Graphical Exchange Mechanisms

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Abstract

Consider an exchange mechanism which accepts "diversified" offers of various commodities and redistributes everything it receives. We impose certain conditions of fairness and convenience on such a mechanism and show that it admits unique prices, which equalize the value of offers and returns for each individual.

We next define the complexity of a mechanism in terms of certain integers τ_{ij}, π_{ij} and k_i that represent the time required to exchange *i* for *j*, the difficulty in determining the exchange ratio, and the dimension of the message space. We show that there are a *finite* number of minimally complex mechanisms, in each of which all trade is conducted through markets for commodity pairs.

Finally we consider minimal mechanisms with smallest worst-case complexities $\tau = \max \tau_{ij}$ and $\pi = \max \pi_{ij}$. For m > 3 commodities, there are precisely three such mechanisms, one of which has a distinguished commodity – the money – that serves as the sole medium of exchange. As $m \to \infty$ the money mechanism is the only one with bounded (π, τ) .

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1 Introduction

The purpose of this paper is to show how simple criteria of fairness, convenience and complexity can lead to the successive emergence of prices, markets, and money, in a Cournotian setting for commodity exchange. In the process, we arrive at a rationale for money which is purely "mechanistic" in spirit, and complements the existing utilitarian and behavioral literature on the subject (see 1.1).

We consider abstract exchange mechanisms¹, which accept "diversified" offers of various commodities and redistribute everything, and further satisfy five conditions, embodying fairness and convenience that we term *Anonymity*, *Aggregation*, *Invariance*, *Non-dissipation* and *Flexibility*. Although there are infinitely many such mechanisms, our first result is that each admits unique $prices^2$, which lead to *value conservation*, *i.e.* equalize the value of individual offers and returns.

We next define some natural notions of "complexity" for a mechanism, and, in keeping with the idea of convenience, study mechanisms with minimal complexity. This leads to a *finite* class $\mathfrak{M}_{*} \subset \mathfrak{M}_{g}$, where \mathfrak{M}_{g} denotes certain graphical mechanisms that are in one-to-one correspondence with directed, connected graphs on the set of commodities. The directed edge ij may be interpreted in \mathfrak{M}_{*} as a market that provides traders the opportunity to offer i in exchange for j. Prices not only conserve values in \mathfrak{M}_{*} (in fact, in \mathfrak{M}_{g}) but mediate trade in the sense that the return to a trader depends only on his own offer and the prices. In short, prices "decouple" the interaction between traders.

The emergence of prices and markets paves the way for the culmination of the analysis, namely the emergence of *money*. To this end we introduce additional refined criteria of complexity on \mathfrak{M}_* and study the corresponding minimal mechanisms, which we term *strongly* minimal. It turns out that there are only *three* strongly minimal mechanisms, up to a relabeling of commodities. In one of these, a single commodity emerges endogenously

¹For us a Cournotian "mechanism" is a formal device that enables everyman to trade, with the simple expedient of offering commodities and without having to account for his precise motivation or even bothering to pretend that he has one. (See section 1.1.3.1 of [24] for a spirited defence of the use of "mechanism" in this plain English meaning of the word.) To forestall confusion, we emphasize that our usage is different from that of the recent mechanism-design literature, where the word has acquired a specialized connotation.

²i.e., consistent exchange-rates across commodity pairs.

as *money* and mediates trade among decentralized markets for the other commodities. Moreover, with a moderate increase in the number of commodities, the money mechanism quickly supersedes the other two in a very precise sense.

Finally let us mention that, while this paper is a companion to [9], the two are meant to be readable independently. This has necessitated some overlap but it is minimal. To be precise, the conditions on the mechanism (with the exception of Flexibility) appear in [9], so does Proposition 5, and the rest is disjoint.

1.1 Related Literature

The emergence of money as a medium of exchange has been a matter of considerable discussion in economics. One approach, following Jevons [16], has focused on search-theoretic models that involve repeated and random bilateral meeting between agents (see, *e.g.*, [2], [15], [17], [19], [20], [23], [26], [42] and the references therein). Another line of inquiry is based on partial or general equilibrium models with various kinds of frictions in trade, such as transactions costs or limited trading opportunities (see, *e.g.*, [10], [11], [12], [13], [14], [27], [28], [40], [41], [43]). These models turn on notions of rational expectations and utility-maximizing behavior of the agents in equilibrium. In contrast, as was said, our analysis is based purely on the mechanism of trade is and independent of the characteristics of the agents, such as their endowments or utilities.

It is worth emphasizing that our analysis is quite agnostic regarding the choice of any particular money³, being only at pains to point out the urgency of appointing *some* money. For a discussion of different criteria entailed in the choice of a suitable "commodity money" such as its portability, verifiability, divisibility and durability; or, alternatively, the backing of the state requisite to sustain "fiat money", see, *e.g.*, [1], [16], [18], [21], [22], [25], [34]; and, for a recent survey on both kinds of money, see [39] and [40]. It would be interesting to incorporate some of these criteria, as well as the utilitarian considerations for money, into our mechanistic framework.

³Our model can equally accomodate fiat money or commodity money, depending on how preferences are introduced. Indeed, *all* we suppose is that the *m* items being traded are distinct from each other. In particular, offers and returns could just be quotes (think of e-commerce!), instead of actual shipment of goods.

This paper is intimately related to [9]. Let us briefly recount the model there. We make the hypothesis in [9] that any offer of a commodity *i* specifies some other commodity *j* which is being sought in exchange for *i*. Thus, drawing a directed arc *ij* for every such offer permitted in the mechanism, we obtain a (directed, connected⁴) graph *G*. There are infinitely many mechanisms for any given *G*, but it is shown in [9] that exactly one of them is categorically determined by *four* of our five conditions, namely *Anonymity*, *Aggregation, Invariance, Non-dissipation* (all except *Flexibility*). This is the graphical "*G*-mechanism" mentioned earlier; so that the class \mathfrak{M}_g is precisely the one generated as *G* varies over all directed, connected graphs on the fixed node-set of commodities. Our refined complexity criteria apply equally to \mathfrak{M}_g as to its subset \mathfrak{M}_* , and we show that \mathfrak{M}_g has the *same* three strongly minimal mechanisms as \mathfrak{M}_* . This is the main conclusion of [9] and it constitutes a key step in the proof of the emergence of money in this paper.

