# Bilateral trading and incomplete information: Price convergence in a small market.

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**Abstract**

We study a model of decentralised bilateral interactions in a small market where one of the sellers has private information about her value. In addition to this seller with private information, there are two identical buyers and another seller, whose valuation is commonly known to be in between the two possible valuations of the seller with private information. We consider an infinite horizon game with simultaneous one-sided offers and simultaneous responses. We construct one particular PBE of the game and show that, as the discount factor goes to 1, prices in all transactions converge to the same value. We then show that this is the case with *any* stationary equilibrium of the game. That is, the asymptotic outcome is *unique* across all stationary equilibria.

**JEL Classification Numbers:** $C78$, $D82$

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# 1 Introduction

This paper studies a small market in which one of the players has private information about her valuation. As such, it is a first step in combining the literature on (bilateral) trading with incomplete information with that on market outcomes obtained through decentralised bilateral bargaining.

We shall discuss the relevant literature in detail later on in the introduction. Here we summarise the motivation for studying this problem.

One of the most important features in the study of bargaining is the role of alternative bargaining partners in determining the trading outcome. There have been several different approaches to modelling the effect of such alternatives on an existing negotiation, starting with treating alternatives to the current bargaining game as exogenously given and always available. Accounts of negotiation directed towards practitioners and policy-oriented academics, like Raiffa's masterly "The Art and Science of Negotiation",([30]) have emphasised the key role of the "Best Alternative to the Negotiated Agreement" and mentioned the role of searching for such alternatives in preparing for negotiations. We could also take such alternatives into account by explicitly considering strategic choice of bargaining partners.

Proceeding more or less in parallel, there has been considerable work on bargaining with incomplete information. The major success of this work has been the complete analysis of the bargaining game in which the seller has private information about the minimum offer she is willing to accept. The probability distribution from which the seller's reservation price is drawn is common knowledge. The buyer makes repeated offers which the seller can accept or reject; each rejection takes the game to another period and time is discounted at a common rate by both parties.[1] With the roles of the seller and buyer reversed, this has also been part of the development of the foundations of dynamic monopoly and the Coase conjecture.[2]

Our model here analyses a small market in which players make strategic choices of whom to make offers to and (for sellers), whose offer to accept. Thus, though trades remain bilateral, a buyer can choose to make an offer to a different seller than the one who rejected his last offer and the seller can entertain an offer from some buyer whom she has not bargained with before. These alternatives are internal to the model of a small market, rather than given as

[1] Other, more complicated, models of bargaining have also been formulated (for example, [10]), with two-sided offers and two-sided incomplete information, but these have not usually yielded the clean results of the game with one-sided offers and one-sided incomplete information.

[2] The "Coase conjecture" relevant here is the bargaining version of the dynamic monopoly problem (as outlined in Gul and Sonnenschein ([21]), Gul, Sonnenschein and Wilson([22]) and the review paper of Ausubel, Cramton and Deneckere ([3])), namely that if an uninformed seller (who is the only player making offers) has a valuation strictly below the privately informed buyer's lowest possible valuation, the unique sequential equilibrium as the seller is allowed to make offers frequently, has a price that converges almost immediately to the lowest buyer valuation. In the current work, we show that even if the bilateral bargaining is embedded in a small market, a similar conclusion holds for all stationary equilibria-hence an extended Coase conjecture holds.

part of the environment. What we do is as follows: We take the basic problem of a seller with private information and an uninformed buyer and add another buyer-seller pair. The new seller's valuation is commonly known and is different from the possible valuations of the seller with private information. The buyers' valuations are identical and commonly known. Specifically, the privately informed seller's valuation can either be $L$ or $H$ ($H > L \geq 0$) and the new seller's valuation is $M$ such that $M \in (L, H)$. Each seller has one good and each buyer wants at most one good. This is the simplest extension of the basic model that gives rise to alternatives for each player, though only *one* buyer can deviate from the incomplete information bargaining to choose the other seller (if this other seller accepts the offer), since each seller only has one good to sell.[3]

In our model, buyers make offers *simultaneously*, each buyer choosing only one seller.[4] Sellers also respond simultaneously, accepting at most one offer. A buyer whose offer is accepted by a seller leaves the market with the seller and the remaining players play the one-sided offers game with or without asymmetric information. We consider the case when buyers' offers are public, so that the continuation strategies can condition on both offers in a given period and the set of players remaining.[5] We first **construct** a particular stationary equilibrium. The **main result** of our analysis shows that in the incomplete information game, **any stationary equilibrium** must have certain specific qualitative features. As agents become patient enough, these qualitative features enable us to show that **all price offers in any stationary equilibrium converge to the highest possible value of the seller with private information** ($H$). In the two-player case, there is a unique sequential equilibrium for the "gap"[6] case. However, there could be non-stationary equilibria with different outcomes in the four-player, public offers case. We note that the discussion of the features of the stationary equilibrium is concerned with the properties of proposed actions occurring with positive probability on the equilibrium path. However, in order to show existence, the properties of outcome paths following deviations become important.

The particular equilibrium we construct is in (non-degenerate) randomized behavioural strategies. As agents become patient enough, in equilibrium competition always takes place for the seller whose valuation is commonly known. The *equilibrium* behaviour of beliefs is similar to the two-player asymmetric information game. However, the off-path behaviour sustaining any equilibrium

[3]What do the seller's valuations represent? (The buyers' valuations are clear enough.) We could consider a seller who can produce a good, if contracted to do so, at a private cost of $H$ or $L$ and pays no cost otherwise. Or we could consider the value she gets from keeping the object as $H$ or $L$. Thus, supposing her value is $L$, if she accepts a price offer $p$ with probability $\alpha$, her payoff is $L(1-\alpha) + (p)\alpha = (p-L)\alpha + L$. Hence, one can think of $(p - L)$ as the net expected benefit to the seller from selling the good at price $p$. For the purpose of making the decision on whether to accept or reject, the two interpretations give identical results.

[4]Simultaneous offers extensive forms capture best the essence of competition.

[5]We also discuss private offers in the extensions. This is the case when only the proposer and the recipient of an offer know what it is and the only public information is the set of players remaining in the game.

[6]The *gap* case implies the situation when the highest possible value of the seller is strictly below the valuation of the buyer.

is different and has to take into account many more possible deviations.[7]

The intuition behind our result can be explained as follows. When it is known with certainty that the privately informed seller's valuation is $L$, from the complete information result ([6]) we know that the transaction price cannot be more than $M$. Once uncertainty about the privately informed seller's valuation is introduced, the Coasian force comes into play. In the absence of any competition for the seller with valuation $M$, for high discount factor, due to the Coasian force the reservation price for the privately informed seller rises above $M$. This implies for high discount factor, in equilibrium both the buyers will try to get matched to the seller with valuation $M$. Since offers are simultaneously made, this results in competition among the buyers to get matched to the seller with valuation $M$. However, in competing for the seller with valuation $M$, no buyer will go above the price that reduces his expected payoff beyond what he could get in the two-player game with the privately informed seller for the specific value of $\pi$. As $\delta \to 1$, the buyers' payoff from the two-player game with the privately informed seller goes to $v - H$. Hence, the price offered to the seller with valuation $M$ also goes to $H$. Though this is the intuition, the validity of this rests on proving for example that the seller with valuation $M$ in any stationary equilibrium accepts an offer immediately. These and other details to verify the intuition are shown in the formal proofs.

Since the seller with valuation $M$ never trades with a delay, competition ensures that for high values of the discount factor the price offered to $S_M$ converges to the price offered to the seller with private information. When the discount factor approaches one, Coasian force implies that the price offered to the privately informed seller approaches $H$. Hence, the price offered to the seller with valuation $M$ also approaches $H$. This explains our result.

Our asymptotic convergence result above has a flavour of the Coase Conjecture and could be considered as a contribution to the discussion of its robustness to the setting of a small market.

**Related literature:** The modern interest in this approach dates back to the seminal work of Rubinstein and Wolinsky ([31], [32]), Binmore and Herrero ([4]) and Gale ([18]),[19]). These papers, under complete information, mostly deal with random matching in large anonymous markets, though Rubinstein and Wolinsky (1990) is an exception. Chatterjee and Dutta ([8]) consider strategic matching in an infinite horizon model with two buyers and two sellers and Rubinstein bargaining but with complete information.

In ([6]), we have analysed markets with equal finite numbers of buyers and sellers, under complete information. The bargaining is with one-sided offers. We show that in any stationary equilibrium, as agents become patient enough, prices in all transactions converge to a single value. In the current work, we show convergence of prices in an incomplete information framework.[8]

[7]The result of this paper is not confined to uncertainty described by two types of seller. Even if the privately informed seller's valuation is drawn from a continuous distribution on $(L, H]$, it can be shown that the asymptotic convergence to $H$ still holds as the unique limiting stationary equilibrium outcome. Please refer to [7] for a formal analysis.

[8]In the complete information case (see [6]), we get a similar result. This however, does not imply that the

Chatterjee and Dutta ([9]) address a setting similar to ours, namely a market with two buyers and two sellers in which one of the sellers has private information about her value. However, their model of the market is different in that though only the buyers make offers, they do so sequentially with the buyer moving second observing the offer made by the first buyer. Sellers move after the buyers' offers have been made. In their model it is not the case that prices in all stationary equilibria converge to the same value. This is true for one stationary equilibrium they analyse but there exists another in which the seller without information gets a low payoff and the price for that seller does not go to $H$. The intuition for this is that the sequential nature of the offer process somewhat affects the competition between the buyers. As ([9]) points out, the authors did not pursue simultaneous offers because of considerations of analytical tractability. We find, however, that simultaneous offers supplies the missing element that makes a Coase-like conjecture hold for all stationary equilibrium outcomes in this informational environment and with this kind of competition.

A paper analysing outside options in asymmetric information bargaining is that by Gantner ([20]), who considers such outside options in the Chatterjee and Samuelson ([10]) model. Our model differs from hers in the choice of the basic bargaining model and in the explicit analysis of a small market (There is competition for alternatives too, in our model but not in hers). In a recent paper, Board and Pycia ([5]) have shown that in a model of bilateral bargaining with incomplete information and one-sided offers, the Coase conjecture does not hold in the presence of outside options. They consider a specific two-player bargaining model with a buyer and a seller and the buyer is privately informed. The buyer can always take a deterministic outside option. The present model is about price determination through bilateral bargaining in a small market. We show that if the alternatives to the current negotiation are modelled as strategic choices and there is competition for the alternatives among agents on one side of the market, a result analogous to the Coase conjecture reappears.

A strand of the bargaining literature based on a player having some probability of being an inflexible behavioural type[9] has also addressed different notions of alternatives to the current negotiation. Atakan and Ekmekci ([1]) and Compte and Jehiel ([13]) adopt different approaches to model outside options. While [1] uses a random matching process among players on different sides of the market, [13] considers an outside option that is exogenously given. [1] shows that in

analyses are the same. In the bilateral bargaining game with complete information where the seller has valuation $H$, the price is $H$; if it is $L$, the price is $L$. From this fact, it is non-trivial to guess that as the discount factor goes to 1 and the probability of a $H$ seller being positive, the price goes to $H$ (This explains the large number of papers on this bilateral case). With four players, even with only one seller's value being unknown, the problem is compounded by the presence of the other alternatives. We leave out the construction of the equilibrium itself, which requires some careful consideration of appropriate beliefs. Without this construction, of course, the equilibrium path cannot be known to be such, so the fact that two equilibrium paths end up looking similar doesn't mean that the *equilibria* are the same.

[9]A player is said to be of an inflexible behavioural type if he is completely rigid about his offers and demands.

equilibrium, a “Coasian force”leads players to inefficiently delay in revealing their type. However, in [13] this force is absent.

Our work builds on the well-known approaches/results in the one-sided asymmetric information bargaining of Sobel and Takahashi ([33]), Fudenberg, Levine and Tirole ([16]) and Ausubel and Deneckere ([2]).

There are papers in very different contexts that have some of the features of this model. For example, Swinkels [34] considers a discriminatory auction with multiple goods, private values (and one seller) and shows convergence to a competitive equilibrium price for fixed supply as the number of bidders and objects becomes large. We keep the numbers small, at two on each side of the market. Kaya and Liu ([24])[10] analyse a sequential bargaining model of price formation with one long lived buyer and an infinite sequence of sellers. They show that unobservability of past negotiations enhances the Coasian effect in equilibrium. In the present work, we show that the Coasian phenomenon can exist even when all negotiations are publicly observable. Finally, there are several other papers on searching for outside options, for example, Chikte and Deshmukh ([12]), Muthoo ([27]), Lee ([25]), Chatterjee and Lee ([11]) (This has private information about outside options). We do not discuss these in detail because they are not directly comparable to our work.

**Outline of rest of the paper:** The rest of the paper is organised as follows. Section 2 discusses the model in detail. The qualitative nature of the equilibrium, its detailed derivation and the asymptotic characteristics are given in section 3, which is the heart of the paper. Section 4 discusses the possibility of other equilibria, as well as the private offers case. Finally, Section 5 concludes the paper.

# 2 The Model

## 2.1 Players and payoffs

The setup we consider has two uninformed homogeneous buyers and two heterogeneous sellers. Buyers ($B_1$ and $B_2$ ) have a common valuation of $v$ for the good (the maximum willingness to pay for a unit of the indivisible good). There are two sellers. Each of the sellers owns one unit of the indivisible good. Sellers differ in their valuations. The first seller ($S_M$) has a reservation value of $M$ which is commonly known. The other seller ($S_I$) has a reservation value that is private

[10]Their work builds on the work of Horner and Vieille ([23]), which studies a model of one long-run seller with a sequence of short-run buyers. They examine how observability of negotiations affects the probability of reaching an agreement. They however, do not explicitly compare the equilibrium price paths between the observability and the non-observability case.

information to her. $S_I$'s valuation is either $L$ or $H$, where,

$$v > H > M > L$$

It is commonly known by all players that the probability that $S_I$ has a reservation value of $L$ is $\pi \in [0,1)$. It is worthwhile to mention that $M \in [L,H]$ constitutes the only interesting case. If, $M < L$ (or $M > H$) then one has no uncertainty about which seller has the lowest reservation value. Although our model analyses the case of $M \in (L,H)$, the same asymptotic result will be true for $M \in [L,H]$.

Players have a common discount factor $\delta \in (0,1)$. If a buyer agrees on a price $p^j$ with seller $S_j$ at a time point $t$, then the buyer has an expected discounted payoff of $\delta^{t-1}(v - p^j)$. The seller's discounted payoff is $\delta^{t-1}(p^j - u_j)$, where $u_j$ is the valuation of seller $S_j$.

## 2.2 The extensive form

This is an infinite horizon, multi-player bargaining game with one-sided offers and discounting. The extensive form is as follows:

At each time point $t = 1, 2, ..$, offers are made simultaneously by the buyers. The offers are targeted. This means an offer by a buyer consists of a seller's name (that is $S_I$ or $S_M$) and a price at which the buyer is willing to buy the object from the seller he has chosen. Each buyer can make only one offer per period. Two informational structures can be considered; one in which each seller observes all offers made (*public targeted offers*) and the one (*private targeted offers*) in which each seller observes only the offers she gets (Similarly for the buyers, after the offers have been made-in the private offers case each buyer knows his own offer and can observe who leaves the market). In the main analysis we shall focus on the first and consider the latter in a subsequent section on extensions. A seller can accept at most one of the offers she receives. Acceptances or rejections are simultaneous. Once an offer is accepted, the trade is concluded and the trading pair leaves the game. Leaving the game is publicly observable. The remaining players proceed to the next period in which buyers again make price offers to the sellers. As is standard in these games, time elapses between rejections and new offers.

# 3 Equilibrium

We will look for *Perfect Bayes Equilibrium* [17] of the above described extensive form. This requires sequential rationality at every stage of the game given beliefs and the beliefs being compatible with Bayes' rule whenever possible, on and off the equilibrium path. We will focus on *stationary* equilibria.[11] These are the equilibria where strategies on the equilibrium path depend on the history only through the updated value of $\pi$ (the probability that $S_I$'s valuation is $L$) and the number of remaining seller-buyer pairs. Thus, at each time point, buyers' offers depend only on the number of players remaining and the value of $\pi$. The sellers' responses depend on the number of players remaining, the value of $\pi$ and the current offers made by the buyers.

