Title
Common belief of weak-dominance rationality in strategic-form games: A qualitative analysis

Authors
Bonanno, Giacomo
Tsakas, Elias

Publication Date
2018-11-01

DOI
10.1016/j.geb.2018.09.003

Peer reviewed

Qualitative analysis of common belief of rationality in strategic-form games*

GIACOMO BONANNO†
University of California, Davis

ELIAS TSAKAS‡
Maastricht University

October 2, 2017

Abstract

In this paper we study common belief of rationality in strategic-form games with ordinal utilities, employing a qualitative doxastic model of beliefs. We characterize the three main solution concepts for such games, viz., Iterated Deletion of Strictly Dominated Strategies (IDSDS), Iterated Deletion of Börgers-dominated Strategies (IDBS) and Iterated Deletion of Inferior Strategy Profiles (IDIP), by means of gradually restrictive properties imposed on the doxastic models. As a corollary, we prove that IDIP refines IDBS, which refines IDSDS.

1. Introduction

Traditionally, game-theoretic analysis has been based on the assumption that the game under consideration is common knowledge among the players. That is, besides asking that the rules of the game (i.e., the set of players, the set of strategies and the set of outcomes for each strategy profile) are commonly known, we typically assume that *the players have vNM preferences* and that *these preferences are also commonly known*.[1] Under these assumptions, rationality and common belief of rationality characterizes correlated rationalizability, i.e., the strategy profiles that survive Iterated Deletion of Strictly Dominated Strategies are exactly those that can be rationally played under common belief of rationality (e.g., see Brandenburger and Dekel, 1987; Tan and Werlang, 1988).

While it is certainly reasonable to assume that the rules of the game are commonly known, the last two assumptions seem harder to justify at the outset. The issue with the preferences being commonly known has already been addressed by Harsanyi (1967-68) and the extensive literature on incomplete information games that followed his seminal contribution. Within Harsanyi's extended

---

*We are indebted to two anonymous referees and an associate editor for their helpful comments and suggestions. Elias thanks the Department of Economics at UC Davis for its hospitality while working on this project.

†Department of Economics, University of California, Davis, CA 95616-8578, USA; Homepage: http://faculty.econ.ucdavis.edu/faculty/bonanno/; E-mail: gfbonanno@ucdavis.edu

‡Department of Economics (AE1), Maastricht University, P.O. Box 616, 6200 MD, Maastricht, The Netherlands; Homepage: http://www.elias-tsakas.com/home.html; E-mail: e.tsakas@maastrichtuniversity.nl

[1] These assumptions are consistent with extending Savage's standard decision-theoretic framework to an interactive setting (e.g., see Epstein and Wang, 1996).

model, rationality and common belief of rationality characterizes interim correlated rationalizability (e.g., see Dekel et al., 2007; Ely and Peski, 2006). However, in Harsanyi's program, preferences are still assumed to be vNM and therefore the utilities of the game outcomes remain cardinal.

There have been several attempts to relax this last assumption by considering ordinal utilities. Relaxing this assumption can be motivated not only from a theoretical, but also from an applied point of view, given that in lab experiments we typically test predictions made by solution concepts for games with ordinal utilities, such as pure-strategy Nash Equilibrium, or pure-strategy Iterated Deletion of Strictly (resp., Weakly) Dominated Strategies. From a theoretical standpoint, the main consequence of sticking to ordinal utilities is that we have to replace the usual models of probabilistic beliefs with Kripke structures, and thus abandon the standard notion of Bayesian rationality. Depending on the notion of rationality that is adopted, there are various solution concepts characterized by rationality and common belief in rationality, e.g., Iterated Deletion of Strictly Dominated Strategies (the pure strategy version of correlated rationalizability à la Brandenburger and Dekel, 1987; Tan and Werlang, 1988, henceforth IDSDS), Iterated Deletion of Börgers-dominated Strategies (Börgers, 1993, henceforth IDBS), Iterated Deletion of Inferior Strategy Profiles (the pure strategy version of strong rationalizability à la Stalnaker, 1994, henceforth IDIP).

In this paper we investigate the content of the notion of common belief of rationality in strategic-form games with ordinal utilities within a *qualitative* context. In particular, we consider qualitative doxastic models, which consist, for each player, of a belief operator (represented by a $KD45$ Kripke structure) and a null operator (represented by a state-dependent collection of null events). Within this model, we manage to characterize each of the aforementioned solution concepts *in terms of restrictions on our two operators, and without needing to vary the notion of rationality that we employ.* In particular, we prove that IDSDS is characterized by common belief in rationality in a very broad class of models (Remark 1); IDBS is characterized by common belief in rationality if we restrict attention to full-support beliefs (Theorem 1); and finally, IDIP is characterized by common belief in rationality if we further restrict attention to correct full-support beliefs (Theorem 2).[2]

With the previous three results at hand, not only do we manage to put under the same umbrella the three main solution concepts for games with ordinal utilities that have been studied in the literature, but we also manage to prove that they monotonically refine each other (see Corollary 1). In particular, the fact that we impose stronger and stronger restrictions on our epistemic characterizations, implies that IDIP refines IDBS, while IDBS refines IDSDS. In fact, while the second relationship (between IDSDS and IDBS) seems to be perhaps straightforward, the one between IDBS and IDIP is far from trivial.

Qualitative doxastic models are quite permissive, in that they induce for each state an incomplete likelihood relation that does not specify the relative likelihood between any two non-null events. In this sense, our model weakens earlier models on qualitative beliefs (e.g., see de Finetti, 1949; Koopman, 1940), which typically rely on complete likelihood relations.[3] Furthermore, notice the duality between our approach and the one taken by preference-based models of beliefs in games (e.g., Di Tillio, 2008), which starts with a preference relation over acts and derives the collection of Savage-null events, contrary to our model where the primitive is the collection of null events. Interestingly, in our case one could in principle derive a preference relation over acts, which would typically be

[2] In the discussion section we also introduce a new solution concept (Iterated Deletion of Uniformly Weakly Dominated Strategies; henceforth IDUS), which is characterized by the benchmark doxastic model (see Remark 2).

[3] Qualitative beliefs have been extensively studied in the literature since the early contributions of de Finetti (1949) and Koopman (1940). Most papers in the literature have focused on whether a qualitative likelihood relation can be represented by a probability measure (Kraft et al., 1959; Mackenzie, 2017; Scott, 1964; Scott and Suppes, 1958; Villegas, 1967) and on the respective logical foundations (Gärdenfors, 1975; Segerberg, 1971; van der Hoek, 1996). For an early overview of qualitative beliefs see Fishburn (1986). To the best of our knowledge there has not been any attempt to embed qualitative probability in a game-theoretic model.

incomplete given that we do not specify the likelihood relation between any two non-null events.

The paper is structured as follows: In Section 2 we introduce the notion qualitative doxastic model of a strategic-form game; in Section 3 we define our notion of rationality and prove our three characterization results; in Section 4 we present some additional results; Section 5 concludes; all proof are relegated to the Appendices.

2. Qualitative models of ordinal games

2.1. The underlying ordinal game

A *finite strategic-form game with ordinal payoffs* is a quintuple $G = \langle I, (S_i)_{i\in I}, O, z, (\succeq_i)_{i\in I}\rangle$, where $I = \{1, 2, \ldots, n\}$ is a finite set of *players*, S_i is a finite set of *strategies* (or actions) of player $i \in I$ with $S = S_1 \times \cdots \times S_n$ being the set of strategy profiles, O is a finite set of *outcomes*, $z : S \to O$ is a function that associates with every strategy profile $s = (s_1, \ldots, s_n) \in S$ an outcome $z(s) \in O$, $\succeq_i$ is player i's *ordinal ranking* of the outcomes, i.e., a binary relation on O which is complete (i.e., for all $o, o' \in O$, $o \succeq_i o'$ or $o' \succeq_i o$) and transitive (i.e., for all $o, o', o'' \in O$, if $o \succeq_i o'$ and $o' \succeq_i o''$ then $o \succeq_i o''$). The interpretation of $o \succeq_i o'$ is that player i considers outcome o to be at least as good as outcome o'.

Games are often represented in *reduced form* by replacing the triple $\langle O, z, (\succeq_i)_{i\in I}\rangle$ with a list $(\pi_i)_{i\in I}$ of *payoff functions*, where $\pi_i : S \to \mathbb{R}$ is any real-valued function that satisfies the property that, for all $s, s' \in S$, $\pi_i(s) \geq \pi_i(s')$ if and only if $z(s) \succeq_i z(s')$. In the following we will adopt this more succinct representation of strategic-form games. It is important to note, however, that the payoff functions are taken to be purely ordinal and one could replace π_i with any other function obtained by composing π_i with an arbitrary strictly increasing function on the set of real numbers.[4]

A strategic-form game provides only a partial description of an interactive situation, since it does not specify what choices the players make, nor what beliefs they have about their opponents' choices. A specification of these missing elements is obtained by introducing the notion of a "model of the game", which represents a possible context in which the game is played. We first do this in terms of operators on events and then provide a semantic characterization.

2.2. Belief and null operators

The players' beliefs are represented by means of a finite model $\langle \Omega, (\mathbb{B}_i)_{i\in I}\rangle$, where Ω is a finite set of *states* (or possible worlds). As usual, 2^Ω denotes the collection of all subsets of Ω (i.e., events), while $\neg E := \Omega \setminus E$ denotes the complement of E for each event $E \subseteq \Omega$. Moreover, for every player $i \in I$, $\mathbb{B}_i : 2^\Omega \to 2^\Omega$ is the *belief operator* that associates each $E \subseteq \Omega$ with the set of states $\mathbb{B}_i E$ where E is believed by $i \in I$. The belief operator is assumed to satisfy the following standard properties, for every $E \subseteq \Omega$ and every $i \in I$:

(K_1) Consistency: $\mathbb{B}_i E \subseteq \neg\mathbb{B}_i \neg E$,

(K_2) Positive Introspection: $\mathbb{B}_i E \subseteq \mathbb{B}_i \mathbb{B}_i E$,

(K_3) Negative Introspection: $\neg\mathbb{B}_i E \subseteq \mathbb{B}_i \neg\mathbb{B}_i E$.

