# Graphical Exchange Mechanisms 

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#### Abstract

Consider an exchange mechanism which accepts "diversified" offers of various commodities and redistributes everything it receives. We impose certain conditions of fairness and convenience on such a mechanism and show that it admits unique prices, which equalize the value of offers and returns for each individual.

We next define the complexity of a mechanism in terms of certain integers $\tau_{i j}, \pi_{i j}$ and $k_{i}$ that represent the time required to exchange $i$ for $j$, the difficulty in determining the exchange ratio, and the dimension of the message space. We show that there are a finite number of minimally complex mechanisms, in each of which all trade is conducted through markets for commodity pairs.

Finally we consider minimal mechanisms with smallest worst-case complexities $\tau=\max \tau_{i j}$ and $\pi=\max \pi_{i j}$. For $m>3$ commodities, there are precisely three such mechanisms, one of which has a distinguished commodity - the money - that serves as the sole medium of exchange. As $m \rightarrow \infty$ the money mechanism is the only one with bounded $(\pi, \tau)$.

JEL Classification: C70, C72, C79, D44, D63, D82. Keywords: exchange mechanism, minimal complexity, prices, markets, money.

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## 1 Introduction

The purpose of this paper is to show how simple criteria of fairness, convenience and complexity can lead to the successive emergence of prices, markets, and money, in a Cournotian setting for commodity exchange. In the process, we arrive at a rationale for money which is purely "mechanistic" in spirit, and complements the existing utilitarian and behavioral literature on the subject (see 1.1).

We consider abstract exchange mechanisms 1 , which accept "diversified" offers of various commodities and redistribute everything, and further satisfy five conditions, embodying fairness and convenience that we term Anonymity, Aggregation, Invariance, Non-dissipation and Flexibility. Although there are infinitely many such mechanisms, our first result is that each admits unique prices 2 , which lead to value conservation, i.e. equalize the value of individual offers and returns.

We next define some natural notions of "complexity" for a mechanism, and, in keeping with the idea of convenience, study mechanisms with minimal complexity. This leads to a finite class $\mathfrak{M}_{*} \subset \mathfrak{M}_{g}$, where $\mathfrak{M}_{g}$ denotes certain graphical mechanisms that are in one-to-one correspondence with directed, connected graphs on the set of commodities. The directed edge $i j$ may be interpreted in $\mathfrak{M}_{*}$ as a market that provides traders the opportunity to offer $i$ in exchange for $j$. Prices not only conserve values in $\mathfrak{M}_{*}$ (in fact, in $\mathfrak{M}_{g}$ ) but mediate trade in the sense that the return to a trader depends only on his own offer and the prices. In short, prices "decouple" the interaction between traders.

The emergence of prices and markets paves the way for the culmination of the analysis, namely the emergence of money. To this end we introduce additional refined criteria of complexity on $\mathfrak{M}_{*}$ and study the corresponding minimal mechanisms, which we term strongly minimal. It turns out that there are only three strongly minimal mechanisms, up to a relabeling of commodities. In one of these, a single commodity emerges endogenously

[^1]as money and mediates trade among decentralized markets for the other commodities. Moreover, with a moderate increase in the number of commodities, the money mechanism quickly supersedes the other two in a very precise sense.

Finally let us mention that, while this paper is a companion to [9], the two are meant to be readable independently. This has necessitated some overlap but it is minimal. To be precise, the conditions on the mechanism (with the exception of Flexibility) appear in [9, so does Proposition 5, and the rest is disjoint.

### 1.1 Related Literature

The emergence of money as a medium of exchange has been a matter of considerable discussion in economics. One approach, following Jevons [16], has focused on search-theoretic models that involve repeated and random bilateral meeting between agents (see, e.g., [2], [15], [17], [19], [20], [23], [26], [42] and the references therein). Another line of inquiry is based on partial or general equilibrium models with various kinds of frictions in trade, such as transactions costs or limited trading opportunities (see, e.g., [10], [11], [12], [13], [14], [27], [28], 40], 41], [43]). These models turn on notions of rational expectations and utility-maximizing behavior of the agents in equilibrium. In contrast, as was said, our analysis is based purely on the mechanism of trade is and independent of the characteristics of the agents, such as their endowments or utilities.

It is worth emphasizing that our analysis is quite agnostic regarding the choice of any particular money $3^{3}$, being only at pains to point out the urgency of appointing some money. For a discussion of different criteria entailed in the choice of a suitable "commodity money" such as its portability, verifiability, divisibility and durability; or, alternatively, the backing of the state requisite to sustain "fiat money", see, e,g., [1], [16], [18], [21], [22], [25], [34]; and, for a recent survey on both kinds of money, see [39] and [40]. It would be interesting to incorporate some of these criteria, as well as the utilitarian considerations for money, into our mechanistic framework.

[^2]This paper is intimately related to [9]. Let us briefly recount the model there. We make the hypothesis in [9] that any offer of a commodity $i$ specifies some other commodity $j$ which is being sought in exchange for $i$. Thus, drawing a directed arc $i j$ for every such offer permitted in the mechanism, we obtain a (directed, connected $\sqrt{4}$ ) graph $G$. There are infinitely many mechanisms for any given $G$, but it is shown in [9] that exactly one of them is categorically determined by four of our five conditions, namely Anonymity, Aggregation, Invariance, Non-dissipation (all except Flexibility). This is the graphical " $G$-mechanism" mentioned earlier; so that the class $\mathfrak{M}_{g}$ is precisely the one generated as $G$ varies over all directed, connected graphs on the fixed node-set of commodities. Our refined complexity criteria apply equally to $\mathfrak{M}_{g}$ as to its subset $\mathfrak{M}_{*}$, and we show that $\mathfrak{M}_{g}$ has the same three strongly minimal mechanisms as $\mathfrak{M}_{*}$. This is the main conclusion of [9] and it constitutes a key step in the proof of the emergence of money in this paper.

Our current paper thus puts the analysis of [9] on a more general footing. We start with a domain which is much richer than $\mathfrak{M}_{g}$, and which permits traders to indulge in "cheap talk" in order to diversify their offers in any commodity $i$. Furthermore, we allow for the possibility that these offers may get complicated bundles of commodities in return. But Flexibility guarantees that there exist special messages in the cheap talk through which a trader can "unbundle" his return, i.e., messages that enable him to get only $j$ in exchange for $i$, whenever $j$ is obtainable - albeit in conjunction with other commodities - via an offer of $i$. Thus flexibility embeds a graph $G$, as a sharply delineated language, within the tangle of cheap talk. We essentially show that our complexity criteria cuts away the tangle, leaving behind only the graphical $G$-mechanisms in $\mathfrak{M}_{*} \subset \mathfrak{M}_{g}$. Thus we deduce the existence of markets (i.e., the edges of the graph G) from an abstract standpoint, instead of postulating them as in [9]. The exact structure of the subdomain $\mathfrak{M}_{*}$ of $\mathfrak{M}_{g}$ is not explored by us, as that would detract from the primary purpose of this paper, which is to arrive expeditiously at the money mechanism.