## 3.1 The Benchmark Case: Complete information

Before we proceed to the analysis of the incomplete information framework, we state the results of the above extensive form with complete information. A formal analysis of the complete information framework has been done in [6].

Suppose the valuation of $S_I$ is commonly known to be $H$. In that case there exists a unique[12] stationary equilibrium (an equilibrium in which buyers' offers depend only on the set of players present and the sellers' responses depend on the set of players present and the current offers made by the buyers) in which one of the buyers (say $B_1$) makes offers to both the sellers with positive probability and the other buyer ($B_2$) makes offers to $S_M$ only. Suppose $E(p)$ represents the expected maximum price offer to $S_M$ in equilibrium when all four players are present in the market. Assuming that there exists a unique $p_l \in (M, H)$ such that,

$$p_l - M = \delta(E(p) - M)^{13}$$

, the equilibrium is as follows:

1. $B_1$ offers $H$ to $S_I$ with probability $q$. With the complementary probability he makes offers to $S_M$. While offering to $S_M$, $B_1$ randomises his offers using an absolutely continuous distribution function $F_1(.)$ with $[p_l, H]$ as the support. $F_1$ is such that $F_1(H) = 1$ and $F_1(p_l) > 0$. This implies that $B_1$ puts a mass point at $p_l$.

2. $B_2$ offers $M$ to $S_M$ with probability $q'$. With the complementary probability his offers to $S_M$ are randomised using an absolutely continuous distribution function $F_2(.)$ with $[p_l, H]$ as the support. $F_2(.)$ is such that $F_2(p_l) = 0$ and $F_2(H) = 1$.

It is shown in [6] that this $p_l$ exists and is unique. Also, the outcome implied by the above equilibrium play constitutes the unique stationary equilibrium outcome and as $\delta \to 1$,

$$q \to 0\ ,\ q' \to 0 \text{ and } p_l \to H$$

This means that as market frictions go away, we tend to get a uniform price in different buyer-

[11]Please see section 5 for discussion of non-stationary equilibria.

[12]Up to the choice of $B_1$ and $B_2$.

[13]Given the nature of the equilibrium it is evident that $M(p_l)$ is the minimum acceptable price for $S_M$ when she gets one(two) offer(s).

seller matches. If $S_I$'s type is commonly known to be $L$, the asymptotic convergence is to $M$. Thus, one might conjecture that for high values of $\pi$ the convergence is to $M$ and for low values to $H$. In this paper, however, we show convergence to $H$, even with incomplete information. The analysis is different, since the current value of $\pi$ plays a major role.

## 3.2 Equilibrium of the one-sided incomplete information game with two players

In the equilibrium of the four-player game, if a buyer-seller pair leaves the market after an agreement, then the two-player games that could arise can be of two types. Either it is one of the buyers remaining with the seller $S_M$ or it is one of the buyers remaining with seller $S_I$. In the former case, the unique equilibrium is the buyer making an offer of $M$ to $S_M$. In the latter case, we have a two-player game with one-sided asymmetric information, where the buyer makes all the offers. In the current subsection, we discuss the equilibrium (as described in, for example, [15]) of this game.

The setting is as follows: There is a buyer with valuation $v$, which is common knowledge. The seller's valuation can either be $H$ or $L$ where $v > H > L = 0$.[14] At each period, conditional on no agreement being reached till then, the buyer makes the offer and the seller (privately informed) responds to it by accepting or rejecting. If the offer is rejected then the value of $\pi$ is updated using Bayes' rule and the game moves on to the next period when the buyer again makes an offer. This process continues until an agreement is reached. The equilibrium of this game is as follows.

For a given $\delta$, we can construct an increasing sequence of probabilities, $d(\delta) = \{0, d_1, ....., d_t, ....\}$ such that if the current belief $\pi \in [d_t, d_{t+1})$, then the game ends in equilibrium in at most $t$ periods from now. Here $\pi$ is the updated belief implied by the play for the game so far.[15] The buyer in this case offers $p^u = \delta^t H$. The $H$ type seller rejects this offer with probability 1. The $L$ type seller rejects this offer with a probability that implies, through Bayes' Rule, that the updated value of the belief $\pi^u = d_{t-1}$. The cutoff points $d_t$'s are such that the buyer is indifferent between offering $\delta^t H$ and continuing the game for a maximum of $t$ periods from now or offering $\delta^{t-1}H$ and continuing the game for a maximum of $t-1$ periods from now. Thus, here $t$ means that the game will last for at most $t$ periods from now. The maximum number of periods (for any $\pi \in (0, 1)$) for which the game can last is given by $N(\delta)$. It is shown in [15] that this $N(\delta)$ is uniformly bounded above by a finite number $N^*$ as $\delta \to 1$. This $N^*$ is independent of $\pi$.

The off-path behaviour that sustains the above equilibrium is described in [7].

Given a $\pi$, the expected payoff to the buyer $v_B(\pi)$ is calculated as follows:

[14] $L = 0$ is assumed to simplify notation and calculations.

[15] An alternate notation can be to denote the current belief as $\pi_t$. This might confuse the reader to think that this denotes the belief at the current time point $t$. However, here $t$ refers to the maximum number of periods for which the game can continue on the equilibrium path.

For $\pi \in [0, d_1)$, the two-player game with one-sided asymmetric information involves the same offer and response as the complete information game between a buyer of valuation $v$ and a seller of valuation $H$. Thus, we have

$$v_B(\pi) = v - H \text{ for } \pi \in [0, d_1)$$

For $\pi \in [d_t, d_{t+1})$, ($t \geq 1$ ), we have,

$$v_B(\pi) = (v - \delta^t H)a(\pi, \delta) + (1 - a(\pi, \delta))\delta(v_B(d_{t-1})) \tag{1}$$

where $a(\pi, \delta)$ is the equilibrium acceptance probability of the offer $\delta^t H$.

These values will be crucial for the construction of the equilibrium of the four-player game. In the following subsection, we construct a particular stationary equilibrium of the four-player game with incomplete information.

### 3.3 Equilibrium of the four-player game with incomplete information.

In this subsection, we construct a particular stationary PBE of the four-player game with incomplete information. This is summarised in the following proposition.

**Proposition 1** *There exists a $\delta^* \in (0, 1)$ such that if $\delta > \delta^*$, then for all $\pi \in [0, 1)$, the equilibrium with following features exists.*

*(i) One of the buyers (say $B_1$) will make offers to both $S_I$ and $S_M$ with positive probability. The other buyer $B_2$ will make offers only to $S_M$.*

*(ii) $B_2$ while making offers to $S_M$, will put a mass point at $p_l'(\pi)$ and will have an absolutely continuous distribution of offers from $p_l(\pi)$ to $\bar{p}(\pi)$ where $p_l'(\pi)$ ($p_l(\pi)$) is the minimum acceptable price to $S_M$ when she gets one (two) offer(s). For a given $\pi$, $\bar{p}(\pi)$ is the upper bound of the price offer $S_M$ can get in the described equilibrium ($p_l'(\pi) < p_l(\pi) < \bar{p}(\pi)$). $B_1$, when making offers to $S_M$, will have an absolutely continuous (conditional) distribution of offers from $p_l(\pi)$ to $\bar{p}(\pi)$, putting a mass point at $p_l(\pi)$. $\bar{p}(\pi)$ is such that $(v - \bar{p}(\pi)) = v_B(\pi)$.*[16]

[16]From the two player game, given a $\delta$, if $\pi > d_1$, we can find a $t > 1$ such that $\pi \in [d_t, d_{t+1})$. Then, $\bar{p}(\pi) > \delta^t H$. This is because

$$v_B(\pi) = (v - \delta^t H)a(\pi, \delta) + (1 - a(\pi, \delta))\delta v_B(d_{t-1})$$

We must have $v_B(d_{t-1}) < v - \delta^t H$. Suppose not. Then $v_B(d_{t-1}) \geq v - \delta^t H$. We have

$$v_B(d_{t-1}) = (v - \delta^{t-1} H)a(d_{t-1}, \delta) + (1 - a(d_{t-1}, \delta))\delta v_B(d_{t-2})$$

Since $(v - \delta^{t-1} H) < (v - \delta^t H)$, we must have $v_B(d_{t-2}) > (v - \delta^t H)$. Iterating in this manner, we will arrive at the *(iii) $B_1$, when making offers to $S_I$ on the equilibrium path, makes the same offer, as a function of the current belief about the type of the privately informed seller, as in the two-player game with one-sided asymmetric information.*

*(iv) $S_I$'s response behaviour on the equilibrium path is identical to that in the two-player game. $S_M$, in the event she gets only one offer, accepts any offer greater than or equal to $p_l^{'}(\pi)$ and rejects otherwise. In the event she gets two offers, $S_M$ accepts any offer greater than or equal to $p_l(\pi)$ and rejects otherwise.*

*(v) Each buyer in equilibrium obtains a payoff of $v_B(\pi)$.*

The mass points and the distribution of buyers' offers will depend upon $\pi$ though we show that these distributions will collapse in the limit (as $\delta \to 1$).

*The descriptions above imply that if the discount factor is high enough, then irrespective of the probability with which the privately informed seller has valuation $L$, in the equilibrium constructed, buyers always compete for the seller whose valuation is known to be $M$. In the following subsection, we will show that this will be the case for any stationary equilibrium of the four-player game with incomplete information.*

We provide here an informal account of the proof, the details of which are in appendix (A). The first step is to consider the effect of buyer competition for the sellers. Our complete information analysis suggests that if the sellers have values $H$ and $M$, competition for the $M$ seller will drive prices to $H$, whilst if these values are $M$ and $L$, competition will be for the $L$ seller. One might therefore expect, in an environment where one seller value is unknown to the buyers, that competition will be for $S_I$ if the probability of $S_I$'s value being $L$ is sufficiently high. As it turns out, this is not the case for high enough $\delta$. This is because the seller with private information is able to obtain an offer close to $H$ by a finite number of rejections, as in the bilateral bargaining model with incomplete information.[17] This is a fundamental difference between the complete and incomplete information settings. We formalise this by proving a *competition lemma*, terminology borrowed from a similar result in the different model of Chatterjee and Dutta [9].[18] To state it

conclusion that $v - H > (v - \delta^t H)$. Since, $\delta < 1$, this is an impossibility. Hence, $v_B(d_{t-1}) < v - \delta^t H \Rightarrow v_B(\pi) < (v - \delta^t H)$. This implies $\bar{p}(\pi) > \delta^t H$.

[17]Note however, this does not follow directly from the bilateral bargaining model: the finiteness property for any stationary equilibrium has to be explicitly proved. This is done as a part of the proof of Theorem 1.

[18]Their version of the competition lemma shows that as $\delta \to 1$, the price offered to the seller with private information will determine the highest price offered to the seller whose valuation is common knowledge. In our model and not in theirs, the seller without private information can get two offers in equilibrium and hence, the competition lemma has been appropriately modified.

more formally, let $\bar{p}_t$ be the price offer such that the payoff to the buyer from getting it accepted is the same as the buyer's equilibrium expected payoff in the two-player game with $S_I$, when on the equilibrium path the game will last for at most $t$ more periods. Hence,

$$\bar{p}_t = v - [(v - \delta^t H)\alpha(\pi) + (1 - \alpha(\pi))\delta(v - \bar{p}_{t-1})]$$

Here $\alpha(\pi)$ is the probability with which $S_I$ accepts an equilibrium offer in the two player game with one sided asymmetric information. However, as shown in the formal proof, this lemma works for any $\alpha(\pi) \in (0, 1)$. Let $p_t^{'}$ be the upper bound of the minimum acceptable price offer to $S_M$, when she gets only one offer in the conjectured equilibrium. The *competition lemma* shows that if $\delta$ is high enough then $\bar{p}_t > p_t^{'}$.[19]

The second step is to derive the equilibrium. We assume that a $\delta^*$ as described in the proposition exists. Then, we consider a $\delta > \delta^*$ and derive the sequence $d_\tau(\delta) = \{0, d_1, d_2, ..., d_t, ..\}$, which is identical with the similar sequence in the two-player game with asymmetric information. We can find a $t \geq 0$ ($t \leq N^*$) such that $\pi \in [d_t, d_{t+1})$. From the analysis of the two player game with one sided asymmetric information we can recall that $N^*$ is the upper bound on the maximum equilibrium duration (for any $\pi$) of the two player game with one sided asymmetric information and it is finite. Next, we compute $v_B(\pi)$, which is the expected payoff to a buyer in a two-player game with the seller with private information. We define $\bar{p}(\pi)$ as

$$\bar{p}(\pi) = v - v_B(\pi)$$

Let $p_l^{'}(\pi)$ be the minimum acceptable price offer to seller $S_M$ in the event she gets only one offer in equilibrium. From the *competition lemma* we know that $\bar{p}(\pi) > p_l^{'}(\pi)$. Assuming existence of a $p_l(\pi) \in (p_l^{'}(\pi), \bar{p}(\pi))$ such that $p_l(\pi)$ is the minimum acceptable price to $S_M$ in the event she gets *two* offers in equilibrium, we derive the distribution of offers and the probabilities at the mass points from the indifference condition of the buyers. This completes the construction of the equilibrium. Strategies off the equilibrium path that sustain the on-path play are described in appendix (A). In this particular equilibrium, in the presence of all four players, $S_M$ on the equilibrium path always gets an offer and she accepts it immediately. This implies on the equilibrium path, if there is a rejection in period 1, from the second period onwards we get a two-player game with incomplete information. We will now prove the existence of $\delta^*$. This is done below in the third and the fourth step.

The third step involves proving a lemma which we term as the *basis lemma*. This lemma exploits the fact that for any $\delta \in (0, 1)$, $d_1$ is independent of $\delta$ and if $\pi \in [0, d_1)$, the two player

[19] Although this lemma is in the context of the conjectured equilibrium, from our uniqueness result in the latter part of the paper we can infer that this lemma will apply to any stationary equilibrium of the game.

game with a buyer and the privately informed seller is identical to the two-player game of a buyer with valuation $v$ and the seller with valuation $H$. The benchmark results now allow us to conclude that for $\pi \in [0, d_1)$, an equilibrium as conjectured in the current proposition exists.

In the final step, we prove a *recursion lemma*. Suppose for a $T$ $(1 \leq T \leq N^*)$, there exists a $\bar{\delta}$ such that for all $\delta$ exceeding $\bar{\delta}$, the conjectured equilibrium exists for all $\pi \in [0, d_T)$. The *recursion lemma* argues that we can find another threshold for $\delta$, namely $\delta_T^* \geq \bar{\delta}$ such that if $\delta$ exceeds this threshold then the conjectured equilibrium exists for all $\pi \in [d_T, d_{T+1})$.

$N^*$ is the upper bound on the maximum equilibrium duration (for any $\pi \in (0, 1)$) of the two player game with one sided asymmetric information. From the analysis of the two player game with one sided asymmetric information, we know that $N^*$ is finite. This implies we can find a maximum of the thresholds of $\delta$ defined in the *recursion lemma*. We denote this threshold by $\delta^*$.[20] Thus, if the discount factor exceeds $\delta^*$, the *basis lemma* and the *recursion lemma* together imply that for all $\pi \in [0, 1)$, the equilibrium as conjectured in the proposition exists. This completes the informal account of the proof of the proposition.

The following proposition describes the limiting equilibrium outcome as the discount factor $\delta$ goes to 1.

**Proposition 2** *As $\delta \to 1$, all price offers in the stationary equilibrium described in proposition 1 converge to $H$.*

**Proof.** The formal proof is relegated to appendix (B). Here, we provide an informal argument. To show this we argue that as $\delta \to 1$, the probability at the mass point at $p_l'(\pi)$ converges to 0, and $p_l(\pi)$ converges to $\bar{p}(\pi)$. Thus, all distributions of offers collapse. Also, the probability with which $B_1$ makes offers to $S_I$ goes to 0. As $\delta \to 1$, from the two-player game with one-sided asymmetric information, we know that $v_B(\pi)$ goes to $v - H$. This implies $\bar{p}(\pi) \to H$. Hence, as $\delta \to 1$, all price offers converge to $H$. The intuition behind this asymptotic result can be understood as follows. From the two player game with asymmetric equilibrium, we know that as $\delta \to 1$, for any $\pi \in (0, 1)$, the price offered to $S_I$ converges to $H$. On the other hand as $\delta \to 1$, the cost of waiting becomes negligible. Thus as $\delta \to 1$, $S_M$ almost surely gets two offers. This results in the convergence of the price offers to $S_M$ to the equilibrium price offered to $S_I$. ■

We show in appendix (F), that this equilibrium constructed, with some modifications can be extended to a setup with $N$ homogeneous buyers with valuation $v$, $N - 1$ sellers with valuation $M$ (commonly known) and one seller with private information.

