y_n \in L([0,1])$ be such that $X_1 \leq_{\alpha,\beta} Y_1, \ldots, X_n \leq_{\alpha,\beta} Y_n$ and $X_1 \leq_{\alpha,\beta} \ldots \leq_{\alpha,\beta} X_n$, $Y_1 \leq_{\alpha,\beta} \ldots \leq_{\alpha,\beta} Y_n$. By (i) we have $C_{\sum,m,d}(X_1, \ldots, X_n) \leq_{\alpha,\beta} C_{\sum,m,d}(Y_1, \ldots, Y_n)$, hence Equation (16) holds. Now consider, for any $k \in [n]$, a fuzzy measure m^k given by:

$$m^k(A) = \begin{cases} 1, & \text{if } |A| \geq n - k + 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then $m^k(\{r, r+1, \ldots, n\}) = 1$ for each $r \leq k$ and $m^k(\{r, r+1, \ldots, n\}) = 0$ for each $r > k$, which imply

$$C_{\sum,m^k,d}(X_1, \ldots, X_n) = d(0_L, X_1)\, m^k(\{1, \ldots, n\}) + d(X_1, X_2)\, m^k(\{2, \ldots, n\}) + \ldots + d(X_{n-1}, X_n)\, m^k(\{n\}) =$$

$$= d(0_L, X_1) + d(X_1, X_2) + \ldots + d(X_{k-1}, X_k)$$

and similarly

$$C_{\sum, m^k, d}(Y_1, \ldots, Y_n) = d(0_L, Y_1) + d(Y_1, Y_2) + \ldots + d(Y_{k-1}, Y_k).$$

Finally,

$$d(0_L, X_1) + d(X_1, X_2) + \ldots + d(X_{k-1}, X_k) \leq_{\alpha,\beta} d(0_L, Y_1) + d(Y_1, Y_2) + \ldots + d(Y_{k-1}, Y_k),$$

for all $k \in [n]$. $\square$

Remark 5.6. It is worth to emphasize that in Theorem 5.5 fuzzy measures whose outputs are numbers were considered. Observe, that the condition (ii) stated in Theorem 5.5 does not generally secure the monotonicity of $C_{\sum,m,d}$ if m is an interval-valued fuzzy measure, that is for $m : 2^{[n]} \to L([0,1])$. The reason is that $X \leq_{\alpha,\beta} Y$ does not imply $X \cdot Z \leq_{\alpha,\beta} Y \cdot Z$ if Z is not a degenerate interval, as is shown in the following counterexample: Let $\alpha = 0.75$, $X = [0.1, 0.8]$, $Y = [0.2, 0.6]$ and $Z = [0.2, 0.9]$, then $K_\alpha(X) = 0.275$, $K_\alpha(Y) = 0.3$, $K_\alpha(X \cdot Z) = 0.195$, $K_\alpha(Y \cdot Z) = 0.165$, hence $X <_{\alpha,\beta} Y$ but $Y \cdot Z <_{\alpha,\beta} X \cdot Z$.

It is not easy to find an IV dissimilarity function that satisfies the condition (ii) of Theorem 5.5. Even a very intuitive and natural construction described in the following example violates (ii).

Example 5.7. We show that the IV dissimilarity function given in Proposition 4.18 induced by $\delta_d(x,y) = |x-y|$ and Arithmetic mean as M_d does not satisfies the condition (ii) of Theorem 5.5. Let $\alpha = 0.5$, $X_1 = [0, 0.2]$, $X_2 = [0.1, 0.5]$, $X_3 = [0.8, 1]$ and $Y_1 = [0, 0.2]$, $Y_2 = [0.6, 0.8]$, $Y_3 = [0.8, 1]$, i.e. $X_1 \leq_{\alpha,\beta} Y_1$, $X_2 \leq_{\alpha,\beta} Y_2$ and $X_3 \leq_{\alpha,\beta} Y_3$. Then

$$\begin{array}{ccc}
K_\alpha(d(0_L, X_1)) = 0.1 & \lambda_\alpha(d(0_L, X_1)) = 0.5 & d(0_L, X_1) = [0.05, 0.15] \\
K_\alpha(d(X_1, X_2)) = 0.2 & \lambda_\alpha(d(X_1, X_2)) = 0.8\overline{3} & d(X_1, X_2) = [0.0\overline{3}, 0.3\overline{6}] \\
K_\alpha(d(X_2, X_3)) = 0.6 & \lambda_\alpha(d(X_2, X_3)) = 0.8\overline{3} & d(X_2, X_3) = [0.2\overline{6}, 0.9\overline{3}] \\
K_\alpha(d(0_L, Y_1)) = 0.1 & \lambda_\alpha(d(0_L, Y_1)) = 0.5 & d(0_L, Y_1) = [0.05, 0.15] \\
K_\alpha(d(Y_1, Y_2)) = 0.6 & \lambda_\alpha(d(Y_1, Y_2)) = 0.\overline{6} & d(Y_1, Y_2) = [0.\overline{3}, 0.8\overline{6}] \\
K_\alpha(d(Y_2, Y_3)) = 0.2 & \lambda_\alpha(d(Y_2, Y_3)) = 0.\overline{6} & d(Y_2, Y_3) = [0.0\overline{6}, 0.\overline{3}].
\end{array}$$

Hence, for $\beta > 0.5$, it holds

$$d(0_L, X_1) + d(X_1, X_2) + d(X_1, X_2) = [0.35, 1.45] >_{\alpha,\beta} [0.45, 1.35] = d(0_L, Y_1) + d(Y_1, Y_2) + d(Y_1, Y_2)$$

which means that for the fuzzy measure given by $m(\emptyset) = 0$ and $m(A) = 1$ otherwise, we obtain

$$C_{\sum,m,d}(X_1, X_2, X_3) >_{\alpha,\beta} C_{\sum,m,d}(Y_1, Y_2, Y_3),$$

i.e. the monotonicity of $C_{\sum,m,d}$ is violated.

Construction of an IV dissimilarity function that satisfies the condition (ii) of Theorem 5.5 is given next. Note that the range of the IV dissimilarity function is narrowed to $[0,1]$.

Proposition 5.8. *Let $\alpha, \beta \in [0,1]$ with $\alpha \neq \beta$ and let $\delta_1, \delta_2 : [0,1]^2 \to [0,1]$ be dissimilarity functions such that $\delta_1(x,y) \leq \delta_2(x,y)$ for all $x, y \in [0,1]$. Then $d \colon L([0,1])^2 \to L([0,1])$ given by:*

$$d(X,Y) = \big[\delta_1\big(K_\alpha(X), K_\alpha(Y)\big), \delta_2\big(K_\alpha(X), K_\alpha(Y)\big)\big] \tag{17}$$

is an IV dissimilarity function with respect to $\leq_{\alpha,\beta}$. Moreover, for each positive integer n, if $\delta = \delta_1 = \delta_2$ and δ satisfies:

- *for all $k \in [n]$ it holds*

$$\delta(0,x_1)+\delta(x_1,x_2)+\ldots+\delta(x_{k-1},x_k) \leq_{\alpha,\beta} \delta(0,y_1)+\delta(y_1,y_2)+\ldots+\delta(y_{k-1},y_k) \tag{18}$$

for all $x_1,\ldots,x_k,y_1,\ldots,y_k \in [0,1]$ such that $x_1 \leq \ldots \leq x_k$, $y_1 \leq \ldots \leq y_k$ and $x_1 \leq y_1,\ldots,x_k \leq y_k$,

then d satisfies the condition (ii) of Theorem 5.5.