Later in the paper we restrict attention to belief operators that rule out erroneous beliefs, by also requiring that, for every $E \subseteq \Omega$ and every $i \in I$,

[4] This is in contrast to von Neumann-Morgenstern utility functions where certain properties (e.g., risk attitudes) are preserved only under positive affine transformations.

(K_4) Truth Axiom: $\mathbb{B}_i E \subseteq E$.

While the belief operator $\mathbb{B}_i$ captures what player i is certain of, a second operator $\mathbb{N}_i$ is introduced to distinguish between events that are deemed to be impossible and events that are considered to be possible but judged to be null (i.e., infinitesimally likely). Formally, the null operator $\mathbb{N}_i : 2^\Omega \to 2^\Omega$ associates each event $E \subseteq \Omega$ with the set of states $\mathbb{N}_i E$ where E is deemed null by $i \in I$. The null operator is assumed to satisfy the following properties, for every $E, F \subseteq \Omega$ and every $\omega \in \Omega$ and every $i \in I$:

(L_1) Measurability: (a) $\mathbb{N}_i E \subseteq \mathbb{B}_i \mathbb{N}_i E$ and (b) $\neg\mathbb{N}_i E \subseteq \mathbb{B}_i \neg\mathbb{N}_i E$,

(L_2) Relationship to Belief: (a) $\mathbb{B}_i E \subseteq \neg\mathbb{N}_i E$ and (b) $\mathbb{B}_i E \subseteq \mathbb{N}_i \neg E$,

(L_3) Monotonicity: if $E \subseteq F$ then $\mathbb{N}_i E \supseteq \mathbb{N}_i F$,

(L_4) Distribution: $\mathbb{N}_i E \cap \mathbb{N}_i F \subseteq \mathbb{N}_i (E \cup F)$.

The interpretation of the previous axioms is straightforward: (L_1) says that if an event is (resp. is not) null then it is believed to be (resp., not to be) null; $(L_2(a))$ says that a player cannot believe a null event, while $(L_2(b))$ says that the complement of a believed event is null; (L_3) says that an implication of a non-null event is also non-null; finally (L_4) says that the union of two null events is also null. Later in the paper we explore the consequences of removing (L_4) from our system (see Section 3.1).

2.3. Doxastic models of games

So far we have introduced the notion of a frame rather abstractly, viz., we have not assigned a meaning to each event $E \subseteq \Omega$. Let us now do so, by introducing a strategy function $\sigma_i : \Omega \to S_i$ for each player $i \in I$. Then, each state $\omega \in \Omega$ is associated with the strategy profile $\sigma(\omega) = (\sigma_1(\omega), \ldots, \sigma_n(\omega))$. Moreover, we denote by $\sigma_{-i}(\omega)$ the profile of strategies played, at ω, by the players other than i, that is, $\sigma_{-i}(\omega) = (\sigma_1(\omega), \ldots, \sigma_{i-1}(\omega), \sigma_{i+1}(\omega), \ldots, \sigma_n(\omega))$; thus the entire profile, $\sigma(\omega)$, can also be denoted by $(\sigma_i(\omega), \sigma_{-i}(\omega))$. For an arbitrary $s_i \in S_i$, we define the event $||s_i|| := \{\omega \in \Omega : \sigma_i(\omega) = s_i\}$. Then we impose the following standard (measurability) property, for every $i \in I$ and every $s_i \in S_i$:

(Σ_0) Knowing your own strategy: $||s_i|| = \mathbb{B}_i ||s_i||$.

That is, each player knows her own strategy at every state.

Definition 1. Given a strategic-form game with ordinal payoffs $G = \langle I, (S_i, \pi_i)_{i\in I}\rangle$ a *qualitative doxastic model of* G is a tuple $M = \langle \Omega, (\mathbb{B}_i)_{i\in I}, (\mathbb{N}_i)_{i\in I}, (\sigma_i)_{i\in I}\rangle$, where Ω is finite, $\mathbb{B}_i$ satisfies $(K_1) - (K_3)$, and σ_i is a strategy function satisfying (Σ_0). Then, we define:

$\mathcal{M}_1$: the class of models where $\mathbb{N}_i$ satisfies $(L_1) - (L_4)$.

2.4. Semantic characterization of the operators

2.4.1. Belief operator

It is very common in the literature to characterize belief operators with binary relations in Kripke frames. Recall that a Kripke frame is a tuple $\langle \Omega, (\mathcal{B}_i)_{i\in I}\rangle$, such that $\mathcal{B}_i$ is a binary relation on Ω that describes i's doxastic accessibility at each state. In particular, $\mathcal{B}_i(\omega) = \{\omega' \in \Omega : \omega \mathcal{B}_i \omega'\}$ contains all states considered possible by player i at state ω. In the game-theoretic literature, it is more common to view $\mathcal{B}_i$ as a function that associates with every state $\omega \in \Omega$ a set of states $\mathcal{B}_i(\omega) \subseteq \Omega$ and to call such a function a *possibility correspondence* or *information correspondence* (e.g., Brandenburger and Keisler, 2006). Of course, the two views (binary relation and possibility correspondence) are equivalent.[5]

A Kripke frame is said to be $KD45$ whenever the relation $\mathcal{B}_i$ is *serial* (i.e., for all $\omega \in \Omega$, $\mathcal{B}_i(\omega) \neq \varnothing$), *transitive* (i.e., if $\omega' \in \mathcal{B}_i(\omega)$ then $\mathcal{B}_i(\omega') \subseteq \mathcal{B}_i(\omega)$) and *euclidean* (i.e., if $\omega' \in \mathcal{B}_i(\omega)$ then $\mathcal{B}_i(\omega) \subseteq \mathcal{B}_i(\omega')$). Obviously, by transitivity and euclideanness, we obtain that $KD45$ Kripke frames satisfy $\mathcal{B}_i(\omega') = \mathcal{B}_i(\omega)$ for every $\omega' \in \mathcal{B}_i(\omega)$. A Kripke frame is $S5$ whenever it is $KD45$ and $\mathcal{B}_i$ is reflexive (i.e., $\omega \in \mathcal{B}_i(\omega)$ for all $\omega \in \Omega$). In this case, we typically use the term "knowledge" instead of "belief". It is straightforward to see that in an $S5$ Kripke frame, $\mathcal{B}_i$ is an equivalence relation. A belief operator $\mathbb{B}_i$ is characterized by a binary relation $\mathcal{B}_i$ if

$$\mathbb{B}_i E = \{\omega \in \Omega : \mathcal{B}_i(\omega) \subseteq E\}$$

for every $E \subseteq \Omega$. It is well-known that a belief operator satisfies $(K_1) - (K_3)$ (resp., $(K_1) - (K_4)$) if and only if it is characterized by a binary relation in a $KD45$ (resp., $S5$) Kripke frame.

2.4.2. Null operator

The null operator can also be characterized semantically, not in terms of a binary relation on Ω, but in terms of a function $\mathcal{N}_i : \Omega \to 2^{2^\Omega}$ that associates with each state $\omega \in \Omega$ the set $\mathcal{N}_i(\omega)$ of events that player i considers null at state ω. Such a function is known in modal logic as a *neighborhood function* (see, for example, Pacuit (2017)). A null operator $\mathbb{N}_i$ is characterized by a neighborhood function $\mathcal{N}_i : \Omega \to 2^{2^\Omega}$ if, for every $\omega \in \Omega$ and every event $E \subseteq \Omega$,

$$\mathbb{N}_i E = \{\omega \in \Omega : E \in \mathcal{N}_i(\omega)\}.$$

The measurability axioms (L_1) correspond to the property

$$\text{if } \omega' \in \mathcal{B}_i(\omega) \text{ then } \mathcal{N}_i(\omega') = \mathcal{N}_i(\omega).$$ [6]

Axiom $(L_2(a))$, namely $\mathbb{B}_i E \subseteq \neg\mathbb{N}_i E$, corresponds to the property

$$\text{if } \mathcal{B}_i(\omega) \subseteq E \text{ then } E \notin \mathcal{N}_i(\omega),$$

which says that every event that is believed is not null,[7] and axiom $(L_2(b))$, namely $\mathbb{B}_i E \subseteq \mathbb{N}_i \neg E$, corresponds to the property

$$\text{if } \mathcal{B}_i(\omega) \subseteq E \text{ then } \neg E \in \mathcal{N}_i(\omega),$$

[5]For more details on Kripke frames see, e.g., Aumann (1999); Battigalli and Bonanno (1999); Chellas (1980); van Ditmarsch et al. (2015); Fagin et al. (1995); Hughes and Cresswell (1968); Kripke (1959).

[6]Axiom $(L_1(a))$, namely $\mathbb{N}_i E \subseteq \mathbb{B}_i \mathbb{N}_i E$, is characterized by the property

$$\text{if } \omega' \in \mathcal{B}_i(\omega) \text{ then } \mathcal{N}_i(\omega) \subseteq \mathcal{N}_i(\omega'),$$

while axiom $(L_1(b))$, namely $\neg\mathbb{N}_i E \subseteq \mathbb{B}_i \neg\mathbb{N}_i E$, is characterized by the propert

$$\text{if } \omega' \in \mathcal{B}_i(\omega) \text{ then } \mathcal{N}_i(\omega) \supseteq \mathcal{N}_i(\omega').$$

[7] Note that this property implies that, for every $\omega \in \Omega$, $\mathcal{B}_i(\omega) \notin \mathcal{N}_i(\omega)$ (take E to be $\mathcal{B}_i(\omega)$), which – in turn – implies that $\Omega \notin \mathcal{N}_i(\omega)$.

which says that the complement of an event that is believed is a null event.[8] The Monotonicity axiom, namely $E \subseteq F \Rightarrow \mathbb{N}_i E \supseteq \mathbb{N}_i F$, is characterized by the property

$$\text{if } E \subseteq F \text{ and } F \in \mathcal{N}_i(\omega) \text{ then } E \in \mathcal{N}_i(\omega),$$

and the distribution axiom, namely $\mathbb{N}_i E \cap \mathbb{N}_i F \subseteq \mathbb{N}_i(E \cap F)$, is characterized by the property

$$\text{if } E, F \in \mathcal{N}_i(\omega) \text{ then } (E \cup F) \in \mathcal{N}_i(\omega).$$

It is worth noting that the null operator induces an incomplete *qualitative likelihood relation* $\trianglerighteq_i^\omega \subseteq 2^\Omega \times 2^\Omega$ for each state $\omega \in \Omega$ and each $i \in I$, viz., $E \trianglerighteq_i^\omega F$ if and only if $\omega \in \mathbb{N}_i F$.[9] In other words, any two null events are deemed equally likely, while a non-null event is deemed strictly more likely than a null event. However, $\trianglerighteq_i^\omega$ remains silent when it comes to comparing the likelihood of two non-null events. The latter is the main difference between our minimalistic approach (based on specifying only the null events) and the one typically taken in the large literature on qualitative likelihood relations.