In the following sub-section, we show that any stationary equilibrium of the four-player game with incomplete information has the same characteristics as described above and as $\delta \to 1$, all price offers in equilibrium converge to $H$.

[20] This implies that we have verified the assumption of the existence of such a $\delta^*$.

## 3.4 Uniqueness of the asymptotic equilibrium outcome

In this subsection, we show that prices in all stationary equilibrium outcomes must converge to the same value as $\delta \to 1$. This shows that there exists a unique limiting stationary equilibrium outcome.

It turns out to be convenient if we adopt the following sequence of stages in proving this result. Let $\Pi$ be the set of beliefs such that for any belief belonging to this set, a stationary equilibrium exists where, on the equilibrium path, both buyers offer only to the seller with private information. Let $\Phi(\Pi)$ be the set of such equilibria, in case there is more than one with this feature. Let $\Phi^c(\Pi)$ be the set of all stationary equilibria where on the equilibrium path we do not have both buyers making offers solely to $S_I$ when all four players are present.

The first stage proves the main result for any equilibrium belonging to the set $\Phi^c(\Pi)$. In the next stage, we show that the set $\Pi$ is empty. Thus, these two stages together imply the main result of this subsection.

In stage 1, we restrict ourselves to the set $\Phi^c(\Pi)$. The argument follows from the following steps.

First, we rule out the possibility of having any stationary equilibrium with the feature that both buyers, on the equilibrium path, offer only to seller $S_M$. We show that if such an equilibrium exists, then it involves both buyers putting a mass point at the lower bound of the common support. This is not possible in equilibrium.

In the next step, we show that in any stationary equilibrium, $S_M$ always accepts an equilibrium offer immediately. This is irrespective of whether $S_M$ gets one offer or two offers on the equilibrium path. The intuition for these results is as follows. If $S_M$ gets two offers, then rejection of both does not lead to any change in the belief and hence, the expected equilibrium payoff in the continuation game remains unchanged. In a stationary equilibrium this implies that these offers always yield a payoff of zero. This is inconsistent with these being equilibrium offers. If $S_M$ gets only one offer, we use the standard argument in bargaining that it is better for the proposer to make an offer that gives the receiver a payoff equal to the discounted expected continuation payoff than waiting for the next period and discounting his own payoff. The formal proof[21] requires some explicit calculations to verify these intuitions.

The third step considers the revision of beliefs about the privately informed seller. In any equilibrium, which involves the privately informed seller rejecting offers, she must also accept equilibrium offers with some positive probability to generate these revisions of beliefs. We show that for any $\delta \in (0, 1)$ and for any $\pi \in (0, 1)$, conditional on getting offers, the seller with private information by rejecting equilibrium offers can generate a path of beliefs such that she is offered $H$ in finite time. Hence, the second and the third step together imply that in any equilibrium, the

[21] This is done in detail in appendix C.

game ends in finite time.

Next, we show that if players are patient enough, then in equilibrium both buyers cannot offer to both sellers with positive probability. If this were to happen then we can show that one of the buyers will deviate to make an unacceptable offer to the privately informed seller in order to increase his probability of being in a two-player game with $S_M$ next period. Also, we show that every equilibrium has the feature that both buyers make offers to $S_M$ with positive probability. This follows from the fact that if only one buyer makes offers to $S_M$, then from the third step we can conclude that the buyer offering to $S_I$ can profitably deviate.

These conclusions allow us to infer that in any stationary equilibrium in the set $\Phi^c(\Pi)$, one buyer makes offers only to $S_M$ and the other buyer randomises between making offers to $S_I$ and $S_M$.

In the last part of the argument, we can show that as $\delta \to 1$, all price offers converge to $H$. This happens because in equilibrium, $S_M$ always accepts an offer immediately, the buyer (say $B_1$) offering to $S_I$ is essentially facing a two-player game with the seller with private information, in which we know the offers converge to $H$ as $\delta \to 1$. In order to prevent a deviation by $B_1$, price offers in equilibrium to $S_M$ should also converge to $H$.

These results are summarised in the following proposition.

**Proposition 3** *Consider the set of stationary equilibria of the four-player game such that any equilibrium belonging to this set has the property that both buyers do not make offers only to the privately informed seller ($S_I$) on the equilibrium path. As the discount factor $\delta \to 1$, all price offers in any equilibrium belonging to this set converge to $H$.*

The detailed proof of this proposition is formally outlined in Appendix C.

We will now argue that there does not exist any belief such that a stationary equilibrium is possible, where on the equilibrium path, both buyers offer only to the seller with private information. The following lemma shows this.

**Lemma 1** *Let $\Pi$ be the set of beliefs such that for $\pi \in \Pi$, it is possible to have a stationary equilibrium where both buyers offer only to $S_I$. The set $\Pi$ is empty.*

**Proof.** Suppose $\Pi$ is non-empty. Consider any $\pi \in \Pi$. At this belief, it is possible to have a stationary equilibrium such that both buyers on the equilibrium path offer only to $S_I$. First, we will argue that the highest price offer in such a case is always strictly less than $H$ for all $\delta \in (0, 1)$. This is because, in the continuation game $S_M$ either faces a four-player game or a two player game. In the former case, either the equilibrium of the continuation game is in $\Phi(\Pi)$ or in $\Phi^c(\Pi)$. Hence, the maximum price offer she can get is bounded above by $H$. In the latter case the maximum price offer she can get is $M$. Since the seller with private information always accepts an offer in equilibrium with positive probability, the latter case can occur with positive probability. This implies that if $\mathcal{E}_\pi$ is the expected continuation payoff to $S_M$, then $\mathcal{E}_\pi < H - M$. Let $\bar{p}_\pi$ be such that $\bar{p}_\pi - M = \delta\mathcal{E}_\pi < \delta(H - M)$. Hence, $\bar{p}_\pi < H$ for all $\delta \in (0, 1)$. If the maximum price offered to $S_I$ exceeds $\bar{p}_\pi$, then the buyer making the highest offer to $S_I$ can profitably deviate by making an offer to $S_M$.

Let $\bar{p}$ be the largest price offer, for any $\pi \in \Pi$, in any such equilibrium.[22] (It is without loss of generality to assume that the maximum exists. This is because a supremum always exists and we can consider prices in the $\epsilon$-neighbourhood of this supremum). Clearly, as argued above, $\bar{p} < H$. As explained earlier, no equilibrium can involve offers that are rejected by both $L$ and $H$ types of $S_I$. Therefore, the $L$ type must accept an offer with positive probability. This implies (by Bayes' Theorem and $\delta < 1$) that the sequence of prices must be increasing. Consider the offer of $\bar{p}$. There are two possibilities. Either the equilibrium of the continuation game (given the updated belief conditional on $\bar{p}$ being rejected) is in $\Phi(\Pi)$ or it is in $\Phi^c(\Pi)$. In the former case, $S_I$ should accept the offer with probability 1 and the updated belief is $\pi = 0$, where the equilibrium price offer must be $H > \bar{p}$. This means there exists a profitable deviation for the $L$-type $S_I$, for $\delta$ sufficiently high. For the latter case, if $\delta$ is high then from proposition (3) we know that for any stationary equilibrium all offers converge to $H$ as $\delta \to 1$. Once again, this implies the existence of a profitable deviation for the $L$-type $S_I$. Hence, we cannot have $\Pi$ non-empty. This concludes the proof. ■

It is to be noted that since $\Pi$ is empty, there are no equilibria with both buyers offering only to $S_I$. Thus, $\Phi^c(\Pi)$ is the set of all stationary equilibria of the four-player game with incomplete information. The main result of this subsection is summarised in the theorem below.

**Theorem 1** *In any arbitrary stationary equilibrium of the four-player game, as the discount factor goes to* 1*, price offers in all transactions converge to* $H$ *for all values of the prior* $\pi \in [0, 1)$.

**Proof.** The proof of the theorem follows directly from proposition (3) and lemma (1). ■

In the next subsection, we discuss some extensions by considering non-stationary equilibria and private offers.

# 4 Extensions

In this section we consider some possible extensions by considering a non-stationary equilibrium. We then discuss private offers.

[22] One can easily check that it is not possible to have $\bar{p} \to H$ as $\delta \to 1$. Suppose it does. From the results already established, we can infer that in such an equilibrium, if $\bar{p} \to H$ as $\delta \to 1$, the probability of acceptance of $\bar{p}$ should always be bounded away from zero. Hence, $\mathcal{E}_\pi < H - M$ and it is bounded away from $H - M$ as $\delta \to 1$. Thus, one of the buyers can profitably deviate.

## 4.1 A non-stationary equilibrium

We show that with public offers we can have a non-stationary equilibrium, so that the equilibrium constructed in the previous sections is not unique. This is based on using the stationary equilibrium as a punishment (the essence is similar to the *pooling equilibrium with positive profits* in [28]). The strategies sustaining this are described below. The strategies will constitute an equilibrium for sufficiently high $\delta$, as is also the case for the stationary equilibrium.

Suppose for a given $\pi$, both the buyers offer $M$ to $S_M$. $S_M$ accepts this offer by selecting each seller with probability $\frac{1}{2}$. If any buyer deviates, for example by offering to $S_I$ or making a higher offer to $M$, then all players revert to the stationary equilibrium strategies described above. If $S_M$ gets the equilibrium offer of $M$ from the buyers and rejects both of them then the buyers make the same offers in the next period and the seller $S_M$ makes the same responses as in the current period.

Given the buyers adhere to their equilibrium strategies, the continuation payoff to $S_M$ from rejecting all offers she gets is zero. So she has no incentive to deviate. Next, if one of the buyers offers slightly higher than $M$ to $S_M$ then it is optimal for her to reject both the offers. This is because on rejection next period players will revert to the stationary equilibrium play described above. Hence, her continuation payoff is $\delta(E_\pi(p) - M)$, which is higher than the payoff from accepting.

Finally, each buyer obtains an equilibrium payoff of $\frac{1}{2}(v - M) + \frac{1}{2}\delta v_B(\pi)$. If a buyer deviates then, according to the strategies specified, $S_M$ should reject the higher offer if the payoff from accepting it is strictly less than the continuation payoff from rejecting(which is the one period discounted value of the payoff from stationary equilibrium). Hence, if a buyer wants $S_M$ to accept an offer higher than $M$ then his offer $p^{'}$ should satisfy,

$$p^{'} = \delta E_\pi(p) + (1 - \delta)M$$

The payoff of the deviating buyer will then be $\delta(v - E_\pi(p)) + (1 - \delta)(v - M)$. As $\delta \to 1$, $\delta(v - E_\pi(p)) + (1 - \delta)(v - M) \approx \delta(v - \bar{p}(\pi) + (1 - \delta)(v - M)$
$= \delta v_B(\pi) + (1 - \delta)(v - M)$.

For $\delta = 1$ this expression is strictly less than $\frac{1}{2}(v - M) + \frac{1}{2}\delta v_B(\pi)$, as $(v - M) > \delta v_B(\pi)$. Hence, for sufficiently high values of $\delta$ this will also be true. Also, if a buyer deviates and makes an offer in the range $(M, p^{'})$ then it will be rejected by $S_M$. The continuation payoff of the buyer will then be $\delta v_B(\pi) < \frac{1}{2}(v - M) + \frac{1}{2}\delta v_B(\pi)$. Hence, we show that neither buyer has any incentive to deviate.

## 4.2 Private offers

In this subsection, we consider a variant of the extensive form of the four-player incomplete information game in which offers are private to the recipient and the proposer. This means in each period a seller observes only the offer(s) she gets and a buyer does not know what offers are made by the other buyer or received by the sellers.

The equilibrium notion here is that of a *Public Weak Perfect Bayesian Equilibrium (PWPBE).* That is, in equilibrium, strategies can condition only on the public history (which is the set of players remaining in a particular period), and the public belief. In case of public targeted offers, while proving that the stationary equilibrium outcome is unique, we did not use the fact that each seller while responding observes the other seller's offer. This implies that any stationary equilibrium of the public targeted offers game can potentially be a particular Public Weak Perfect Bayesian Equilibrium of the game with private targeted offers. If we suitably define the play off the equilibrium path, then we can sustain the equilibrium characterised for the four-player incomplete information game with public offers as a PWPBE of the game with private offers. The following proposition describes this.

**Proposition 4** *There exists a $\delta'' < 1$ such that for all $\delta > \delta''$, there exists a PWPBE of the four-player incomplete information game with private offers. The equilibrium has the following features:*

*(i) On the equilibrium path, the play is identical to the equilibrium described in proposition 1.*

*(ii) Off the equilibrium path, the play is defined in appendix (E)*

**Proof.** We refer to the proof of proposition (1) in order to ensure that on the equilibrium path, no player has any incentive to deviate. Appendix(E) shows that behaviour described for off the equilibrium path constitute mutual optimal play by the players. ■

The main difference between private and public offers lies in the nature of the deviations that are observable. In particular, a buyer cannot observe a deviation in offers made by the other buyer. In this model, we specifically need to take into account three kinds of deviations by the players and the associated observables. While this is outlined formally in appendix (E), here we provide an informal discussion.

All these three deviations are from the buyer ($B_2$) who is supposed to make offers only to $S_M$. First, we consider the situation when $B_2$ makes an unacceptable offer to the seller $S_M$. The second case is when $B_2$ makes an offer to the privately informed seller but it is less than the equilibrium offer. Finally, we need to consider the situation when $B_2$ deviates and makes an offer to the privately informed seller but the offer is higher than the equilibrium offer. Any of these deviations matters only if subsequently all four players are still present, while equilibrium play would have resulted in at least one pair ($S_M$ and a buyer) leaving. This means that the buyer

($B_1$) who is making offers to both sellers with positive probability has made an equilibrium offer to the privately informed seller that has been rejected. In that case, if all players are present, $B_1$ can detect that some deviation has taken place. However, he does not know the exact kind of deviation. In such a situation, the updated belief of all uninformed players is the same as it would have been after an equilibrium offer getting rejected by $S_I$.

It is to be observed that given the off-the equilibrium path belief of $B_1$, in the third case ($B_2$ makes a higher-than-equilibrium offer to $S_I$), the deviating buyer becomes more pessimistic, compared to $B_1$, about the valuation of the privately informed seller (he now believes it to be $H$ since an offer higher than the one that makes the $L$ type indifferent has been rejected). However, the deviating buyer $B_2$, who offers only to $S_M$ on the equilibrium path, does not do worse by playing an equilibrium strategy with actions corresponding to the belief of $B_1$. This justifies the off the equilibrium path belief of $B_2$.

Finally, observe that there also exists a fourth kind of deviation where buyer $B_1$ can make an offer to $S_M$ such that the offer is less than $p_l(\pi)$. In that case, $B_1$'s offer will never get accepted this period. Next period, $B_1$ either faces a continuation game with $S_I$ or a four-player game. The latter happens when this period, $B_2$ offers $p_l^{'}(\pi)$ to $S_M$. In that case, all uninformed players' beliefs remain unchanged at $\pi$. Thus, $B_1$ knows that next period the expected payoff is bounded above by $v_B(\pi)$. Hence, the expected payoff to $B_1$ from making such an out-of-equilibrium offer is bounded above by $\delta v_B(\pi)$. This shows that it is not profitable for $B_1$ to make such an out-of-equilibrium offer. Since, this argument is not calculation intensive, we do not consider this case explicitly in our formal description of the off the equilibrium path behaviour in appendix (E).

We now argue that no non-stationary equilibrium with public offers can be sustained as an equilibrium with private offers. To begin with, it is to be noted that it is not possible to sustain the equilibrium described in the previous subsection in a model with private offers. This is because the equilibrium depends crucially on "overbidding" by one of the buyers to $S_M$ being detected by the other buyer and seller, who then condition their future play on this deviation. But with private offers, this deviation is not detectable, so the switch to the "punishment phase" is not possible.