Our current paper thus puts the analysis of [9] on a more general footing. We start with a domain which is much richer than \mathfrak{M}_q , and which permits traders to indulge in "cheap talk" in order to diversify their offers in any commodity *i*. Furthermore, we allow for the possibility that these offers may get complicated *bundles* of commodities in return. But *Flexibility* guarantees that there exist special messages in the cheap talk through which a trader can "unbundle" his return, i.e., messages that enable him to get only j in exchange for i, whenever j is obtainable — albeit in conjunction with other commodities — via an offer of i. Thus flexibility embeds a graph G, as a sharply delineated language, within the tangle of cheap talk. We essentially show that our complexity criteria cuts away the tangle, leaving behind only the graphical G-mechanisms in $\mathfrak{M}_* \subset \mathfrak{M}_q$. Thus we deduce the existence of markets (i.e., the edges of the graph G) from an abstract standpoint, instead of postulating them as in [9]. The exact structure of the subdomain \mathfrak{M}_* of \mathfrak{M}_q is not explored by us, as that would detract from the primary purpose of this paper, which is to arrive expeditiously at the money mechanism.

The other precursor of this paper, at both the technical and conceptual level, is [6], where a mechanism produces not only trades but also prices, based upon everyone's offers; and where *price mediation* is *postulated* as a condition. The main result of [6] is that \mathfrak{M}_g is characterized by *Anonymity*, *Aggregation*, *Invariance*, *Price Mediation* and *Accessibility* (the last repre-

⁴connected, because we require that it should be possible to convert any commodity to another by iterated trading.

senting a weak form of continuity). In contrast, here we deduce the existence of prices, as well as their crucial role in mediating trade, based on considerations of a different sort, as outlined in the introduction and formalized in section 4 below. Moreover, the analysis in [6] stopped at the characterization of \mathfrak{M}_g and did not delve into any further selection among the mechanisms.

2 Exchange Mechanisms

Definition 1 A pre-mechanism consists of the following data:

- 1. a commodity space $C = \mathbb{R}^m_+$;
- 2. an action space $S = \mathbb{R}^K_+$ where $K = K_1 \amalg \cdots \amalg K_m$ is a finite set;
- 3. for each integer $n \ge 1$, a return map $\rho^n : S(n) \to C^n$ where

$$S(n) = \{ (a^1, \dots, a^n) \in S^n : a^1 + \dots + a^n \in S_+ \}, \quad S_+ = \mathbb{R}_{++}^K.$$
(1)

We refer to the elements of K_i as *i*-indices, and for $a \in S$ we define $\overline{a} \in C$ by summing over the various *i*-indices for each *i*; thus we have

$$\overline{a} = (\alpha_1, \ldots, \alpha_m)$$
 where $\alpha_i = \sum_{h \in K_i} a_h$.

Definition 2 An exchange mechanism is a pre-mechanism which satisfies:

if
$$(r^1, \dots, r^n) = \rho^n (a^1, \dots, a^n)$$
 then $r^1 + \dots + r^n = \overline{a^1} + \dots + \overline{a^n}$. (2)

The interpretation is as follows. There is an underlying set $\{1, \ldots, m\}$ of commodities⁵, and for each *i*-index $h \in K_i$, the component a_h of an action a represents an offer of commodity i. Thus we also refer to a as an offer, and the various *i*-indices serve to "diversify" the offer in i. An exchange mechanism enables trade as follows: having received n offers (a^1, \ldots, a^n) , which are "positive on the aggregate" (1), it redistributes the commodities according to ρ^n . Condition (2) means that commodities are conserved.

As mentioned already in section 1.1, one possible interpretation of the indices is as a common language in which traders may communicate with the mechanism M. The language is completely abstract: no structure is

⁵It will always be clear from the context whether i is the name of a commodity, or that of an individual, or just an integer.

imposed on it except that it be of finite size. The elements of K_i may be thought of as costless messages ("cheap talk") that accompany offers in *i*. Another interpretation of K_i could be that it represents different times (or, places) when (or, where) the offer of *i* is sent. The reader can no doubt think of still more interpretations. It is to make room for all these that we have used the neutral term "index". However, in the context of graphical mechanisms which emerge out of our analysis (see section 4), it turns out that *i*-indices have a concrete economic interpretation as certain commodity pairs *ij*, representing "markets" where *i* can be exchanged for *j*.

2.1 Conditions on the Mechanisms

In order to describe our conditions, we need some notation. We first define scaling actions of $\lambda \in \mathbb{R}^m_{++}$ on $r \in C$ and $a \in S$ via

$$(\lambda * r)_i = \lambda_i r_i$$
 for all i , $(\lambda * a)_h = \lambda_i a_h$ for all $h \in K_i$.

Let $\rho : S(2) \to C$ denote the first component of the two trader return function ρ^2 . If $h \in K_i$ is an *i*-index then the unit vector $e_h \in S$ is an offer in commodity *i* alone. The *j*-th component $\rho_j(e_h, a)$ of $\rho(e_h, a)$ is the return of commodity *j* to trader 1, when he offers e_h and the other offers *a*.

Definition 3 We say $h \in K_i$ is an ij-index if for some a we have

 $\rho_j \left(e_h, a \right) > 0.$

We say an *ij*-index h is pure if for all a we have

$$\rho_k(e_h, a) = 0 \text{ for all } k \neq j.$$

Our five conditions on an exchange mechanism, termed Anonymity, Aggregation, Invariance, Non-dissipation, and Flexibility, are as follows.

Condition 4 If $\rho^n(a^1,\ldots,a^n) = (r^1,\ldots,r^n)$ then we have

- 1. $\rho^n \left(a^{\sigma(1)}, \dots, a^{\sigma(n)} \right) = \left(r^{\sigma(1)}, \dots, r^{\sigma(n)} \right)$ for any permutation σ . 2. $\rho^{n-1} \left(a^1, \dots, a^{n-2}, a^{n-1} + a^n \right) = (r^1, \dots, r^{n-2}, r^{n-1} + r^n).$
- 3. $\rho^n (\lambda * a^1, \dots, \lambda * a^n) = (\lambda * r^1, \dots, \lambda * r^n)$ for all $\lambda \in \mathbb{R}^m_{++}$.