2.4.3. Relationship to the standard approach

We now explain how to relate the "orthodox" approach, based on probabilistic beliefs, and our more general, qualitative, approach. When beliefs are represented by probability distributions, one defines a function $p_i : \Omega \to \Delta(\Omega)$ (with $\Delta(\Omega)$ being the set of probability distribution over Ω) where $p_{i,\omega}$ (we use the notation $p_{i,\omega}$ rather than $p_i(\omega)$) are the probabilistic beliefs of player i at state ω. One then imposes the restriction that

$$\text{if } p_{i,\omega}(\omega') > 0 \text{ then } p_{i,\omega'} = p_{i,\omega}$$

to capture the fact that the player knows his own beliefs. The event $\|p_{i,\omega}\| := \{\omega' \in \Omega : p_{i,\omega'} = p_{i,\omega}\}$ is called a *type of player i*. For the probabilistic case, our $\mathcal{B}_i(\omega)$ coincides with $\|p_{i,\omega}\|$ and $\mathcal{N}_i(\omega)$ coincides with the set of zero-probability events (that is, $E \in \mathcal{N}_i(\omega)$ if and only if $p_{i,\omega}(E) = 0$). Note that under this interpretation, the standard approach implies that the belief operator satisfies the Truth axiom (K_4), viz., that the belief relation $\mathcal{B}_i$ is reflexive.

3. Common belief of rationality

Fix a player i and two strategies $a, b \in S_i$ of player i. We denote by $\|b \geq a\|$ the event that strategy b yields at least as high a payoff for player i as strategy a, that is, $\|b \geq a\| = \{\omega \in \Omega : \pi_i(b, \sigma_{-i}(\omega)) \geq \pi_i(a, \sigma_{-i}(\omega))\}$. Similarly, $\|b > a\| = \{\omega \in \Omega : \pi_i(b, \sigma_{-i}(\omega)) > \pi_i(a, \sigma_{-i}(\omega))\}$ is the event that strategy b yields a strictly higher payoff for player i than strategy a.

Definition 2. Player i is *rational* at state ω whenever, for all $b \in S_i$,

$$\text{if } \omega \in \mathbb{B}_i \|b \geq \sigma_i(\omega)\| \text{ then } \omega \in \mathbb{N}_i \|b > \sigma_i(\omega)\|. \tag{1}$$

Let $\mathbf{R}_i \subseteq \Omega$ be the event that player i is rational and $\mathbf{R} = \bigcap_{i \in I} \mathbf{R}_i$ be the event that all players are rational.

[8] Note that this property implies that, for every $\omega \in \Omega$, $\varnothing \in \mathcal{N}_i(\omega)$ (take E to be Ω).

[9] Early contributions in this literature focused on the problem of "representing a qualitative likelihood relation with a probability measure" (e.g., see Fishburn, 1986). It is not difficult to verify that $(L_1) - (L_4)$ do not suffice for a probability-measure representation to be obtained, e.g., the well-known example of Kraft et al. (1959) satisfies our properties and yet the likelihood relation cannot be probabilistically represented. In fact, even in the presence of additional properties – that we will impose in the upcoming sections – our likelihood relations will not always be represented by a probability measure.

Intuitively, if at ω player i believes that b yields at least as high a payoff as the chosen strategy $\sigma_i(\omega)$ at every state deemed possible, then the event that b yields a *strictly higher* payoff than $\sigma_i(\omega)$ is a null event for player i at ω.

We want to investigate the implications of common belief of rationality. Given an event E, let $\mathbb{B}_I E = \bigcap_{i \in I} \mathbb{B}_i E$ denote the event that all the players believe E. Then the event that E is commonly believed, denoted by $\mathbb{CB}E$, is defined as the infinite intersection $\mathbb{CB}E = \mathbb{B}_I E \cap \mathbb{B}_I \mathbb{B}_I E \cap \mathbb{B}_I \mathbb{B}_I \mathbb{B}_I E \cap \cdots$, that is, the event that everybody believes E, and everybody believes that everybody believes E, and everybody believes that everybody believes that everybody believes E, and so on. It is well-known that, for every state ω and every event E, $\omega \in \mathbb{CB}E$ if and only if $\mathcal{B}^*(\omega) \subseteq E$, where $\mathcal{B}^*(\omega)$ is the transitive closure of $\bigcup_{i \in I} \mathcal{B}_i(\omega)$.[10] We are interested in the event that there is common belief of rationality, henceforth denoted by $\mathbb{CB}\mathbf{R}$. In particular, we ask the question: *which strategy profiles are compatible with states in* $\mathbb{CB}\mathbf{R}$*?*

Definition 3. We say that common belief of rationality in a class of models $\mathcal{M}$ *(epistemically) characterizes* the set $S^* \subseteq S$ of strategy profiles whenever the following two conditions hold:

(A) in every model $M \in \mathcal{M}$, if $\omega \in \mathbb{CB}\mathbf{R}$ then $\sigma(\omega) \in S^*$,

(B) for every $s \in S^*$, there exists a model $M \in \mathcal{M}$ and a state ω in that model such that $\sigma(\omega) = s$ and $\omega \in \mathbb{CB}\mathbf{R}$.

In the following sections, we will epistemically characterize three well-known solution concepts for ordinal strategic-form games by means of common belief of rationality, by successively imposing stronger properties on the models of qualitative beliefs. That way, (i) we will place these different solution concepts under the same umbrella of common belief of rationality, and (ii) we will formally order the solution concepts in terms of the strategy profiles that they predict.

3.1. Iterated Deletion of Strictly Dominated Strategies

We begin with the best-known solution concept, namely the Iterated Deletion of Strictly Dominated Strategies, whose relationship to the notion of common belief of rationality has been explored extensively in the literature.[11]

A strategy $a \in S_i$ of player i is strictly dominated if there is another strategy $b \in S_i$ such that $\pi(b, s_{-i}) > \pi(a, s_{-i})$ for every strategy profile $s_{-i} \in S_{-i}$ of the players other than i, where as usual $S_{-i} = S_1 \times \cdots \times S_{i-1} \times S_{i+1} \times \cdots \times S_n$. Iterated Deletion of Strictly Dominated Strategies (IDSDS) is the following algorithm: reduce the game by deleting, for each player, all the strategies that are strictly dominated and then repeat the procedure in the reduced game, and so on, until there are no strictly dominated strategies left. Formally, the procedure is defined as follows:[12]

Definition 4. Given a strategic-form game with ordinal payoffs $G = \langle I, (S_i, \pi_i)_{i \in I} \rangle$, recursively define the sequence of reduced games $\{G^0, G^1, \ldots, G^m, \ldots\}$ as follows: for each $i \in I$,

[10] $\mathcal{B}^*$ is thus defined as follows: $\omega' \in \mathcal{B}^*(\omega)$ if and only if there is a sequence $\{\omega_1, \ldots, \omega_m\}$ in Ω and a sequence $\{i_1, \ldots, i_{m-1}\}$ in I such that (1) $\omega_1 = \omega$, (2) $\omega_m = \omega'$, and (3) for every $j = 1, \ldots, m-1$, $\omega_{j+1} \in \mathcal{B}_{i_j}(\omega_j)$.

[11] The pioneering contributions are Bernheim (1984) and Pearce (1984), followed by the characterization provided by Tan and Werlang (1988) for probabilistic beliefs. More recent characterizations for pure-strategy domination (which is what we focus on in this paper) can be found in Apt and Zvesper (2010) and Chen et al (2007).

[12] This is the pure-strategy version of the procedure commonly considered in the literature, which allows for domination by a mixed strategy. Recall that we have restricted attention to *ordinal* payoffs and thus a pure strategy of player i can be strictly dominated only by another pure strategy; in other words, *domination by a mixed strategy is not meaningful in this context.*

(4.1) let $S_i^0 = S_i$, and let $D_i^0 \subsetneq S_i^0$ be the set of i's strategies that are strictly dominated in $G^0 = G$;

(4.2) for each $m \geq 1$ let G^{m-1} be the reduced game with strategy sets S_i^{m-1}, and define $S_i^m = S_i^{m-1} \setminus D_i^{m-1}$, where $D_i^{m-1} \subsetneq S_i^{m-1}$ is the set of i's strategies that are strictly dominated in G^{m-1}.

Let $S_i^\infty = \bigcap_{m=0}^\infty S_i^m$. The strategy profiles in $S^\infty = S_1^\infty \times \cdots \times S_n^\infty$ are those surviving IDSDS.

Obviously, since the strategy sets are finite, there exists an integer r such that $S^\infty = S^k$ for every $k \geq r$, i.e., the procedure terminates after finitely many steps. Moreover, it is straightforward to verify that $S^\infty \neq \varnothing$.

IDSDS is the most permissive solution concept for strategic form games with ordinal payoffs, often allowing for seemingly counter-intuitive strategies. To see this, consider the game in Figure 1, where no strategy is strictly dominated and thus IDSDS does not eliminate any strategy, i.e., $S^\infty = \{a, b, c\} \times \{d, e\}$. Although strategy a is not strictly dominated, it is natural to ask: *how can we justify strategy a being a rational choice for by Player 1?*

Player 2

		d	e
	a	1 , 1	1 , 0
Player 1	b	2 , 0	1 , 1
	c	1 , 1	2 , 0

Figure 1: IDSDS and models without the Distribution Axiom.