To be more general, any non-stationary equilibrium depends on the expected future changes due to publicly observed deviations. With private offers, the only deviation that is observed results in either some subset of players leaving or all players remaining. In the first case, the continuation game will be a 2 player game. These continuation games have unique equilibria. This implies that in any non-stationary equilibrium, there has to be a possible off-path play where all four players remain. However, in that case, the player who deviated needs to be detected for proper specification of continuation play after a deviation. This is not possible with private offers.

# 5 Conclusion

To summarise, we have first constructed a stationary PBE, thus demonstrating existence. For this equilibrium, as $\delta \to 1$, the price in all transactions converge to $H$ and the game ends "almost immediately". We then show that any stationary equilibrium must have qualitative characteristics similar to the equilibrium we construct, so that the unique stationary PBE outcome has the property that, as $\delta \to 1$, the prices in both transactions go to the highest seller value $H$, with the buyers making offers. This is reminiscent of the Coase Conjecture, though the setting here is that of a stylised small market, rather than one of bilateral bargaining or such bargaining with an exogenously fixed outside option. We also show that the outcome of the particular stationary equilibrium constructed, can also be supported as the outcome of a Public Weak Perfect Bayesian Equilibrium of the game with private offers.

Unlike the bilateral bargaining case, non-stationary equilibria exist in this model with very different characteristics and we show one such.

In our future research we intend to address the issue of having two privately informed sellers and to extend this model to more agents on both sides of the market.

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# Appendix

## A Equilibrium of the four-player game with incomplete information

**Proof.** We prove this proposition in steps. First, we construct the equilibrium as described in the proposition for a given value of $\pi$ by assuming existence. This implies we assume that a $\delta^*$ as described in the proposition exists. Later on we will prove this existence result.

To formally construct the equilibrium for different values of $\pi$, we need the following lemma which we label as the *Competition Lemma*, following the terminology of [9], though they proved it for a different model.

Consider the following sequences for $t \geq 1$:

$$\bar{p}_t = v - [(v - \delta^t H)\alpha + (1-\alpha)\delta(v - \bar{p}_{t-1})] \tag{2}$$

$$p_t^{'} = M + \delta(1-\alpha)(\bar{p}_{t-1} - M) \tag{3}$$

where $\alpha \in (0,1)$ and $\bar{p}_0 = H$.

**Lemma 2** *There exists a $\delta^{'} \in (0,1)$, such that for $\delta > \delta^{'}$ and for all $t \in \{1, ....N(\delta)\}$, we have,*

$$\bar{p}_t > p_t^{'}$$

**Proof.**

$$\begin{aligned} \bar{p}_t - p_t^{'} &= v - [(v - \delta^t H)\alpha + (1-\alpha)\delta(v - \bar{p}_{t-1}))] - M \\ &\quad -\delta(1-\alpha)(\bar{p}_{t-1} - M) \end{aligned}$$

$$= (1-\delta)(v - M) + \alpha(\delta^t H - \delta M - (1-\delta)v)$$

From the analysis of the two player game with one-sided asymmetric information, we know that given a $\delta \in (0,1)$, the maximum number of periods (for any $\pi \in (0,1)$) for which the game can last on the equilibrium path is given by $N(\delta)$. As $\delta \to 1$, $N(\delta)$ is uniformly bounded above by $N^*$ which is finite. Hence, $N^*$ is the upper bound on the maximum equilibrium duration of the two player game with one sided asymmetric information.

We need to show that the second term in the above expression is always positive. Note that the coefficient of $\alpha$ is increasing in $\delta$ and is positive at $\delta = 1$. For $t = N^*$, $\exists\ \delta^{'} < 1$ such that the term is positive whenever $\delta > \delta^{'}$. Since this is true for $t = N^*$, it will be true for all lower values of $t$.

For any $\delta < 1$, $N(\delta) \leq N^*$. Hence, for all $t \in \{1, ....N(\delta)\}$,

$$\bar{p}_t > p_t^{'}$$

whenever $\delta > \delta^{'}$.

This concludes the proof of the lemma.

■

Fix a $\delta > \delta^*$. Suppose the current prior is $\pi \in (0,1)$.[23] Consider the sequence $d_\tau(\delta) = \{0, d_1, d_2, ...d_t..\}$, which is identical with the similar sequence in the two-player game with asymmetric information. This implies that there exists a $t \geq 0$ ( $t \leq N^*$ ) such that $\pi \in [d_t, d_{t+1})$. Next, we evaluate $v_B(\pi)$ (from the two player game). Define $\bar{p}(\pi)$ as,

$$\bar{p}(\pi) = v - v_B(\pi)$$

Define $p_l^{'}(\pi)$ as,

$$p_l^{'}(\pi) = M + \delta(1 - a(\pi))[E_{d_{t-1}}(p) - M] \tag{4}$$

where $E_{d_{t-1}}(p)$ represents the expected price offer to $S_M$ in equilibrium when the probability that $S_I$ is of the low type is $d_{t-1}$. $a(\pi)$ is the probability with which the seller with private information accepts an equilibrium offer. From (4), we can posit that in equilibrium, $p_l^{'}(\pi)$ is the minimum acceptable price for $S_M$, if she gets only one offer.

For $\pi = d_{t-1}$, the maximum price offer to $S_M$ (according to the conjectured equilibrium) is $\bar{p}(d_{t-1})$. This implies that $E_{d_{t-1}}(p) \leq \bar{p}(d_{t-1})$.

Since $a(\pi) \in (0,1)$, from Lemma (2) we can infer that $\bar{p}(\pi) > p_l^{'}(\pi)$. Suppose there exists a $p_l(\pi) \in (p_l^{'}(\pi), \bar{p}(\pi))$ such that,

$$p_l(\pi) = (1 - \delta)M + \delta E_\pi(p)$$

We can see that $p_l(\pi)$ represents the minimum acceptable price offer for $S_M$ in the event she gets two offers (Note that if $S_M$ rejects both offers, the game goes to the next period with $\pi$ remaining the same).

From the conjectured equilibrium behaviour, we derive the following:

1. $B_1$ makes offers to $S_I$ with probability $q(\pi)$. According to the conjectured equilibrium, $B_2$ puts a mass point at $p_l^{'}(\pi)$ while making offers to $S_M$. Hence, $B_2$'s expected payoff from making an offer of $p_l^{'}(\pi)$ to $S_M$ must be equal to the equilibrium payoff. This gives us

$$(v - p_l^{'}(\pi))q(\pi) + (1 - q(\pi))\delta v_B(\pi) = v_B(\pi)$$

From the above we obtain

$$q(\pi) = \frac{v_B(\pi)(1-\delta)}{(v - p_l^{'}(\pi)) - \delta v_B(\pi)} \tag{5}$$

$B_1$ offers $\delta^t H$ to $S_I$. With probability $(1 - q(\pi))$, he makes offers to $S_M$. The conditional distribution of offers to $S_M$, given $B_1$ makes an offer to this seller is obtained as follows.

From the indifference relation of $B_2$ while making offers to $S_M$ in the range $[p_l(\pi), \bar{p})(\pi)]$, we have

$$(v - s)[q(\pi) + (1 - q(\pi))F_1^\pi(s)] + (1 - q(\pi))(1 - F_1^\pi(s))\delta v_B(\pi) = v_B(\pi)$$

This gives us

$$F_1^\pi(s) = \frac{v_B(\pi)[1 - \delta(1 - q(\pi))] - q(\pi)(v - s)}{(1 - q(\pi))[v - s - \delta v_B(\pi)]} \tag{6}$$

We can check that $F_1^\pi(p_l(\pi)) > 0$ and $F_1^\pi(\bar{p}(\pi)) = 1$. This confirms that $B_1$ puts a mass point at $p_l(\pi)$.

2. $B_2$ offers $p_l^{'}(\pi)$ to $S_M$ with probability $q^{'}(\pi)$. In equilibrium, we know that $B_1$ puts a mass point at $p_l(\pi)$. Hence, the expected payoff to $B_1$ from making an offer of $p_l(\pi)$ to $S_M$ must be equal to the equilibrium payoff. This gives us

$$(v - p_l(\pi))q^{'}(\pi) + (1 - q^{'}(\pi))\delta v_B(\pi) = v_B(\pi)$$

[23] $\pi = 0$ is the complete information case with a $H$ seller.

From the above we obtain

$$q^{'}(\pi) = \frac{v_B(\pi)(1-\delta)}{(v - p_l(\pi)) - \delta v_B(\pi)} \tag{7}$$

With probability $(1 - q^{'}(\pi))$, $B_2$ makes offers to $S_M$ by randomizing his offers in the support $[p_l(\pi), \bar{p}(\pi)]$. The conditional distribution of offers is obtained as follows.

From $B_1$'s indifference relation while making offers to $S_M$ in the range $[p_l(\pi), \bar{p}(\pi)]$, we have

$$(v-s)[q^{'}(\pi) + (1-q^{'}(\pi))F_2^{\pi}(s)] + (1-q^{'}(\pi))(1-F_2^{\pi}(s))\delta v_B(\pi) = v_B(\pi)$$

This gives us

$$F_2^{\pi}(s) = \frac{v_B(\pi)[1-\delta(1-q^{'}(\pi))] - q^{'}(\pi)(v-s)}{(1-q^{'}(\pi))[v-s-\delta v_B(\pi)]} \tag{8}$$

This completes the derivation of the equilibrium. We will now state the off the equilibrium path behaviour that sustain the equilibrium play.

**Off the equilibrium path behaviour:**

Suppose $B_2$ adheres to his equilibrium strategy. Then the off-path behaviour of $B_1$ and that of $L$-type $S_I$, while $B_1$ makes an offer greater than $\delta^t H$ to $S_I$, are the same as in the 2-player game with incomplete information. If $B_1$'s offer to $S_I$ is less than $\delta^t H$ then the off-path behaviour of the $L$-type $S_I$ is described in the following manner. If $B_2$'s offer to $S_M$ is in the range $[p_l(\pi), \bar{p}(\pi)]$, then the $L$-type $S_I$ behaves in the same way as in the 2-player game. Consider the case when $B_2$ offers $p_l^{'}(\pi)$ to $S_M$. If the offer to $S_I$ from $B_1$ is less than $[q(\pi)\delta^{t+1}H + (1-q(\pi))\delta^{t+2}H]$, then both types reject the offer and $S_M$ also rejects the offer. If the offer is greater than $[q(\pi)\delta^{t+1}H + (1-q(\pi))\delta^{t+2}H]$ but less than $[q(d_t)\delta^t H + (1-q(d_t))\delta^{t+1}H]$ then the $L$-type $S_I$ rejects it with a probability such that the updated belief is $d_t$. $S_M$ also rejects the offer. In the following period(s), the offer to $S_I$ is either $\delta^{t-1}H$ or a randomised offer between $\delta^t H$ or $\delta^{t-1}H$. Randomisations are done in a manner such that $S_I$ is indifferent between accepting or rejecting the offer. If the offer is greater than $[q(d_t)\delta^t H + (1-q(d_t))\delta^{t+1}H]$, the offer is accepted by $L$-type $S_I$ with a probability such that the updated belief is $d_{t-1}$. $S_M$ randomises between accepting and rejecting the offer of $p_l^{'}(\pi)$. In the following period(s), offer to $S_I$ is either $\delta^{t-1}H$ or a randomised offer between $\delta^{t-1}H$ and $\delta^{t-2}H$. These randomisations are such that $S_I$ is indifferent between accepting and rejecting the offer. For high values of $\delta$, $B_1$ does not have any incentive to deviate.

Next, suppose $B_2$ makes an unacceptable offer to $S_M$ (which is observable to $S_I$), and $B_1$ makes an equilibrium offer to $S_I$. The $L$-type $S_I$ rejects this offer with a probability that takes the updated belief to $d_{t-1}$. If $S_I$ rejects this equilibrium offer and next period both the buyers make offers to $S_M$, then two periods from now, the remaining buyer offers $\delta^{t-2}H$ (the buyer is indifferent between offering $\delta^{t-1}H$ and $\delta^{t-2}H$ at $\pi = d_{t-1}$) to $S_I$. Thus, the expected continuation payoff to $S_I$ from rejection is $\delta(q(d_{t-1})\delta^{t-1}H + \delta(1-q(d_{t-1}))\delta^{t-2}H) = \delta^t H$. This implies that the $L$-type $S_I$ is indifferent between accepting and rejecting an offer of $\delta^t H$ if he observes $S_M$ to get an unacceptable offer.

Now consider the case when $B_2$ deviates and makes an offer to $S_I$. It is assumed that if $S_I$ gets two offers then she disregards the lower offer.

Suppose $B_1$ makes an equilibrium offer to $S_I$ and $B_2$ deviates and offers something less than $\delta^t H$ to $S_I$. $S_I$'s probability of accepting the equilibrium offer (which is the higher offer in this case) remains the same. If $S_I$ rejects the higher offer (which in this case is the offer of $\delta^t H$ from $B_1$) and next period both the buyers make offers to $S_M$, then two periods from now, the remaining buyer offers $\delta^{t-2}H$ to $S_I$.

If $B_2$ deviates and offers $p^o \in (\delta^t H, \delta^{t-1}H]$ to $S_I$, then $S_I$ rejects this with a probability that takes the updated belief to $d_{t-2}$. If $S_I$ rejects this offer then next period if $B_1$ offers to $S_I$, he offers $\delta^{t-2}H$. If both $B_1$ and $B_2$ make offers to $S_M$ then two periods from now the remaining buyer randomises between offering $\delta^{t-2}H$ and $\delta^{t-3}H$ to $S_I$

(conditional on $S_I$ being present). Randomisations are done in a manner to ensure that the expected continuation payoff to $S_I$ from rejection is $p^o$. It is easy to check that for high values of $\delta$, this can always be done. Lastly, if $B_2$ deviates and offers to $S_I$ and $B_1$ offers to $S_M$ (according to his equilibrium strategy), then the off-path specifications are the same as in the 2-player game with incomplete information.

We will now show that $B_2$ has no incentive to deviate. Suppose he makes an unacceptable offer to $S_M$. His expected discounted payoff from deviation is given by,

$$\mathcal{D} = q(\pi)[\delta\{a(\pi)(v - M) + (1 - a(\pi))v_B(d_{t-1})\}] + (1 - q(\pi))\delta v_B(\pi) \tag{9}$$

From (4) we know that,

$$p_l^{'}(\pi) < M + \delta(1 - a(\pi))[\bar{p}(d_{t-1}) - M]$$

as $E_{d_{t-1}} < \bar{p}(d_{t-1})$. Hence we have,

$$p_l^{'}(\pi) < M + \delta(1 - a(\pi))[(v - M) - (v - \bar{p}(d_{t-1}))]$$

Rearranging the above terms we get,

$$(v - p_l^{'}(\pi)) > \delta\{a(\pi)(v - M) + (1 - a(\pi))v_B(d_{t-1})\} + (1 - \delta)(v - M) \tag{10}$$

By comparing (9) and (10) we have,

$$q(\pi)(v - p_l^{'}(\pi)) + (1 - q(\pi))\delta v_B(\pi) > \mathcal{D}$$

The L.H.S of the above relation is $B_2$'s equilibrium payoff, as he puts a mass point at $p_l^{'}(\pi)$. Hence, he has no incentive to make an unacceptable offer to $S_M$.

Next, suppose $B_2$ deviates and makes an offer of $p^o$ to $S_I$ such that $p^o \in (\delta^t H, \delta^{t-1} H]$. $B_2$'s payoff from deviation is:

$$\Gamma_H = q(\pi)[(v - p^o)a^{'}(\pi) + (1 - a^{'}(\pi))\delta v_B(d_{t-2})] + (1 - q(\pi))[(v - p^o)a(\pi) + (1 - a(\pi))\delta v_B(d_{t-1})]$$

where $a^{'}(\pi)$ is the probability with which $B_2$'s offer is accepted by $S_I$ in the event when both $B_1$ and $B_2$ make offers to $S_I$ and $B_2$'s offer is in the range $(\delta^t H, \delta^{t-1} H]$. From our above specification it is clear that $a^{'}(\pi) > a(\pi)$, where $a(\pi)$ is the acceptance probability of an equilibrium offer to $S_I$. This is very intuitive. In the contingency when $B_1$ makes an equilibrium offer to $S_M$ and $B_2$'s out of the equilibrium offer to $S_I$ is in the range $(\delta^t H, \delta^{t-1} H]$, the acceptance probability is equal to $a(\pi)$. This is the equilibrium acceptance probability. In this case if the $L$-type $S_I$ rejects an offer, then next period he will get an offer with probability 1. However, if both $B_1$ and $B_2$ make offers to $S_I$ and $B_2$'s offer is in the range $(\delta^t H, \delta^{t-1} H]$ then the $L$-type $S_I$ accepts this offer with a higher probability. This is because, on rejection, there is a positive probability that $S_I$ might not get an offer in the next period. This explains why $a^{'}(\pi) > a(\pi)$.