The other precursor of this paper, at both the technical and conceptual level, is [6], where a mechanism produces not only trades but also prices, based upon everyone's offers; and where price mediation is postulated as a condition. The main result of [6] is that $\mathfrak{M}_{g}$ is characterized by Anonymity, Aggregation, Invariance, Price Mediation and Accessibility (the last repre-

[^3]senting a weak form of continuity). In contrast, here we deduce the existence of prices, as well as their crucial role in mediating trade, based on considerations of a different sort, as outlined in the introduction and formalized in section [ 4 below. Moreover, the analysis in [6] stopped at the characterization of $\mathfrak{M}_{g}$ and did not delve into any further selection among the mechanisms.

## 2 Exchange Mechanisms

Definition $1 A$ pre-mechanism consists of the following data:

1. a commodity space $C=\mathbb{R}_{+}^{m}$;
2. an action space $S=\mathbb{R}_{+}^{K}$ where $K=K_{1} \amalg \cdots \amalg K_{m}$ is a finite set;
3. for each integer $n \geq 1$, a return $\operatorname{map} \rho^{n}: S(n) \rightarrow C^{n}$ where

$$
\begin{equation*}
S(n)=\left\{\left(a^{1}, \ldots, a^{n}\right) \in S^{n}: a^{1}+\cdots+a^{n} \in S_{+}\right\}, \quad S_{+}=\mathbb{R}_{++}^{K} . \tag{1}
\end{equation*}
$$

We refer to the elements of $K_{i}$ as $i$-indices, and for $a \in S$ we define $\bar{a} \in C$ by summing over the various $i$-indices for each $i$; thus we have

$$
\bar{a}=\left(\alpha_{1}, \ldots, \alpha_{m}\right) \text { where } \alpha_{i}=\sum_{h \in K_{i}} a_{h} .
$$

Definition 2 An exchange mechanism is a pre-mechanism which satisfies:

$$
\begin{equation*}
\text { if }\left(r^{1}, \ldots, r^{n}\right)=\rho^{n}\left(a^{1}, \ldots, a^{n}\right) \text { then } r^{1}+\cdots+r^{n}=\overline{a^{1}}+\cdots+\overline{a^{n}} . \tag{2}
\end{equation*}
$$

The interpretation is as follows. There is an underlying set $\{1, \ldots, m\}$ of commodities ${ }^{5}$, and for each $i$-index $h \in K_{i}$, the component $a_{h}$ of an action $a$ represents an offer of commodity $i$. Thus we also refer to $a$ as an offer, and the various $i$-indices serve to "diversify" the offer in $i$. An exchange mechanism enables trade as follows: having received $n$ offers $\left(a^{1}, \ldots, a^{n}\right)$, which are "positive on the aggregate" (11), it redistributes the commodities according to $\rho^{n}$. Condition (2) means that commodities are conserved.

As mentioned already in section 1.1, one possible interpretation of the indices is as a common language in which traders may communicate with the mechanism $M$. The language is completely abstract: no structure is

[^4]imposed on it except that it be of finite size. The elements of $K_{i}$ may be thought of as costless messages ("cheap talk") that accompany offers in $i$. Another interpretation of $K_{i}$ could be that it represents different times (or, places) when (or, where) the offer of $i$ is sent. The reader can no doubt think of still more interpretations. It is to make room for all these that we have used the neutral term "index". However, in the context of graphical mechanisms which emerge out of our analysis (see section (4), it turns out that $i$-indices have a concrete economic interpretation as certain commodity pairs $i j$, representing "markets" where $i$ can be exchanged for $j$.

### 2.1 Conditions on the Mechanisms

In order to describe our conditions, we need some notation. We first define scaling actions of $\lambda \in \mathbb{R}_{++}^{m}$ on $r \in C$ and $a \in S$ via

$$
(\lambda * r)_{i}=\lambda_{i} r_{i} \text { for all } i, \quad(\lambda * a)_{h}=\lambda_{i} a_{h} \text { for all } h \in K_{i} .
$$

Let $\rho: S(2) \rightarrow C$ denote the first component of the two trader return function $\rho^{2}$. If $h \in K_{i}$ is an $i$-index then the unit vector $e_{h} \in S$ is an offer in commodity $i$ alone. The $j$-th component $\rho_{j}\left(e_{h}, a\right)$ of $\rho\left(e_{h}, a\right)$ is the return of commodity $j$ to trader 1 , when he offers $e_{h}$ and the other offers $a$.

Definition 3 We say $h \in K_{i}$ is an $i j$-index if for some a we have

$$
\rho_{j}\left(e_{h}, a\right)>0
$$

We say an ij-index $h$ is pure if for all a we have

$$
\rho_{k}\left(e_{h}, a\right)=0 \text { for all } k \neq j .
$$

Our five conditions on an exchange mechanism, termed Anonymity, $A g$ gregation, Invariance, Non-dissipation, and Flexibility, are as follows.

Condition 4 If $\rho^{n}\left(a^{1}, \ldots, a^{n}\right)=\left(r^{1}, \ldots, r^{n}\right)$ then we have

1. $\rho^{n}\left(a^{\sigma(1)}, \ldots, a^{\sigma(n)}\right)=\left(r^{\sigma(1)}, \ldots, r^{\sigma(n)}\right)$ for any permutation $\sigma$.
2. $\rho^{n-1}\left(a^{1}, \ldots, a^{n-2}, a^{n-1}+a^{n}\right)=\left(r^{1}, \ldots, r^{n-2}, r^{n-1}+r^{n}\right)$.
3. $\rho^{n}\left(\lambda * a^{1}, \ldots, \lambda * a^{n}\right)=\left(\lambda * r^{1}, \ldots, \lambda * r^{n}\right)$ for all $\lambda \in \mathbb{R}_{++}^{m}$.
4. For each $i, r^{i}-\overline{a^{i}}$ is either 0 or it has a strictly positive component.
5. For all $i j$, if there is an $i j$-index then there is a pure $i j$-index.

For such a mechanism $M$, by Anonymity and Aggregation we have
$\rho^{n}\left(a^{1}, \ldots, a^{n}\right)=\left(\rho\left(a^{1}, a^{-1}\right), \ldots, \rho\left(a^{n}, a^{-n}\right)\right)$ where $a^{-i}=a^{1}+\cdots+a^{n}-a^{i}$.
Thus $M$ is uniquely determined by $\rho$, and also by the trade and net trade functions, which are defined as follows:

$$
r(a, b)=\rho(a, b-a), \quad \nu(a, b)=r(a, b)-\bar{a} .
$$

These latter functions have domain $\left\{(a, b) \in S \times S_{+}: a \leq b\right\}$.
Proposition $5 \nu$ admits a unique extension to $S \times S_{+}$satisfying

$$
\nu(t a, b)=t \nu(a, b), \quad \nu(a, t b)=\nu(a, b) \text { for all } t>0 .
$$

Proof. See Lemma 1 of [6]. Although [6] considers a more restrictive class of mechanisms, we note that the proof of Lemma 1 there only uses Anonymity, Aggregation, and Invariance.

In view of the above result, we will drop the restriction $a \leq b$ for $\nu(a, b)$.