Let us consider an arbitrary qualitative doxastic model $M \in \mathcal{M}_1$, with $D := \{\omega \in \Omega : \sigma_2(\omega) = d\} = \|b > a\|$ and $E := \{\omega \in \Omega : \sigma_2(\omega) = e\} = \|c > a\|$; thus $\Omega = \|b > a\| \cup \|c > a\|$. Assume that there exists some $\omega \in \mathbf{R}_1$ such that $\sigma_1(\omega) = a$. Clearly, $\|b \geq a\| = \|c \geq a\| = \Omega$, implying that $\omega \in \mathbb{B}_1\|b \geq a\| = \mathbb{B}_1\|c \geq a\|$. Hence, by $\omega \in \mathbf{R}_1$, it is necessarily the case that $\omega \in \mathbb{N}_1\|b > a\| \cap \mathbb{N}_1\|c > a\|$, which by (L_4) implies $\omega \in \mathbb{N}_1\Omega$. The latter clearly contradicts (L_2) (see Footnote 7). Hence, no model in $\mathcal{M}_1$ can sustain a as a rational strategy.

Given that our notion of rationality is anyway weak, it seems natural to further weaken the axioms of $\mathcal{M}_1$, in order to characterize IDSDS. Indeed, we define:

$\mathcal{M}_0$: the class of models where $\mathbb{N}_i$ satisfies $(L_1) - (L_3)$.

Obviously, $\mathcal{M}_0$ generalizes $\mathcal{M}_1$ by relaxing the Distribution Axiom (L_4) (see Definition 1). Then, the following characterization result is obtained.

Remark 1 (Characterization of IDSDS). *If the belief operators satisfy $(K_1) - (K_3)$ and the null operators satisfy merely $(L_1)-(L_3)$, then common belief of rationality characterizes IDSDS. Formally,*

(A_0) *in every model $M \in \mathcal{M}_0$, if $\omega \in \mathbb{CB}\mathbf{R}$ then $\sigma(\omega) \in S^\infty$,*

(B_0) *for every $s \in S^\infty$, there exists a model $M \in \mathcal{M}_0$ and a state ω in that model such that $\sigma(\omega) = s$ and $\omega \in \mathbb{CB}\mathbf{R}$.*

The proof of Remark 1 is given in the Appendix. Let us point out that we present the result of this section as a remark, rather than a theorem, since it relies on assumptions that may be viewed as too weak to be appealing, as illustrated below.

Consider the following model $M \in \mathcal{M}_0$ of the game of Figure 1: $\Omega = \{\omega_0, \omega_1\}$, $\mathcal{B}_i(\omega_0) = \mathcal{B}_i(\omega_1) = \Omega$, $\mathcal{N}_i(\omega_0) = \mathcal{N}_i(\omega_1) = \{\varnothing, \{\omega_0\}, \{\omega_1\}\}$, $\sigma_1(\omega_0) = \sigma_1(\omega_1) = a$, $\sigma_2(\omega_0) = d$ and $\sigma_2(\omega_1) = e$, noticing that $\mathbf{R}_1 = \Omega$. In fact, the reason a is sustained as a rational strategy at some ω is that every strict subset of $\mathcal{B}_1(\omega)$ is a null event, viz., $\|b > a\| = \{\omega_0\} \in \mathcal{N}_1(\omega)$ and $\|c > a\| = \{\omega_1\} \in \mathcal{N}_1(\omega)$. In other words, what is unappealing about this characterization is that it is possible that a strategy that survives IDSDS is compatible with common belief of rationality only at states ω where the only non-null event is $\mathcal{B}_i(\omega)$ itself.[13]

This last fact should be seen as a shortcoming of the notion of strict dominance employed above. We further discuss this issue later in the paper, in relation to the different refinements of IDSDS.

3.2. Iterated Deletion of Börgers-dominated Strategies

Börgers (1993) introduced a refined notion of pure-strategy dominance. In particular, let $a, b \in S_i$ be two pure strategies of player i, and let $X_{-i} \subseteq S_{-i}$ be a non-empty set of strategy-profiles of the players other than i (note that X_{-i} need not have a product structure). We say that b *weakly dominates* a *relative to* X_{-i} whenever: (1) $\pi_i(b, x_{-i}) \geq \pi_i(a, x_{-i})$ for all $x_{-i} \in X_{-i}$, and (2) there exists some $\hat{x}_{-i} \in X_{-i}$ such that $\pi_i(b, \hat{x}_{-i}) > \pi_i(a, \hat{x}_{-i})$. Then, a pure strategy $a \in S_i$ is *Börgers-dominated* (henceforth *B-dominated*) if for every non-empty subset $X_{-i} \subseteq S_{-i}$ there exists a strategy $b \in S_i$ (which is allowed to vary with X_{-i}) such that b weakly dominates a relative to X_{-i}. Iterated Deletion of B-dominated Strategies (IDBS) is the following algorithm: reduce the game by deleting, for each player, all the strategies that are B-dominated and then repeat the procedure in the reduced game, and so on, until there are no B-dominated strategies left.

Definition 5. Given a strategic-form game with ordinal payoffs $G = \langle I, (S_i, \pi_i)_{i \in I} \rangle$, recursively define the sequence of reduced games $\{G^0, G^1, \ldots, G^m, \ldots\}$ as follows: for each $i \in I$,

(5.1) let $B_i^0 = S_i$, and let $E_i^0 \subsetneq B_i^0$ be the set of i's strategies that are B-dominated in $G^0 = G$;

(5.2) for each $m \geq 1$ let G^{m-1} be the reduced game with strategy sets B_i^{m-1}, and define $B_i^m = B_i^{m-1} \setminus E_i^{m-1}$, where $E_i^{m-1} \subsetneq B_i^{m-1}$ is the set of i's strategies that are B-dominated in G^{m-1}.

Let $B_i^\infty = \bigcap_{m=0}^\infty B_i^m$. The strategy profiles in $B^\infty = B_1^\infty \times \cdots \times B_n^\infty$ are those surviving IDBS.

Similarly to IDSDS, since the strategy sets are finite, there exists an integer r such that $B^\infty = B^k$ for every $k \geq r$, i.e., the procedure terminates after finitely many steps. Furthermore, it is straightforward to verify that $B^\infty \neq \varnothing$.

For example, in the game of Figure 1, strategy a of Player 1 is B-dominated. Indeed, a is weakly dominated by b relative to $\{d\}$ and also relative to $\{d, e\}$, and it is weakly dominated by c relative to $\{e\}$. However, as we saw above, unless additional restrictions (besides $(L_1) - (L_3)$) are imposed on the null operator, there exists a model and a state within this model such that strategy a is rational according to Definition 2.

The first natural step is to restrict attention to qualitative doxastic models in $\mathcal{M}_1$, i.e., to additionally impose (L_4). However, as it turns out, $\mathbb{CB}\mathbf{R}$ in $\mathcal{M}_1$ does not characterize IDBS. Indeed,

[13] Such a situation is not compatible with the Distribution Axiom (L_4). To see this, first of all recall that, by Axiom $L_2(a)$, $\mathcal{B}_i(\omega) \notin \mathcal{N}_i(\omega)$ (see Footnote 7). Let E be any *proper* subset of $\mathcal{B}_i(\omega)$. If $\mathcal{B}_i(\omega)$ is the only non-null event, then $E \in \mathcal{N}_i(\omega)$ and $(\neg E \cap \mathcal{B}_i(\omega)) \in \mathcal{N}_i(\omega)$. By the Distribution Axiom, $\mathcal{B}_i(\omega) = E \cup (\neg E \cap \mathcal{B}_i(\omega)) \in \mathcal{N}_i(\omega)$, yielding a contradiction.

consider the game in Figure 2, which is a variant of the game that we presented in the previous section, noticing that strategy a is Börgers-dominated, viz., $B^\infty = \{b, c\} \times \{d, e\}$.

		Player 2	
		d	e
	a	1 , 1	1 , 0
Player 1	b	2 , 0	1 , 1
	c	0 , 1	2 , 0

Figure 2: IDBS and models without full-support beliefs.

Consider the following model $M \in \mathcal{M}_1$: $\Omega = \{\omega_0, \omega_1\}$, $\mathcal{B}_1(\omega_0) = \mathcal{B}_1(\omega_1) = \Omega$, $\mathcal{N}_1(\omega_0) = \mathcal{N}_1(\omega_1) = \{\varnothing, \{\omega_0\}\}$, $\sigma_1(\omega_0) = \sigma_1(\omega_1) = a$, $\sigma_2(\omega_0) = d$ and $\sigma_2(\omega_1) = e$. Observe that $\mathbf{R}_1 = \Omega$, which follows from $\mathbb{B}_1\|b \geq a\| = \mathbb{N}_1\|b > a\| = \Omega$ and $\mathbb{B}_1\|c \geq a\| = \varnothing$. Hence, even though a is Börgers-dominated, it is rational at a state in M, implying that additional restrictions need to be imposed on our doxastic models.

In what follows we shall restrict attention to models with cautious players, i.e., with players who have full-support beliefs.

Definition 6. A finite qualitative doxastic model of a strategic-form game with ordinal payoffs has *full support* if, for every $i \in I$ and every $E \subseteq \Omega$,

(L_5) $\mathbb{N}_i E \subseteq \mathbb{B}_i \neg E$.

Let $\mathcal{M}_2 \subsetneq \mathcal{M}_1$ denote the class of finite full-support qualitative doxastic models.

Property (L_5) says that only impossible events are deemed null. In other words, for every $\omega \in \Omega$, if $\mathcal{B}_i(\omega) \cap E \neq \varnothing$ then $E \notin \mathcal{N}_i(\omega)$. Notice that (L_5) is the converse of $(L_2(b))$, i.e., in full-support models, an event is null if and only if its negations is believed.

Theorem 1 (Characterization of IDBS)**.** *If the belief operators satisfy* $(K_1) - (K_3)$ *and the null operators satisfy* $(L_1) - (L_5)$*, then common belief of rationality characterizes IDBS. Formally,*

(A_2) *in every model* $M \in \mathcal{M}_2$*, if* $\omega \in \mathbb{CB}\mathbf{R}$ *then* $\sigma(\omega) \in B^\infty$*,*

(B_2) *for every* $s \in B^\infty$*, there exists a model* $M \in \mathcal{M}_2$ *and a state* ω *in that model such that* $\sigma(\omega) = s$ *and* $\omega \in \mathbb{CB}\mathbf{R}$*.*

Note that, in order to "rationalize" a strategy profile in B^∞, *it may be necessary for a player to have erroneous beliefs.* To see this, consider the game in Figure 3, where $B^\infty = S$, that is, IDBS does not eliminate any strategy; in particular, $(a, d) \in B^\infty$.[14] Consider an arbitrary full-support model of this game and a state ω_0 such that $\sigma(\omega_0) = (a, d)$. Since, for every $s_2 \in \{c, d\}$, $\pi_1(b, s_2) \geq \pi_1(a, s_2)$, $\|b \geq a\| = \Omega$ and thus $\mathcal{B}_1(\omega_0) \subseteq \|b \geq a\|$. That is, $\omega_0 \in \mathbb{B}_1\|b \geq a\|$. Hence, if Player 1 is rational at

[14] For Player 1, a is weakly dominated by b relative to $\{d\}$ and $\{c, d\}$ but not relative to $\{c\}$ and for Player 2 d is weakly dominated by c relative to $\{a\}$ but not relative to $\{b\}$ or $\{a, b\}$.