- 4. For each i, $r^i \overline{a^i}$ is either 0 or it has a strictly positive component.
- 5. For all ij, if there is an ij-index then there is a pure ij-index.

For such a mechanism M, by Anonymity and Aggregation we have

$$\rho^{n}(a^{1},\ldots,a^{n}) = (\rho(a^{1},a^{-1}),\ldots,\rho(a^{n},a^{-n})) \text{ where } a^{-i} = a^{1} + \cdots + a^{n} - a^{i}.$$

Thus M is uniquely determined by ρ , and also by the *trade* and *net trade* functions, which are defined as follows:

$$r(a,b) =
ho(a,b-a), \quad
u(a,b) = r(a,b) - \overline{a}.$$

These latter functions have domain $\{(a, b) \in S \times S_+ : a \leq b\}$.

Proposition 5 ν admits a unique extension to $S \times S_+$ satisfying

$$\nu(ta, b) = t\nu(a, b), \quad \nu(a, tb) = \nu(a, b) \text{ for all } t > 0.$$

Proof. See Lemma 1 of [6]. Although [6] considers a more restrictive class of mechanisms, we note that the proof of Lemma 1 there only uses Anonymity, Aggregation, and Invariance. \blacksquare

In view of the above result, we will drop the restriction $a \leq b$ for $\nu(a, b)$.

2.2 Comments on the Conditions

The first condition is *Anonymity*; it stipulates that the mechanism be blind to all characteristics of a trader other than his offer. The second condition is *Aggregation*, it asserts that if any trader pretends to be two different persons by splitting his offer, the returns to the others is unaffected. *Aggregation* does not imply that if two individuals were to merge, they would be unable to enhance their "oligopolistic power". For despite the aggregation condition, the merged individuals are free to *coordinate* their actions by jointly picking a point in the Cartesian product of their action spaces. Indeed all the mechanisms we obtain display this "oligopolistic effect", even though they also satisfy *Aggregation*. The two conditions embody fairness, enabling free entry for any new participant on non-discriminatory terms, and thereby making the mechanism more "inclusive". They also contribute to convenience, if either of these conditions were violated, trade would become a cumbersome affair: each individual would need to keep track of the full distribution of offers across the entire population, and then figure out how to diversify his own offers in response.

The third condition is *Invariance*; its main content is that the maps ρ^n which comprise the mechanism are invariant under a change of units in which commodities are measured. This makes the mechanism much simpler to operate in: one does not need to keep track of seven pounds or seven kilograms or seven tons, just the numeral 7 will do. It is worthy of note that the cuneiform tablets of ancient Sumeria, which are some of the earliest examples of written language and arithmetic, are in large part devoted to records and receipts pertaining to economic transactions. *Invariance* postulates the "numericity" property of the ρ^n making them independent of the underlying choice of units, and this goes to the very heart of the quantitative measurement of commodities. In its absence, one would need to figure out how the maps are altered when units change, as they are prone to do, especially in a dynamic economy. This would make the mechanism cumbersome to use.

The fourth condition is *Non-dissipation*; it says that no trader's return can be less commodity-wise than his offer. If it were violated, such unfortunate traders would find it grossly unfair and tend to abandon the mechanism. In conjunction with *Aggregation, Anonymity*, and the conservation of commodities, this immediately implies *no-arbitrage*:

for any a, b neither $\nu(a, b) \geqq 0$ nor $\nu(a, b) \leqq 0$.

To see this, note that in view of Proposition 5 we need consider only the case $a \leq b$ and rule out $\nu(a, b) \geqq 0$. Denote c = b - a. Then

$$\nu(a,b) + \nu(c,b) = \nu(a+c,b) = \nu(b,b) = 0,$$

where the first equality follows from Aggregation, and the last from conservation of commodities. But then $\nu(a, b) \ge 0$ implies $\nu(c, b) \le 0$, contradicting Non-dissipation.

The fifth and final condition is *Flexibility*, it reflects the perspective of a trader who wishes to interact with the mechanism to exchange a single commodity *i* for some other commodity *j*. If *h* is an *ij*-index then we have $\rho_j(e_h, a) > 0$, which means that the trader can get a positive amount of commodity *j* for a suitable offer by the other(s). However if there is no pure *ij*-index then the trader may be forced to accept commodity *j* bundled with other commodities. *Flexibility* guarantees that there are "enough" pure indices to enable individuals to "unbundle" their returns. The mechanism may well admit complex trading opportunities, such as swaps of commodity bundles, that coexist with these indices; the former comprising, so to speak, a tangled web around the latter. It is our complexity criteria below which eliminate the web and allow only the pure ij-indices to survive (as markets of i for j), see Theorem 12.

3 Complexity

We now discuss three notions of *complexity* for a mechanism M. The first, and simplest, is

$$k_i = k_i(M) = |K_i(M)|$$

which is the dimension of the offer space for commodity i, and which we refer to as the *i*-index complexity.

The next two notions are defined from standpoint of a "binary" ij-trader⁶ who interfaces with M in order to exchange commodity i for exclusively commodity j. We focus on two basic concerns for such a trader: first, how long will it take him to effect the exchange; and, second, how difficult will it be for him to figure out the terms of exchange? The first concern leads to the notion of "time complexity", and the second to that of "price complexity".

We fix some notation; let e_i denote the *i*-th unit vector.

Definition 6 A vector v is an i-vector if $v = se_i$ for some real number s > 0; and an $\bar{i}j$ -vector if $v = -se_i + te_j$ for some real s, t > 0.

3.1 Time Complexity

Definition 7 Given two commodity bundles $v, w \in C$ we will say that v can be converted to w, and we write $v \to w$ if there exist a, b such that

$$w = v + \nu(a, b) \text{ and } \overline{a} \leq v.$$

 $^{^{6}}$ We focus on bilateral trades between pairs of commodities because they form an iterative basis for all trade. This is so on account of *prices* (exchange rates) that will shortly be shown to emerge and govern all trade.