### 2.2 Comments on the Conditions

The first condition is Anonymity; it stipulates that the mechanism be blind to all characteristics of a trader other than his offer. The second condition is Aggregation, it asserts that if any trader pretends to be two different persons by splitting his offer, the returns to the others is unaffected. Aggregation does not imply that if two individuals were to merge, they would be unable to enhance their "oligopolistic power". For despite the aggregation condition, the merged individuals are free to coordinate their actions by jointly picking a point in the Cartesian product of their action spaces. Indeed all the mechanisms we obtain display this "oligopolistic effect", even though they also satisfy Aggregation. The two conditions embody fairness, enabling free entry for any new participant on non-discriminatory terms, and thereby making the mechanism more "inclusive". They also contribute to convenience, if either of these conditions were violated, trade would become a cumbersome affair: each individual would need to keep track of the full distribution of
offers across the entire population, and then figure out how to diversify his own offers in response.

The third condition is Invariance; its main content is that the maps $\rho^{n}$ which comprise the mechanism are invariant under a change of units in which commodities are measured. This makes the mechanism much simpler to operate in: one does not need to keep track of seven pounds or seven kilograms or seven tons, just the numeral 7 will do. It is worthy of note that the cuneiform tablets of ancient Sumeria, which are some of the earliest examples of written language and arithmetic, are in large part devoted to records and receipts pertaining to economic transactions. Invariance postulates the "numericity" property of the $\rho^{n}$ making them independent of the underlying choice of units, and this goes to the very heart of the quantitative measurement of commodities. In its absence, one would need to figure out how the maps are altered when units change, as they are prone to do, especially in a dynamic economy. This would make the mechanism cumbersome to use.

The fourth condition is Non-dissipation; it says that no trader's return can be less commodity-wise than his offer. If it were violated, such unfortunate traders would find it grossly unfair and tend to abandon the mechanism. In conjunction with Aggregation, Anonymity, and the conservation of commodities, this immediately implies no-arbitrage:

$$
\text { for any } a, b \text { neither } \nu(a, b) \nsupseteq 0 \text { nor } \nu(a, b) \nsupseteq 0 \text {. }
$$

To see this, note that in view of Proposition 5 we need consider only the case $a \leq b$ and rule out $\nu(a, b) \not \geqq 0$. Denote $c=b-a$. Then

$$
\nu(a, b)+\nu(c, b)=\nu(a+c, b)=\nu(b, b)=0
$$

where the first equality follows from Aggregation, and the last from conservation of commodities. But then $\nu(a, b) \nsupseteq 0$ implies $\nu(c, b) \nsupseteq 0$, contradicting Non-dissipation.

The fifth and final condition is Flexibility, it reflects the perspective of a trader who wishes to interact with the mechanism to exchange a single commodity $i$ for some other commodity $j$. If $h$ is an $i j$-index then we have $\rho_{j}\left(e_{h}, a\right)>0$, which means that the trader can get a positive amount of commodity $j$ for a suitable offer by the other(s). However if there is no pure $i j$-index then the trader may be forced to accept commodity $j$ bundled with other commodities. Flexibility guarantees that there are "enough" pure indices to enable individuals to "unbundle" their returns. The mechanism
may well admit complex trading opportunities, such as swaps of commodity bundles, that coexist with these indices; the former comprising, so to speak, a tangled web around the latter. It is our complexity criteria below which eliminate the web and allow only the pure $i j$-indices to survive (as markets of $i$ for $j$ ), see Theorem 12 ,

## 3 Complexity

We now discuss three notions of complexity for a mechanism $M$. The first, and simplest, is

$$
k_{i}=k_{i}(M)=\left|K_{i}(M)\right|
$$

which is the dimension of the offer space for commodity $i$, and which we refer to as the $i$-index complexity.

The next two notions are defined from standpoint of a "binary" $i j$-trader ${ }^{6}$ who interfaces with $M$ in order to exchange commodity $i$ for exclusively commodity $j$. We focus on two basic concerns for such a trader: first, how long will it take him to effect the exchange; and, second, how difficult will it be for him to figure out the terms of exchange? The first concern leads to the notion of "time complexity", and the second to that of "price complexity".

We fix some notation; let $e_{i}$ denote the $i$-th unit vector.
Definition $6 A$ vector $v$ is an $i$-vector if $v=s e_{i}$ for some real number $s>0$; and an $\bar{\imath} j$-vector if $v=-s e_{i}+t e_{j}$ for some real $s, t>0$.

### 3.1 Time Complexity

Definition 7 Given two commodity bundles $v, w \in C$ we will say that $v$ can be converted to $w$, and we write $v \rightarrow w$ if there exist $a, b$ such that

$$
w=v+\nu(a, b) \text { and } \bar{a} \leq v .
$$

[^5]Let $\tau(v, w, M)$ denote the smallest "time" $t$ for which there is a sequence ${ }^{7}$

$$
v \rightarrow v^{1} \rightarrow \cdots \rightarrow v^{t-1} \rightarrow w .
$$

We define the $i j$-time complexity and (maximum) time complexity as follows 8

$$
\tau_{i j}(M):=\tau\left(e_{i}, e_{j}, M\right), \quad \tau(M):=\max _{i \neq j}\left\{\tau_{i j}(M)\right\}
$$

We say that a mechanism $M$ is connected if $\tau(M)<\infty$.
Definition $8 \mathfrak{M}(m)$ is the class of all connected mechanisms with commodity set $\{1, \ldots, m\}$, that satisfy Anonymity, Aggregation, Invariance, Nondissipation and Flexibility.

When the commodity set $\{1, \ldots, m\}$ is understood, we shall often suppress $m$ and write $\mathfrak{M}=\mathfrak{M}(m)$.

### 3.2 The Emergence of Prices

Let $\mathbb{R}^{m} / \sim$ be the set of rays in $\mathbb{R}_{++}^{m}$, representing prices. It turns out that price ${ }^{10}$ emerge in connected mechanisms; and the values, under these prices, of offers and returns are conserved for every trader.

Theorem 9 Let $M \in \mathfrak{M}$ with associated net trade function $\nu$. Then there is a unique map $p: \mathbb{R}_{++}^{K} \rightarrow \mathbb{R}_{++}^{m} / \sim$ satisfying value conservation 11 : $p(b)$. $\nu(a, b)=0$.

[^6]Even though $p(b)$ is only defined up to an overall scalar multiple, for each pair $i, j$ we get a well-defined price ratio function

$$
p_{i j}: S_{+} \mapsto \mathbb{R}_{++} ; \quad p_{i j}(b)=\frac{p_{i}(b)}{p_{j}(b)}
$$

Theorem 9 has the following immediate consequence.
Corollary 10 Suppose $\nu(a, b)$ is an $\bar{\imath} j$-vector. Then $\frac{\nu_{i}(a, b)}{\nu_{j}(a, b)}=-p_{i j}(b)$.

### 3.3 Price Complexity

Note that a binary $i j$-trader is only interested in net trades $\nu(a, b)$ that are $\bar{\imath} j$-vectors. By the previous corollary, the exchange ratio $\frac{\nu_{i}(a, b)}{\nu_{j}(a, b)}$ is independent of the action $a$ producing the $\bar{\imath} j$-trade, and depends only on $p_{i j}(b)$. Therefore such a trader is interested only in those components of $b$ which "influence" the function $p_{i j}(b)$.