Since $p^o > p_l^{'}(\pi)$[24] and $\bar{p}(d_{t-2}) > p_l^{'}(\pi)$[25], we have

$$v - p_l^{'}(\pi) > (v - p^o)a^{'}(\pi) + (1 - a^{'}(\pi))\delta v_B(d_{t-2}) \tag{11}$$

[24] For sufficiently high values of $\delta$ this will always be the case.

[25] Since $\bar{p}(d_{t-2}) > \bar{p}(\pi) > p_l^{'}(\pi)$.

Also, since $p^o > \delta^t H$, we have
$$(v - p^o)a(\pi) + (1 - a(\pi))\delta v_B(d_{t-1}) < v_B(\pi)$$
The expression $[(v - p^o)a(\pi) + (1 - a(\pi))\delta v_B(d_{t-1}) - \delta v_B(\pi)]$ is strictly negative for $\delta = 1$. From continuity, we can say that for sufficiently high values of $\delta$, $(v - p^o)a(\pi) + (1 - a(\pi))\delta v_B(d_{t-1}) < \delta v_B(\pi)$. This implies that

$$(v - p_l^{'}(\pi))q(\pi) + (1 - q(\pi))\delta v_B(\pi) > \Gamma_H$$

The L.H.S of the above inequality is the equilibrium payoff of $B_2$. Similarly if $B_2$ deviates and makes an offer to $S_I$ such that his offer $p^0$ is in the range $[\delta^{t+1}H, \delta^t H)$, the payoff from deviation is

$$\Gamma_L = q(\pi)[\delta\{a(\pi)(v - M) + (1 - a(\pi))v_B(d_{t-1})\}]$$

$$+(1 - q(\pi))[(v - p^0)a^{''}(\pi) + (1 - a^{''}(\pi))\delta v_B(d_t)]$$

From the 2-player game we know that $[(v - p^0)a^{''}(\pi) + (1 - a^{''}(\pi))\delta v_B(d_t)] < v_B(\pi)$. Also, from the previous analysis we can posit that $(v - p_l^{'}(\pi)) > \delta\{a(\pi)(v - M) + (1 - a(\pi))v_B(d_{t-1})\}$. Thus, for sufficiently high values of $\delta$, $(v - p_l^{'}(\pi))q(\pi) + (1 - q(\pi))\delta v_B(\pi) > \Gamma_L$.

Hence, $B_2$ has no incentive to deviate and make an offer to $S_I$. This completes the specification of off the equilibrium path behaviour.

We shall now prove the existence of the equilibrium.

**Existence:**

We show that there exists a $\delta^*$ such that $\delta^{'} < \delta^* < 1$ and for all $\delta > \delta^*$, the equilibrium constructed above exists for all values of $\pi \in [0, 1)$. This is shown by proving the following two lemmas.

**Lemma 3** *If $\pi \in [0, d_1)$, then the equilibrium of the game is identical to that of the benchmark case.*

**Proof.** From the equilibrium of the two player game with one sided asymmetric information, we know that for $\pi \in [0, d_1)$, buyer always offers $H$ to the seller and the seller accepts this with probability 1. Hence, this game is identical to the game between a buyer of valuation $v$ and a seller of valuation $H$, with the buyer making the offers. Thus, in the four-player game, we will have an equilibrium which is identical to the one described in the benchmark case. It is to be observed that $d_1$ is independent of $\delta$. We conclude the proof by assigning the following values:

$$p_l^{'}(\pi) = M \text{ and } \bar{p}(\pi) = H \text{ for } \pi \in [0, d_1)$$

■

The above lemma implies that if $\pi \to 0$, the four player game with incomplete information ends in at most two periods.

**Lemma 4** *If there exists a $\bar{\delta} \in (\delta^{'}, 1)$ such that for $\delta \geq \bar{\delta}$ and for all $t \leq T$ ($T \leq N^*$) an equilibrium as described in the current proposition exists for $\pi \in [0, d_t(\delta))$, then there exists a $\delta_T^* \geq \bar{\delta}$ such that, for all $\delta \in (\delta_T^*, 1)$, such an equilibrium also exists for $\pi \in [d_T(\delta), d_{T+1}(\delta))$. As mentioned earlier, $N^*$ is the upper bound on the maximum equilibrium duration of the two player game with one sided asymmetric information as $\delta \to 1$.*

**Proof.** From the analysis of the two player game with one sided asymmetric information we know that $N^*$ is finite and it is independent of $\pi$. We need to show that there exists a $\delta_T^* \geq \bar{\delta}$, such that for all $\delta > \delta_T^*$ and for all $\pi \in [d_T(\delta), d_{T+1}(\delta))$, there exists a $p_l(\pi) \in (p_l^{'}(\pi), \bar{p}(\pi))$ with

$$p_l(\pi) = (1 - \delta)M + \delta E_\pi(p)$$

From now on we will write $d_T$ instead of $d_T(\delta)$. For each $\delta \in (\delta^{'}, 1)$, we can construct $d(\delta)$ and the equilibrium strategies as above for any $p_l = x \in (p_l^{'}(\pi), \bar{p}(\pi))$. Function $G(x)$ is constructed as

$$G(x) = x - [\delta E_\pi^x(p) + (1-\delta)M]$$

We can infer from ([6]) that the function $G(.)$ is monotonically increasing in $x$. Since, $E_\pi^x(p) < \bar{p}(\pi)$,

$$\lim_{x \to \bar{p}(\pi)} G(x) > 0$$

Next, we have

$$G(p_l^{'}(\pi)) = p_l^{'}(\pi) - [\delta E_\pi^{p_l^{'}(\pi)}(p) + (1-\delta)M]$$

By definition, $E_\pi^{p_l^{'}(\pi)}(p) > p_l^{'}(\pi)$. So for $\delta = 1$, $G(p_l^{'}(\pi))) < 0$. Since $G(.)$ is a continuous function, there exists a $\delta_T^* \geq \bar{\delta}$, such that for all $\delta > \delta_T^*$, $G(p_l^{'}(\pi))) < 0$. By invoking the *Intermediate Value Theorem* we can say that for a given $\delta > \delta_T^*$, there exists a unique $x^*(\pi) \in (p_l^{'}(\pi), \bar{p}(\pi))$ such that $G(x^*(\pi)) = 0$. This $x^*(\pi)$ is our required $p_l(\pi)$.

This concludes the proof. ■

From Lemma (3), we know that for any $\delta \in (0,1)$, an equilibrium exists for $\pi \in [0, d_1)$.[26] Using Lemma (4), we can obtain $\delta_t^*$ for all $t \in \{1, 2, ..., N^*\}$. $\delta^*$ is defined as:

$$\delta^* = \max_{1,..,N^*} \delta_T^*$$

We can do this because $N^*$ is finite. Lemma (3) and (4) together now guarantee that whenever $\delta > \delta^*$, the equilibrium as described in the current proposition exists for all $\pi \in [0, 1)$.

This concludes the proof of the proposition. ■

# B Asymptotic result

We will show that in the equilibrium characterised in Proposition 1, as $\delta \to 1$, price offers in all transactions converge to $H$.

We first show that as $\delta \to 1$, $p_l^{'}(\pi)$ reaches a limit which is strictly less than $H$. To show this we prove the following lemma.

**Lemma 5** *For a given $\pi > d_1$, the acceptance probability $a(\pi, \delta)$ of an equilibrium offer is increasing in $\delta$ and has a limit $\bar{a}(\pi)$ which is less than* 1.

**Proof.** The acceptance probability $a(\pi, \delta)$ of an equilibrium offer is equal to $\pi\beta(\pi, \delta)$, where $\beta(\pi, \delta)$ is the probability with which the $L$-type $S_I$ accepts an equilibrium offer. From the updating rule we know that $\beta(\pi, \delta)$ is such that the following relation is satisfied:

$$\frac{\pi(1 - \beta(\pi, \delta))}{\pi(1 - \beta(\pi, \delta)) + (1 - \pi)} = d_{t-1}(\delta)$$

From the above expression, we get

$$\beta(\pi, \delta) = \frac{\pi - d_{t-1}(\delta)}{\pi(1 - d_{t-1}(\delta))}$$

[26]Note that $d_1$ is independent of $\delta$.

Since $d_t's$ have a limit as $\delta$ goes to 1, so does $\beta(\pi,\delta)$. Therefore, $a(\pi,\delta)$ also has a limit $\bar{a}(\pi)$ which is less than 1 for $\pi \in (0,1)$. ■

The above lemma implies that as $\delta \to 1$, $p_l^{'}(\pi)$ reaches a limit which is strictly less than $H$ and greater than $M$.

Since $v_B(\pi) \to (v-H)$ as $\delta \to 1$, from (5) we have

$$q(\pi) \to 0 \text{ as } \delta \to 1$$

From (6) we have,

$$1 - F_1^{\pi}(s) = \frac{\bar{p}(\pi) - s}{(1-q(\pi))[v - s - \delta v_B(\pi)]}$$

We have shown that $q(\pi) \to 0$ as $\delta \to 1$. Hence, as $\delta \to 1$, for $s$ arbitrarily close to $\bar{p}(\pi)$, we have

$$1 - F_1^{\pi}(s) \approx \frac{\bar{p}(\pi) - s}{\bar{p}(\pi) - s} = 1$$

Hence, the distribution collapses and $p_l(\pi) \to \bar{p}(\pi)$. From the expression of $p_l(\pi)$, we know that $p_l(\pi) \to E_\pi(p)$ as $\delta$ goes to 1. Thus, we can conclude that as $\delta$ goes to 1, $E_\pi(p) \to H$ for all values of $\pi$.

Since $G(\bar{p}(\pi)) > 0$, there will be a threshold of $\delta$ such that for all $\delta$ higher than that threshold, we have $G(\delta\bar{p}(\pi)) > 0$. Thus, $p_l(\pi)$ is bounded above by $\delta\bar{p}(\pi)$ for high values of $\delta$. From (7), we can then infer that

$$q^{'}(\pi) = \frac{1}{\frac{v}{v_B(\pi)} + \frac{\delta\bar{p}(\pi) - p_l(\pi)}{(1-\delta)v_B(\pi)}}$$

Since $p_l(\pi)$ is bounded above by $\delta\bar{p}(\pi)$ for high values of $\delta$, $q^{'}(\pi) \to 0$ as $\delta$ goes to 1.

Thus, we conclude that as $\delta$ goes to 1, prices in all transactions in the equilibrium constructed go to $H$.

# C Proof of proposition 3

**Proof.** We prove this proposition in steps, through a series of lemmas. First, we show that for any equilibrium belonging to the set of equilibria considered, the following lemma holds.

**Lemma 6** *For any $\pi \in (0,1)$, it is never possible to have a stationary equilibrium in the set of equilibria considered such that both buyers offer only to $S_M$ on the equilibrium path.*

**Proof.** Suppose it is the case that there exists a stationary equilibrium in the game with four players such that both buyers offer only to $S_M$. Both buyers should have a distribution of offers to $S_M$ with a common support[27] $[\underline{s}(\pi), \bar{s}(\pi)]$. The payoff to each buyer should then be $(v - \bar{s}(\pi)) = v_4(\pi)$(say). Let $v_B(\pi)$ be the payoff obtained by a buyer when his offer to $S_M$ gets rejected. This is the payoff a buyer obtains by making offers to the privately informed seller in a two player game.[28]

[27] If the upper bounds are not equal, then the buyer with the higher upper bound can profitably deviate. On the other hand, if the lower bounds are different, then the buyer with the smaller lower bound can profitably deviate.

[28] If there exists a stationary equilibrium where both buyers offer only to $S_M$, then the lower bound of the common support of offers is not less than the minimum acceptable price to $S_M$ in the candidate stationary equilibrium. To see this, let $p_2^2(\pi) = (1-\delta)M + \delta E_p^2(\pi)$. Suppose the lower bound of the support is strictly less than $p_2^2(\pi)$. Let $z(\pi)$ be the probability with which each buyer's offer is strictly less than $p_2^2(\pi)$. If $v_4^2(\pi)$ is the payoff to the buyers in this candidate equilibrium, the expected payoff to the buyer from making an offer strictly less than $p_2^2(\pi)$

Consider any $s \in [\underline{s}(\pi), \bar{s}(\pi)]$ and one of the buyers (say $B_1$). If the distributions of the offers are given by $F_i$ for buyer $i$, then we have

$$(v - s)F_2(s) + (1 - F_2(s))\delta v_B(\pi) = v - \bar{s}(\pi)$$

This follows from the buyer $B_1$'s indifference condition.

Since in equilibrium, the above needs to be true for any $s \in [\underline{s}(\pi), \bar{s}(\pi)]$, we must have $v - \bar{s}(\pi) > \delta v_B(\pi)$. The above equality then gives us

$$F_2(s) = \frac{(v - \bar{s}(\pi)) - \delta v_B(\pi)}{(v - s) - \delta v_B(\pi)}$$

Since $v - \bar{s}(\pi) > \delta v_B(\pi)$, for $s \in [\underline{s}(\pi), \bar{s}(\pi))$, we have $v - s > v - \bar{s}(\pi) > \delta v_B(\pi)$. This implies

$$F_2(\underline{s}(\pi)) > 0$$

Similarly, we can show that

$$F_1(\underline{s}(\pi)) > 0$$

In equilibrium, it is not possible for both the buyers to put mass points at the lower bound of the support. Hence, $S_M$ cannot get two offers with probability 1. This concludes the proof of the lemma. ■

For any equilibrium belonging to the set of equilibria we are considering, we know that $S_M$ must get at least one offer with positive probability. The above lemma implies that $S_I$ also gets at least one offer with a positive probability. We will now argue that for any equilibrium in the set of equilibria considered, $S_M$ always accepts an equilibrium offer immediately. This is irrespective of whether $S_M$ gets one offer or two offers.

To show this formally, consider such an equilibrium. We first define the following. Given a $\pi$, let $p_i(\pi)$ be the minimum acceptable price to the seller $S_M$ in the event she gets $i$ ($i = 1, 2$) offer(s) in the considered equilibrium. We have

$$p_1(\pi) - M = (1 - (\alpha(\pi))\delta[E_p(\tilde{\pi}) - M]$$

$E_p(\tilde{\pi})$ is the price corresponding to the expected equilibrium payoff to the seller $S_M$ in the event she rejects the offer and the privately informed seller does not accept the offer. It is evident that when the seller $S_M$ is getting one offer, the seller with private information is also getting an offer. Here $\alpha(\pi)$ is the probability with which the privately informed seller accepts the offer and $\tilde{\pi}$ is the updated belief.

Similarly, we have

$$p_2(\pi) - M = \delta[E_p(\pi) - M]$$

where $E_p(\pi)$ is the price corresponding to the expected equilibrium payoff to $S_M$ in the event she rejects both offers. In lemma 12 (appendix (D)), we argue that $E_p(\pi) > M$. The following lemma has the consequence that $S_M$ always accepts an equilibrium offer (or highest of the equilibrium offers) immediately.

**Lemma 7** *For any $\pi < 1$, if we restrict ourselves to the set of equilibria considered, then in any arbitrary equilibrium, it is never possible for a buyer to make an offer to $S_M$, which is strictly less than* $\min\{p_1(\pi), p_2(\pi)\}$.