ω_0 (according to Definition 2) then $\|b > a\| \cap \mathcal{B}_1(\omega_0) = \varnothing$.[15] Thus, $\sigma_2(\omega) = c$ for all $\omega \in \mathcal{B}_1(\omega_0)$. In particular, it must be that $\omega_0 \notin \mathcal{B}_1(\omega_0)$. Thus at state ω_0 Player 2 actually plays d but Player 1 – who plays a – must erroneously believe that Player 2 is playing c. In the next section we investigate

		Player 2	
		c	d
Player 1	a	1 , 1	1 , 0
	b	1 , 0	2 , 2

Figure 3: IDBS and full-support models with erroneous beliefs.

the consequences of ruling out false beliefs, while maintaining caution, i.e. full-support beliefs.

3.3. Iterated Deletion of Inferior Strategy Profiles

The following algorithm is the pure-strategy version of a procedure first introduced by Stalnaker (1994) and further studied in Bonanno (2008); Bonanno and Nehring (1998); Hillas and Samet (2014); Trost (2013). Unlike the procedures considered above (viz., IDSDS and IDBS), this procedure deletes entire strategy profiles, rather than individual strategies. In particular, let $X \subseteq S$ be a set of strategy profiles (not necessarily having a product structure). A strategy profile $x \in X$ is *inferior relative to X* if there exist a player i and a strategy $s_i \in S_i$ of player i (i.e., s_i need not belong to the projection of X onto S_i) such that (1) $\pi_i(s_i, x_{-i}) > \pi_i(x_i, x_{-i})$, and (2) for all $s_{-i} \in S_{-i}$, either $(x_i, s_{-i}) \notin X$ or $\pi_i(s_i, s_{-i}) \geq \pi_i(x_i, s_{-i})$. Iterated Deletion of Inferior Profiles (IDIP) is the following algorithm: reduce the game by deleting all the inferior strategy profiles and then repeat the procedure by eliminating inferior profiles relative to the strategy profiles that have not been eliminated so far, until there are no inferior profiles left. Formally, the algorithm is defined as follows:

Definition 7. Given a strategic-form game with ordinal payoffs $G = \langle I, (S_i, \pi_i)_{i \in I} \rangle$, recursively define the sequence of sets of strategy profiles $\{T^0, T^1, \ldots, T^m, \ldots\}$ as follows:

(7.1) let $T^0 = S$, and let $I^0 \subsetneq T^0$ be the set of inferior strategy profiles relative to T^0;

(7.2) for each $m \geq 1$ let $T^m = T^{m-1} \setminus I^{m-1}$, where $I^{m-1} \subsetneq T^{m-1}$ is the set of strategy profiles in T^{m-1} that are inferior relative to T^{m-1}.

Then $T^\infty = \bigcap_{m=0}^{\infty} T^m$ denotes the strategy profiles surviving IDIP.

Once again, since the strategy sets are finite, there exists an integer r such that $T^\infty = T^k$ for every $k \geq r$, i.e., the procedure terminates after finitely many steps. Besides, it is straightforward to verify that $T^\infty \neq \varnothing$.

As an illustration of this procedure, consider the game in Figure 4. In this game (a, d) is inferior relative to $T^0 = S$ since $\pi_1(b, d) > \pi_1(a, d)$ and $\pi_1(b, c) = \pi_1(a, c)$ (and $(a, c) \in S$). No other strategy profile is inferior relative to T^0 and thus $I^0 = \{(a, d)\}$ so that $T^1 = \{(a, c), (b, c), (b, d)\}$. Now (b, d) is inferior relative to T^1 since $\pi_2(b, c) > \pi_2(b, d)$ and $(a, d) \notin T^1$. No other strategy profile is inferior

[15] Suppose that $\omega_0 \in \mathbf{R}_1$. Then, by Definition 2, since $\omega_0 \in \mathbb{B}_1\|b \geq a\|$, it must be that $\|b > a\| \in \mathcal{N}_1(\omega_0)$. Fix an arbitrary $\omega \in \mathcal{B}_1(\omega_0)$ and suppose that $\omega \in \|b > a\|$, noting that, by (Σ_0), $\sigma_1(\omega) = \sigma_1(\omega_0) = a$. By the assumption of full support, $\{\omega\} \notin \mathcal{N}_1(\omega_0)$ and thus, by (L_2), $\|b > a\| \notin \mathcal{N}_1(\omega_0)$, yielding a contradiction.

		Player 2	
		c	d
Player 1	a	1 , 1	1 , 2
	b	1 , 1	2 , 0

Figure 4: IDIP.

relative to T^1 and thus $I^1 = \{(b,d)\}$ so that $T^2 = \{(a,c),(b,c)\}$. Now no strategy profile is inferior relative to T^2 so that $T^\infty = T^2$.

We now turn to investigating the consequences of ruling out false beliefs. At state ω player i has correct beliefs if ω is one of the states that player i considers possible at ω, that is, if $\omega \in \mathcal{B}_i(\omega)$.

Definition 8. A finite qualitative doxastic model of a strategic-form game with ordinal payoffs *rules out false beliefs* if, for every $i \in I$ the belief operator $\mathbb{B}_i$ satisfies the Truth Axiom (K_4). Let $\mathcal{M}_3 \subsetneq \mathcal{M}_2$ denote the class of finite full-support qualitative doxastic models that rule out false beliefs.

Theorem 2 (Characterization of IDIP)**.** *If the belief operators satisfy* $(K_1)-(K_4)$ *and the null operators satisfy* $(L_1)-(L_5)$*, then common belief of rationality characterizes IDIP. Formally,*

(A_3) *in every model* $M \in \mathcal{M}_3$*, if* $\omega \in \mathbb{CB}\mathbf{R}$ *then* $\sigma(\omega) \in T^\infty$*,*

(B_3) *for every* $s \in T^\infty$*, there exists a model* $M \in \mathcal{M}_3$ *and a state* ω *in that model such that* $\sigma(\omega) = s$ *and* $\omega \in \mathbb{CB}\mathbf{R}$*.*

Property (K_4) says that no player can have false beliefs. This is actually stronger than simply requiring that it is commonly believed that every player has correct beliefs. In fact, in order to get a characterization of the set T^∞, common belief that all players have correct beliefs is not sufficient (see Section 4.3).

4. Discussion

4.1. Iterated Deletion of Uniformly Weakly Dominated Strategies

We have characterized three well-known solution concepts for games with ordinal payoffs (viz., IDSDS, IDBS and IDIP) by means of restrictions imposed on the belief operator and the null operator. Interestingly, none of these solution concepts is characterized by our benchmark system, $\mathcal{M}_1$. In this section, we do so, by introducing a new elimination solution concept that coarsens IDBS.

We say that a strategy $a \in S_i$ is *uniformly weakly dominated* (henceforth *U-dominated*) if for every $X_{-i} \subseteq S_{-i}$ and every $s_{-i} \in X_{-i}$ there exists some $b \in S_i$ (which is allowed to vary with X_{-i} and s_{-i}) such that (1) $\pi_i(b, x_{-i}) \geq \pi_i(a, x_{-i})$ for all $x_{-i} \in X_{-i}$, and (2) $\pi_i(b, s_{-i}) > \pi_i(a, s_{-i})$. For instance, strategy a in Figure 1 is U-dominated. On the other hand, in Figure 2, the same strategy a is not: take for instance $X_{-i} = \{d, e\}$ and $s_{-i} = e$, and observe that b violates condition (2), while c violates condition (1), viz., $\pi_1(b,e) \not> \pi_1(a,e)$ and $\pi_1(c,d) \not\geq \pi_1(a,d)$.

Iterated Deletion of U-dominated Strategies (IDUS) is the following algorithm: reduce the game by eliminating, for each player, all the strategies that are U-dominated, and then repeat the procedure in the reduced game, and so on until there are no U-dominated strategies left.

Definition 9. Given a strategic-form game with ordinal payoffs $G = \langle I, (S_i, \pi_i)_{i \in I} \rangle$, recursively define the sequence of reduced games $\{G^0, G^1, \dots, G^m, \dots\}$ as follows: for each $i \in I$,

(9.1) let $U_i^0 = S_i$, and let $W_i^0 \subsetneq U_i^0$ be the set of i's strategies that are U-dominated in $G^0 = G$;

(9.2) for each $m \geq 1$ let G^{m-1} be the reduced game with strategy sets U_i^{m-1}, and define $U_i^m = U_i^{m-1} \setminus W_i^{m-1}$, where $W_i^{m-1} \subsetneq U_i^{m-1}$ is the set of i's strategies that are U-dominated in G^{m-1}.

Let $U_i^\infty = \bigcap_{m=0}^{\infty} U_i^m$. The strategy profiles in $U^\infty = U_1^\infty \times \cdots \times U_n^\infty$ are those surviving IDUS.

Like in the case of IDSDS and IDBS, by the fact that the strategy sets are finite, it follows that the algorithm terminates after finitely many steps, i.e., there is some integer $r \geq 0$ such that $U^\infty = U^k$ for all $k \geq r$. Hence, it is straightforward that $U^\infty \neq \varnothing$.

Remark 2 (Characterization of IDUS)**.** *If the belief operators satisfy $(K_1) - (K_3)$ and the null operators satisfy $(L_1) - (L_4)$, then common belief of rationality characterizes IDUS. Formally,*

(A_1) *in every model $M \in \mathcal{M}_1$, if $\omega \in \mathbb{CB}\mathbf{R}$ then $\sigma(\omega) \in U^\infty$,*

(B_1) *for every $s \in U^\infty$, there exists a model $M \in \mathcal{M}_1$ and a state ω in that model such that $\sigma(\omega) = s$ and $\omega \in \mathbb{CB}\mathbf{R}$.*

It follows from the previous result that the main difference between IDBS and IDUS is that the former requires every strict subset of $\mathcal{B}_i(\omega)$ to be non-null, whereas the latter requires at least one strict subset of $\mathcal{B}_i(\omega)$ to be non-null.