To make this notion precise, say that component $i$ is influential for a function $f\left(x_{1}, \ldots, x_{l}\right)$ if there are two inputs $x, x^{\prime}$, differing only in the $i$ th place, such that $f(x) \neq f\left(x^{\prime}\right)$. Define the ij-price complexity $\pi_{i j}(M)$ to be the number of influential components of the function $p_{i j}$. Also define the (maximum) price complexity by

$$
\pi(M):=\max \left\{\pi_{i j}(M): i \neq j\right\}
$$

## 4 The Emergence of Markets: G-Mechanisms

### 4.1 Directed Graphs

In this paper by a graph we mean a directed simple graph. Such a graph $G$ consists of a finite vertex set $V_{G}$, togther with an edge set $E_{G} \subseteq V_{G} \times V_{G}$ that does not contain any loops, i.e., edges of the form $i i$. For simplicity we shall often write $i \in G, i j \in G$ in place of $i \in V_{G}, i j \in E_{G}$ but there should be no confusion.

By a path $i i_{1} i_{2} \ldots i_{k} j$ from $i$ to $j$ we mean a nonempty sequence of edges in $G$ of the form

$$
i i_{1}, i_{1} i_{2}, \ldots, i_{k-1} i_{k}, i_{k} j
$$

If $k=0$ then the path consists of the single edge $i j$, otherwise we insist that the intermediate vertices $i_{1}, \ldots, i_{k}$ be distinct from each other and from the endpoints $i, j$. However we do allow $i=j$, in which case the path is called a cycle. We say that $G$ is connected if for any two vertices $i \neq j$ there is a path from $i$ to $j$.

### 4.2 G-mechanisms

Let $G$ be a connected graph with vertex set $\{1, \ldots, m\}$. Following [6] one may associate to $G$ a mechanism $M_{G} \in \mathfrak{M}=\mathfrak{M}(m)$ as follows. We let $K_{i}$ be the set of outgoing edges at vertex $i$, and regard $v \in S$ as a matrix $\left(v_{i j}\right)$ with $v_{i j}$ understood to be 0 if $i j \notin G$. To define $r(a, b)$ we need the following elementary result (see, e.g. [6]).

Lemma 11 For $b \in S_{+}$, there is a unique ray $p=p(b)$ in $\mathbb{R}_{++}^{m} / \sim$ satisfying

$$
\begin{equation*}
\sum_{i} p_{i} b_{i j}=\sum_{i} p_{j} b_{j i} \text { for all } j . \tag{3}
\end{equation*}
$$

Now for $(a, b) \in S \times S_{+}$we set $p=p(b)$ as in (3) and define $r(a, b)$ by

$$
\begin{equation*}
r_{i}(a, b)=p_{i}^{-1}\left(\sum_{j} p_{j} a_{j i}\right) \text { for all } i \tag{4}
\end{equation*}
$$

We remark that the left side of (3) is the total value of all the goods "chasing" good $j$, while the right side is the total value of good $j$ on offer.

Mechanisms of the form $M_{G}$ will be called (connected) G-mechanisms, and we write $\mathfrak{M}_{g}=\mathfrak{M}_{g}(m)$ for the totality of such mechanisms. It is worth noting that $\mathfrak{M}_{g}$ is a finite set. Moreover, the formula (4) for the return function of a $G$-mechanism immediately implies

$$
\begin{equation*}
p(b)=p(c) \Longrightarrow r(a, b)=r(a, c) \text { for all } a \in S ; b, c \in S_{+} \tag{5}
\end{equation*}
$$

In [6] this property was referred to as price mediation and, in conjunction with other axioms, shown to characterize $\mathfrak{M}_{g}$.

To sum up, these graphical $G$-mechanisms have very special structure. All the indices are pure, i.e. each edge $i j$ of $G$ represents a pure $i j$-index and can be interpreted as a market to exchange $i$ for $j$; furthermore, as we just saw, prices mediate trade in $M_{G}$ in the following strong sense: the return to a trader depends only on his own offer and the price $\mathbb{1}^{12}$ ((see equation

[^7](5)). Thus prices play the full-fledged role of a "decoupling device" in any $G$-mechanism.

It is worth emphasizing that the markets of $G$-mechanisms are, in general, not decentralized in that the exchange rate $p_{i} / p_{j}$ may depend on offers of commodities other than $i$ and $j$, at various edges in the graph.

### 4.3 Minimal Mechanisms

Given $M$ and $M^{\prime}$ in $\mathfrak{M}=\mathfrak{M}(m)$ with complexities $\tau_{i j}, \pi_{i j}, k_{i}$ and $\tau_{i j}^{\prime}, \pi_{i j}^{\prime}, k_{i}^{\prime}$ respectively, we say that $M$ is no more complex than $M^{\prime}$ and write $M \preceq M^{\prime}$ if for all $i, j$

$$
\tau_{i j} \leq \tau_{i j}^{\prime}, \quad \pi_{i j} \leq \pi_{i j}^{\prime}, \quad k_{i} \leq k_{i}^{\prime}
$$

Clearly $\preceq$ is reflexive and transitive, and hence constitutes a quasiorder on $\mathfrak{M}$. We let $\mathfrak{M}_{*}=\mathfrak{M}_{*}(m)$ denote the set of $\preceq$-minimal elements of $\mathfrak{M}$.

Theorem 12 Minimal mechanisms are $G$-mechanisms: $\mathfrak{M}_{*} \subset \mathfrak{M}_{g}$.

## 5 The Emergence of Money

Let us, from now on, identify two mechanisms if one can be obtained from the other by relabeling commodities. There are three mechanisms of special interest to us in $\mathfrak{M}_{g}(m)$ called the star, cycle, and complete mechanisms; with the following edge-sets:

| $G$ | Star | Cycle | Complete |
| :---: | :---: | :---: | :---: |
| $E_{G}$ | $\{m i, i m: i<m\}$ | $\{12,23, \ldots, m 1\}$ | $\{i j: i \neq j\}$ |

Notice that the central vertex $m$ of the graph underlying the star mechanism plays the role of money, and is the sole medium of exchang ${ }^{13}$.

Although the set $\mathfrak{M}_{*}$ is finite, it can be quite large and we will not attempt to characterize it here. Instead we consider the "worst-case complexities" $\pi(M)=\max \pi_{i j}(M)$ and $\tau(M)=\max \tau_{i j}(M)$, and the corresponding quasiorder on $\mathfrak{M}$, namely: $M \preceq_{w} M^{\prime}$ if

$$
\tau(M) \leq \tau\left(M^{\prime}\right), \quad \pi(M) \leq \pi\left(M^{\prime}\right)
$$

[^8]If $\tilde{\mathfrak{M}}$ is a subset of $\mathfrak{M}$ one can consider the minimal elements of $\tilde{\mathfrak{M}}$ with respect to the quasiorder $\preceq_{w}$ restricted to $\tilde{\mathfrak{M}}$; these will be referred to as strongly minimal mechanisms of $\tilde{\mathfrak{M}}$.

Theorem 13 If $m>3$ then the three special mechanisms are precisely the strongly minimal mechanisms of both $\mathfrak{M}_{*}(m)$ and $\mathfrak{M}_{g}(m)$. Their complexities are

|  | Star | Cycle | Complete |
| :---: | :---: | :---: | :---: |
| $\pi(M)$ | 4 | 2 | $m(m-1)$ |
| $\tau(M)$ | 2 | $m-1$ | 1 |

The array clearly exhibits the superiority of the star mechanism. As the number of commodities $m$ increases, the other two will beat star slightly in one component, but will lose by a huge margin to star in the other component, with the upshot that the star is the overall winner:

Theorem 14 For any strictly positive $\lambda$ and $\mu$, there exists $m_{0}$ such that the star mechanism is the unique maximizer of $\lambda \pi(M)+\mu \tau(M)$ on $\mathfrak{M}_{*}(m)$ and on $\mathfrak{M}_{g}(m)$ whenever $m \geq m_{0}$.