**Proof.** Suppose the conclusion of the lemma does not hold, so there is such an equilibrium. Let the payoff to the

is $z(\pi)\delta v_4^2(\pi) + (1 - z(\pi))\delta v_B(\pi)$. In equilibrium, we must have $z(\pi)\delta v_4^2(\pi) + (1 - z(\pi))\delta v_B(\pi) = v_4^2(\pi)$. Either $v_B(\pi) > v_4^2(\pi)$ or $v_B(\pi) \leq v_4^2(\pi)$. In the former case the equality does not hold for values of $\delta$ close to 1 and in the later case the equality does not hold for any value of $\delta < 1$.

buyers from this candidate equilibrium of the four-player game be $v_4(\pi)$. In lemma 13 (appendix (D)), we argue that $v_4(\pi) < v - p_2(\pi)$. Let $v_B(\pi)$ be the payoff the buyer gets by making offers to $S_I$ in a two-player game.

Consider the buyer who makes the lowest offer to $S_M$. We label this buyer as $B_1$ and the lowest offer as $\underline{p(\pi)}$, where $\underline{p(\pi)} < \min\{p_1(\pi), p_2(\pi)\}$. Let $q(\pi)$ be the probability with which the other buyer makes an offer to the seller $S_I$. Let $\gamma(\pi)$ be the probability with which the other buyer, conditional on making offers to the seller $S_M$, makes an offer which is less than $p_2(\pi)$. Finally, $\alpha(\pi)$ is the probability with which the privately informed seller accepts an offer if the other buyer makes an offer to her. Since $B_1$'s offer of $\underline{p(\pi)}$ to $S_M$ is always rejected, the payoff to $B_1$ from making such an offer is

$$\{q(\pi)\delta\{\alpha(\pi)(v-M)+(1-\alpha(\pi))(v-E_p^b(\tilde{\pi}))\}+(1-q(\pi))\delta\{\gamma(\pi)v_4(\pi)+(1-\gamma(\pi))v_B(\pi)\}$$

where $E_p^b(\tilde{\pi})$ is such that $(v-E_p^b(\tilde{\pi}))$ is the expected equilibrium payoff to the buyer if the updated belief is $\tilde{\pi}$. We first argue that $(v-E_p^b(\tilde{\pi}))$ is less than or equal to $(v-E_p(\tilde{\pi}))$. This is because since $(E_p(\tilde{\pi})-M)$ is the expected equilibrium payoff to the seller $S_M$ when the belief is $\tilde{\pi}$, there is at least one price offer by the buyer, which is greater than or equal to $E_p(\tilde{\pi})$. Hence, we have

$$\delta\{\alpha(\pi)(v-M)+(1-\alpha(\pi))(v-E_p^b(\tilde{\pi}))\} \leq \delta\{\alpha(\pi)(v-M)+(1-\alpha(\pi))(v-E_p(\tilde{\pi}))\}$$

$$\Rightarrow (v-p_1(\pi))-\delta\{\alpha(\pi)(v-M)+(1-\alpha(\pi))(v-E_p^b(\tilde{\pi}))\}$$

$$\geq (v-p_1(\pi))-\delta\{\alpha(\pi)(v-M)+(1-\alpha(\pi))(v-E_p(\tilde{\pi}))\}$$

Since, $(v-p_1(\pi))-\delta\{\alpha(\pi)(v-M)+(1-\alpha(\pi))(v-E_p(\tilde{\pi}))\}=(1-\delta)(v-M)>0$, we have

$$(v-p_1(\pi))-\delta\{\alpha(\pi)(v-M)+(1-\alpha(\pi))(v-E_p^b(\tilde{\pi}))\}>0$$

There are two possibilities. Either $p_1(\pi) < p_2(\pi)$ or $p_2(\pi) < p_1(\pi)$. If $p_2(\pi) > p_1(\pi)$, then the buyer can profitably deviate by making an offer of $p_1(\pi)$. The payoff from making such an offer is

$$q(\pi)(v-p_1(\pi))+(1-q(\pi))\{\gamma(\pi)\delta v_4(\pi)+(1-\gamma(\pi))\delta v_B(\pi)\}$$

Since $(v-p_1(\pi))-\delta\{\alpha(\pi)(v-M)+(1-\alpha)(v-E_p^b(\tilde{\pi}))\}>0$, we can infer that this constitutes a profitable deviation by the buyer.

Next, consider the case when $p_2(\pi) < p_1(\pi)$. In this situation, the buyer can profitably deviate by making an offer of $p_2(\pi)$. The payoff from making such an offer is

$$\{q(\pi)\delta\{\alpha(\pi)(v-M)+(1-\alpha(\pi))(v-E_p^b(\tilde{\pi}))\}+(1-q(\pi))\{\gamma(\pi)(v-p_2(\pi))+(1-\gamma(\pi))\delta v_B(\pi)\}$$

Since $v_4(\pi) < (v-p_2(\pi))$, this constitutes a profitable deviation by the buyer.

This concludes the proof of the lemma. ∎

There are two immediate conclusions from the above lemma. First, if $p_2(\pi) < p_1(\pi)$, it can be shown that if $\delta$ is high enough, then in equilibrium, no buyer should offer anything less than $p_1(\pi)$. To show this, suppose at least one of the buyers makes an offer which is less than $p_1(\pi)$ and consider the buyer who makes the lowest offer to $S_M$. Let $\gamma_1(\pi)$ be the probability with which the other buyer, conditional on making offers to $S_M$, makes an offer which is less than $p_1(\pi)$. The payoff to the buyer by making the lowest offer to $S_M$ is

$$\{q(\pi)\delta\{\alpha(\pi)(v-M)+(1-\alpha)(v-E_p^b(\tilde{\pi}))\}+(1-q(\pi))\delta\{v_B(\pi)\}$$

However, if he makes an offer of $p_1(\pi)$ then the payoff is

$$\{q(\pi)(v - p_1(\pi)) + (1 - q(\pi))\{\gamma_1(\pi)(v - p_1(\pi)) + (1 - \gamma_1(\pi))\delta v_B(\pi)\}$$

We know that as $\delta \to 1$, $v_B(\pi) \to v - H$. Since $p_1(\pi) < H$, this implies that for high $\delta$, $\gamma_1(\pi)(v - p_1(\pi)) + (1 - \gamma_1(\pi))\delta v_B(\pi) > \delta v_B(\pi)$. Hence, for high $\delta$, this constitutes a profitable deviation by the buyer.

Secondly, if $p_1(\pi) < p_2(\pi)$, then only one buyer can make an offer with positive probability that is less than $p_2(\pi)$. This is because, any buyer who makes an offer to $S_M$ in the range $(p_1(\pi), p_2(\pi))$ can get the offer accepted when the seller $S_M$ gets only one offer. In that case the offer can still get accepted if it is lowered and that will not alter the outcomes following the rejection of the offer. Hence, the buyer can profitably deviate by making a lower offer. Thus, in equilibrium if a buyer has to offer anything less than $p_2(\pi)$ to the seller $S_M$, then it has to be equal to $p_1(\pi)$. However, in equilibrium both buyers cannot put mass points at $p_1(\pi)$. This shows that only one buyer can make an offer to $S_M$ which is strictly less than $p_2(\pi)$.

*Hence, we have argued that all offers to $S_M$ are always greater than or equal to $p_1(\pi)$ and in the event $S_M$ gets two offers, both offers are never below $p_2(\pi)$. This shows that $S_M$ always accepts an equilibrium offer immediately.*

We will now show that for any equilibrium in the set of equilibria considered, the seller with private information by rejecting equilibrium offers for a finite number of periods can take the posterior to 0. This is shown in the following lemma.

**Lemma 8** *Suppose we restrict ourselves to the set of equilibria considered. Given a $\pi$ and $\delta$, there exists a $T_\pi(\delta) > 0$ such that conditional on getting offers, the privately informed seller can get an offer of $H$ in $T_\pi(\delta)$ periods from now by rejecting all offers she gets in between. $T_\pi(\delta)$ depends on the sequence of equilibrium offers and corresponding strategies of the responders in the candidate equilibrium. $T_\pi(\delta)$ is uniformly bounded above as $\delta \to 1$.*

**Proof.** To prove the first part of the lemma, we show that in the candidate equilibrium, rejection of offers by the privately informed seller can never lead to an upward revision of the belief.[29] If it does, then it implies that the offer is such that the $H$-type $S_I$ accepts the offer with a positive probability and the $L$-type $S_I$ rejects it with a positive probability. Since the $H$-type accepts the offer with a positive probability, this means that the offer must be greater than or equal to $H$ (let this offer be equal to $p_h \geq H$ ) and we have

$$p_h - H \geq \delta(E^{'} - H)$$

where $E^{'}$ is the price corresponding to the expected equilibrium payoff to the $H$-type $S_I$ next period. Then,

$$p_h \geq \delta(E^{'}) + (1 - \delta)H \Rightarrow p_h - L \geq \delta(E^{'}) + (1 - \delta)H - L$$

$$\Rightarrow p_h - L \geq \delta(E^{'} - L) + (1 - \delta)(H - L) > \delta(E^{'} - L)$$

This shows that the $L$- type $S_I$ should accept $p_h$ with probability[30] 1. This is a contradiction to our supposition that the $L$-type $S_I$ rejects with some positive probability. Thus, the belief revision following a rejection must be

[29] We consider updating in equilibrium. Since this is about a candidate equilibrium, out of equilibrium events could only arise from non-equilibrium offers made by the buyers. However, if we were to follow the definition of the PBE, then no player's action should be treated as containing information about things which that player does not know (*no-signalling-what-you-don't-know*). Hence, these out of equilibrium events cannot lead to change in beliefs.

[30] This follows from the fact that from next period onwards, $L$-type $S_I$ can always adopt the optimal strategy of the $H$-type $S_I$. Hence, following a rejection of the offer $p_h$, the expected equilibrium payoff to the $L$-type $S_I$ is $\leq E^{'} - L$.

in the downward direction. It cannot be zero since in that case it implies that both types reject with probability 1. This is not possible in equilibrium.

Thus, in equilibrium, $S_I$ ($L$-type) should always accept an offer with a positive probability. This proves the first part of the lemma.

To show that the number of rejections required to get an offer of $H$ is uniformly bounded above as $\delta \to 1$, we need to show that it cannot happen that the acceptance probabilities of any sequence of equilibrium offers to $S_I$ are not uniformly bounded below as $\delta \to 1$.

In the equilibrium considered, if only one buyer makes offers to $S_I$, then the claim of the lemma holds. This is because of the fact that $S_M$ always accepts an equilibrium offer immediately and hence, $S_I$ on rejecting an offer knows that the continuation game will be a two-player game with one-sided asymmetric information. Thus, by invoking the finiteness result of the two-player game with one-sided asymmetric information, we know that $S_I$ can take the posterior to 0 by rejecting equilibrium offers for a finite number of periods.

Consider equilibria where more than one buyer makes offers to $S_I$. Given the set of equilibria we have considered and the results already proved, we can posit that in such a case, either one of the buyers is making offers only to $S_I$ and the other is randomising between making offers to $S_I$ and $S_M$, or both buyers are offering to both sellers with positive probabilities.

Let $p_l$ be the minimum offer which gets accepted by $S_I$ with positive probability in an equilibrium where two buyers offer to $S_I$ with positive probability. We will now argue that there exists a possible outcome such that $S_I$ gets only one offer and the offer is equal to $p_l$. When one of the buyers is making offers to $S_I$ only, then $p_l$ must be the lower bound of the support of his offers . In the second case, when both buyers with positive probability make offers to $S_I$ and $S_M$, with positive probability $S_I$ gets only one offer. Thus, there exists an instance that $S_I$ gets the offer of $p_l$ only.

When $S_I$ gets the offer of $p_l$ only, then she knows that by rejecting that she gets back a two- player game, which has the finiteness property. Thus, there exists a $\tilde{T}(\delta) > 0$ such that $S_I$ is indifferent between getting $p_l$ now and $H$ in $\tilde{T}(\delta)$ periods from now. This implies

$$p_l - L = \delta^{\tilde{T}(\delta)}(H - L)$$

From the finiteness property of the two player game with one sided asymmetric information, we know that $\tilde{T}(\delta)$ is uniformly bounded above as $\delta \to 1$.

Suppose there is a sequence of equilibrium offers such that the acceptance probabilities of the offers are not bounded below as $\delta \to 1$. Let $p$ be the initial offer of that particular sequence. $p \geq p_l$. For a given $\delta$, let $T(\delta) > 0$ be such that, given the acceptance probabilities of the sequence of offers, by rejecting $p$ and subsequent equilibrium offers, $S_I$ can get $H$ in $T(\delta)$ periods from now. Hence, the $L$-type $S_I$ should be indifferent between getting $p$ now and $H$ in $T(\delta)$ time periods from now. As per our supposition, $T(\delta)$ is not uniformly bounded above as $\delta \to 1$.

Then, we can find a $\delta^h < 1$ such that for all $\delta \in (\delta^h, 1)$, we have $T(\delta) > \tilde{T}(\delta)$. This gives us

$$\delta^{T(\delta)}(H - L) < \delta^{\tilde{T}(\delta)}(H - L) = p_l - L \leq p - L$$

Hence, the $L$-type $S_I$ is not indifferent between getting $p$ now and $H$ in $T(\delta)$ time periods from now, contrary to our assumption.

Hence, as $\delta \to 1$, probabilities of acceptance of any sequence of equilibrium offers are bounded below. This concludes the proof of the lemma. ■

The above lemma shows that any stationary equilibrium in the set of equilibria considered possess the finiteness property. We will now show that we cannot have both buyers offering to both sellers with positive probability.

This is argued in the following lemma.

**Lemma 9** *In any equilibrium belonging to the set of equilibria considered, if players are patient enough then both buyers cannot make offers to both sellers with positive probability.*

**Proof.**

From the arguments of lemma (8), we know that in an arbitrary stationary equilibrium, any offer made to the privately informed seller should get accepted by the low type with a positive probability bounded away from 0. Suppose there exists a stationary equilibrium of the four-player game where both buyers offer to both sellers with a positive probability. Hence, in equilibrium, if the privately informed seller gets offer(s), then she either gets two offers or one offer. Since $S_M$ always accepts an offer in equilibrium immediately, $S_I$ knows that on rejecting an offer(s) she will get another offer in at most two periods from now. Hence, from lemma (8) we infer that if the privately informed seller gets one offer, then the $L$-type $S_I$ can expect to get an offer of $H$ in at most $T_1(\pi) > 0$ time periods from now, by rejecting all offers she gets in between. Similarly, if the seller with private information gets two offers then the $L$-type $S_I$ by rejecting both offers can expect to get an offer of $H$ in at most $T_2(\pi) > 0$ time periods from now by rejecting all offers she gets in between. As we have argued in lemma (8), both $T_1(\pi)$ and $T_2(\pi)$ are bounded above as $\delta \to 1$. Thus, any offer $s$ to the privately informed seller in equilibrium should satisfy

$$s \geq \delta^{T_1(\pi)} H + (1 - \delta^{T_1(\pi)}) L \equiv s_1(\delta)$$

and

$$s \geq \delta^{T_2(\pi)} H + (1 - \delta^{T_2(\pi)}) L \equiv s_2(\delta)$$

It is clear from the above that as $\delta \to 1$, both $s_1(\delta) \to H$ and $s_2(\delta) \to H$. Hence, if there is a support of offers to $S_I$ in equilibrium, then the support should collapse as $\delta \to 1$.

We will now argue that for $\delta$ high enough but $\delta < 1$, the support in equilibrium cannot have two or more points.

Suppose it is possible that the support of offers to $S_I$ has two or more points. This implies that the upper bound and the lower bound of the support are different from each other. Let $\underline{s}(\pi)$ and $\bar{s}(\pi)$ be the lower and upper bound of the support respectively.

Consider a buyer who is making an offer to $S_I$. This buyer must be indifferent between making an offer of $\underline{s}(\pi)$ and $\bar{s}(\pi)$. Let $q(\pi)$ be the probability with which the other buyer makes an offer to $S_I$. Since in equilibrium $S_M$ always accepts an offer immediately, the payoff from making an offer of $\underline{s}(\pi)$ to $S_I$ is

$$\begin{aligned} \Pi_{\underline{s}(\pi)} &= (1 - q(\pi))[\alpha_{\underline{s}(\pi)}(v - \underline{s}(\pi)) + (1 - \alpha_{\underline{s}(\pi)})\delta v_B(\pi^{'})] \\ &\quad + q(\pi) E_s\{[\beta_\pi^s \delta(v - M) + (1 - \beta_\pi^s)\delta v_4(\pi_s^{''})]\} \end{aligned}$$

$\alpha_{\underline{s}(\pi)}$ is the acceptance probability of $\underline{s}(\pi)$ when $S_I$ gets the offer of $\underline{s}(\pi)$ only. $\beta_\pi^s$ is the acceptance probability of the offer $s$ to $S_I$ when she gets two offers. $v_B(.)$ and $v_4(.)$ are the buyer's payoffs from the two-player incomplete information game and the four-player incomplete information game respectively. For the second term of the right-hand side, we have taken an expectation because when two offers are made, this buyer's offer of $\underline{s}(\pi)$ to $S_I$ never gets accepted and the payoff then depends on the offer made by the other buyer. When $S_I$ gets only one offer and rejects an offer of $\underline{s}(\pi)$, then the updated belief is $\pi^{'}$; $\pi_s^{''}$ denotes the updated belief when $S_I$ rejects an offer of $s \in (\underline{s}(\pi), \bar{s}(\pi)]$ and she gets two offers.