Let $\tau(v, w, M)$ denote the smallest "time" t for which there is a sequence⁷

$$v \to v^1 \to \dots \to v^{t-1} \to w$$

We define the *ij-time complexity* and (maximum) time complexity as follows⁸

$$\tau_{ij}(M) := \tau(e_i, e_j, M), \quad \tau(M) := \max_{i \neq j} \{\tau_{ij}(M)\}.$$

We say that a mechanism M is connected if $\tau(M) < \infty$.

Definition 8 $\mathfrak{M}(m)$ is the class of all connected mechanisms with commodity set $\{1, \ldots, m\}$, that satisfy Anonymity, Aggregation, Invariance, Nondissipation and Flexibility.

When the commodity set $\{1, \ldots, m\}$ is understood, we shall often suppress m and write $\mathfrak{M} = \mathfrak{M}(m)$.

3.2 The Emergence of Prices

Let \mathbb{R}^m_{++}/\sim be the set of rays in \mathbb{R}^m_{++} , representing prices⁹. It turns out that prices¹⁰ emerge in connected mechanisms; and the values, under these prices, of offers and returns are conserved for every trader.

Theorem 9 Let $M \in \mathfrak{M}$ with associated net trade function ν . Then there is a unique map $p : \mathbb{R}_{++}^K \to \mathbb{R}_{++}^m / \sim satisfying$ value conservation¹¹: $p(b) \cdot \nu(a, b) = 0$.

⁷Notice that at the moment we permit the market state to vary across the t transitions of the sequence $v \to v^1 \to \cdots \to v^{t-1} \to w$. But even if we were to restrict attention to the case in which the same $b \in S_+$ should represent the market state across all transitions, the time complexity would of M will remain unaffected. This follows from Lemma 15

⁸It follows by *Invariance* that in the definition one can replace the unit vectors e_i, e_j by arbitrary *i*- and *j*- vectors.

⁹A ray p represents a price vector up to overall multiplication by a positive scalar; the ratios p_i/p_j represent well-defined consistent exchange rates across all pairs ij of commodities.

¹⁰Our analysis remains intact if there is a continuum of traders (see Section 7 of [9]). In this case, an individual's action has no effect on the exchange rate. Otherwise it affects the aggregate offer (i.e., the state of the market) and thereby the exchange rate, which is but to be expected in an oligopolistic framework.

¹¹Note that value conservation is perforce true on the aggregate since commodities are neither created nor destroyed by the mechanism, only redistributed. What is shown here is that it holds at the individual level, i.e., the mechanism does not assign "profitable" trades to some at the expense of others.

Even though p(b) is only defined up to an overall scalar multiple, for each pair i, j we get a well-defined price ratio function

$$p_{ij}: S_+ \mapsto \mathbb{R}_{++}; \qquad p_{ij}(b) = \frac{p_i(b)}{p_j(b)}$$

Theorem 9 has the following immediate consequence.

Corollary 10 Suppose $\nu(a, b)$ is an $\overline{i}j$ -vector. Then $\frac{\nu_i(a, b)}{\nu_j(a, b)} = -p_{ij}(b)$.

3.3 Price Complexity

Note that a binary *ij*-trader is only interested in net trades $\nu(a, b)$ that are $\bar{\imath}j$ -vectors. By the previous corollary, the exchange ratio $\frac{\nu_i(a,b)}{\nu_j(a,b)}$ is independent of the action *a* producing the $\bar{\imath}j$ -trade, and depends only on $p_{ij}(b)$. Therefore such a trader is interested only in those components of *b* which "influence" the function $p_{ij}(b)$.

To make this notion precise, say that component *i* is *influential* for a function $f(x_1, \ldots, x_l)$ if there are two inputs x, x', differing only in the *i*th place, such that $f(x) \neq f(x')$. Define the *ij-price complexity* $\pi_{ij}(M)$ to be the number of influential components of the function p_{ij} . Also define the (maximum) price complexity by

$$\pi(M) := \max\left\{\pi_{ij}(M) : i \neq j\right\}$$

4 The Emergence of Markets: G-Mechanisms

4.1 Directed Graphs

In this paper by a graph we mean a *directed simple graph*. Such a graph G consists of a finite *vertex* set V_G , together with an *edge* set $E_G \subseteq V_G \times V_G$ that does not contain any loops, *i.e.*, edges of the form *ii*. For simplicity we shall often write $i \in G$, $ij \in G$ in place of $i \in V_G$, $ij \in E_G$ but there should be no confusion.

By a *path* $ii_1i_2...i_kj$ from *i* to *j* we mean a nonempty sequence of edges in *G* of the form

$$ii_1, i_1i_2, \ldots, i_{k-1}i_k, i_kj.$$

If k = 0 then the path consists of the single edge ij, otherwise we insist that the *intermediate* vertices i_1, \ldots, i_k be distinct from each other and from the endpoints i, j. However we do allow i = j, in which case the path is called a *cycle*. We say that G is *connected* if for any two vertices $i \neq j$ there is a *path* from i to j.

4.2 G-mechanisms

Let G be a connected graph with vertex set $\{1, \ldots, m\}$. Following [6] one may associate to G a mechanism $M_G \in \mathfrak{M} = \mathfrak{M}(m)$ as follows. We let K_i be the set of outgoing edges at vertex i, and regard $v \in S$ as a matrix (v_{ij}) with v_{ij} understood to be 0 if $ij \notin G$. To define r(a, b) we need the following elementary result (see, e.g. [6]).

Lemma 11 For $b \in S_+$, there is a unique ray p = p(b) in \mathbb{R}^m_{++} / \sim satisfying

$$\sum_{i} p_i b_{ij} = \sum_{i} p_j b_{ji} \text{ for all } j.$$
(3)

Now for $(a, b) \in S \times S_+$ we set p = p(b) as in (3) and define r(a, b) by

$$r_i(a,b) = p_i^{-1} \left(\sum_j p_j a_{ji} \right) \text{ for all } i.$$
(4)

We remark that the left side of (3) is the total value of all the goods "chasing" good j, while the right side is the total value of good j on offer.

Mechanisms of the form M_G will be called (connected) *G*-mechanisms, and we write $\mathfrak{M}_g = \mathfrak{M}_g(m)$ for the totality of such mechanisms. It is worth noting that \mathfrak{M}_g is a *finite* set. Moreover, the formula (4) for the return function of a *G*-mechanism immediately implies

$$p(b) = p(c) \Longrightarrow r(a, b) = r(a, c) \text{ for all } a \in S; b, c \in S_+$$
(5)

In [6] this property was referred to as *price mediation* and, in conjunction with other axioms, shown to characterize \mathfrak{M}_{g} .