In the star mechanism ${ }^{14}$, the pair of edges $\mathrm{im}, \mathrm{mi}$ constitutes a bilateral market between $i$ and $m$ for all $i \neq m$. Thus the central node $m$ plays the role of money, mediating trade between various markets. Furthermore these markets are decentralized in that the trade at any market is independent of the offers at all other markets.

## 6 Proof of Theorem 9

We fix a mechanism $M$ in $\mathfrak{M}$ with net trade function $\nu(a, b)$. Consider the set of pairs $(i, j)$ for which there is at least one pure $i j$-index in $K$, and fix a subset $P \subset K$ which contains exactly one $i j$-index for each such pair. Let $S_{P} \subset S$ denote the set of $P$-offers, i.e. those $a$ satisfying $a_{h}=0$ for $h \notin P$, and further define the set of $P$-offers "subordinate" to $v$ as follows:

$$
S_{P}(v)=\left\{a \in S_{P}: \bar{a} \leq v\right\}
$$

[^9]Given a vector $v \in S$ we write $\langle v\rangle$ for the class of vectors with the same sign as $v$, thus $w \in\langle v\rangle$ if each component $w_{i}$ has the same $\operatorname{sign}(+,-, 0)$ as $v_{i}$.

Lemma 15 Let $v, w \in S$ then the following are equivalent.

1. There is an $a \in S_{P}(v)$ such that $v+\nu(a, b) \in\langle w\rangle$ for some $b \in S_{+}$
2. There is an $a \in S_{P}(v)$ such that $v+\nu(a, b) \in\langle w\rangle$ for all $b \in S_{+}$
3. For each $u \in\langle v\rangle$ there is an $a \in S_{P}(u)$ such that $u+\nu(a, b) \in\langle w\rangle$ for all $b \in S_{+}$

Proof. It is evident that (3) implies (2), and (2) implies (1). We now show that (1) implies (3). Suppose $v, a, b, w$ satisfy (1). Given $u \in\langle v\rangle$ and $b_{*} \in S_{+}$, we need to find $a_{*} \in S_{P}(u)$ such that $u, a_{*}, b_{*}, w$ satisfy (3). Since $u$ and $v$ have the same signs there exist positive scalars $\lambda_{i}$ such that $u_{i}=\lambda_{i} v_{i}$ for all $i$. Define $a_{*}$ by $\left(a_{*}\right)_{i}=\lambda_{i} a_{i}$, where (recall) $a_{i}$ is the vector obtained from $a$ by restricting to the $K_{i}$-components. Now we have

$$
\begin{aligned}
v+\nu(a, b) & =(v-\bar{a})+r(a, b) \\
u+\nu\left(a_{*}, b_{*}\right) & =\left(u-\overline{a_{*}}\right)+r\left(a_{*}, b_{*}\right)
\end{aligned}
$$

By construction of $a_{*}$ we have $(v-\bar{a})_{i}=\lambda_{i}\left(u-\overline{a_{*}}\right)_{i}$ for all $i$, and hence $\langle v-\bar{a}\rangle=\left\langle u-\overline{a_{*}}\right\rangle$. Also since $a$ and $a_{*}$ are $P$-offers, by Aggregation and Invariance we have $\langle r(a, b)\rangle=\left\langle r\left(a, b_{*}\right)\right\rangle=\left\langle r\left(a_{*}, b_{*}\right)\right\rangle$. We note that if $x, y$ are non-negative vectors then $\langle x+y\rangle$ is uniquely determined by $\langle x\rangle$ and $\langle y\rangle$, thus we get

$$
\left\langle u+\nu\left(a_{*}, b_{*}\right)\right\rangle=\langle v+\nu(a, b)\rangle=\langle w\rangle
$$

which establishes (3).
We note that Lemma 15 (3) only depends on $\langle v\rangle$ and $\langle w\rangle$ and we will write $\langle v\rangle \rightarrow\langle w\rangle$ if it holds.

Lemma 16 For any $(a, b) \in S \times S_{+}$there is $a_{*} \in S_{P}(\bar{a})$ such that

$$
\begin{equation*}
\langle r(a, b)\rangle=\left\langle\bar{a}+\nu\left(a_{*}, b\right)\right\rangle . \tag{6}
\end{equation*}
$$

Proof. By Aggregation, it suffices to prove this when $a$ is a $K_{i}$-offer for some $i$. By Flexibility there is some $a_{*} \in S_{P}(\bar{a})$ such that $r_{i}\left(a_{*}, b\right)=0$, while $r_{j}\left(a_{*}, b\right)$ has the same sign as $r_{j}(a, b)$ for all $j \neq i$. We write

$$
\bar{a}+\nu\left(a_{*}, b\right)=\left(\bar{a}-\overline{a_{*}}\right)+r\left(a_{*}, b\right)
$$

and note that since $a_{*}$ is a pure $K_{i}$-offer, the sign of $r\left(a_{*}, b\right)$ does not change if we rescale $a_{*}$. If $r_{i}(a, b)=0$ we scale up $a_{*}$ to ensure $\overline{a_{*}}=\bar{a}$, while if $r_{j}(a, b)>0$ then we scale down $a_{*}$ to ensure $\overline{a_{*}} \ddagger \bar{a}$; in each case the rescaled $a_{*}$ satisfies (6).

Lemma $17 v^{1} \rightarrow \cdots \rightarrow v^{t}$ implies $\left\langle v^{1}\right\rangle \rightarrow \cdots \rightarrow\left\langle v^{t}\right\rangle$.
Proof. It suffices to show that $v \rightarrow w$ implies $\langle v\rangle \rightarrow\langle w\rangle$. Now by definition

$$
w=v+\nu(a, b) \text { for some }(a, b) \in S \times S_{+} \text {with } \bar{a} \leq v
$$

If $a_{*}$ is as in (6) then the identities

$$
\begin{aligned}
v+\nu\left(a_{*}, b\right) & =(v-\bar{a})+\left(\bar{a}+\nu\left(a_{*}, b\right)\right) \\
v+\nu(a, b) & =(v-\bar{a})+r(a, b)
\end{aligned}
$$

imply $\left\langle v+\nu\left(a_{*}, b\right)\right\rangle=\langle w\rangle$, whence $\langle v\rangle \rightarrow\langle w\rangle$ by Lemma 15 (1).
Proposition 18 For $b \in S_{+}$and any $i \neq j$ there is $a \in S_{P}$ such that $\nu(a, b)$ is an $\bar{\imath} j$-vector.