Proof. First observe that d is an IV dissimilarity function with respect to $\leq_{\alpha,\beta}$ since:

1. the symmetry od d follows from the symmetry of δ_1 and δ_2;
2. $d(0_L,1_L) = \big[\delta_1(0,1),\delta_2(0,1)\big] = 1_L$;
3. $d(X,X) = \big[\delta_1\big(K_\alpha(X),K_\alpha(X)\big),\delta_2\big(K_\alpha(X),K_\alpha(X)\big)\big] = 0_L$;
4. the monotonicity of d follows from the monotonicity of δ_1 and δ_2.

Finally, if $\delta_1=\delta_2$, the results of $d(X,Y)$ is always a degenerate interval, so the rest is obvious. □

Example 5.9. Note that δ satisfies the condition from Proposition 5.8 given by Equation (18) whenever for each $k \in [n]$ there exists a non-decreasing function $f_k : [0,1] \to [0,1]$ such that $f_k(x) = d(0,x_1)+d(x_1,x_2)+\ldots+d(x_{k-1},x_k)$ for all $x_1,\ldots,x_k \in [0,1]$ such that $x_1 \leq \ldots \leq x_k = x$. See [23] for more details. For instance, $\delta(x,y)=|x-y|$, $\delta(x,y)=|\sqrt{x}-\sqrt{y}|$ and $\delta(x,y)=|x^2-y^2|$ satisfy the condition (with functions $f_k(x)=x$, $f_k(x)=\sqrt{x}$ and $f_k(x)=x^2$, respectively).

It is easy to check that for monotone aggregation functions the idempotency and averagingness is equivalent, hence the following result is immediate.

Corollary 5.10. *Let n be a positive integer and $\alpha,\beta \in [0,1]$ with $\alpha \neq \beta$. Let $C_{\sum,m,d}$ be an n-ary discrete IV Choquet integral given by Equation (15) where $m : 2^{[n]} \to [0,1]$ be a fuzzy measure and the IV dissimilarity function d satisfies the condition (ii) of Theorem 5.5. Then the following assertions are equivalent:*

(i) For all $X_1,\ldots,X_n \in L([0,1])$ it holds:

$$\min\{X_1,\ldots,X_n\} \leq_{\alpha,\beta} C_{\sum,m,d}(X_1,\ldots,X_n) \leq_{\alpha,\beta} \max\{X_1,\ldots,X_n\}.$$

(ii) $d(X,0_L)=X$ for all $X \in L([0,1])$.

Proof. Directly follows from Theorem 5.5 and Theorem 5.4. □

5.2. Discrete interval-valued Choquet integral with respect to $\bigoplus$

Now we are going to build a discrete IV Choquet integral w.r.t. the orders $\leq_{\alpha,\beta}$ replacing the sum by a new operation that is more in line with the intuition behind the $\leq_{\alpha,\beta}$ orders. First, a specific expression of the operation S in Definition 3.1, namely $\bigoplus$, is introduced.

Theorem 5.11. *Let n be a positive integer, $\alpha,\beta \in]0,1[$ and $M_s : \bigcup_{k=1}^{n} [0,1]^k \to [0,1]$ be an extended aggregation function. Let $d : (L([0,1]))^2 \to L([0,1])$ be an IV dissimilarity function on $L([0,1])$ w.r.t. $\leq_{\alpha,\beta}$ given in Proposition 4.18 where $\delta_d : [0,1]^2 \to [0,1]$ satisfies the condition $(P1)$. Let $\bigoplus : \bigcup_{k=1}^{n} (L([0,1]))^k \to L([0,\infty[)$ be the IV function defined by, for all $k \in [n]$, $Y_1,\ldots,Y_k \in L([0,1])$,*

$$\bigoplus_{i=1}^{k}(Y_i) = Z, \qquad \textit{where} \qquad \begin{cases} K_\alpha(Z) = \sum_{i=1}^{k} K_\alpha(Y_i); \\ \lambda_\alpha(Z) = M_s(\lambda_\alpha(Y_1),\ldots,\lambda_\alpha(Y_k)), & \textit{if } K_\alpha(Z) \in [0,1]; \\ w(Z)=1, & \textit{if } K_\alpha(Z) > 1. \end{cases} \tag{19}$$

Then for all $X_1,\ldots,X_k \in L([0,1])$ such that $0_L = X_0 \leq_{\alpha,\beta} X_1 \leq_{\alpha,\beta} \ldots \leq_{\alpha,\beta} X_k \leq_{\alpha,\beta} 1_L$, it holds $\bigoplus_{i=1}^{k}\big(d(X_i,X_{i-1})\big) \in L([0,1])$. Moreover, the function $\bigoplus$ is non-decreasing w.r.t. $\leq_{\alpha,\beta}$ in each variable.

Proof. Clearly, for each $k \in [n]$, $\bigoplus_{i=1}^{k} \big(d(X_i, X_{i-1})\big)$ is an interval and $\bigoplus_{i=1}^{k} \big(d(X_i, X_{i-1})\big) \geq_{\alpha,\beta} 0_L$. It remains to prove that $\bigoplus_{i=1}^{k} \big(d(X_i, X_{i-1})\big) \leq_{\alpha,\beta} 1_L$. Since $X_1 \leq_{\alpha,\beta} \ldots \leq_{\alpha,\beta} X_k$, we have $K_\alpha(0_L) \leq K_\alpha(X_1) \leq \ldots \leq K_\alpha(X_k)$, hence

$$K_\alpha\left(\bigoplus_{i=1}^{k} \big(d(X_i, X_{i-1})\big)\right) = \sum_{i=1}^{k} K_\alpha\big(d(X_i, X_{i-1})\big) = \sum_{i=1}^{k} \delta_d\big(K_\alpha(X_i), K_\alpha(X_{i-1})\big) \leq 1$$

where the last inequality follows from $(P1)$. Consequently $\bigoplus_{i=1}^{k} \big(d(X_i, X_{i-1})\big) \leq_{\alpha,\beta} 1_L$, hence $\bigoplus_{i=1}^{k} \big(d(X_i, X_{i-1})\big) \in L([0,1])$.

The non-decreasingness of $\bigoplus$ follows from Equation (19) and the monotonicity of M_s. ☐

Lemma 5.12. *Let $\bigoplus : \bigcup_{k=1}^{n} (L([0,1]))^k \to L([0,\infty[)$ be the IV function induced by $M_s : \bigcup_{k=1}^{n} [0,1]^k \to [0,1]$ as in Theorem 5.11. Then 0_L is the extended neutral element of $\bigoplus$ if and only if 0 is the extended neutral element of M_s.*

Proof. The proof is obvious. ☐

Corollary 5.13. *Let n be a positive integer and $\alpha, \beta \in]0,1[$. Let $m : 2^{[n]} \to L([0,1])$ be an IV fuzzy measure on $[n]$ w.r.t. $\leq_{\alpha,\beta}$, $d : (L([0,1]))^2 \to L([0,1])$ an IV dissimilarity function on $L([0,1])$ w.r.t. $\leq_{\alpha,\beta}$ and $\bigoplus : \bigcup_{k=1}^{n} (L([0,1]))^k \to L([0,\infty[)$ be the IV function defined in Theorem 5.11 and induced by a function M_s with extended neutral element 0. Then the mapping $C_{\bigoplus,m,d} : (L([0,1]))^n \to L([0,\infty[)$ defined by*

$$C_{\bigoplus,m,d}(X_1, \ldots, X_n) = \bigoplus_{i=1}^{n} d(X_{\sigma(i)}, X_{\sigma(i-1)}) m\left(A_{\sigma(i)}\right)$$

where σ is a permutation on $[n]$ with $X_{\sigma(1)} \leq_{\alpha,\beta} \ldots \leq_{\alpha,\beta} X_{\sigma(n)}$, with the convention $X_{\sigma(0)} = 0_L$ and $A_{\sigma(i)} := \{\sigma(i), \ldots, \sigma(n)\}$, is a discrete IV Choquet integral associated with m and d w.r.t. $\leq_{\alpha,\beta}$.