To sum up, these graphical G-mechanisms have very special structure. All the indices are pure, *i.e.* each edge ij of G represents a pure ij-index and can be interpreted as a *market* to exchange i for j; furthermore, as we just saw, prices mediate trade in M_G in the following strong sense: the return to a trader depends only on his own offer and the prices¹²((see equation

¹²Price mediation in fact follows from value conservation *once* all indices are pure.

(5)). Thus prices play the full-fledged role of a "decoupling device" in any G-mechanism.

It is worth emphasizing that the markets of G-mechanisms are, in general, not decentralized in that the exchange rate p_i/p_j may depend on offers of commodities other than *i* and *j*, at various edges in the graph.

4.3 Minimal Mechanisms

Given M and M' in $\mathfrak{M} = \mathfrak{M}(m)$ with complexities τ_{ij}, π_{ij}, k_i and $\tau'_{ij}, \pi'_{ij}, k'_i$ respectively, we say that M is no more complex than M' and write $M \leq M'$ if for all i, j

 $\tau_{ij} \le \tau'_{ij}, \quad \pi_{ij} \le \pi'_{ij}, \quad k_i \le k'_i.$

Clearly \leq is reflexive and transitive, and hence constitutes a quasiorder on \mathfrak{M} . We let $\mathfrak{M}_* = \mathfrak{M}_*(m)$ denote the set of \leq -minimal elements of \mathfrak{M} .

Theorem 12 Minimal mechanisms are G-mechanisms: $\mathfrak{M}_* \subset \mathfrak{M}_q$.

5 The Emergence of Money

Let us, from now on, identify two mechanisms if one can be obtained from the other by relabeling commodities. There are three mechanisms of special interest to us in $\mathfrak{M}_g(m)$ called the *star*, *cycle*, and *complete mechanisms*; with the following edge-sets:

G	Star	Cycle	Complete
E_G	$\{mi, im : i < m\}$	$\{12, 23, \ldots, m1\}$	$\{ij:i\neq j\}$

Notice that the central vertex m of the graph underlying the star mechanism plays the role of money, and is the sole medium of exchange¹³.

Although the set \mathfrak{M}_* is finite, it can be quite large and we will not attempt to characterize it here. Instead we consider the "worst-case complexities" $\pi(M) = \max \pi_{ij}(M)$ and $\tau(M) = \max \tau_{ij}(M)$, and the corresponding quasiorder on \mathfrak{M} , namely: $M \preceq_w M'$ if

$$\tau(M) \le \tau(M'), \quad \pi(M) \le \pi(M').$$

¹³This is reminiscent of "spontaneous symmetry breaking" in physics. The *ex ante* symmetry between commodities, assumed in our model, is carried over to the cycle and complete mechanisms. It breaks down only in the star mechanism, giving rise to money.

If \mathfrak{M} is a subset of \mathfrak{M} one can consider the minimal elements of \mathfrak{M} with respect to the quasiorder $\leq_w restricted$ to \mathfrak{M} ; these will be referred to as strongly minimal mechanisms of \mathfrak{M} .

Theorem 13 If m > 3 then the three special mechanisms are precisely the strongly minimal mechanisms of both $\mathfrak{M}_*(m)$ and $\mathfrak{M}_g(m)$. Their complexities are

	Star	Cycle	Complete
$\pi(M)$	4	2	m(m-1)
$\tau(M)$	2	m - 1	1

The array clearly exhibits the superiority of the star mechanism. As the number of commodities m increases, the other two will be star slightly in one component, but will lose by a huge margin to star in the other component, with the upshot that the star is the overall winner:

Theorem 14 For any strictly positive λ and μ , there exists m_0 such that the star mechanism is the unique maximizer of $\lambda \pi(M) + \mu \tau(M)$ on $\mathfrak{M}_*(m)$ and on $\mathfrak{M}_g(m)$ whenever $m \geq m_0$.

In the star mechanism¹⁴, the *pair* of edges im, mi constitutes a *bilateral* market between i and m for all $i \neq m$. Thus the central node m plays the role of money, mediating trade between various markets. Furthermore these markets are *decentralized* in that the trade at any market is independent of the offers at all *other* markets.

6 Proof of Theorem 9

We fix a mechanism M in \mathfrak{M} with net trade function $\nu(a, b)$. Consider the set of pairs (i, j) for which there is at least one pure ij-index in K, and fix a subset $P \subset K$ which contains *exactly* one ij-index for each such pair. Let $S_P \subset S$ denote the set of P-offers, i.e. those a satisfying $a_h = 0$ for $h \notin P$, and further define the set of P-offers "subordinate" to v as follows:

$$S_P(v) = \{a \in S_P : \overline{a} \le v\}$$

¹⁴The star mechanism (also known as the "Shapley-Shubik mechanism", see [35], [36], [37]) has been much-studied in the literature in different contexts, see, *e.g.*, [3], [4], [5], [7], [8], [29], [30], [31], , [39], [38], [39], [40]. The complete mechanism (also known as "Shapley's windows mechanism") is analysed in [33]. All other *G*-mechanisms are discussed in [6].

Given a vector $v \in S$ we write $\langle v \rangle$ for the class of vectors with the same sign as v, thus $w \in \langle v \rangle$ if each component w_i has the same sign (+, -, 0) as v_i .

Lemma 15 Let $v, w \in S$ then the following are equivalent.