Proof. Let $v$ be an $i$-vector and let $t=\tau_{i j}(M)$ then by definition we have a sequence

$$
v \rightarrow v^{1} \rightarrow \cdots \rightarrow v^{t-1}=w
$$

where $w$ is a $j$-vector. By the previous lemma we get

$$
\langle v\rangle \rightarrow\left\langle v^{1}\right\rangle \rightarrow \cdots \rightarrow\left\langle v^{t-1}\right\rangle \rightarrow\langle w\rangle
$$

By Lemma 15 (3) this means we can find sequences

$$
u^{i} \in\left\langle v^{i}\right\rangle, a^{i} \in S_{P}\left(u^{i}\right) \text { for } i=0, \ldots, t-1
$$

such that $u^{i}+\nu\left(a^{i}, b\right)=u^{i+1}$. If $a=\sum a^{i}$ then we have $a \in S_{P}$ and

$$
\nu(a, b)=\sum \nu\left(a^{i}, b\right)=u^{t}-u^{1}
$$

which is an $\bar{\imath} j$-vector.
It will be convenient to write an $\bar{\imath} j$-vector in the form $(-x, y)$ after suppressing the other components. In the context of the above proposition if $\nu(a, b)=(-x, y)$ then by linearity $\nu(a / x, b)=(-1, y / x)$, and we will say that the offer $a$ (or $a / x$ ) achieves an $i j$-exchange ratio of $y / x$ at $b$.

Lemma 19 If $a^{\prime}, a^{\prime \prime}$ achieve ij-exchange ratios $\alpha^{\prime}, \alpha^{\prime \prime}$ at $b$, then $\alpha^{\prime}=\alpha^{\prime \prime}$.
Proof. By the previous proposition there exists an $a$ such that $\nu(a, b)$ is a $\bar{j} i$-vector; if $\alpha$ is the corresponding exchange ratio then by rescaling $a, a^{\prime}, a^{\prime \prime}$ we may assume that

$$
\nu(a, b)=(1,-\alpha), \nu\left(a^{\prime}, b\right)=\left(-1, \alpha^{\prime}\right), \nu\left(a^{\prime \prime}, b\right)=\left(-1, \alpha^{\prime \prime}\right) .
$$

By Proposition 5 we get

$$
\nu\left(a+a^{\prime}, b\right)=\left(0, \alpha-\alpha^{\prime}\right)
$$

Now by Non-dissipation we get $\alpha \geq \alpha^{\prime}$, and exchanging the roles of $i$ and $j$ we conclude that $\alpha^{\prime} \geq \alpha$ and hence that $\alpha=\alpha^{\prime}$. Arguing similarly we get $\alpha=\alpha^{\prime \prime}$ and hence that $\alpha^{\prime}=\alpha^{\prime \prime}$.

Proof of Theorem 9. Fix $b \in S_{+}$and consider the vector

$$
p=\left(1, p_{2}, \ldots, p_{m}\right)
$$

where $p_{j}^{-1}$ is the $1 j$-exchange ratio at $b$, as in the previous lemma. We will show that $p$ satisfies the conditions of Theorem 9, i.e. that

$$
\begin{equation*}
p \cdot \nu(a, b)=0 \text { for all } a . \tag{7}
\end{equation*}
$$

We argue by induction on the number $d(a, b)$ of non-zero components of $\nu(a, b)$ in positions $2, \ldots, m$. If $d(a, b)=0$ then $\nu(a, b)=0$ by Nondissipation and (7) is obvious. If $d(a, b)=1$ then $\nu(a, b)$ is either an $\overline{1} j$ vector or a $\bar{j} 1$ vector, which by the definition of $p_{j}$ and the previous lemma is necessarily of the form

$$
\left(-x, x p_{j}^{-1}\right) \text { or }\left(x,-x p_{j}^{-1}\right) ;
$$

for such vectors (7) is immediate. Now suppose $d(a, b)=d>1$ and fix $j$ such that $\nu_{j}(a, b) \neq 0$. Then we can choose $a^{\prime}$ such that $\nu\left(a^{\prime}, b\right)$ is a $\overline{1} j$ or a $\bar{j} 1$ - vector such that $\nu_{j}(a, b)=-\nu_{j}\left(a^{\prime}, b\right)$. It follows that $d\left(a+a^{\prime}, b\right)<d$ and by linearity we get

$$
p \cdot \nu(a, b)=p \cdot \nu\left(a+a^{\prime}, b\right)-p \cdot \nu\left(a^{\prime}, b\right) .
$$

By the inductive hypothesis the right side is zero, hence so is the left side.
Finally the uniqueness of the price function is obvious, because the return function of the mechanism dictates how many units of $j$ may be obtained for one unit of $i$, yielding just one possible candidate for the exchange rate for every pair $i j$.

## 7 Proof of Theorem 12

We say a matrix $X$ is an $S \times T$ matrix if its rows and columns are indexed by finite sets $S$ and $T$ respectively; if $Y$ is a $T \times U$ matrix then the product $X Y$ is a well-defined $S \times U$ matrix. For the set $[n]=\{1, \ldots, n\}$ we will speak of $n \times T$ matrices instead of $[n] \times T$ matrices, etc.

Let $M \in \mathfrak{M}(m)$ and write $K_{i}=K_{i}(M)$ and $K=\coprod_{i} K_{i}$ as usual. For any vector $v \in \mathbb{R}_{++}^{m}$, let $D_{v}$ denote the $m \times m$ diagonal matrix $\operatorname{diag}\left\{v_{1}, \ldots, v_{m}\right\}$, and let $E_{v}$ denote the $K \times K$ "extended" diagonal matrix whose $K_{i}$-diagonal entries are all $v_{i}$. Also let $A$ be the $m \times K$ "auxiliary" matrix whose $K_{1}$ columns are $(1,0, \ldots, 0)^{t}, K_{2}$-columns are $(0,1,0, \ldots, 0)^{t}$, etc.

Lemma $20 M$ is uniquely determined by a map $b \mapsto N_{b}$ from $S_{+}$to the space of non-negative $m \times K$ column-stochastic matrices as follows.

1. The price ray $p=p(b)$ is obtained as the unique solution of

$$
\begin{equation*}
C_{b} p=\Delta_{b} p \tag{8}
\end{equation*}
$$

where $\Delta_{b}=A D_{b} A^{t}$ is the diagonal matrix of column sums of $C_{b}=$ $N_{b} D_{b} A^{t}$
2. The return function is given by

$$
\begin{equation*}
r(a, b)=M_{b} a \text { where } M_{b}=D_{p}^{-1} N_{b} E_{p} \tag{9}
\end{equation*}
$$

Proof. Let $p=p(b)$ be the price function whose existence is guaranteed by Theorem 9. We will first prove formula 9 for $r(a, b)$ and then prove formula 8. By Proposition 55, the return function of the mechanism $M$ is of the form $r(a, b)=M_{b} a$, where $b \mapsto M_{b}$ is a map from $S_{+}$to the space of non-negative $m \times k$ matrices satisfying

$$
M_{b} b=A b
$$

and the identity

$$
\begin{equation*}
M_{E_{v} b}=D_{v} M_{b} E_{v}^{-1} \text { for all } v \in R_{++}^{m} . \tag{10}
\end{equation*}
$$

(The non-negativity $M_{b}$ follows from that of $r(a, b)$. The first display holds by conservation of commodities and the second by Invariance.) Define

$$
b^{\prime}=E_{p} b, N_{b}=M_{b^{\prime}} .
$$

By Invariance it follows that $p\left(b^{\prime}\right)=1$. Also each column of $N_{b}=M_{b^{\prime}}$ is the return to the offer of a single unit in some commodity. Since all prices are 1 at $b^{\prime}$, Theorem 9 implies that each column of $N_{b}$ sums to 1, i.e. $N_{b}$ is column stochastic. Now by (10) we get

$$
N_{b}=M_{E_{p} b}=D_{p} M_{b} E_{p}^{-1},
$$

whence $M_{b}=D_{p}^{-1} N_{b} E_{p}$ as desired
Now combining (9) and $M_{b} b=A b$, with the identity $D_{p} A=A E_{p}$ we have

$$
N_{b} E_{p} b=D_{p} M_{b} b=D_{p} A b=A E_{p} b .
$$

Using the identity $E_{p} b=D_{b} A^{t} p$ we can rewrite this as

$$
N_{b} D_{b} A^{t} p=A D_{p} A^{t} b
$$

which is precisely (8).
Lemma 21 Let $N_{b}$ in Lemma 20 and let $h \in K_{i}$ be a pure $i j$-index.