Similarly, the payoff from offering $\bar{s}(\pi)$ is

$$\Pi_{\bar{s}(\pi)} = (1-q(\pi))[\alpha_{\bar{s}(\pi)}(v-\bar{s}(\pi)) + (1-\alpha_{\bar{s}(\pi)})\delta v_B(\pi^{'''})] + q(\pi)[\beta_{2\pi}(v-\bar{s}(\pi)) + (1-\beta_{2\pi})\delta v_4(\pi^4)]$$

Here $\pi^{'''}$ is the updated belief when $S_I$ gets one offer and rejects an offer of $\bar{s}(\pi)$. When $S_I$ gets two offers and rejects an offer of $\bar{s}(\pi)$, the updated belief is denoted by $\pi^4$. Note that if at all $S_I$ accepts an offer, she always accepts the offer of $\bar{s}(\pi)$, if made. The quantity $\alpha_{\bar{s}(\pi)}$ is the probability with which the offer of $\bar{s}(\pi)$ is accepted by $S_I$ when she gets one offer. When $S_I$ gets two offers, then the offer of $\bar{s}(\pi)$ gets accepted with probability $\beta_{2\pi}$.

As argued above, $\bar{s}(\pi) \to H$ and $\underline{s}(\pi) \to H$ as $\delta \to 1$. This implies that $v_4(\pi) \to (v-H)$ as $\delta \to 1$. From the result of the two-player one-sided asymmetric information game, we know that $v_B(\pi) \to H$ as $\delta \to 1$. Since $v-M > v-H$, we have $\Pi_{\underline{S}(\pi)} > \Pi_{\bar{s}(\pi)}$ as $\delta \to 1$. From lemma (8) we can infer that both $\beta^s_\pi$ and $\beta_{2\pi}$ are positive. Hence, there exists a threshold for $\delta$ such that if $\delta$ crosses that threshold, $\Pi_{\underline{S}(\pi)} > \Pi_{\bar{s}(\pi)}$. This is not possible in equilibrium. Thus, for high $\delta$, the support of offers can have only one point. The same arguments hold for the other buyer as well. Hence, each buyer while offering to $S_I$ has a one-point support. Next, we establish that both buyers should make the same offer. If they make different offers, then as explained before, for $\delta$ high enough the buyer making the higher offer can profitably deviate by making the lower offer. However, in equilibrium it is not possible to have both buyers making the same offer to $S_I$.[31]

Hence, when agents are patient enough, in equilibrium both buyers cannot offer to both sellers with a positive probability. This concludes the proof of the lemma. ■

In the following lemma we show that in any stationary equilibrium of the four player game, as players get patient enough, $S_M$ always gets offers from two buyers with a positive probability.

**Lemma 10** *In any stationary equilibrium belonging to the set of equilibria considered, there exists a threshold of $\delta$ such that if $\delta$ exceeds that threshold, both buyers make offers to $S_M$ with positive probability.*

**Proof.**

Suppose there exists a stationary equilibrium where $S_M$ gets offers from only one buyer, say $B_1$. There are two possibilities. Either $S_I$ gets an offer from $B_2$ only or from both $B_1$ and $B_2$ with positive probability.

Consider the first possibility. From lemma (8), we know that in a stationary equilbrium, the privately informed seller should always accept an equilibrium offer with some positive probability. Thus, as $\delta \to 1$, the equilibrium payoff to $B_2$ approaches $v-H$ from above. On the other hand, $B_1$ knows that the privately informed seller is going to accept with some probability and hence under that situation if his offer to $S_M$ gets rejected, he gets a two-player game with $S_M$. The payoff to $B_1$ from that two-player game is $(v-M)$. Thus, $B_1$'s continuation payoff from his offer getting rejected by $S_M$ is strictly greater than $(v-H)$. This implies that the equilibrium payoff to $B_1$ should also be strictly greater than $(v-H)$. Hence, as $\delta$ exceeds a threshold, $B_2$ finds it profitable to deviate and make offers to $S_M$.

In the latter case, we know that $B_1$ offers to both $S_I$ and $S_M$ with positive probability and $B_2$ makes offers only to $S_I$. Therefore, using the result of lemma (8), if $B_1$ has to get an offer accepted by $S_I$, then for high values of $\delta$ that offer should be close to $H$ and thus the payoff to $B_1$ from making offers to $S_I$ should be close to $(v-H)$. However, as argued, the payoff to $B_1$ from making offers to $S_M$ is strictly greater than $(v-H)$. In equilibrium, the buyer has to be indifferent between making offers to $S_I$ and $S_M$. Hence, it is not possible to have a stationary equilibrium where $S_M$ gets offers from only one buyer. This concludes the proof. ■

[31] These arguments would also work even if the supports were not taken to be symmetric. In that case, let $\underline{s}(\pi)$ be the minimum of the lower bounds and $\bar{s}(\pi)$ be the maximum of the upper bounds. If these are associated with the same buyer, then same arguments hold. If not, then the buyer with the higher upper bound can proftibly deviate by shifting its mass to $\underline{s}(\pi)$.

From the characteristics of the restricted set of equilibria being considered, we know that $S_M$ always gets an offer with a positive probability. The above lemma then allows us to infer that, in any stationary equilibrium of the four player game, both buyers should offer to $S_M$ with positive probability. From our arguments and hypothesis, we know that both buyers cannot make offers to only one seller ($S_I$ or $S_M$) and both buyers cannot randomise between making offers to both sellers. Hence, we can infer that one of the buyers has to make offers to $S_M$ only and the other buyer should randomise between making offers to $S_I$ and $S_M$.

The following lemma now shows that for any $\pi \in [0,1)$, any equilibrium in this restricted set possesses the characteristic that the price offers to all sellers approach $H$ as $\delta \to 1$.

**Lemma 11** *For a given $\pi$, in any hypothesised equilibrium, price offers to all sellers go to $H$ as $\delta \to 1$.*

**Proof.** Let $\bar{s}(\pi)$ be the upper bound of the support[32] of offers to $S_M$.

$S_M$ always accepts an equilibrium offer immediately. Hence, if the $L$-type $S_I$ rejects an equilibrium offer, she gets back a two-player game with one-sided asymmetric information. Thus, the buyer offering to $S_I$ in a period must offer at least $p^e$ such that

$$p^e - L = \delta(H - \epsilon - L) \Rightarrow p^e = (1-\delta)L + \delta(H - \epsilon)$$

where $\epsilon > 0$ and $\epsilon \to 0$ as $\delta \to 1$.

Consider $B_1$, who is randomising between making offers to $S_I$ and $S_M$. When offering to $S_I$, $B_1$ must offer $p^e$ and it must be the case that

$$(v - p^e)\alpha(\pi) + (1 - \alpha(\pi))\delta\{v - (H - \epsilon)\} = v - \bar{s}(\pi)$$

where $\alpha(\pi)$ is the probability with which the offer is accepted by the seller with private information. This follows from the fact that $B_1$ must be indifferent between offering to $S_I$ and $S_M$. The L.H.S of the above equality is the payoff to $B_1$ from offering to $S_I$ and the R.H.S is the payoff to him from offering to $S_M$. Since in any hypothesized equilibrium, $S_M$ always gets an offer in period 1 and $S_M$ accepts an equilibrium offer immediately, $S_I$, by rejecting an equilibrium offer, always gets back a two-player game with one-sided asymmetric information. Hence, the payoff to the buyer from offering to $S_I$ is the same as in the two-player game with one-sided asymmetric information. This implies that

$$(v - p^e)\alpha(\pi) + (1 - \alpha(\pi))\delta\{v - (H - \epsilon)\} = v_B(\pi)$$

Thus, we can conclude that $v_B(\pi) = v - \bar{s}(\pi)$.

We will now show that the upper bound of the support of offers to $S_M$ is strictly greater than $p^e$. We have

$$(v - p^e) - \delta\{v - (H - \epsilon)\} = v(1-\delta) + \delta(H - \epsilon) - \delta(H - \epsilon) - (1-\delta)L$$

$$= (1-\delta)(v - L) > 0$$

for $\delta < 1$. This implies that

$$v - p^e > \delta\{v - (H - \epsilon)\}$$

Since $(v - \bar{s}(\pi))$ is a convex combination of $v - p^e$ and $\delta\{v - (H - \epsilon)\}$, we have

$$v - p^e > v - \bar{s}(\pi) \Rightarrow \bar{s}(\pi) > p^e$$

[32] The upper bound of support of offers to $S_M$ for both buyers should be the same. Else, the buyer with the higher upper bound can profitably deviate.

Next, we will argue that as $\delta \to 1$, the support of offers to $S_M$ from any buyer is bounded below by $p^e$. Consider a buyer who makes an offer of $p^e$ to $S_M$ in equilibrium. Then, if $q^e$ is the probability with which this offer gets accepted, we have

$$(v - p^e)q^e + (1 - q^e)\delta v_B(\pi) = v - \bar{s}(\pi)$$

This follows since $S_M$ always accepts an offer in equilibrium immediately, this buyer's offer to $S_M$ gets rejected only when the other buyer also makes an offer to $S_M$.

This gives us,

$$q^e = \frac{(v - \bar{s}(\pi)) - \delta v_B(\pi)}{(v - p^e) - \delta v_B(\pi)} = \frac{(1-\delta)(v - \bar{s}(\pi))}{(v - p^e) - \delta(v - \bar{s}(\pi))}$$

$$\Rightarrow q^e = \frac{1}{\frac{v}{v - \bar{s}(\pi)} + \frac{\delta \bar{s}(\pi) - p^e}{(1-\delta)(v - \bar{s}(\pi))}}$$

and

$$q^e \to 0 \text{ as } \delta \to 1$$

This shows that in equilibrium, as $\delta \to 1$, any offer to $S_M$ that is less than or equal to $p^e$ always gets rejected. Since we have argued earlier that in equilibrium, no buyer should make an offer to $S_M$ that she always rejects, we can infer that the support of offers to $S_M$ from any buyer is bounded below by $p^e$ as $\delta$ approaches 1. Hence, in any arbitrary stationary equilibrium of this kind, the price offers to all sellers are bounded below by $p^e$ as $\delta$ approaches 1. However, as $\delta \to 1$, $p^e \to H$. Hence, as $\delta \to 1$, the support of offers to $S_M$ from any buyer collapses and hence price offers to all sellers converge to $H$. ■

Thus, we have shown that for any stationary equilibrium in the set of equilibria considered, one of the buyers randomises between making offers to $S_M$ and $S_I$ and the other buyer makes offers to $S_M$ only. Further, as $\delta \to 1$, price offers in all transactions in these stationary equilibria go to $H$. This concludes the proof of the proposition.

■

# D

## D.1

**Lemma 12** $E_p(\pi) > M$

**Proof.**

Suppose not. This means that $E_p(\pi) = M$. The implication of this is that the seller $S_M$ in equilibrium only gets offer(s) equal to $M$. Thus, in that case $S_M$ can get offers from one buyer only and the offer is always equal to $M$. $S_M$ always accepts this offer immediately as by rejecting she cannot get anything more. Hence, the equilibrium payoff of this buyer is $v - M$. The other buyer is making offers to $S_I$ only. Since, $S_M$ immediately accepts an offer, the payoff to her is never greater than $v_B(\pi)$ where $v_B(\pi)$ is the payoff to a buyer in the two-player game with one sided asymmetric information. As $\delta \to 1$, $v_B(\pi) \to H$. Hence, this buyer can profitably deviate by offering to $S_M$. Thus, we must have $E_p(\pi) > M$. This concludes the proof of the lemma.

■

## D.2

**Lemma 13** $(v - p_2(\pi)) > v_4(\pi)$

**Proof.**

Since $E_p(\pi) - M$ is the expected payoff to the seller $S_M$ in equilibrium, there is at least one price offer by the buyer which is greater than or equal to $E_p(\pi)$. Hence, we must have $v_4(\pi) \leq (v - E_p(\pi))$. This gives us

$$(v - p_2(\pi)) = \delta(v - E_p(\pi)) + (1-\delta)(v - M) > (v - E_p(\pi)) \geq v_4(\pi)$$

This concludes the proof of the lemma.

■

# E Off-path behaviour with private offers

The off-path behaviour of the players in the case of private offers, to sustain the equilibrium described in Proposition 4, is described as follows.

Specifically we need to focus on the following three deviations.

(i) $B_2$ makes an unacceptable offer to $S_M$.

(ii) $B_2$ makes an offer of $p^o$ to $S_I$ such that $p^o < \delta^t H$.

(iii) $B_2$ makes an offer of $p^o$ to $S_I$ such that $p^o > \delta^t H$.

We denote the above three events by $E_1$, $E_2$ and $E_3$ respectively. A deviation is detected when following some offers made by the buyers it is observed that all four players are present, while equilibrium play would have resulted in at least one pair ($S_M$ and a buyer) leaving. Hence, it is detected that one of $E_1$, $E_2$ or $E_3$ has occurred.

Off-the-equilibrium path, the belief should be such that the equilibrium play should be sustained. Once a deviation is detected as stated above, the beliefs of all uninformed players become $d_{t-1}$.[33]

If $E_1$ or $E_2$ occurs and $B_1$ makes an equilibrium offer to $S_I$, then $S_I$'s probability of accepting the equilibrium offer remains the same.

The $L$-type $S_I$ accepts an offer higher than $\delta^t H$ with probability 1 if she gets two offers. If she gets only one offer then the probability of her acceptance of out-of-equilibrium offers is the same as in the two-player game with incomplete information.

[33] We can justify these out-of-equilibrium beliefs by supposing that each of $E_1$, $E_2$ and $E_3$ has associated with it a mistake probability depending on a parameter $\lambda > 0$ and the beliefs are given by the relative likelihoods as $\lambda$ goes to zero. This is, of course, not a full consistency analysis because a *mistake* is to a class of deviations rather than to each individual deviation. Suppose $B_2$ sticks to his equilibrium strategy with probability $1 - (\lambda + \lambda^2 + \lambda^3)$ with $\lambda > 0$. $E_1$, $E_2$ and $E_3$ occur with probabilities $\lambda$, $\lambda^2$ and $\lambda^3$ respectively. Consider the buyer $B_1$. Suppose $B_1$ makes an equilibrium offer to $S_I$ and it gets rejected. Although offers are private, each player can observe the number of players remaining. Thus, next period, if $B_1$ finds that all four players are present he infers that this is due to an out-of-equilibrium play by $B_2$. Using Bayes' rule, he attaches probabilities $\frac{1}{1+\lambda+\lambda^2}$, $\frac{\lambda}{1+\lambda+\lambda^2}$ and $\frac{\lambda^2}{1+\lambda+\lambda^2}$ to $E_1$, $E_2$ and $E_3$ respectively. This will give us a sequence of beliefs. As $\lambda \to 0$, this sequence of beliefs converges to $d_{t-1}$. Consider seller $S_M$. If she gets an out-of-equilibrium offer then she knows for sure that it is due to $E_1$. If $S_M$ does not get an offer and finds out that all four players are present, using Bayes' rule she attaches probabilities 0, $\frac{1}{1+\lambda}$ and $\frac{\lambda}{1+\lambda}$ to $E_1$, $E_2$ and $E_3$ respectively. Again, as $\lambda \to 0$, the sequence of beliefs converges to $d_{t-1}$. Finally, in the case of $E_1$ or $E_2$ the beliefs of $B_1$ and $B_2$ coincide. However, in the case of $E_3$ they differ. Suppose $E_3$ occurs and $B_1$'s equilibrium offer to $S_I$ gets rejected. Then next period all four players will be present and given $L$-type $S_I$'s behaviour, the belief of $B_2$ will be $\pi = 0$ and that of $B_1$ will be $\pi = d_{t-1}$. In that contingency it is an optimal response of $B_2$ to offer to $S_M$ as in the case with belief equal to $d_{t-1}$. This is because he knows that $B_1$ is playing his equilibrium strategy with the belief $d_{t-1}$.