Proof. In line with Remark 3.2 and Lemma 5.12, the discrete IV Choquet integral is well defined, since it is a special case of d_G-Choquet integral given in Definition 3.1. ☐

Corollary 5.14. *Under the assumptions of Corollary 5.13, if d is an IV dissimilarity function given in Proposition 4.18 and induced by δ_d satisfying the condition $(P1)$ then*

$$C_{\bigoplus,m,d}(X_1, \ldots, X_n) \in L([0,1])$$

for all $X_1, \ldots, X_n \in L([0,1])$.

Remark 5.15. In line with Remark 4.13 and Remark 4.19, the IV discrete Choquet integral on $L([0,1])$ in Corollary 5.13 can be expressed by the explicit form as follows. To shorten the notation we set $\mathbf{K_1} = \delta_d(K_\alpha(X_{\sigma(i)}), K_\alpha(X_{\sigma(i-1)}))$, $\mathbf{L_1} = M_d(\lambda_\alpha(X_{\sigma(i)}), \lambda_\alpha(X_{\sigma(i-1)}))$, $\mathbf{K_2} = \sum_{i=1}^{n} K_\alpha(Y_i)$ and $\mathbf{L_2} = M_s(\lambda_\alpha(Y_1), \lambda_\alpha(Y_n))$. Then,

$$C_{\bigoplus,m,d}(X_1, \ldots, X_n) = U,$$

where

$$\underline{U} = \mathbf{K_2} - \alpha \mathbf{L_2} \min\left\{\frac{\mathbf{K_2}}{\alpha}, \frac{1-\mathbf{K_2}}{1-\alpha}\right\} \qquad \text{and} \qquad \overline{U} = \frac{\mathbf{K_2}}{\alpha} - \frac{1-\alpha}{\alpha}\underline{U}$$

and

$$\underline{Y_i} = \underline{m\left(A_{\sigma(i)}\right)} \left(\mathbf{K_1} - \alpha \mathbf{L_1} \min\left\{\frac{\mathbf{K_1}}{\alpha}, \frac{1-\mathbf{K_1}}{1-\alpha}\right\}\right) \qquad \text{and} \qquad \overline{Y_i} = \overline{m\left(A_{\sigma(i)}\right)} \left(\frac{\mathbf{K_1}}{\alpha} - \frac{1-\alpha}{\alpha}\underline{Y_i}\right)$$

We give two examples of discrete IV Choquet integral. The first one, Example 5.16, with respect to a fuzzy measure (with range in $[0,1]$) and the second one, Example 5.17, with respect to an IV fuzzy measure (with range in $L([0,1])$).

Example 5.16. (i) Let $\alpha = \frac{1}{3}$, the operation $\bigoplus$ be given as in Theorem 5.11 where M_s is the arithmetic mean, IV dissimilarity function d be given as in Proposition 4.18 where $\delta_d(x,y) = |x-y|$ and $M_d(x,y) = \frac{x+y}{2}$ for all $x,y \in [0,1]$, and let IV fuzzy measure m be given as in Theorem 4.3 for M_m defined as in Proposition 4.12 where $M_{\mathrm{M1}}, M_{\mathrm{M2}}$ are the arithmetic means. Let us denote

$$C_{\bigoplus,m,d}\Big([0.1,0.7],[0.6,0.9],[0,0.3]\Big) = Z.$$

Then

$$Z = \bigoplus\Big(d\left([0,0.3],[0,0]\right), d\left([0.1,0.7],[0,0.3]\right) m\left(\{1,2\}\right), d\left([0.6,0.9],[0.1,0.7]\right) m\left(\{2\}\right)\Big)$$

and since $d\left([0,0.3],[0,0]\right) = \left[\frac{1}{20},\frac{4}{20}\right]$, $d\left([0.1,0.7],[0,0.3]\right) = \left[\frac{1}{30},\frac{16}{30}\right]$, $d\left([0.6,0.9],[0.1,0.7]\right) = \left[\frac{1}{5},\frac{4}{5}\right]$, $m\left(\{1,2\}\right) = \left[\frac{2}{3},\frac{2}{3}\right]$ and $m\left(\{2\}\right) = \left[\frac{1}{3},\frac{1}{3}\right]$, we obtain $K_\alpha(Z) = \frac{11}{30}$, $\lambda_\alpha(Z) = \frac{11}{18}$ and $w(Z) = \frac{209}{360}$, hence

$$C_{\bigoplus,m,d}\Big([0.1,0.7],[0.6,0.9],[0,0.3]\Big) = \left[\frac{187}{1080},\frac{814}{1080}\right].$$

(ii) We compare $C_{\bigoplus,m,d}$ with $C_{\sum,m,d}$ for the same input and under the same conditions:

$$C_{\sum,m,d}\Big([0.1,0.7],[0.6,0.9],[0,0.3]\Big) = \left[\frac{1}{20},\frac{4}{20}\right] + \left[\frac{1}{30},\frac{16}{30}\right]\left[\frac{2}{3},\frac{2}{3}\right] + \left[\frac{1}{5},\frac{4}{5}\right]\left[\frac{1}{3},\frac{1}{3}\right] = \left[\frac{150}{1080},\frac{888}{1080}\right].$$

Example 5.17. We construct a discrete IV Choquet integral according to Corollary 5.13 ($\bigoplus$ induced by an extended aggregation function M) associated with an IV fuzzy measure given in Proposition 4.9 (induced by a fuzzy measure μ) and IV dissimilarity function given in Proposition 4.18 (induced by a strictly monotone dissimilarity function δ_d satisfying $(P1)$ and a symmetric aggregation function M_d) with respect to $\leq_{\alpha,\beta}$. Let us denote

$$U_{\sigma(i)} = d(X_{\sigma(i)}, X_{\sigma(i-1)}) \qquad \text{and} \qquad V_{\sigma(i)} = m\left(A_{\sigma(i)}\right).$$

Then

$$U_{\sigma(i)} = \left[K_\alpha(U_{\sigma(i)}) - \alpha w(U_{\sigma(i)}), K_\alpha(U_{\sigma(i)}) + (1-\alpha) w(U_{\sigma(i)})\right],$$

where

$$K_\alpha(U_{\sigma(i)}) = \delta_d\left(\alpha \underline{X_{\sigma(i)}} + (1-\alpha)\overline{X_{\sigma(i)}}, \alpha \underline{X_{\sigma(i-1)}} + (1-\alpha)\overline{X_{\sigma(i-1)}}\right),$$

$$\lambda_\alpha(U_{\sigma(i)}) = M_d\left(\frac{\overline{X_{\sigma(i)}} - \underline{X_{\sigma(i)}}}{\min\left\{\frac{\alpha \underline{X_{\sigma(i)}} + (1-\alpha)\overline{X_{\sigma(i)}}}{\alpha}, \frac{1-\alpha \underline{X_{\sigma(i)}} - (1-\alpha)\overline{X_{\sigma(i)}}}{1-\alpha}\right\}}, \frac{\overline{X_{\sigma(i-1)}} - \underline{X_{\sigma(i-1)}}}{\min\left\{\frac{\alpha \underline{X_{\sigma(i-1)}} + (1-\alpha)\overline{X_{\sigma(i-1)}}}{\alpha}, \frac{1-\alpha \underline{X_{\sigma(i-1)}} - (1-\alpha)\overline{X_{\sigma(i-1)}}}{1-\alpha}\right\}}\right)$$

and

$$w(U_{\sigma(i)}) = \lambda_\alpha(U_{\sigma(i)}) \min\left\{\frac{K_\alpha(U_{\sigma(i)})}{\alpha}, \frac{1-K_\alpha(U_{\sigma(i)})}{1-\alpha}\right\}.$$