1. The $h$-th column of $N_{b}$ is the $j$-th unit vector $e_{j}$, independent of $b$.
2. Every $K_{i}$-column of $N_{b}$ is a linear combination of the "pure" $K_{i}$-columns.

Proof. By definition there is an $h$-offer $a$ such that $r(a, b)=M_{b} a$ is a $j$-return vector. This means that the $h$-th column of $M_{b}$ has a non-zero entry only in its $j$-th component. Since $N_{b}$ is obtained from $M_{b}$ by rescaling entries this is also true of $N_{b}$. By column stochasticity the $h$-th column of $N_{b}$ must be $e_{j}$.

For the second part, let $h^{\prime} \in K_{i}$ be an $i$-index, let $v, w$ be the $h^{\prime}$-th columns of $N_{b}$ and $M_{b}$, and suppose the $j$-th component of $v$ (and hence of $w)$ is non-zero. It suffices to show that in this case the mechanism has a pure $i j$-index. However if $a$ is an $h^{\prime}$-offer then $r(a, b)=M_{b} a$ is a multiple of $w$, and thus the assertion follows from Flexibility.

Let $G$ be the graph in which we connect $i$ to $j$ if $M$ has a pure $i j$-index. Since $M$ is connected, Lemma 17 implies that $G$ is connected, and we let $M^{\prime}=M_{G}$ denote the corresponding $G$-mechanism. We will identify the $i$ indices $K_{i}^{\prime}$ of $M^{\prime}$ as a subset of $K_{i}$. If $M$ has several pure $i j$-indices for a given $j$ then this involves a choice, however the choice will play no role in the
subsequent discussion. We will refer to $M^{\prime}$ as the embedded $G$-mechanism of $M$.

To continue we need a result from [32]. Let $G$ be any connected directed graph on $\{1, \ldots, n\}$ with weights $z_{i j}$ attached to edges $i j \in G$. We write $Z=\left(z_{i j}\right)$ for the $n \times n$ matrix of edge weights of $G$, setting $z_{i j}=0$ if $i j \notin G$. We also define

$$
\delta_{j}=\sum_{i} z_{i j}, \quad \Delta_{Z}=\operatorname{diag}\left(\delta_{1}, \ldots, \delta_{n}\right),
$$

so that $\Delta$ is the diagonal matrix of column sums of $Z$. We define the weight of a subgraph $\Gamma$ to be the product of its edge weights, thus

$$
w_{\Gamma}(z)=\prod_{i j \in E_{\Gamma}} z_{i j}
$$

We define an $i$-tree in $G$ to be a (directed) subgraph $T$ with $n$ vertices and $n-1$ edges, and the futher property that $T$ contains a path from $j$ to $i$ for every $j \neq i$. We write $\mathcal{T}_{i}$ for the set of $i$-trees in $G$, and define

$$
w_{i}=\sum_{\Gamma \in \mathcal{T}_{i}} w_{\Gamma}(z), \quad w=\left(w_{1}, \ldots, w_{n}\right)^{t}
$$

The following lemma from [32] is critical and paves the way for the rest of the analysis.

Lemma 22 If $Z, \Delta_{Z}, w$ are as above then one has $Z w=\Delta_{Z} w$.
We can now prove a key property of embedded $G$-mechanisms.
Proposition 23 If a price ratio depends on some variable in $M^{\prime}$, then it does so in $M$.

Proof. The pure columns of $N_{b}$ are fixed unit vectors, independent of $b$. By assumption there is a bijection between the pure variables and the nonzero entries $c_{r s}(b)$ of the matrix $C_{b}$. We denote the pure components of $b$ by $x=\left(x_{r s}\right)$ and the remaining mixed components by $y=\left(y_{k}\right)$. Then by the definition of $C_{b}$ we have an expression of the form

$$
\begin{equation*}
c_{r s}(b)=x_{r s}+\sum_{k} \varepsilon_{k}(b) y_{k} ; 0 \leq \varepsilon_{k}(b) \leq 1 . \tag{11}
\end{equation*}
$$

By formula (8) and Lemma 22, the prices $p$ in $M$ and $M^{\prime}$ are weighted sums of trees with edge weights $c_{r s}$ and $x_{r s}$ repectively. Let $p(x, y)$ denote the
price vector in $M$ at $b=(x, y)$ and let $p(x)$ denote the price vector in $M^{\prime}$ at $x$. Then by (11) we get

$$
p(x)=\lim _{y \rightarrow 0} p(x, y)
$$

We now fix a pair of commodities $i, j$ and let $\pi(x, y)$ and $\pi(x)$ denote the price ratios $p_{i} / p_{j}$ in $M$ and $M^{\prime}$ respectively, then we have

$$
\pi(x)=\lim _{y \rightarrow 0} \pi(x, y)
$$

Thus if $\pi(x)$ depends on some $x$-component, so must $\pi(x, y)$.
Proof of Theorem 12. By lemma 17, lemma 21 and the previous proposition (respectively), we have:

$$
\tau_{i j}\left(M^{\prime}\right)=\tau_{i j}(M), k\left(M^{\prime}\right) \leq k(M), \pi_{i j}\left(M^{\prime}\right) \leq \pi_{i j}(M)
$$

If $M$ is minimal then equality must hold throughout. Hence we get $k\left(M^{\prime}\right)=$ $k(M)$ and so $M=M^{\prime}$ is a $G$-mechanism..

## 8 Proof of Theorem 13

First let us recall some basic order-theoretic notions.
Definition $24 A$ quasiorder $\precsim$ on a set $X$ is a binary relation that is reflexive ( $x \precsim x$ ) and transitive:

$$
x \precsim y, y \precsim z \Longrightarrow x \precsim z .
$$

We write $x \prec y$ if $x \precsim y$ holds but $y \precsim x$ does not hold. We say that $x$ is $\precsim-m i n i m a l ~ i f ~ t h e r e ~ i s ~ n o ~ y ~ i n ~ X ~ s u c h ~ y \prec x$, equivalently if for all $y \in X$

$$
y \precsim x \Longrightarrow x \precsim y .
$$

We write $X_{\prec}$ for the set of $\precsim-m i n i m a l ~ e l e m e n t s ~ o f ~ X . ~ W e ~ s a y ~ t h a t ~ \precsim ~ i s ~$ a well-quasiorder (wqo) if there does not exist an infinite descending chain

$$
\cdots \prec x_{n} \prec \cdots \prec x_{2} \prec x_{1} .
$$

Note that if $\precsim$ is a wqo on $X$ and $Y \subset X$ then the restriction of $\precsim$ defines a wqo on $Y$. In general the minimal elements $Y_{\prec}$ can be quite different from $X_{\prec}$, however we have the following elementary result.