- 1. There is an $a \in S_P(v)$ such that $v + \nu(a, b) \in \langle w \rangle$ for some $b \in S_+$
- 2. There is an $a \in S_P(v)$ such that $v + \nu(a, b) \in \langle w \rangle$ for all $b \in S_+$
- 3. For each $u \in \langle v \rangle$ there is an $a \in S_P(u)$ such that $u + \nu(a, b) \in \langle w \rangle$ for all $b \in S_+$

Proof. It is evident that (3) implies (2), and (2) implies (1). We now show that (1) implies (3). Suppose v, a, b, w satisfy (1). Given $u \in \langle v \rangle$ and $b_* \in S_+$, we need to find $a_* \in S_P(u)$ such that u, a_*, b_*, w satisfy (3). Since uand v have the same signs there exist positive scalars λ_i such that $u_i = \lambda_i v_i$ for all i. Define a_* by $(a_*)_i = \lambda_i a_i$, where (recall) a_i is the vector obtained from a by restricting to the K_i -components. Now we have

$$v + \nu (a, b) = (v - \overline{a}) + r (a, b)$$
$$u + \nu (a_*, b_*) = (u - \overline{a_*}) + r (a_*, b_*)$$

By construction of a_* we have $(v - \overline{a})_i = \lambda_i (u - \overline{a_*})_i$ for all *i*, and hence $\langle v - \overline{a} \rangle = \langle u - \overline{a_*} \rangle$. Also since *a* and a_* are *P*-offers, by Aggregation and Invariance we have $\langle r(a,b) \rangle = \langle r(a,b_*) \rangle = \langle r(a_*,b_*) \rangle$. We note that if x, y are non-negative vectors then $\langle x + y \rangle$ is uniquely determined by $\langle x \rangle$ and $\langle y \rangle$, thus we get

$$\langle u + \nu (a_*, b_*) \rangle = \langle v + \nu (a, b) \rangle = \langle w \rangle$$

which establishes (3).

We note that Lemma 15 (3) only depends on $\langle v \rangle$ and $\langle w \rangle$ and we will write $\langle v \rangle \rightarrow \langle w \rangle$ if it holds.

Lemma 16 For any $(a, b) \in S \times S_+$ there is $a_* \in S_P(\overline{a})$ such that

$$\langle r(a,b) \rangle = \langle \overline{a} + \nu(a_*,b) \rangle.$$
 (6)

Proof. By Aggregation, it suffices to prove this when a is a K_i -offer for some i. By *Flexibility* there is some $a_* \in S_P(\overline{a})$ such that $r_i(a_*, b) = 0$, while $r_i(a_*, b)$ has the same sign as $r_i(a, b)$ for all $j \neq i$. We write

$$\overline{a} + \nu (a_*, b) = (\overline{a} - \overline{a_*}) + r (a_*, b)$$

and note that since a_* is a pure K_i -offer, the sign of $r(a_*, b)$ does not change if we rescale a_* . If $r_i(a, b) = 0$ we scale up a_* to ensure $\overline{a_*} = \overline{a}$, while if $r_j(a, b) > 0$ then we scale down a_* to ensure $\overline{a_*} \leq \overline{a}$; in each case the rescaled a_* satisfies (6).

Lemma 17 $v^1 \to \cdots \to v^t$ implies $\langle v^1 \rangle \to \cdots \to \langle v^t \rangle$.

Proof. It suffices to show that $v \to w$ implies $\langle v \rangle \to \langle w \rangle$. Now by definition

$$w = v + \nu(a, b)$$
 for some $(a, b) \in S \times S_+$ with $\overline{a} \leq v$.

If a_* is as in (6) then the identities

$$v + \nu (a_*, b) = (v - \overline{a}) + (\overline{a} + \nu (a_*, b))$$
$$v + \nu (a, b) = (v - \overline{a}) + r (a, b)$$

imply $\langle v + \nu (a_*, b) \rangle = \langle w \rangle$, whence $\langle v \rangle \rightarrow \langle w \rangle$ by Lemma 15 (1).

Proposition 18 For $b \in S_+$ and any $i \neq j$ there is $a \in S_P$ such that $\nu(a, b)$ is an $\overline{i}j$ -vector.

Proof. Let v be an *i*-vector and let $t = \tau_{ij}(M)$ then by definition we have a sequence

$$v \to v^1 \to \dots \to v^{t-1} = w$$

where w is a *j*-vector. By the previous lemma we get

$$\langle v \rangle \to \langle v^1 \rangle \to \dots \to \langle v^{t-1} \rangle \to \langle w \rangle$$

By Lemma 15 (3) this means we can find sequences

$$u^{i} \in \langle v^{i} \rangle, a^{i} \in S_{P}(u^{i}) \text{ for } i = 0, \dots, t-1$$

such that $u^{i} + \nu(a^{i}, b) = u^{i+1}$. If $a = \sum a^{i}$ then we have $a \in S_{P}$ and

$$\nu(a,b) = \sum \nu(a^i,b) = u^t - u^1$$

which is an $\overline{i}j$ -vector.

It will be convenient to write an $\bar{i}j$ -vector in the form (-x, y) after suppressing the other components. In the context of the above proposition if $\nu(a,b) = (-x,y)$ then by linearity $\nu(a/x,b) = (-1,y/x)$, and we will say that the offer a (or a/x) achieves an ij-exchange ratio of y/x at b.

Lemma 19 If a', a'' achieve ij-exchange ratios α' , α'' at b, then $\alpha' = \alpha''$.

Proof. By the previous proposition there exists an a such that $\nu(a, b)$ is a ji-vector; if α is the corresponding exchange ratio then by rescaling a, a', a'' we may assume that

$$\nu(a,b) = (1, -\alpha), \nu(a', b) = (-1, \alpha'), \nu(a'', b) = (-1, \alpha'').$$

By Proposition 5 we get

$$\nu \left(a + a', b \right) = \left(0, \alpha - \alpha' \right)$$

Now by *Non-dissipation* we get $\alpha \geq \alpha'$, and exchanging the roles of *i* and *j* we conclude that $\alpha' \geq \alpha$ and hence that $\alpha = \alpha'$. Arguing similarly we get $\alpha = \alpha''$ and hence that $\alpha' = \alpha''$.