If $E_1$ or $E_2$ occurs and $B_1$ makes an equilibrium offer to $S_I$, then two periods from now (conditional on the fact that the game continues until then), if $B_2$ is the remaining buyer he offers $\delta^{t-2}H$ to $S_I$. If $E_3$ occurs and all players are observed to be present, then next period $B_2$'s offer to $S_M$ has the same support and follows the same distribution as in the case with belief equal to $d_{t-1}$. In any off-path contingency, if $B_1$ is the last buyer remaining (two periods from now) then he offers $\delta^{t-2}H$ to $S_I$.

We will now argue that the off-path behaviour constitutes a sequentially optimal response by the players.

First, we will argue that the $L$-type $S_I$ finds it optimal to accept an offer higher than $\delta^t H$ with probability 1, if she gets two offers. This is because in the event when she gets two offers she knows that rejection will lead the buyer $B_1$ to play according to the belief $d_{t-1}$ and, two periods from now, the remaining buyer will offer $\delta^{t-2}H$ to $S_I$. Thus, her continuation payoff from rejection is

$$\delta\{\delta^{t-1}Hq(d_{t-1}) + \delta(1-q(d_{t-1}))\delta^{t-2}H\} = \delta\{\delta^{t-1}H\} = \delta^t H$$

Hence she finds it optimal to accept an offer higher than $\delta^t H$ with probability 1.

Next, we need to check that $B_2$ has no incentive to deviate and make an offer of $p^o$ to $S_I$ such that $p^o > \delta^t H$.

Suppose $B_2$ deviates and makes an offer of $p^o$ to $S_I$ such that $p^o > \delta^t H$. With probability $q(\pi)$, $S_I$ will get two offers. In that case, $B_2's$ offer will be accepted with probability $\pi$ and with the complementary probability $(1-\pi)$, the offer gets rejected and hence, as specified above the updated belief is $d_{t-1}$. With probability $(1-q(\pi))$, $S_I$ will get only one offer. Thus, $B_2$'s expected payoff from making such an out-of-equilibrium offer is

$$q(\pi)[\pi(v-p^o) + (1-\pi)\delta v_B(d_{t-1})] + (1-q(\pi))[(v-p^o)a(\pi) + (1-a(\pi))\delta v_B(d_{t-1})]$$

As stated above, here $a(\pi)$ is the probability of acceptance of the out of equilibrium offer as in the two player case. Hence, for high values of $\delta$ we have $(v-p^o)a(\pi) + (1-a(\pi))\delta v_B(d_{t-1}) < \delta v_B(\pi)$. Since $p_l^{'}(\pi)$ reaches a limit which is strictly lower than $H$ and $p^o > \delta^t H$, for high values of $\delta$ we have $(v - p_l^{'}(\pi)) > \{\pi(v-p^o) + (1-\pi)\delta v_B(d_{t-1})\}$. Thus[34],

$$\begin{aligned} v_B(\pi) &= (v - p_l^{'}(\pi))q(\pi) + (1-q(\pi))\delta v_B(\pi) \\ &> q(\pi)[\pi(v-p^o) + (1-\pi)\delta v_B(d_{t-1})] + (1-q(\pi))[(v-p^o)a(\pi) + (1-a(\pi))\delta v_B(d_{t-1})] \end{aligned}$$

Hence, $B_2$ has no incentive to deviate and make an offer of $p^o > \delta^t H$ to $S_I$.

Lastly, to show that $B_2$ has no incentive to deviate and make an unacceptable offer to $S_M$ or offer $p^0$ to $S_I$ such that $p^0 < \delta^t H$, we refer to the analysis described in appendix A.

# F Extension of the equilibrium to $N$ buyers and $N$ sellers case

We will show that similar to the particular equilibrium constructed for the basic model with public offers, we can construct an equilibrium for a model with public offers where there are $N$ buyers and $N$ sellers with $N > 2$. $N-1$ sellers (denoted as $S_1$,..,$S_{N-1}$) have valuation equal to $M$ and this is commonly known. The remaining seller's (denoted as $S_I$) valuation is private information to her and it can either be $H$ or $L$. As before, all other players have a common prior about the valuation of $S_I$. All buyers have a valuation of $v$. The following proposition describes

[34] This is because $B_2$ puts a mass point at $p_l^{'}(\pi)$.

the equilibrium.

**Proposition 5** *For high values of $\delta$,*

*(i) Buyer $B_i$ $(i = 1, ., N-1)$ makes offers only to seller $S_i$. He offers $p_l^{'}(\pi)$ to $S_i$ with probability $q^{'}(\pi)$. This implies that $B_i$ puts a mass point at $p_l^{'}(\pi)$. With probability $(1 - q^{'}(\pi))$, he has a continuous distribution of offers from $p_l(\pi)$ to $\bar{p}(\pi)$. The conditional distribution of offers is*

$$F_i^{\pi}(s) = \frac{v_B(\pi)(1 - \delta(1 - q^{'}(\pi))) - (v - s)q^{'}(\pi)}{(1 - q^{'}(\pi))\{(v - s) - \delta v_B(\pi)\}}$$

*(ii) Buyer $B_N$ offers to all sellers with positive probability. With probability $q(\pi)$, he makes offers to $S_I$. His offers to $S_I$ are same as in the two-player game with asymmetric information. With probability $\frac{(1-q(\pi))}{N-1}$, he makes offers to each of the sellers with valuation $M$. For each of these sellers, he has a conditional distribution of offers from $p_l(\pi)$ to $\bar{p}(\pi)$, putting a mass point at $p_l(\pi)$. The conditional distribution of offers is given by*

$$F_N^{\pi}(s) = \frac{v_B(\pi)\{1 - \frac{\delta(1-q(\pi))}{N-1}\} - (v - s)\{q(\pi) + (1 - q(\pi))\frac{N-2}{N-1}\}}{\frac{(1-q(\pi))}{N-1}\{(v - s) - \delta v_B(\pi)\}}$$

*(iii) $S_I$'s probability of accepting an equilibrium offer is same as in the two-player game with incomplete information.*

*(iv) Seller $S_i$ $(i = 1, .., N-1)$ accepts any offer greater than or equal to $p_l^{'}(\pi)$ when $B_N$ offers to $S_I$. Otherwise, she rejects. If $B_N$ is not offering to $S_I$, then $S_i$ accepts any offer greater than or equal to $p_l(\pi)$. Otherwise, she rejects.*

*(v) $\bar{p}(\pi)$, $p_l(\pi)$, $p_l^{'}(\pi)$, $q(\pi)$ and $q^{'}(\pi)$ are the same as in the equilibrium constructed in the four-player game of the paper.*

*(vi) All buyers obtain a payoff of $v_B(\pi)$, the payoff a buyer gets in the two-player game with incomplete information.*

**Proof.**

We know from our analysis this far that for high values of $\delta$, $p_l^{'}(\pi) < p_l(\pi)$. Consider a buyer $B_i$ $(i = 1, 2.., N-1)$. He has a continuous distribution of offers from $p_l(\pi)$ to $\bar{p}(\pi)$. From $B_i$'s indifference condition, for all $s \in [p_l(\pi), \bar{p}(\pi)]$ we have

$$(v - s)[q(\pi) + (1 - q(\pi))\frac{N-2}{N-1} + \frac{(1 - q(\pi))}{N-1}F_N^{\pi}(s)] + \frac{(1 - q(\pi))}{N-1}(1 - F_N^{\pi}(s))\delta v_B(\pi) = v_B(\pi)$$

This gives us

$$F_N^{\pi}(s) = \frac{v_B(\pi)\{1 - \frac{\delta(1-q(\pi))}{N-1}\} - (v - s)\{q(\pi) + (1 - q(\pi))\frac{N-2}{N-1}\}}{\frac{(1-q(\pi))}{N-1}\{(v - s) - \delta v_B(\pi)\}}$$

From the above expression of $F_N^{\pi}(s)$, we have

$$F_N^{\pi}(\bar{p}(\pi)) = \frac{v_B(\pi)[(1 - q(\pi)) - \frac{N-2}{N-1}(1 - q(\pi)) - \frac{\delta(1-q(\pi))}{N-1}]}{v_B(\pi)(\frac{(1-\delta)(1-q(\pi))}{N-1}}$$

$$\Rightarrow F_N^{\pi}(\bar{p}(\pi)) = \frac{v_B(\pi)[\frac{(1-\delta)(1-q(\pi))}{N-1}]}{v_B(\pi)[\frac{(1-\delta)(1-q(\pi))}{N-1}]} = 1$$

Also, $F_N^{\pi'}(s) > 0$. As $\delta \to 1$, we know that $p_l(\pi) \to \bar{p}(\pi)$. Hence, for high values of $\delta$, we will have $F_N^{\pi}(p_l(\pi)) > 0$. This confirms that $B_N$ while offering to $S_i$, puts a mass point at $p_l(\pi)$.

Since, $B_i$ puts a mass point at $p_l^{'}(\pi)$, we have

$$(v - p_l^{'}(\pi))q(\pi) + (1 - q(\pi))\delta v_B(\pi) = v_B(\pi)$$

Let us explain the above equation. $S_i$ accepts the offer of $p_l^{'}(\pi)$ only when $B_N$ offers to $S_I$. In that case the continuation game $S_i$ will face by rejecting the offer can either be a two-player game with the same buyer or a four-player game as in the main model of the paper. Since, $p_l^{'}(\pi)$ is computed on that basis, $S_i$ will accept the offer. In all other cases, $S_i$ by rejecting all offers faces a four-player game as in the main model of the paper. In that case, her minimum acceptable price is $p_l(\pi)$. Thus, we have

$$q(\pi) = \frac{v_B(\pi)(1-\delta)}{(v - p_l^{'}(\pi)) - \delta v_B(\pi)}$$

This shows that $q(\pi)$ is the same as in the equilibrium constructed earlier for the four-player game.

$B_N$ makes offers to all $S_i$ $(i = 1, 2, .., N-1)$. From his indifference condition, for all $s \in (p_l(\pi), \bar{p}(\pi)]$, we have

$$(v - s)[q^{'}(\pi) + (1 - q^{'}(\pi))F_i^{\pi}(s)] + (1 - q^{'}(\pi))(1 - F_i^{\pi}(s))\delta v_B(\pi) = v_B(\pi)$$

$$\Rightarrow F_i^{\pi}(s) = \frac{v_B(\pi)(1 - \delta(1 - q^{'}(\pi))) - (v - s)q^{'}(\pi)}{(1 - q^{'}(\pi))\{(v - s) - \delta v_B(\pi)\}}$$

This is same as $F_2^{\pi}(s)$ in the equilibrium described in the paper for the four-player game.

$B_N$ puts a mass point at $p = p_l(\pi)$. This gives us

$$(v - p_l(\pi))q^{'}(\pi) + (1 - q^{'}(\pi))\delta v_B(\pi) = v_B(\pi)$$

$$\Rightarrow q^{'}(\pi) = \frac{v_B(\pi)(1-\delta)}{(v - p_l(\pi)) - \delta v_B(\pi)}$$

This shows that $q^{'}(\pi)$ is the same as in the equilibrium constructed earlier for the four-player game.

This completes our equilibrium path description. We know that as $\delta \to 1$, $q(\pi) \to 0$, $q^{'}(\pi) \to 0$, $p_l(\pi) \to H$ and $\bar{p}(\pi) \to H$. This shows that as $\delta \to 1$, in the equilibrium constructed in the present proposition, all price offers converge to $H$ for any value of $\pi$. It is to be observed that if we put $N = 2$ in the expression of $F_N^{\pi}$, it becomes equal to $F_1^{\pi}$, the distribution of offers from $B_1$ to $S_M$ in the equilibrium constructed in proposition (1).

Regarding off the equilibrium path, apart from what already have been described in appendix (A) for the equilibrium constructed in proposition (1), the additional thing we need to ensure is that no $B_i$ $(i = 1, 2, .., N-1)$ has any incentive to make offer to a seller $S_j$ $(j = 1, 2.., N-1,\ j \neq i)$. We show below that there is no such incentive.

From the analysis of the four-player game, we know that

$$q(\pi)\{a(\pi,\delta)\delta(v - M) + (1 - a(\pi,\delta))\delta v_B(d_{t-1})\} + (1 - q(\pi))\delta v_B(\pi) < v_B(\pi)$$

The above follows from the fact that in the four-player game, $B_2$ has no incentive to make an unacceptable offer to $S_M$. L.H.S of the above expression is the expected payoff to $B_i$ $(i = 1, 2, ..., N-1)$ from making an unacceptable offer to $S_i$ in the present case as well. This ensures that $B_i$ has no incentive to make an offer less than $p_l^{'}(\pi)$ to $S_i$.

Suppose $B_1$ who in equilibrium offers only to $S_1$, deviates and offers to $S_2$. First, we argue that making an offer less than $p_l^{'}(\pi)$ is equivalent to making an unacceptable offer to $S_1$. This is because given the strategies of $B_2$ and $B_N$, the highest price offer to $S_2$ is always greater than or equal to $p_l^{'}(\pi)$.

Next, we will argue that for high values of $\delta$, $B_1$ has no incentive to make an offer of $s$ to $B_2$ such that $s \in (p_l^{'}(\pi), p_l(\pi)]$. If such an offer is never accepted by $S_2$, then as before $B_1$ should not have any incentive to make that offer. A necessary condition for such an offer to get accepted is to have $B_2$ offering $p_l^{'}(\pi)$ to $S_2$. This happens with probability $q^{'}(\pi)$. As $\delta \to 1$, $q^{'}(\pi) \to 0$. Hence, the probability of acceptance of the offer of $s$ from $B_1$ to $S_2$ goes to zero. This implies that for high values of $\delta$, such a deviation is not profitable.

Finally, we will argue that $B_1$ has no incentive to make an offer of $s$ to $S_2$ such that $s \in (p_l(\pi), \bar{p}(\pi))$. For any $s \in (p_l(\pi), \bar{p}(\pi))$, from $B_N$'s indifference condition while making offers to $S_2$, we have

$$(v-s)[q^{'}(\pi) + (1-q^{'}(\pi))F_2^{\pi}(s)] + (1-q^{'}(\pi))(1-F_2^{\pi}(s))\delta v_B(\pi) = v_B(\pi)$$

The expected payoff of $B_1$ from making an out of equilibrium offer of $s \in (p_l(\pi), \bar{p}(\pi))$ to $S_2$ is

$$q(\pi)[\{q^{'}(\pi) + (1-q^{'}(\pi))F_2^{\pi}(s)\}(v-s) + (1-q^{'}(\pi))(1-F_2^{\pi}(s))\{a(\pi,\delta)\delta(v-M) + (1-a(\pi,\delta))\delta v_B(d_{t-1})\}]$$

$$+(1-q(\pi))\{(v-s)P^A + (1-P^A)\delta v_B(\pi)\}$$

$P^A$ is the probability of acceptance of the offer of $s$ when $B_N$ does not make an offer to $S_I$. Since in that case with a positive probability $B_N$ can make an offer higher than $s$ to $S_2$, we must have

$$P^A < \{q^{'}(\pi) + (1-q^{'}(\pi))F_2^{\pi}(s)\}$$

This implies that

$$(v-s)P^A + (1-P^A)\delta v_B(\pi) < (v-s)[q^{'}(\pi) + (1-q^{'}(\pi)F_2^{\pi}(s)] + (1-q^{'}(\pi))(1-F_2^{\pi}(s))\delta v_B(\pi) = v_B(\pi)$$

We know that $q(\pi) \to 0$ as $\delta \to 1$. This implies that for high values of $\delta$, the expected payoff of $B_1$ from offering $s \in (p_l(\pi), \bar{p}(\pi))$ to $S_2$ is strictly less than $v_B(\pi)$. This shows that $B_1$ has no incentive to make this offer.

This concludes the proof of the proposition.

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