Lemma 25 If $(X, \precsim)$ is a wqo and $X_{\prec} \subset Y \subset X$ then $X_{\prec}=Y_{\prec}$.
Proof. Any minimal element of $X$ that happens to lie in $Y$ is clearly minimal in $Y$. Thus $X_{*} \subset Y$ implies $X_{\prec} \subset Y_{\prec}$. On the other hand if $z$ is a non-minimal element of $X$ then $x \prec z$ for some $x \in X_{\prec}$, otherwise we could construct an infinite descending chain in $X$ starting with $z$. In particular any $z \in Y \backslash X_{\prec}$ satisfies $x \prec z$ for some $x \in X_{\prec} \subset Y$, hence $z$ is not minimal in $Y$.

It is easy to check that both the quasiorders $\preceq$ and $\preceq_{w}$ that we have introduced on $\mathfrak{M}$ are wqo's; and therefore in the proof below we will apply the previous lemma to them.

Proof of Theorem 13. Let $\mathfrak{S}$ denote the set consisting of the three special mechanisms. We need to show that $\left(\mathfrak{M}_{g}\right)_{\prec_{w}}=\mathfrak{S}$ and $\left(\mathfrak{M}_{*}\right)_{\prec_{w}}=\mathfrak{S}$.

It is shown in [9] that $\left(\mathfrak{M}_{g}\right)_{\prec_{w}}=\mathfrak{S}$.
We now prove $\left(\mathfrak{M}_{*}\right)_{\prec_{w}}=\mathfrak{S}$. Since $\left(\mathfrak{M}_{g}\right)_{\prec_{w}}=\mathfrak{S}$, by Lemma 25 applied to the wqo $\preceq_{w}$ it suffices to show that $\mathfrak{S} \subset \mathfrak{M}_{*}$. We further note that

$$
\begin{equation*}
\mathfrak{M}_{*}=\left(\mathfrak{M}_{g}\right)_{\prec} . \tag{12}
\end{equation*}
$$

Indeed $\mathfrak{M}_{*}=\mathfrak{M}_{\prec}$ by definition, and $\mathfrak{M}_{*} \subset \mathfrak{M}_{g}$ by Theorem 12, now (12) follows from Lemma 25 applied to the wqo $\preceq$. Thus it suffices to prove that

$$
\begin{equation*}
\mathfrak{S} \subset\left(\mathfrak{M}_{g}\right)_{\prec} \tag{13}
\end{equation*}
$$

i.e., that each of the three special mechanisms is $\preceq$-minimal in $\mathfrak{M}_{g}$.

The $\preceq$-minimality is obvious for the complete graph since any other graph would have some $\tau_{i j}>1$, and also for the cycle since any other graph would have some $k_{i}>1$. To establish $\preceq$-minimality for the star graph, it suffices to show that any non-star graph $G$ has either some $\pi_{i j} \geq 5$ or some $\tau_{i j} \geq 3$. For this we note that $\pi \geq 5$ holds by Theorem 16 of [9] if $G$ is not a rose or a chorded cycle; while $\tau \geq 3$ holds trivially for non-star roses and by Lemma 37 of 9 for chorded cycles. This completes the proof of (13) and hence of $\left(\mathfrak{M}_{*}\right)_{\prec_{w}}=\mathfrak{S}$.

Remark 26 For $m=3$, Lemma 37 of [9] does not hold and we have an additional strongly minimal mechanism with $(\tau, \pi)=(2,4)$, namely the chorded triangle


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[^1]:    ${ }^{1}$ For us a Cournotian "mechanism" is a formal device that enables everyman to trade, with the simple expedient of offering commodities and without having to account for his precise motivation or even bothering to pretend that he has one. (See section 1.1.3.1 of [24] for a spirited defence of the use of "mechanism" in this plain English meaning of the word.) To forestall confusion, we emphasize that our usage is different from that of the recent mechanism-design literature, where the word has acquired a specialized connotation.
    ${ }^{2}$ i.e., consistent exchange-rates across commodity pairs.

[^2]:    ${ }^{3}$ Our model can equally accomodate fiat money or commodity money, depending on how preferences are introduced. Indeed, all we suppose is that the $m$ items being traded are distinct from each other. In particular, offers and returns could just be quotes (think of e-commerce!), instead of actual shipment of goods.

[^3]:    ${ }^{4}$ connected, because we require that it should be possible to convert any commodity to another by iterated trading.

[^4]:    ${ }^{5}$ It will always be clear from the context whether $i$ is the name of a commodity, or that of an individual, or just an integer.

[^5]:    ${ }^{6}$ We focus on bilateral trades between pairs of commodities because they form an iterative basis for all trade. This is so on account of prices (exchange rates) that will shortly be shown to emerge and govern all trade.

[^6]:    ${ }^{7}$ Notice that at the moment we permit the market state to vary across the $t$ transitions of the sequence $v \rightarrow v^{1} \rightarrow \cdots \rightarrow v^{t-1} \rightarrow w$. But even if we were to restrict attention to the case in which the same $b \in S_{+}$should represent the market state across all transitions, the time complexity would of $M$ will remain unaffected. This follows from Lemma 15
    ${ }^{8}$ It follows by Invariance that in the definition one can replace the unit vectors $e_{i}, e_{j}$ by arbitrary $i$ - and $j$ - vectors.
    ${ }^{9}$ A ray $p$ represents a price vector up to overall multiplication by a positive scalar; the ratios $p_{i} / p_{j}$ represent well-defined consistent exchange rates across all pairs $i j$ of commodities.
    ${ }^{10}$ Our analysis remains intact if there is a continuum of traders (see Section 7 of 9 ). In this case, an individual's action has no effect on the exchange rate. Otherwise it affects the aggregate offer (i.e., the state of the market) and thereby the exchange rate, which is but to be expected in an oligopolistic framework.
    ${ }^{11}$ Note that value conservation is perforce true on the aggregate since commodities are neither created nor destroyed by the mechanism, only redistributed. What is shown here is that it holds at the individual level, i.e., the mechanism does not assign "profitable" trades to some at the expense of others.

[^7]:    ${ }^{12}$ Price mediation in fact follows from value conservation once all indices are pure.

[^8]:    ${ }^{13}$ This is reminiscent of "spontaneous symmetry breaking" in physics. The ex ante symmetry between commodities, assumed in our model, is carried over to the cycle and complete mechanisms. It breaks down only in the star mechanism, giving rise to money.

[^9]:    ${ }^{14}$ The star mechanism (also known as the "Shapley-Shubik mechanism", see [35], 36, [37]) has been much-studied in the literature in different contexts, see, e.g., [3] , 4], [5, 7], [8, 29, [30, 31, , 39, [38, [39, 40]. The complete mechanism (also known as "Shapley's windows mechanism") is analysed in [33]. All other $G$-mechanisms are discussed in [6].