Similarly,

$$V_{\sigma(i)} = \left[K_\alpha(V_{\sigma(i)}) - \alpha w(V_{\sigma(i)}), K_\alpha(V_{\sigma(i)}) + (1-\alpha) w(V_{\sigma(i)})\right],$$

where

$$K_\alpha(V_{\sigma(i)}) = \mu\left(A_{\sigma(i)}\right),$$

$$\lambda_\alpha(V_{\sigma(i)}) = \frac{\max\left\{0, \alpha - \mu\left(A_{\sigma(i)}\right)\right\}}{\alpha} + \frac{\max\left\{0, \mu\left(A_{\sigma(i)}\right) - \alpha\right\}\left(1 - \mu\left(A_{\sigma(i)}\right)\right)}{(1-\alpha)\left(\mu\left(A_{\sigma(i)}\right) - \alpha\right)}$$

and

$$w(V_{\sigma(i)}) = \lambda_\alpha(V_{\sigma(i)}) \min\left\{\frac{\mu(A_{\sigma(i)})}{\alpha}, \frac{1-\mu(A_{\sigma(i)})}{1-\alpha}\right\}.$$

Thus,

$$K_\alpha\left(U_{\sigma(i)} \cdot V_{\sigma(i)}\right) = \alpha\Big(K_\alpha(U_{\sigma(i)}) - \alpha w(U_{\sigma(i)})\Big)\Big(K_\alpha(V_{\sigma(i)}) - \alpha w(V_{\sigma(i)})\Big) +$$

$$+(1-\alpha)\Big(K_\alpha(U_{\sigma(i)}) + (1-\alpha)w(U_{\sigma(i)})\Big)\Big(K_\alpha(V_{\sigma(i)}) + (1-\alpha)w(V_{\sigma(i)})\Big).$$

Now, let us denote

$$Z = \bigoplus_{i=1}^{n} \left(U_{\sigma(i)} \cdot V_{\sigma(i)}\right),$$

then

$$K_\alpha(Z) = \sum_{i=1}^{n} \Big(K_\alpha\left(U_{\sigma(i)} \cdot V_{\sigma(i)}\right)\Big),$$

$$\lambda_\alpha(Z) = M_{i=1}^{n}\left(\frac{\overline{U_{\sigma(i)} \cdot V_{\sigma(i)}} - \underline{U_{\sigma(i)} \cdot V_{\sigma(i)}}}{\min\left\{\frac{K_\alpha\left(U_{\sigma(i)}\cdot V_{\sigma(i)}\right)}{\alpha}, \frac{1-K_\alpha\left(U_{\sigma(i)}\cdot V_{\sigma(i)}\right)}{1-\alpha}\right\}}\right),$$

$$w(Z) = \lambda_\alpha(Z) \min\left\{\frac{K_\alpha(Z)}{\alpha}, \frac{1-K_\alpha(Z)}{1-\alpha}\right\}.$$

Hence, finally

$$C_{\bigoplus,m,d}(X_1,\ldots,X_n) = [K_\alpha(Z) - \alpha w(Z), K_\alpha(Z) + (1-\alpha)w(Z)]. \tag{20}$$

The reason why we have chosen the particular operation $\bigoplus$ in Corollary 5.13, that is the operation for construction of an IV discrete Choquet integral, is the following: when we consider degenerate intervals we recover the discrete d-Choquet integral for numbers. This is stated in Proposition 5.18.

Proposition 5.18. *Let n be a positive integer, $\mu : 2^{[n]} \to [0,1]$ be a fuzzy measure, $\delta_d : [0,1]^2 \to [0,1]$ a dissimilarity measure and C_{μ,δ_d} a discrete d-Choquet integral according to Definition 3.1. Under the assumptions of Corollary 5.14, if $m(A) = \mu(A)$ for all $A \subseteq [n]$, then*

$$C_{\bigoplus,m,d}(X_1,\ldots,X_n) = C_{\mu,\delta_d}(x_1,\ldots,x_n)$$

for all $x_1,\ldots,x_n \in [0,1]$ where $X_1 = [x_1,x_1],\ldots,X_n = [x_n,x_n]$.

Proof. Let $X_1 = [x_1,x_1],\ldots,X_n = [x_n,x_n]$ for some $x_1,\ldots,x_n \in [0,1]$ and let $Z = C_{\bigoplus,m,d}(X_1,\ldots,X_n)$. Observe that, by Proposition 4.18, $d(X_i,X_j) = \delta_d(x_i,x_j)$ for all $i,j \in \{1,\ldots,n\}$. Then

$$Z = \bigoplus_{i=1}^{n} d(X_{\sigma(i)}, X_{\sigma(i-1)}) m\left(A_{\sigma(i)}\right) = \bigoplus_{i=1}^{n} \delta_d(x_{\sigma(i)}, x_{\sigma(i-1)}) \mu\left(A_{\sigma(i)}\right)$$

hence

$$K_\alpha(Z) = \sum_{i=1}^{n} \delta_d(x_{\sigma(i)}, x_{\sigma(i-1)}) \mu\left(A_{\sigma(i)}\right) \qquad \text{and} \qquad \lambda_\alpha(Z) = M_1(0,\ldots,0) = 0.$$

Finally,

$$Z = \sum_{i=1}^{n} \delta_d(x_{\sigma(i)}, x_{\sigma(i-1)}) \mu\left(A_{\sigma(i)}\right) = C_{\mu,\delta_d}.$$

□

A discrete IV Choquet integral is idempotent if and only if d has neutral element 0_L.

Theorem 5.19. *Let n be a positive integer. Let $C_{\bigoplus,m,d}$ be an n-ary discrete IV Choquet integral given in Corollary 5.13. Then $C_{\bigoplus,m,d}$ is idempotent for any IV fuzzy measure m on $[n]$ if and only if the dissimilarity function d satisfies $d(X, 0_L) = X$ for all $X \in L([0,1])$.*

Proof. Let $X \in L([0,1])$, let us denote $C_{\mu,\delta}(X,\dots,X) = Z$. The proof follows from the observation:

$$K_\alpha(Z) = \sum_{i=1}^{n} K_\alpha\Big(d\left(X_{\sigma(i)}, X_{\sigma(i-1)}\right) m\left(A_{\sigma(i)}\right)\Big) =$$

$$= K_\alpha\Big(d\left(X, 0_L\right)\Big) + \sum_{i=2}^{n} K_\alpha\Big(d\left(X, X\right) m\left(A_{\sigma(i)}\right)\Big) = K_\alpha\Big(d\left(X, 0_L\right)\Big)$$

and

$$\lambda_\alpha(Z) = M_s\Big(\lambda_\alpha\left(d\left(X, 0_L\right)\right), \lambda_\alpha\left(0_L\right), \dots, \lambda_\alpha\left(0_L\right)\Big) = \lambda_\alpha\left(d\left(X, 0_L\right)\right).$$

□

Corollary 5.20. *Let n be a positive integer. Let $C_{\bigoplus,m,d}$ be an n-ary discrete IV Choquet integral given in Corollary 5.13 where d is given in Proposition 4.18 and induced by δ_d, M_d. Then $C_{\bigoplus,m,d}$ is idempotent for any IV fuzzy measure m on $[n]$ if and only if the $\delta_d(x,0) = x$ and $M_d(x,0) = x$ for all $x \in [0,1]$.*

Proof. The proof follows from Theorem 5.19 and the observation:

$$K_\alpha\big(d(X, 0_L)\big) = \delta_d\big(K_\alpha(X), 0\big),$$

$$\lambda_\alpha\big(d(X, 0_L)\big) = M_d\big(\lambda_\alpha(X), 0\big).$$

□

Regarding the monotonicity (with respect to $\le_{\alpha,\beta}$ order) of $C_{\bigoplus,m,d}$ with respect to a fuzzy measure $m : 2^{[n]} \to [0,1]$ we obtained a similar result as in the case of $C_{\sum,m,d}$.