Proof of Theorem 9. Fix $b \in S_+$ and consider the vector

$$p = (1, p_2, \dots, p_m)$$

where p_j^{-1} is the 1*j*-exchange ratio at *b*, as in the previous lemma. We will show that *p* satisfies the conditions of Theorem 9, *i.e.* that

$$p \cdot \nu (a, b) = 0 \text{ for all } a. \tag{7}$$

We argue by induction on the number d(a, b) of non-zero components of $\nu(a, b)$ in positions $2, \ldots, m$. If d(a, b) = 0 then $\nu(a, b) = 0$ by Nondissipation and (7) is obvious. If d(a, b) = 1 then $\nu(a, b)$ is either an $\bar{1}j$ -vector or a $\bar{j}1$ vector, which by the definition of p_j and the previous lemma is necessarily of the form

$$(-x, xp_j^{-1})$$
 or $(x, -xp_j^{-1})$;

for such vectors (7) is immediate. Now suppose d(a,b) = d > 1 and fix j such that $\nu_j(a,b) \neq 0$. Then we can choose a' such that $\nu(a',b)$ is a $\bar{1}j$ or a $\bar{j}1$ -vector such that $\nu_j(a,b) = -\nu_j(a',b)$. It follows that d(a+a',b) < d and by linearity we get

$$p \cdot \nu (a, b) = p \cdot \nu (a + a', b) - p \cdot \nu (a', b).$$

By the inductive hypothesis the right side is zero, hence so is the left side.

Finally the uniqueness of the price function is obvious, because the return function of the mechanism dictates how many units of j may be obtained for one unit of i, yielding just one possible candidate for the exchange rate for every pair ij.

7 Proof of Theorem 12

We say a matrix X is an $S \times T$ matrix if its rows and columns are indexed by finite sets S and T respectively; if Y is a $T \times U$ matrix then the product XY is a well-defined $S \times U$ matrix. For the set $[n] = \{1, \ldots, n\}$ we will speak of $n \times T$ matrices instead of $[n] \times T$ matrices, etc.

Let $M \in \mathfrak{M}(m)$ and write $K_i = K_i(M)$ and $K = \coprod_i K_i$ as usual. For any vector $v \in \mathbb{R}^m_{++}$, let D_v denote the $m \times m$ diagonal matrix $diag\{v_1, \ldots, v_m\}$, and let E_v denote the $K \times K$ "extended" diagonal matrix whose K_i -diagonal entries are all v_i . Also let A be the $m \times K$ "auxiliary" matrix whose K_1 columns are $(1, 0, \ldots, 0)^t$, K_2 -columns are $(0, 1, 0, \ldots, 0)^t$, etc.

Lemma 20 M is uniquely determined by a map $b \mapsto N_b$ from S_+ to the space of non-negative $m \times K$ column-stochastic matrices as follows.

1. The price ray p = p(b) is obtained as the unique solution of

$$C_b p = \Delta_b p \tag{8}$$

where $\Delta_b = AD_bA^t$ is the diagonal matrix of column sums of $C_b = N_bD_bA^t$

2. The return function is given by

$$r(a,b) = M_b a \text{ where } M_b = D_p^{-1} N_b E_p \tag{9}$$

Proof. Let p = p(b) be the price function whose existence is guaranteed by Theorem 9. We will first prove formula 9 for r(a, b) and then prove formula 8. By Proposition 5, the return function of the mechanism M is of the form $r(a, b) = M_b a$, where $b \mapsto M_b$ is a map from S_+ to the space of non-negative $m \times k$ matrices satisfying

$$M_b b = A b$$

and the identity

$$M_{E_vb} = D_v M_b E_v^{-1} \text{ for all } v \in R^m_{++}.$$
(10)

(The non-negativity M_b follows from that of r(a, b)). The first display holds by conservation of commodities and the second by *Invariance*.) Define

$$b' = E_p b, \ N_b = M_{b'}.$$

By *Invariance* it follows that $p(b') = \mathbf{1}$. Also each column of $N_b = M_{b'}$ is the return to the offer of a single unit in some commodity. Since all prices are 1 at b', Theorem 9 implies that each column of N_b sums to 1, *i.e.* N_b is column stochastic. Now by (10) we get

$$N_b = M_{E_p b} = D_p M_b E_p^{-1},$$

whence $M_b = D_p^{-1} N_b E_p$ as desired

Now combining (9) and $M_b b = Ab$, with the identity $D_p A = AE_p$ we have

$$N_b E_p b = D_p M_b b = D_p A b = A E_p b.$$

Using the identity $E_p b = D_b A^t p$ we can rewrite this as

$$N_b D_b A^t p = A D_p A^t b$$

which is precisely (8).

Lemma 21 Let N_b in Lemma 20 and let $h \in K_i$ be a pure ij-index.

- 1. The h-th column of N_b is the j-th unit vector e_j , independent of b.
- 2. Every K_i -column of N_b is a linear combination of the "pure" K_i -columns.

Proof. By definition there is an *h*-offer *a* such that $r(a, b) = M_b a$ is a *j*-return vector. This means that the *h*-th column of M_b has a non-zero entry only in its *j*-th component. Since N_b is obtained from M_b by rescaling entries this is also true of N_b . By column stochasticity the *h*-th column of N_b must be e_j .

For the second part, let $h' \in K_i$ be an *i*-index, let v, w be the h'-th columns of N_b and M_b , and suppose the *j*-th component of v (and hence of w) is non-zero. It suffices to show that in this case the mechanism has a pure *ij*-index. However if a is an h'-offer then $r(a, b) = M_b a$ is a multiple of w, and thus the assertion follows from *Flexibility*.

Let G be the graph in which we connect i to j if M has a pure ij-index. Since M is connected, Lemma 17 implies that G is connected, and we let $M' = M_G$ denote the corresponding G-mechanism. We will identify the *i*indices K'_i of M' as a subset of K_i . If M has several pure ij-indices for a given j then this involves a choice, however the choice will play no role in the subsequent discussion. We will refer to M' as the embedded G-mechanism of M.

To continue we need a result from [32]. Let G be any connected directed graph on $\{1, \ldots, n\}$ with weights z_{ij} attached to edges $ij \in G$. We write $Z = (z_{ij})$ for the $n \times n$ matrix of edge weights of G, setting $z_{ij} = 0$ if $ij \notin G$. We also define

$$\delta_j = \sum_i z_{ij}, \quad \Delta_Z = diag\left(\delta_1, \dots, \delta_n\right),$$

so that Δ is the diagonal matrix of column sums of Z. We define the weight of a subgraph Γ to be the product of its edge weights, thus

$$w_{\Gamma}(z) = \prod_{ij \in E_{\Gamma}} z_{ij}.$$

We define an *i*-tree in G to be a (directed) subgraph T with n vertices and n-1 edges, and the futher property that T contains a path from j to i for every $j \neq i$. We write \mathcal{T}_i for the set of *i*-trees in G, and define

$$w_i = \sum_{\Gamma \in \mathcal{T}_i} w_{\Gamma}(z), \quad w = (w_1, \dots, w_n)^t$$

The following lemma from [32] is critical and paves the way for the rest of the analysis.