4.2. Monotonicity result

A direct implication of our four results (Theorems 1-2 and Remarks 1-2) is the following (monotonicity) result, which proves that IDIP is a refinement of IDBS, which is a refinement of IDUS, which is a refinement of IDSDS.

Corollary 1 (Monotonicity result)**.** $T^\infty \subseteq B^\infty \subseteq U^\infty \subseteq S^\infty$.

The proof follows directly from $\mathcal{M}_3 \subsetneq \mathcal{M}_2 \subsetneq \mathcal{M}_1 \subsetneq \mathcal{M}_0$. The last two parts of the result are not very surprising and can also be proven directly, viz., it can be shown that for every $m \geq 0$ it is the case that $B^m \subseteq U^m \subseteq S^m$. However, this is not the case with the first part of our monotonicity result, which is far from trivial. The difficulty of proving the result stems from the fact that there exist games where $B^m \subsetneq T^m$ for some $m > 0$, as illustrated in the game in Figure 5.

		Player 2	
		d	e
	a	1 , 1	2 , 1
Player 1	b	2 , 1	0 , 1
	c	1 , 1	1 , 1

Figure 5: Monotonicity.

In this game, c is B-dominated, while no other strategy is subsequently eliminated. That is, formally $B^\infty = B^1 = \{a,b\} \times \{d,e\}$. On the other hand, the only inferior strategy profile relative to the entire game is (c,e), and therefore $T^1 \supsetneq B^1$. But then, (c,d) is inferior relative to T^1, thus implying that $B^\infty = B^2 = T^2 = T^\infty$, consistently with the conclusions of our Corollary 1.

4.3. Correct beliefs

As we have already mentioned above, common belief in correct beliefs does not suffice for a strategy that survives IDIP to be played. Formally, let $\mathbf{C}_i = \{\omega \in \Omega : \omega \in \mathcal{B}_i(\omega)\}$ be the event that player i has correct beliefs, and let $\mathbf{C}_\cup = \bigcup_{i\in I} \mathbf{C}_i$ be the event that at least one player has correct beliefs and $\mathbf{C} = \bigcap_{i\in I} \mathbf{C}_i$ the event that all players have correct beliefs. As the following example shows, it is possible that $\omega \in \mathbb{CB}\mathbf{R} \cap \mathbb{CB}\mathbf{C}$ and yet the strategy profile played at ω does not survive IDIP.

Player 2

		c	d
Player 1	a	1 , 1	1 , 0
	b	1 , 0	2 , 2

Figure 6: Common belief in correct beliefs.

Consider the following model of the game in Figure 6: $\Omega = \{\omega_1, \omega_2, \omega_3\}, \mathcal{B}_1(\omega_1) = \{\omega_1\}, \mathcal{B}_1(\omega_2) = \mathcal{B}_1(\omega_3) = \{\omega_3\}, \mathcal{B}_2(\omega_1) = \mathcal{B}_2(\omega_2) = \{\omega_1\}, \mathcal{B}_2(\omega_3) = \{\omega_3\}, \sigma_1(\omega_1) = b, \sigma_1(\omega_2) = \sigma_1(\omega_3) = a, \sigma_2(\omega_1) = \sigma_2(\omega_2) = d$ and $\sigma_2(\omega_3) = c$. Then $\sigma(\omega_2) = (a,d) \notin T^\infty$ and yet $\omega_2 \in \mathbb{CB}\mathbf{R} \cap \mathbb{CB}\mathbf{C}$ (in fact, $\mathcal{B}^*(\omega_2) = \{\omega_1, \omega_3\}, \mathbf{C} = \{\omega_1, \omega_3\}$ and $\mathbf{R} = \Omega$). Note that, in this model, at state ω_2 both players have false beliefs. This is because, although it is common belief at ω_2 that only strategy profiles in T^∞ are played, the strategy profile *actually* played does not belong to T^∞.

Although common belief in correct beliefs does not suffice for IDIP, it guarantees common belief in the event that only strategy profiles in T^∞ are played. Let $\mathbf{T}^\infty = \{\omega \in \Omega : \sigma(\omega) \in T^\infty\}$.

Proposition 1. *If the belief operators satisfy $(K_1) - (K_3)$ and the null operators satisfy $(L_1) - (L_5)$, then common belief of rationality and common belief of correct beliefs imply common belief in IDIP. Formally, $\mathbb{CB}\mathbf{R} \cap \mathbb{CB}\mathbf{C} \subseteq \mathbb{CB}\mathbf{T}^\infty$ in every model $M \in \mathcal{M}_2$.*

The condition that there is common belief that all players have correct beliefs ($\omega \in \mathbb{CB}\mathbf{C}$) is necessary for Proposition 1. To see this, consider the game in Figure 7, where $T^\infty = \{(a,c),(b,c)\}$. Consider the following model of this game: $\Omega = \{\omega_1, \omega_2\}, \mathcal{B}_1(\omega_1) = \mathcal{B}_1(\omega_2) = \{\omega_2\}, \mathcal{B}_2(\omega_1) = \{\omega_1\}, \mathcal{B}_2(\omega_2) = \{\omega_2\}, \sigma_1(\omega_1) = \sigma_1(\omega_2) = a, \sigma_2(\omega_1) = d, \sigma_2(\omega_2) = c$. Then $\mathbf{R} = \mathbb{CB}\mathbf{R} = \Omega$, while $\mathbf{T}^\infty = \{\omega_2\}$ (since $\sigma(\omega_1) = (a,d) \notin T^\infty$). Since $\mathcal{B}^*(\omega_1) = \{\omega_1, \omega_2\}$, $\omega_1 \in \mathbb{CB}\mathbf{R}$ but $\omega_1 \notin \mathbb{CB}\mathbf{T}^\infty$. In this model, at state ω_1 Player 1 has false beliefs ($\mathbf{C}_1 = \{\omega_2\}$) and thus $\omega_1 \notin \mathbb{CB}\mathbf{C}$.

The following Corollary shows that if, to the hypotheses of Proposition 1, we add the further hypothesis that *at least one player* does not have false beliefs, then it follows that the strategy profile *actually played* also belongs to T^∞. Recall that $\mathbf{C}_\cup = \bigcup_{i\in I} \mathbf{C}_i$ is the event that at least one player has correct beliefs.

Corollary 2. *If the belief operators satisfy $(K_1) - (K_3)$ and the null operators satisfy $(L_1) - (L_5)$, then common belief of rationality, common belief of correct beliefs and correct beliefs of at least one player imply IDIP. Formally, $\mathbb{CB}\mathbf{R} \cap \mathbb{CB}\mathbf{C} \cap \mathbf{C}_\cup \subseteq \mathbf{T}^\infty$ in every model $M \in \mathcal{M}_2$.*

		Player 2	
		c	d
Player 1	a	1 , 1	1 , 1
	b	1 , 1	2 , 0

Figure 7: Correct beliefs.

5. Conclusion

In this paper we have studied the behavioral implications of common belief of rationality in strategic-form games with ordinal utilities, using qualitative beliefs. Focusing on ordinal utilities is relevant both theoretically (as we implicitly relax the admittedly unrealistic assumption of commonly known vNM preferences), as well as empirically (as experimental economists typically use solution concepts for games with ordinal payoffs for their benchmark theoretical predictions).

Our main contribution is threefold. Firstly, we systematically embed qualitative doxastic models into a game-theoretic model. Secondly, we manage to characterize three well-known solution concepts for games with ordinal payoffs in terms of common belief of rationality, without needing to vary the notion of rationality that we employ, but rather by gradually strengthening the properties of the doxastic model that we use. Finally, as a consequence of our characterization results, we prove that the aforementioned solution concepts monotonically refine each other, viz., IDIP refines IDBS, which in turn refines IDSDS. Notably, the first refinement result is far from trivial to prove.

It is worth noting that the definition of rationality that we have used (Definition 2) is extremely weak; indeed, it allows one to label as rational a strategy that intuitively ought to be considered as irrational. To see this, consider the pure coordination game of Figure 8 and the following model of it.[16] $\Omega = \{\omega_0, \omega_1\}, \mathcal{B}_1(\omega_0) = \mathcal{B}_1(\omega_1) = \Omega, \mathcal{N}_1(\omega_0) = \mathcal{N}_1(\omega_1) = \{\varnothing, \{\omega_0\}\}, \sigma_1(\omega_0) = \sigma_1(\omega_1) = a, \sigma_2(\omega_0) = c, \sigma_2(\omega_1) = d$. At state ω_1 (and also at ω_0) Player 1 does not rule out the event that

		Player 2	
		c	d
Player 1	a	1 , 1	0 , 0
	b	0 , 0	1 , 1

Figure 8: A coordination game

Player 2 plays c but considers this event null; on the other hand she does not consider the event that Player 2 plays d null. Thus $\omega_1 \in \mathbb{N}_1 \|a > b\|$ while $\omega_1 \notin \mathbb{N}_1 \|b > a\|$. In other words, Player 1 is confident that Player 2 is playing d and should therefore play b. Yet, playing a is rational for Player 1, that is, $\omega_1 \in \mathbf{R_1}$, because $\omega_1 \notin \mathbb{B}_1 \|b \geq a\|$. This example might be seen as suggesting that one should employ a stronger, and intuitively more appealing, definition of rationality. However, it is important to stress that *the weaker the definition of rationality, the stronger the results*. The fact

[16]This example was suggested by an anonymous referee.

that we are able to obtain a characterization of solution concepts such as IDSDS, IDBS, IDUS and IDIP with a such a weak notion of rationality is actually a strength rather than a weakness of our approach.

A. Proofs of Section 3

PROOF OF REMARK 1. (A_0) Fix a strategic-form game with ordinal payoffs and a model $M \in \mathcal{M}_0$ of the game. Suppose that $\omega_1 \in \mathbb{CB}\mathbf{R}$. That is, $\mathcal{B}^*(\omega_1) \subseteq \mathbf{R}$. We want to show that $\sigma(\omega_1) \in S^\infty$. The proof is by induction.