Theorem 5.21. *Let n be a positive integer and $\alpha, \beta \in [0,1]$ with $\alpha \neq \beta$. Let $C_{\bigoplus,m,d} : (L([0,1]))^n \to L([0,\infty[)$ be an n-ary discrete IV Choquet integral given by Corollary 5.13. Then*

(i) For any fuzzy measure $m : 2^{[n]} \to [0,1]$ it holds

$$C_{\bigoplus,m,d}\left(X_1, \dots, X_n\right) \le_{\alpha,\beta} C_{\bigoplus,m,d}\left(Y_1, \dots, Y_n\right)$$

for all $X_1, \dots, X_n, Y_1, \dots, Y_n \in L([0,1])$ such that $X_1 \le_{\alpha,\beta} Y_1, \dots, X_n \le_{\alpha,\beta} Y_n$.

whenever

(ii) For all $k \in [n]$ it holds

$$d(0_L, X_1) + d(X_1, X_2) + \ldots + d(X_{k-1}, X_k) \le_{\alpha,\beta} d(0_L, Y_1) + d(Y_1, Y_2) + \ldots + d(Y_{k-1}, Y_k)$$

for all $X_1, \dots, X_k, Y_1, \dots, Y_k \in L([0,1])$ such that $X_1 \le_{\alpha,\beta} \ldots \le_{\alpha,\beta} X_k$, $Y_1 \le_{\alpha,\beta} \ldots \le_{\alpha,\beta} Y_k$ and $X_1 \le_{\alpha,\beta} Y_1, \dots, X_k \le_{\alpha,\beta} Y_k$.

Proof. First observe that

- $K_\alpha(X) \le K_\alpha(Y)$ implies $K_\alpha(c\,X) \le K_\alpha(c\,Y)$ for all $X, Y \in L([0,1])$ and $c \in [0,1]$;
- $K_\alpha\Big(X_1 + \ldots + X_s\Big) = K_\alpha\left(X_1\right) + \ldots + K_\alpha\left(X_s\right)$ for all $X_1, \dots, X_s \in L([0,1])$ and any positive integer s.

Let $X_1, \ldots, X_n, Y_1, \ldots, Y_n \in L([0,1])$ be such that $X_1 \leq_{\alpha,\beta} Y_1, \ldots, X_n \leq_{\alpha,\beta} Y_n$. Then there exist permutations σ_x, σ_y of $[n]$ such that $X_{\sigma_x(1)} \leq_{\alpha,\beta} \ldots \leq_{\alpha,\beta} X_{\sigma_x(n)}$, $Y_{\sigma_y(1)} \leq_{\alpha,\beta} \ldots \leq_{\alpha,\beta} Y_{\sigma_y(n)}$ and $X_{\sigma_x(1)} \leq_{\alpha,\beta} Y_{\sigma_y(1)}, \ldots, X_{\sigma_x(n)} \leq_{\alpha,\beta} Y_{\sigma_y(n)}$. Let us denote, for any $r \in [n]$, $m(\{r, r+1, \ldots, n\})$ by m_r. From (ii) we have

$$K_\alpha\Big((m_1 - m_2)d\left(0_L, X_{\sigma_x(1)}\right)\Big) \leq K_\alpha\Big((m_1 - m_2)d\left(0_L, Y_{\sigma_y(1)}\right)\Big) \tag{21}$$

and

$$K_\alpha\Big((m_2-m_3)\left(d\left(0_L, X_{\sigma_x(1)}\right) + d\left(X_{\sigma_x(1)}, X_{\sigma_x(2)}\right)\right)\Big) \leq K_\alpha\Big((m_2-m_3)\left(d\left(0_L, Y_{\sigma_y(1)}\right) + d\left(Y_{\sigma_y(1)}, Y_{\sigma_y(2)}\right)\right)\Big), \tag{22}$$

thus

$$K_\alpha\Big((m_1 - m_3)d\left(0_L, X_{\sigma_x(1)}\right) + (m_2 - m_3)d\left(X_{\sigma_x(1)}, X_{\sigma_x(2)}\right)\Big) \leq \tag{23}$$

$$\leq K_\alpha\Big((m_1 - m_3)d\left(0_L, Y_{\sigma_y(1)}\right) + (m_2 - m_3)d\left(Y_{\sigma_y(1)}, Y_{\sigma_y(2)}\right)\Big).$$

Repeating a similar procedure we have

$$K_\alpha\Big((m_1-m_n)d\left(0_L, X_{\sigma_x(1)}\right) + (m_2-m_n)d\left(X_{\sigma_x(1)}, X_{\sigma_x(2)}\right) + \ldots + (m_{n-1}-m_n)d\left(X_{\sigma_x(n-2)}, X_{\sigma_x(n-1)}\right)\Big) \leq \tag{24}$$

$$\leq K_\alpha\Big((m_1 - m_n)d\left(0_L, Y_{\sigma_y(1)}\right) + (m_2 - m_n)d\left(Y_{\sigma_y(1)}, Y_{\sigma_y(2)}\right) + \ldots + (m_{n-1} - m_n)d\left(Y_{\sigma_y(n-2)}, Y_{\sigma_y(n-1)}\right)$$

Finally, since

$$K_\alpha\Big(m_n\left(d\left(0_L, X_{\sigma_x(1)}\right) + d\left(X_{\sigma_x(1)}, X_{\sigma_x(2)}\right) + \ldots + d\left(X_{\sigma_x(n-1)}, X_{\sigma_x(n)}\right)\right)\Big) \leq \tag{25}$$

$$\leq K_\alpha\Big(m_n\left(d\left(0_L, Y_{\sigma_y(1)}\right) + d\left(Y_{\sigma_y(1)}, Y_{\sigma_y(2)}\right) + \ldots + d\left(Y_{\sigma_y(n-1)}, Y_{\sigma_y(n)}\right)\right)\Big),$$

it holds

$$K_\alpha\Big(m_1 d\left(0_L, X_{\sigma_x(1)}\right) + m_2 d\left(X_{\sigma_x(1)}, X_{\sigma_x(2)}\right) + \ldots + m_n d\left(X_{\sigma_x(n-1)}, X_{\sigma_x(n)}\right)\Big) \leq \tag{26}$$

$$\leq K_\alpha\Big(m_1 d\left(0_L, Y_{\sigma_y(1)}\right) + m_2 d\left(Y_{\sigma_y(1)}, Y_{\sigma_y(2)}\right) + \ldots + m_n d\left(Y_{\sigma_y(n-1)}, Y_{\sigma_y(n)}\right)\Big),$$

that is

$$K_\alpha\Big(C_{\bigoplus,m,d}\left(X_1, \ldots, X_n\right)\Big) \leq K_\alpha\Big(C_{\bigoplus,m,d}\left(Y_1, \ldots, Y_n\right)\Big). \tag{27}$$

Now there are two possibilities:

1. There is at least one strict inequality among the Equations (21)–(26), then also the inequality in Equation 27 is strict and the proof is finished.
2. All the Equations (21)–(26) are equalities, then each K_α in (21)–(26) can be replaced by λ_α and we obtain
$$\lambda_\alpha\Big(C_{\bigoplus,m,d}\left(X_1, \ldots, X_n\right)\Big) \leq \lambda_\alpha\Big(C_{\bigoplus,m,d}\left(Y_1, \ldots, Y_n\right)\Big)$$
which completes the proof.