Lemma 22 If Z, Δ_Z, w are as above then one has $Zw = \Delta_Z w$.

We can now prove a key property of embedded G-mechanisms.

Proposition 23 If a price ratio depends on some variable in M', then it does so in M.

Proof. The pure columns of N_b are fixed unit vectors, independent of b. By assumption there is a bijection between the pure variables and the nonzero entries $c_{rs}(b)$ of the matrix C_b . We denote the pure components of b by $x = (x_{rs})$ and the remaining mixed components by $y = (y_k)$. Then by the definition of C_b we have an expression of the form

$$c_{rs}(b) = x_{rs} + \sum_{k} \varepsilon_{k}(b) y_{k}; \ 0 \le \varepsilon_{k}(b) \le 1.$$
(11)

By formula (8) and Lemma 22, the prices p in M and M' are weighted sums of trees with edge weights c_{rs} and x_{rs} repectively. Let p(x, y) denote the price vector in M at b = (x, y) and let p(x) denote the price vector in M' at x. Then by (11) we get

$$p\left(x\right) = \lim_{y \to 0} p\left(x, y\right).$$

We now fix a pair of commodities i, j and let $\pi(x, y)$ and $\pi(x)$ denote the price ratios p_i/p_j in M and M' respectively, then we have

$$\pi\left(x\right) = \lim_{y \to 0} \pi\left(x, y\right)$$

Thus if $\pi(x)$ depends on some x-component, so must $\pi(x, y)$.

Proof of Theorem 12. By lemma 17, lemma 21 and the previous proposition (respectively), we have:

$$\tau_{ij}(M') = \tau_{ij}(M), \ k(M') \le k(M), \ \pi_{ij}(M') \le \pi_{ij}(M)$$

If M is minimal then equality must hold throughout. Hence we get k(M') = k(M) and so M = M' is a G-mechanism.

8 Proof of Theorem 13

First let us recall some basic order-theoretic notions.

Definition 24 A quasiorder \preceq on a set X is a binary relation that is reflexive $(x \preceq x)$ and transitive:

$$x \precsim y, y \precsim z \implies x \precsim z.$$

We write $x \prec y$ if $x \preceq y$ holds but $y \preceq x$ does not hold. We say that x is \preceq -minimal if there is no y in X such $y \prec x$, equivalently if for all $y \in X$

$$y \precsim x \implies x \precsim y.$$

We write X_{\prec} for the set of \preceq -minimal elements of X. We say that \preceq is a well-quasiorder (wqo) if there does not exist an infinite descending chain

$$\cdots \prec x_n \prec \cdots \prec x_2 \prec x_1.$$

Note that if \preceq is a wqo on X and $Y \subset X$ then the restriction of \preceq defines a wqo on Y. In general the minimal elements Y_{\prec} can be quite different from X_{\prec} , however we have the following elementary result. **Lemma 25** If (X, \preceq) is a wqo and $X_{\prec} \subset Y \subset X$ then $X_{\prec} = Y_{\prec}$.

Proof. Any minimal element of X that happens to lie in Y is clearly minimal in Y. Thus $X_* \subset Y$ implies $X_\prec \subset Y_\prec$. On the other hand if z is a non-minimal element of X then $x \prec z$ for some $x \in X_\prec$, otherwise we could construct an infinite descending chain in X starting with z. In particular any $z \in Y \setminus X_\prec$ satisfies $x \prec z$ for some $x \in X_\prec \subset Y$, hence z is not minimal in Y.

It is easy to check that both the quasiorders \leq and \leq_w that we have introduced on \mathfrak{M} are wqo's; and therefore in the proof below we will apply the previous lemma to them.

Proof of Theorem 13. Let \mathfrak{S} denote the set consisting of the three special mechanisms. We need to show that $(\mathfrak{M}_g)_{\prec_w} = \mathfrak{S}$ and $(\mathfrak{M}_*)_{\prec_w} = \mathfrak{S}$. It is shown in [0] that $(\mathfrak{M})_{\sim} = \mathfrak{S}$

It is shown in [9] that $(\mathfrak{M}_g)_{\prec_w} = \mathfrak{S}$. We now prove $(\mathfrak{M}_*)_{\prec_w} = \mathfrak{S}$. Since $(\mathfrak{M}_g)_{\prec_w} = \mathfrak{S}$, by Lemma 25 applied to the wqo \preceq_w it suffices to show that $\mathfrak{S} \subset \mathfrak{M}_*$. We further note that

$$\mathfrak{M}_* = (\mathfrak{M}_g)_{\prec} \,. \tag{12}$$

Indeed $\mathfrak{M}_* = \mathfrak{M}_{\prec}$ by definition, and $\mathfrak{M}_* \subset \mathfrak{M}_g$ by Theorem 12; now (12) follows from Lemma 25 applied to the wqo \preceq . Thus it suffices to prove that

$$\mathfrak{S} \subset (\mathfrak{M}_g)_{\prec} \tag{13}$$

i.e., that each of the three special mechanisms is \preceq -minimal in \mathfrak{M}_{g} .

The \leq -minimality is obvious for the complete graph since any other graph would have some $\tau_{ij} > 1$, and also for the cycle since any other graph would have some $k_i > 1$. To establish \leq -minimality for the star graph, it suffices to show that any non-star graph G has either some $\pi_{ij} \geq 5$ or some $\tau_{ij} \geq 3$. For this we note that $\pi \geq 5$ holds by Theorem 16 of [9] if G is not a rose or a chorded cycle; while $\tau \geq 3$ holds trivially for non-star roses and by Lemma 37 of [9] for chorded cycles. This completes the proof of (13) and hence of $(\mathfrak{M}_*)_{\prec_w} = \mathfrak{S}$.

Remark 26 For m = 3, Lemma 37 of [9] does not hold and we have an additional strongly minimal mechanism with $(\tau, \pi) = (2, 4)$, namely the chorded triangle

•		
\downarrow	K	
•	$\stackrel{\longleftarrow}{\longrightarrow}$	•

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