Initial Step. First we show (by contradiction) that, for every player $i \in I$ and for every $\omega \in \mathcal{B}^*(\omega_1)$, $\sigma_i(\omega) \notin D_i^0$ (see Definition 4). Suppose not. Then there exist a player i and an $\omega_2 \in \mathcal{B}^*(\omega_1)$ such that $\sigma_i(\omega_2) \in D_i^0$, that is, strategy $\sigma_i(\omega_2)$ of player i is strictly dominated in G by some other strategy $\hat{s}_i \in S_i$: for every $s_{-i} \in S_{-i}$, $\pi_i(\hat{s}_i, s_{-i}) > \pi_i(\sigma_i(\omega_2), s_{-i})$. Thus, for every $\omega \in \mathcal{B}_i(\omega_2)$, $\pi_i(\hat{s}_i, \sigma_{-i}(\omega)) > \pi_i(\sigma_i(\omega_2), \sigma_{-i}(\omega))$, that is, $\|\hat{s}_i > \sigma_i(\omega_2)\| \cap \mathcal{B}_i(\omega_2) = \mathcal{B}_i(\omega_2)$. Since $\mathcal{B}_i(\omega_2) \notin \mathcal{N}_i(\omega_2)$ (see Footnote 7), it follows from Definition 2 that $\omega_2 \notin \mathbf{R}_i$, thus contradicting the hypothesis that $\omega_2 \in \mathcal{B}^*(\omega_1) \subseteq \mathbf{R}$ (recall that $\mathbf{R} \subseteq \mathbf{R}_i$). Thus, for every $\omega \in \mathcal{B}^*(\omega_1)$, $\sigma_i(\omega) \in S_i \backslash D_i^0 = S_i^1$.

Inductive Step. Fix an integer $m \geq 1$ and suppose that, for every player $j \in I$ and for every $\omega \in \mathcal{B}^*(\omega_1)$, $\sigma_j(\omega) \in S_j^m$. We want to show (again by contradiction) that, for every player $i \in I$ and for every $\omega \in \mathcal{B}^*(\omega_1)$, $\sigma_i(\omega) \notin D_i^m$. Suppose not. Then there exist a player i and an $\omega_2 \in \mathcal{B}^*(\omega_1)$ such that $\sigma_i(\omega_2) \in D_i^m$, that is, strategy $\sigma_i(\omega_2)$ is strictly dominated in G^m by some other strategy $\tilde{s}_i \in S_i^m$. Since, by hypothesis, for every player j and for every $\omega \in \mathcal{B}^*(\omega_1)$, $\sigma_j(\omega) \in S_j^m$, it follows – since $\mathcal{B}_i(\omega_2) \subseteq \mathcal{B}^*(\omega_2) \subseteq \mathcal{B}^*(\omega_1)$ (the latter inclusion follows from transitivity of $\mathcal{B}^*$) – that, for every $\omega \in \mathcal{B}_i(\omega_2)$, $\pi_i(\tilde{s}_i, \sigma_{-i}(\omega)) > \pi_i(\sigma_i(\omega_2), \sigma_{-i}(\omega))$, that is, $\|\tilde{s}_i > \sigma_i(\omega_2)\| \cap \mathcal{B}_i(\omega_2) = \mathcal{B}_i(\omega_2)$. Since $\mathcal{B}_i(\omega_2) \notin \mathcal{N}_i(\omega_2)$ (see Footnote 7), it follows from Definition 2 that $\omega_2 \notin \mathbf{R}_i$, contradicting the hypothesis that $\omega_2 \in \mathcal{B}^*(\omega_1) \subseteq \mathbf{R}$. Thus, for every player $i \in I$ and for every $\omega \in \mathcal{B}^*(\omega_1)$, $\sigma_i(\omega) \in \bigcap_{m=1}^{\infty} S_i^m = S_i^\infty$. It only remains to show that $\sigma_i(\omega_1) \in S_i^\infty$. Fix an arbitrary $\omega_2 \in \mathcal{B}_i(\omega_1)$. Since $\mathcal{B}_i(\omega_1) \subseteq \mathcal{B}^*(\omega_1)$, $\omega_2 \in \mathcal{B}^*(\omega_1)$. Thus $\sigma_i(\omega_2) \in S_i^\infty$. By (Σ_0), since $\omega_2 \in \mathcal{B}_i(\omega_1)$, $\sigma_i(\omega_2) = \sigma_i(\omega_1)$. Thus $\sigma_i(\omega_1) \in S_i^\infty$.

(B_0) Given a game G construct the following model $M \in \mathcal{M}_0$: $\Omega = S^\infty = S_1^\infty \times \cdots \times S_n^\infty$; for every player i and for every $s \in S^\infty$, $\mathcal{B}_i(s) = \{s' \in S^\infty : s'_i = s_i\}$ (that is, at state s player i considers possible each of the strategy profiles of the other players in S_{-i}^∞, while her strategy is held constant at s_i); $\sigma_i : S^\infty \to S_i$ is defined by $\sigma_i(s) = s_i$ (that is, $\sigma_i(s)$ is the i^{th} coordinate of s); finally, for every $i \in I$ and $s \in S^\infty$, let $\mathcal{N}_i(s) = \{E \in 2^\Omega : E \cap \mathcal{B}_i(s) \neq \mathcal{B}_i(s)\}$. Fix an arbitrary state $s \in S^\infty$ and an arbitrary player i. By definition of S^∞, for every $s'_i \in S_i^\infty$ there exists an $\hat{s}_{-i} \in S_{-i}^\infty$ such that $\pi_i(s_i, \hat{s}_{-i}) \geq \pi_i(s'_i, \hat{s}_{-i})$ (that is, $(s_i, \hat{s}_{-i}) \notin \|s'_i > s_i\|$). By construction, $(s_i, \hat{s}_{-i}) \in \mathcal{B}_i(s)$ so that $\|s'_i > s_i\| \cap \mathcal{B}_i(s) \neq \mathcal{B}_i(s)$ and thus, by construction, $\|s'_i > s_i\| \in \mathcal{N}_i(s)$ so that, by Definition 2, $s \in \mathbf{R}_i$. Since s and i were chosen arbitrarily, if follows that, for every $s \in S^\infty$, $s \in \mathbf{R}$, that is, $\mathbf{R} = S^\infty$ and thus $\mathbb{CB}\mathbf{R} = S^\infty$. □

PROOF OF THEOREM 1. (A_2) Fix a strategic-form game with ordinal payoffs and a model $M \in \mathcal{M}_2$. Suppose that $\omega_1 \in \mathbb{CB}\mathbf{R}$ (that is, $\mathcal{B}^*(\omega_1) \subseteq \mathbf{R}$). We want to show that $\sigma(\omega_1) \in B^\infty$. The proof is by induction.

Initial Step. First we show (by contradiction) that, for every player $i \in I$ and for every $\omega \in \mathcal{B}^*(\omega_1)$, $\sigma_i(\omega) \notin E_i^0$ (see Definition 5). Suppose not. Then there exist a player i and an $\omega_2 \in \mathcal{B}^*(\omega_1)$ such that $\sigma_i(\omega_2) \in E_i^0$, that is, strategy $\sigma_i(\omega_2)$ of player i is B-dominated relative to S_{-i}, i.e., for every

$X_{-i} \subseteq S_{-i}$ there exists a strategy $s_i \in S_i$ such that (1) for all $x_{-i} \in X_{-i}$, $\pi_i(s_i, x_{-i}) \geq \pi_i(\sigma_i(\omega_2), x_{-i})$, and (2) there exists an $\hat{x}_{-i} \in X_{-i}$ such that $\pi_i(s_i, \hat{x}_{-i}) > \pi_i(\sigma_i(\omega_2), \hat{x}_{-i})$. Let $X_{-i} = \sigma_{-i}(\mathcal{B}_i(\omega_2)) = \{s_{-i} \in S_{-i} : s_{-i} = \sigma_{-i}(\omega) \text{ for some } \omega \in \mathcal{B}_i(\omega_2)\}$. Let $s_i \in S_i$ and $\hat{x}_{-i} \in X_{-i}$ satisfy (1) and (2) and let $\hat{\omega} \in \mathcal{B}_i(\omega_2)$ be such that $\sigma_{-i}(\hat{\omega}) = \hat{x}_{-i}$. Then, by (1), $\omega_2 \in \mathbb{B}_i\|s_i \geq \sigma_i(\omega_2)\|$ and by (2) $\|s_i > \sigma_i(\omega_2)\| \cap \mathcal{B}_i(\omega_2) \supseteq \{\hat{\omega}\}$. By (L_5), $\{\hat{\omega}\} \notin \mathcal{N}_i(\omega_2)$ and thus, by (L_2), $\|s_i > \sigma_i(\omega_2)\| \cap \mathcal{B}_i(\omega_2) \notin \mathcal{N}_i(\omega_2)$; hence $\omega_2 \notin \mathbf{R}_i$ (see Definition 2), contradicting the hypothesis that $\omega_1 \in \mathbb{CBR}$ and $\omega_2 \in \mathcal{B}^*(\omega_1)$ (which implies that $\omega_2 \in \mathbf{R} \subseteq \mathbf{R}_i$). Thus we have shown that, for every $\omega \in \mathcal{B}^*(\omega_1)$, $\sigma_i(\omega) \in S_i \backslash E_i^0 = B_i^1$.