□

Example 5.22. Consider the same situation as in Example 5.7. Then, for $\beta > 0.5$, the fuzzy measure given by $m(\emptyset) = 0$ and $m(A) = 1$ otherwise and a strictly monotone aggregation function M_s (from Theorem 5.11), we have

$$K_\alpha\Big(C_{\bigoplus,m,d}\,(X_1, X_2, X_3)\Big) = 0.9 = K_\alpha\Big(C_{\bigoplus,m,d}\,(Y_1, Y_2, Y_3)\Big)$$

and

$$\lambda_\alpha\Big(C_{\sum,m,d}\,(X_1, X_2, X_3)\Big) = M_s(0.5, 0.8\overline{3}, 0.8\overline{3}) > M_s(0.5, 0.\overline{6}, 0.\overline{6}) = \lambda_\alpha\Big(C_{\sum,m,d}\,(Y_1, Y_2, Y_3)\Big)$$

which implies that

$$C_{\bigoplus,m,d}\,(X_1, X_2, X_3) >_{\alpha,\beta} C_{\sum,m,d}\,(Y_1, Y_2, Y_3)\,.$$

So, the monotonicity of $C_{\bigoplus,m,d}$ is violated (in a similar way as that of $C_{\sum,m,d}$).

6. Application: Combining classifiers with the interval-valued Choquet integral

In this section an application of the interval-valued choquet integral for aggregating the decision of multiple classifiers whose predictions are given in terms of interval-valued data is proposed.

6.1. Motivation

In machine learning it is known that, when dealing with classification problems, the accuracy of an individual classifier can be improved by training and combining small variations of the algorithm [37, 38]. These variations can be achieved by the use of different parameters of the algorithm or, as in this work, by considering different subsets of the training set so that each classifier is trained with a different dataset. Once the base classifiers are learned, it is necessary to combine the prediction of each classifier to assess the final class label. Here is where the aggregation procedure takes an important role. The set of classifiers aimed at solving a specific task is usually referred as ensemble.

The aggregation step of an ensemble is usually performed by combining the probabilities (or confidence degrees) of each class provided by each classifier into a single probability value. Thus, the class with the highest probability is predicted. The combination is usually performed by classical aggregation functions, such as the arithmetic mean. However, it has been proven that the use of more sophisticated aggregation functions, such as weighted means, OWA opertors or fuzzy integrals, can lead to an improvement of the classification accuracy of the ensemble [39, 40, 41]. In the case of fuzzy integrals (Choquet or Sugeno), the underlying fuzzy measure is able to capture positive and negative interaction among the classifiers of the ensemble.

However, some classification algorithms provide the probability or confidence degree of a certain class not by a real number, but using more complex structures, such as intervals. This is the case, for example of IVTURS (Interval-Valued fuzzy reasoning method with TUning and Rule Selection) [30]. Then, when considering an ensemble of interval-valued classifiers, the aggregation step is not a trivial task, specially if aggregation functions such as the Choquet integral are considered.

The objective of this section is to evaluate the use of the proposed interval-valued Choquet integral for combining the predictions of an ensemble of IVTURS classifiers, whose predictions are interval-valued. The accuracy of the proposed methodology will be compared with existing alternatives of the interval-valued Choquet integral [26].

6.2. Interval-valued Choquet Integral for Aggregating Classifiers

This section describes how the ensemble of interval-valued classifiers is learned and how the interval-valued Choquet integral is used for combining their outputs.

IVTURS [30] is used as an interval-valued classifier which takes an example with real-valued features as input and uses interval-valued fuzzy sets to model the labels in the fuzzy rule-based classification system. As a consequence, all the fuzzy reasoning method is designed to work with intervals and hence, the confidence for each class is given by an interval. When using a single IVTURS model, the class with the largest interval-valued confidence is predicted. In our case, we will use IVTURS as a baseline classifier for learning a pool of classifiers that will form an ensemble. To learn this ensemble, the well-known Bagging procedure is considered [42]. In Bagging, each classifier is learned with a different dataset created from the same original training dataset with random sampling with replacement (some examples will appear twice or more and others will not appear at all).

Therefore, in the ensemble of Bagging+IVTURS, each base classifier will provide an interval-valued confidence for each example and class. In the case of two class problems, each sample will be associated to N intervals for the class label 0 (b_j^0, $j = 1, \ldots, N$) and N intervals for the class label 1 (b_j^1, $j = 1, \ldots, N$), being N the number of base classifiers in the ensemble. Usually, the confidences for each class (intervals in this case) are aggregated and the class with the largest confidence is assigned to the example. However, in the case of IVTURS the confidences are not normalized among classifiers and hence, normalization is required before aggregating them. In this case, we will follow the normalization proposed in [43], where the intervals are normalized by the upper bounds (the upper bounds of the normalized intervals sum up to one). Thus, we will have that, for each classifier j, $\overline{b}_j^0 + \overline{b}_j^1 = 1$. Finally, to compute the global confidence of each class, the N intervals will be aggregated by the interval-valued Choquet integral proposed in this work, considering $C_{\sum,m,d}$ where d is an IV dissimilarity function with respect to $\leq_{\alpha,\beta}$, $\delta_d(x, y) = |x - y|$ and M_d is the arithmetic mean. The final class label will be the one whose global confidence is the greatest.

When considering the Choquet integral for aggregating data, one of the key aspects is the estimation of the underlying fuzzy measure, since it is the responsible for modeling the interaction among data. Although in this paper the proposed interval-valued Choquet integral is given with respect to a interval-valued fuzzy measure, for the sake of simplicity, we have consider a classical fuzzy measure whose codomain is $[0, 1]$ instead of $L([0, 1])$. In this section, the estimation of the fuzzy measure will be done using the CPM construction method [39], which is specifically designed for classifier aggregation using the Choquet integral. This method estimates each coefficient of the fuzzy measure by considering the accuracy of each possible coalition of classifiers.

Finally, the algorithm for assigning the class label to a test instance is described in Algorithm 1.

Algorithm 1 Classification step using $C_{m,d}$

Input: x: instance to be classified; $b_1^0, \ldots, b_N^0$: IV confidences of class 0; $b_1^1, \ldots, b_N^1$: IV confidences of class 1; $C_{\sum,m,d}$: IV Choquet integral function w.r.t. $\leq_{\alpha,\beta}$
Output: Class label $Class(x)$

Estimates the fuzzy measure $m : 2^{\{1,\ldots,N\}} \to [0, 1]$ using CPM
Normalize the IV confidences for each classifier j: $b_j^0 = \frac{b_j^0}{\overline{b}_j^0+\overline{b}_j^1}$, $b_j^1 = \frac{b_j^1}{\overline{b}_j^0+\overline{b}_j^1}$
Aggregate the confidences of class 0: $Y^0 = C_{\sum,m,d}(b_1^0, \ldots, b_N^0))$
Aggregate the confidences of class 1: $Y^1 = C_{\sum,m,d}(b_1^1, \ldots, b_N^1))$
Predict class with largest confidence according to $\leq_{\alpha,\beta}$

6.3. Experimental framework

For the experiments, 24 binary datasets from the KEEL dataset repository [44] are considered. For each dataset, the total number of examples and number of attributes are presented in Table 1. To estimate the performance of each method in each dataset a 5-fold stratified cross-validation (SCV) is carried out. Hence, the metric for each dataset and method will be obtained by averaging five runs. Accuracy measure will be used to evaluate the performance of each method.

The rest of the parameters used for the experiments are presented in Table 2. Notice that the parameters of IVTURS are the ones recommended by the authors [30]. Likewise, 21 classifiers are used for Bagging to have an moderately large ensemble due to the runtimes required to learn an IVTURS model.

Non-parametric statistical tests will be used to analyze the results obtained, as recommended in the literature [45]. To compare pairs of methods, the Wilcoxon test is considered, whereas the Friedman aligned-ranks test is applied for multiple comparisons tests followed by Holm *post-hoc* test in case of significant differences being found. Statistically significant differences will be considered at 95% confidence (p-value lower than 0.05).

Table 1: Summary of the datasets considered in this study.

Data-sets	#Ex.	#Atts.	#Cl 0	#Cl 1
appendicitis	106	7	85	21
banana	5300	2	2924	2376
bands	365	19	135	230
breast	277	9	196	81
chess	3196	36	1527	1669
crx	653	15	357	296
haberman	306	3	225	81
heart	270	13	150	120
hepatitis	80	19	13	67
housevotes	232	16	124	108
ionosphere	351	33	126	225
mammographic	830	5	427	403
monk-2	432	6	204	228
mushroom	5644	22	3488	2156
phoneme	5404	5	3818	1586
pima	768	8	500	268
ring	7400	20	3664	3736
saheart	462	9	302	160
sonar	208	60	111	97
spectfheart	267	44	55	212
tic-tac-toe	958	9	332	626
titanic	2201	3	1490	711
wdbc	569	30	357	212
wisconsin	683	9	444	239

Table 2: Parameters for IVTURS and BAGGING algorithm.

Algorithm	*Parameters*
IVTURS	Num labels = 5, Min support = 0.05 Min confidence = 0.8, Deph of Trees = 3 K = 2, Max evaluations = 20000 Population size = 50, Alpha = 0.15 Bits per gene = 30, Type of inference = 1
Bagging	Number of final classifiers = 21 Bag size = size of training set

For the definition of the admissible order $\leq_{\alpha,\beta}$ associated with the IV Choquet integral, all the combinations of $\alpha, \beta \in \{0, 1/3, 1/2, 2/3, 1\}$ have been considered. However, notice that according to [24], given $\alpha \in [0, 1[$ ($\alpha \in]0, 1]$), all admissible orders $\leq_{\alpha,beta}$ with $\beta > \alpha$ ($\beta < \alpha$) coincide. We denote this admissible order as $\leq_{\alpha+}$ ($\leq_{\alpha-}$). Then, all the combinations yield to a total of 8 potentially different admissible orders, namely $\leq_{0+}$, $\leq_{1/3-}$, $\leq_{1/3+}$, $\leq_{1/2-}$, $\leq_{1/2+}$, $\leq_{2/3-}$, $\leq_{2/3+}$ and $\leq_{1-}$.
Remark: although most of the expressions of the IV Choquet presented in this paper assume $\alpha \in]0, 1[$, we have consider $\alpha = \epsilon$ and $\alpha = 1 - \epsilon$ for the application.

For comparing Algorithm 1 with other approaches, we have considered exactly the same steps by using an alternative definition of the IV Choquet integral. Specifically, we have considered the IV Choquet given in [26] considering the same admissible orders. It is important to note that this comparison can be done since m in Algorithm 1 is a fuzzy measure and not an interval-valued fuzzy measure. We will denote this IV Choquet integral by C_m.

6.4. Experimental study

The results obtained in terms of accuracy for each IV Choquet integral (the proposed one that is named as $C_{\sum,m,d}$ and C_m) and dataset are presented in Table 3. The best result for each dataset is stressed in **bold-face**. The last row summarizes the results over all datasets showing the average performance.

Overall, small differences in terms of accuracy are found (1% between the maximum and the minimum average values). However, this was expected as only the aggregation is being changed, being all the other components exactly the same (the base classifiers and their IV outputs). The best performers in terms of average accuracy are both IVC-dis with $\alpha = 0.66$. Notice that however, β has also a much lower importance, changing the results only in the case of phoneme dataset. This behaviour is repeated for almost all α values when comparing the effect of different β values. We recall that when using admissible orders based on $\leq_{\alpha,\beta}$, the parameter β is only applied in case of ties with α, which sporadically occurs.

Rather than looking at the results themselves, we will continue our analysis based on non-parametric statistical tests. First, both IV Choquet integrals will be compared when using the same α and β values. To do so, pairwise Wilcoxon tests are run, whose results are presented in Table 4.

Table 3: Average test accuracy results for each dataset and configuration tested.

Dataset	$C_{\sum,m,d}$								C_m							
	$\leq_{0+}$	$\leq_{1/3-}$	$\leq_{1/3+}$	$\leq_{1/2-}$	$\leq_{1/2+}$	$\leq_{2/3-}$	$\leq_{2/3+}$	$\leq_{1-}$	$\leq_{0+}$	$\leq_{1/3-}$	$\leq_{1/3+}$	$\leq_{1/2-}$	$\leq_{1/2+}$	$\leq_{2/3-}$	$\leq_{2/3+}$	$\leq_{1-}$
appendicitis	0.8494	0.8398	0.8398	0.8398	0.8398	0.8489	0.8489	**0.8584**	0.8494	0.8398	0.8398	**0.8584**	**0.8584**	**0.8584**	**0.8584**	**0.8584**
banana	0.8192	**0.8274**	**0.8274**	0.8230	0.8230	0.8226	0.8226	0.8194	0.8189	0.8211	0.8211	0.8215	0.8215	0.8206	0.8206	0.8192
bands	0.6915	0.6971	0.6971	0.7025	0.7025	0.7025	0.7025	0.6863	0.6941	0.6944	0.6944	**0.7052**	**0.7052**	0.6943	0.6943	0.6832
breast	0.7620	**0.7728**	**0.7728**	0.7616	0.7616	0.7656	0.7656	0.7627	0.7584	0.7621	0.7621	0.7618	0.7618	0.7586	0.7586	0.7589
chess	0.9484	0.9643	0.9643	0.9659	0.9659	0.9659	0.9659	**0.9662**	0.9484	0.9637	0.9640	0.9643	0.9643	0.9628	0.9628	0.9640
crx	0.8714	0.8699	0.8699	0.8699	0.8699	**0.8730**	**0.8730**	**0.8730**	0.8637	0.8652	0.8652	0.8668	0.8668	0.8668	0.8668	0.8668
haberman	0.7547	0.7449	0.7449	0.7416	0.7416	0.7416	0.7416	0.7351	**0.7678**	0.7515	0.7515	0.7482	0.7482	0.7449	0.7449	0.7384
heart	**0.8667**	0.8593	0.8593	0.8593	0.8593	0.8556	0.8556	0.8481	0.8630	0.8630	0.8630	0.8556	0.8556	0.8481	0.8481	0.8481
hepatitis	0.8908	0.9042	0.9042	0.9042	0.9042	**0.9175**	**0.9175**	0.9032	0.8899	0.9042	0.9042	0.9042	0.9042	0.8899	0.8899	0.8899
housevotes	0.9646	0.9646	0.9646	0.9602	0.9602	0.9566	0.9566	0.9520	**0.9689**	0.9646	0.9646	0.9646	0.9646	0.9646	0.9646	0.9646
ionosphere	0.9118	0.9118	0.9118	0.9089	0.9089	0.9118	0.9118	**0.9204**	0.9174	0.9146	0.9146	0.9146	0.9146	0.9147	0.9147	0.9118
mammographic	0.8407	0.8383	0.8383	0.8322	0.8322	0.8297	0.8297	0.8196	**0.8433**	0.8431	0.8431	0.8369	0.8369	0.8286	0.8286	0.8237
monk-2	0.9814	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	0.9814	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**
mushroom	0.9959	0.9996	0.9996	**1.0000**	**1.0000**	0.9998	0.9998	0.9991	0.9952	0.9993	0.9993	0.9998	0.9998	0.9995	0.9995	0.9989
phoneme	0.8038	**0.8090**	**0.8090**	0.8079	0.8075	0.8077	0.8079	0.7996	0.8057	0.8077	0.8077	0.8070	0.8068	0.8064	0.8061	0.7981
pima	0.7604	0.7617	0.7617	0.7656	0.7656	0.7604	0.7604	0.7487	0.7643	0.7682	**0.7695**	0.7630	0.7630	0.7630	0.7630	0.7513
ring	0.8441	0.9230	0.9230	0.9374	**0.9376**	0.9369	0.9368	0.8839	0.8461	0.9246	0.9246	0.9343	0.9342	0.9361	0.9361	0.8949
saheart	0.7055	0.7120	0.7120	0.7142	0.7142	0.7099	0.7099	0.6969	0.7120	**0.7164**	**0.7164**	0.7099	0.7099	0.7099	0.7099	0.7076
sonar	0.8367	0.8415	0.8415	0.8463	0.8463	**0.8609**	**0.8609**	0.8560	0.8462	0.8318	0.8318	0.8318	0.8318	0.8367	0.8367	0.8222
spectfheart	0.7751	0.7902	0.7902	0.8015	0.8015	**0.8052**	**0.8052**	0.7864	0.7751	0.7977	0.7977	0.7940	0.7940	0.7978	0.7978	0.7903
tic-tac-toe	0.9248	0.9656	0.9666	0.9770	0.9770	**0.9802**	**0.9802**	0.9760	0.9342	0.9603	0.9603	0.9718	0.9718	0.9760	0.9760	0.9760
titanic	0.7815	0.7883	0.7883	0.7883	0.7883	0.7883	0.7883	**0.7887**	0.7824	0.7860	0.7860	0.7860	0.7860	0.7865	0.7865	0.7865
wdbc	0.9543	0.9649	0.9649	**0.9684**	**0.9684**	**0.9684**	**0.9684**	0.9666	0.9561	0.9631	0.9631	**0.9684**	**0.9684**	**0.9684**	**0.9684**	**0.9684**
wisconsin	0.9708	**0.9752**	**0.9752**	**0.9752**	**0.9752**	0.9738	0.9738	0.9737	0.9722	**0.9752**	**0.9752**	**0.9752**	**0.9752**	0.9738	0.9738	**0.9752**
Average	0.8544	0.8635	0.8636	0.8646	0.8646	**0.8659**	**0.8659**	0.8592	0.8564	0.8632	0.8633	0.8643	0.8643	0.8628	0.8627	0.8582

Table 4: Wilcoxon tests comparing the two IV Choquet integrals with the same $\leq_{\alpha,\beta}$ configurations.

	$\leq_{0+}$	$\leq_{1/3-}$	$\leq_{1/3+}$	$\leq_{1/2-}$	$\leq_{1/2+}$	$\leq_{2/3-}$	$\leq_{2/3+}$	$\leq_{1-}$
Rank+ ($C_{\sum,m,d}$)	79.50	181.00	183.00	195.00	195.00	236.00	236.00	176.00
Rank- (C_m)	245.50	144.00	142.00	130.00	130.00	89.00	89.00	149.00
p-value	**0.0255**	0.6185	0.5812	0.3818	0.3818	**0.0479**	**0.0479**	0.7164

Table 5: Holm's post tests results after Friendman aligned-ranks test comparing the different configurations in each IV Choquet integral.

$\leq_{\alpha,\beta}$	$C_{\sum,m,d}$	C_m
$0-$	142.86 (0.0010*)	129.78 (0.0132*)
$1/3-$	95.80 (1.0000)	87.78 (1.0000)
$1/3+$	95.50 (1.0000)	87.18 (1.0000)
$1/2-$	90.28 (1.0000)	79.64 -
$1/2+$	90.92 (1.0000)	80.10 (1.0000)
$2/3-$	80.72 (1.0000)	101.74 (0.8241)
$2/3+$	80.52 -	102.38 (0.8241)
$1-$	127.40 (0.0251*)	135.40 (0.0046*)

Table 6: Wilcoxon test comparing the best configuration for each IV Choquet integral.

Comparison	Rank+	Rank-	p-value
$C_{\sum,m,d} \leq_{2/3+}$ VS $C_m \leq_{1/2-}$	186.00	139.00	0.5270

Attending at these results, one can observe that in all comparisons except for the extreme case where $\alpha = 0$, the new proposal ($C_{\sum,m,d}$) provides higher number of ranks. Moreover, statistical differences are found when $\alpha = 0.66$. As mentioned, in the case of $\alpha = 0$, C_m performs statistically better, but looking at the average performance, one can understand that this extreme value leads to lower accuracy. In fact, this is what will be studied in our next analysis. Our aims is to check first which is the best configuration for each IV Choquet integral and then, compare the best configurations between them.

To do so, Friedman aligned-ranks test is applied to compare the different configurations for each IV Choquet integral. In both cases, the p-value indicates that significant differences exists among them (0.0023 and 0.0024 for $C_{\sum,m,d}$ and C_m, respectively). Hence, Holm's *post-hoc* test is performed for each comparison to analyze the differences with respect to the control method (the best configuration in each case). The results of these tests are presented in Table 5. The first two columns corresponds to the configuration in the corresponding row. Then, each column represents a comparison among the configurations of $C_{\sum,m,d}$ and C_m, respectively. For each configuration, the ranks obtained (the lower the better) and the p-value of Holm's test (in brackets) are indicated. An asterisk '*' close to the p-value represents that significant differences are found.

Attending at Table 5, it becomes clear that extreme cases are performing worse than the control methods, which are $\leq_{2/3+}$ and $\leq_{1/2-}$ for $C_{\sum,m,d}$ and C_m, respectively. In fact, the extreme cases are the only ones showing statistical differences in favor of the control methods, whereas no significant differences are found with respect to the other configurations. Continuing with the analysis, a Wilcoxon test is carried out to contrast the best alternatives for each IV Choquet integral. The results of this test are presented in Table 6.

The test shows that no significant differences are found between both IV Choquet integrals. However, the proposed $C_{\sum,m,d}$ gets a higher number of ranks, showing that it is performing better in more datasets. Notice also that the new $C_{\sum,m,d}$ could also work with IV fuzzy measures, which C_m cannot handle.

7. Conclusions

In this work we have proposed a generalization of the discrete Choquet integral, called d_G-Choquet integral, replacing, in the first place, the difference between inputs represented by closed subintervals of the unit interval $[0, 1]$ by a dissimilarity function; and we also replace the sum by more general appropriate functions. This generalization extends both the classical Choquet integral and the recently introduced d-Choquet integral. Furthermore, we have introduced an interval-valued fuzzy measure, that, considered in our generalization, has also allowed us to extend to the interval-valued setting the discrete Choquet integrals, keeping the monotonicity properties under suitable conditions.

Although our motivation for introducing the d_G-Choquet integral has been the analysis of interval-valued Choquet integrals, this new class of integrals offers a large variety of possibilities which should be explored in subsequent works. In particular, d_G-Choquet integrals should be related to C_{F1F2}-integrals. This can be of special interest for applications in the nearby future. For this reason, we expect that the illustrative example considered in this paper can be further expanded to develop new algorithms in classification in future research papers.

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