Inductive Step. Fix an integer $m \geq 1$ and suppose that, for every player $j \in I$ and for every $\omega \in \mathcal{B}^*(\omega_1)$, $\sigma_j(\omega) \in B_j^m$, that is, $\mathcal{B}^*(\omega_1) \subseteq B^m$. We want to show (by contradiction) that, for every player $i \in I$ and for every $\omega \in \mathcal{B}^*(\omega_1)$, $\sigma_i(\omega) \notin E_i^m$. Suppose not. Then there exist a player i and an $\omega_2 \in \mathcal{B}^*(\omega_1)$ such that $\sigma_i(\omega_2) \in E_i^m$, that is, strategy $\sigma_i(\omega_2)$ of player i is B-dominated relative to B_{-i}^m: for every $X_{-i} \subseteq B_{-i}^m$ there exists a strategy $s_i \in S_i$ such that (1) for all $x_{-i} \in X_{-i}$, $\pi_i(s_i, x_{-i}) \geq \pi_i(\sigma_i(\omega_2), x_{-i})$ and (2) there exists an $\hat{x}_{-i} \in X_{-i}$ such that $\pi_i(s_i, \hat{x}_{-i}) > \pi_i(\sigma_i(\omega_2), \hat{x}_{-i})$. Let $X_{-i} = \sigma_{-i}(\mathcal{B}_i(\omega_2)) = \{s_{-i} \in S_{-i} : s_{-i} = \sigma_{-i}(\omega) \text{ for some } \omega \in \mathcal{B}_i(\omega_2)\}$. By the induction hypothesis and the fact that $\mathcal{B}_i(\omega_2) \subseteq \mathcal{B}^*(\omega_2) \subseteq \mathcal{B}^*(\omega_1)$ (the latter inclusion follows from transitivity of $\mathcal{B}^*$), $X_{-i} \subseteq B_{-i}^m$. Let $s_i \in S_i$ and $\hat{x}_{-i} \in X_{-i}$ satisfy (1) and (2) and let $\hat{\omega} \in \mathcal{B}_i(\omega_2)$ be such that $\sigma_{-i}(\hat{\omega}) = \hat{x}_{-i}$. Then, by (1), $\omega_2 \in \mathbb{B}_i\|s_i \geq \sigma_i(\omega_2)\|$ and by (2) $\|s_i > \sigma_i(\omega_2)\| \cap \mathcal{B}_i(\omega_2) \supseteq \{\hat{\omega}\}$. By (L_5), $\{\hat{\omega}\} \notin \mathcal{N}_i(\omega_2)$ and thus, by (L_2), $\|s_i > \sigma_i(\omega_2)\| \cap \mathcal{B}_i(\omega_2) \notin \mathcal{N}_i(\omega_2)$; hence $\omega_2 \notin \mathbf{R}_i$ (see Definition 2), contradicting the hypothesis that $\omega_1 \in \mathbb{CBR}$ and $\omega_2 \in \mathcal{B}^*(\omega_1)$ (which implies that $\omega_2 \in \mathbf{R} \subseteq \mathbf{R}_i$). Thus, for every player $i \in I$ and for every $\omega \in \mathcal{B}^*(\omega_1)$, $\sigma_i(\omega) \in \bigcap_{m=1}^{\infty} B_i^m = B_i^\infty$.

It only remains to show that $\sigma_i(\omega_1) \in B_i^\infty$. Take any $\omega_2 \in \mathcal{B}_i(\omega_1)$. Since $\mathcal{B}_i(\omega_1) \subseteq \mathcal{B}^*(\omega_1)$, $\omega_2 \in \mathcal{B}^*(\omega_1)$. Thus $\sigma_i(\omega_2) \in B_i^\infty$. By (Σ_0), since $\omega_2 \in \mathcal{B}_i(\omega_1)$, $\sigma_i(\omega_2) = \sigma_i(\omega_1)$. Thus $\sigma_i(\omega_1) \in B_i^\infty$.

(B_2) Given a game G construct the following model $M \in \mathcal{M}_2$: $\Omega = B^\infty = B_1^\infty \times \cdots \times B_n^\infty$; for every player i and for every $s \in B^\infty$, $\sigma_i : B^\infty \to S_i$ is defined by $\sigma_i(s) = s_i$ (that is, $\sigma_i(s)$ is the i^{th} coordinate of s). To define $\mathcal{B}_i$ first note that, by Definition of B^∞, every $s_i \in B_i^\infty$ is not B-dominated relative to B_{-i}^∞, that is, there exists an $X_{-i}^{s_i} \subseteq B_{-i}^\infty$ (note that this set may vary with s_i, hence the superscript "s_i") such that, for all $s_i' \in S_i$, either there exists an $\hat{x}_{-i} \in X_{-i}^{s_i}$ such that:

$$\pi_i(s_i', \hat{x}_{-i}) < \pi_i(s_i, \hat{x}_{-i}) \tag{A.1}$$

or for all $x_{-i} \in X_{-i}^{s_i}$,

$$\pi_i(s_i', \hat{x}_{-i}) \leq \pi_i(s_i, \hat{x}_{-i}). \tag{A.2}$$

For every $s_i \in B_i^\infty$ fix one such set $X_{-i}^{s_i}$ (there may be several) and define $\mathcal{B}_i(s_i, s_{-i}') = \{s_i\} \times X_{-i}^{s_i}$. By construction, $(s_i, \hat{x}_{-i}) \in \mathcal{B}_i(s)$ and thus, either, by (A.1), $s \notin \mathbb{B}_i\|s_i' \geq s_i\|$ or, by (A.2), $\|s_i' > s_i\| \cap \mathcal{B}_i(s) = \varnothing$. It follows that, for every $i \in I$ and for every $s \in B^\infty$, $s \in \mathbf{R}_i$ and thus $B^\infty = \mathbf{R} = \mathbb{CBR}$. Note that for completeness – although strictly speaking this is not needed – we can add the condition that, for every $i \in I$ and $s \in B^\infty$, $\mathcal{N}_i(s) = \{E \in 2^\Omega : E \cap \mathcal{B}_i(s) = \varnothing\}$. □

The proof of Theorem 2 uses as intermediate results the ones stated in Section 4.3 and proved in Appendix B.

Proof of Theorem 2. (A_3) Given a game, consider a model $M \in \mathcal{M}_3$. Then $\mathbf{C} = \mathbb{CBC} = \mathbf{C}_\cup = \Omega$ (so that $\mathbb{CBR} \cap \mathbb{CBC} \cap \mathbf{C}_\cup = \mathbb{CBR}$). Let $\omega \in \mathbb{CBR}$. Then, by Corollary 2 in Section 4.3, $\omega \in \mathbf{T}^\infty$.

(B_3) Given a game construct the following model of it: $\Omega = T^\infty$; for every player i and for every $s \in T^\infty$, $\mathcal{B}_i(s) = \{s' \in T^\infty : s_i' = s_i\}$ (that is, $s' \in \mathcal{B}_i(s)$ if and only if both s and s' belong to T^∞ and player i's strategy is the same in s and s'); $\sigma_i : T^\infty \to S_i$ is defined by $\sigma_i(s) = s_i$ (that is, $\sigma_i(s)$

is the i^{th} coordinate of s); finally, for all $i \in I$ and $s \in T^\infty$, $\mathcal{N}_i(s) = \{E \in 2^\Omega : E \cap \mathcal{B}_i(s) = \varnothing\}$. Note that each relation $\mathcal{B}_i$ is an equivalence relation. Fix an arbitrary state $s \in T^\infty$ and an arbitrary player i and suppose that, for some $s'_i \in S_i$, $\pi_i(s'_i, s_{-i}) > \pi_i(s_i, s_{-i})$, that is, $s \in \|s'_i > s_i\|$, so that $\|s'_i > s_i\| \cap \mathcal{B}_i(s) \supseteq \{s\} \notin \mathcal{N}_i(s)$. Then, by definition of T^∞, there exists an $\hat{s}_{-i} \in S_{-i}$ such that $(s_i, \hat{s}_{-i}) \in T^\infty$ and $\pi_i(s'_i, \hat{s}_{-i}) < \pi_i(s_i, \hat{s}_{-i})$; by construction, $(s_i, \hat{s}_{-i}) \in \mathcal{B}_i(s)$ so that $s \notin \mathbb{B}_i\|s'_i \geq s_i\|$. Thus, by Definition 2, player i is rational at state s, that is, $s \in \mathbf{R}_i$. Since i and s were chosen arbitrarily, it follows that $\mathbf{R} = T^\infty$. $\square$

B. Proofs of Section 4

Proof of Remark 2. *Preliminary Step.* Let us first formally prove (by contradiction) that within $M \in \mathcal{M}_1$ it is the case that, for all $i \in I$ and for all $\omega \in \Omega$ there exists some $\omega' \in \mathcal{B}_i(\omega)$ such that $\omega \notin \mathbb{N}_i\{\omega'\}$. Suppose otherwise, i.e., let $\omega \in \mathbb{N}_i\{\omega'\}$ for all $\omega' \in \mathcal{B}_i(\omega)$. Then, by (L_4), $\omega \in \mathbb{N}_i\mathcal{B}_i(\omega)$, which contradicts (L_2).

(A_1) Fix a strategic-form game with ordinal payoffs and a model $M \in \mathcal{M}_1$. Suppose that $\omega_1 \in \mathbb{CB}\mathbf{R}$ (that is, $\mathcal{B}^*(\omega_1) \subseteq \mathbf{R}$). We want to show that $\sigma(\omega_1) \in U^\infty$. The proof is by induction.

Initial Step. First we show (by contradiction) that, for every player $i \in I$ and for every $\omega \in \mathcal{B}^*(\omega_1)$, $\sigma_i(\omega) \notin W_i^0$ (see Definition 9). Suppose not. Then there exist a player i and an $\omega_2 \in \mathcal{B}^*(\omega_1)$ such that $\sigma_i(\omega_2) \in W_i^0$, that is, strategy $\sigma_i(\omega_2)$ of player i is U-dominated relative to S_{-i}, i.e., for every $X_{-i} \subseteq S_{-i}$ and every $s_{-i} \in X_{-i}$ there exists a strategy $s_i \in S_i$ such that (1) for all $x_{-i} \in X_{-i}$, $\pi_i(s_i, x_{-i}) \geq \pi_i(\sigma_i(\omega_2), x_{-i})$, and (2) $\pi_i(s_i, s_{-i}) > \pi_i(\sigma_i(\omega_2), s_{-i})$. Set $X_{-i} = \sigma_{-i}(\mathcal{B}_i(\omega_2))$. By our preliminary step, there exists some $\omega' \in \mathcal{B}_i(\omega_2)$ such that $\omega_2 \notin \mathbb{N}_i\{\omega'\}$. Then, set $s_{-i} = \sigma_{-i}(\omega')$. Let $b \in S_i$ be the strategy that satisfies conditions (1) and (2) for $(X_{-i}, s_{-i}) = (\sigma_{-i}(\mathcal{B}_i(\omega_2)), \sigma_{-i}(\omega'))$, implying that $\omega_2 \in \mathbb{B}_i\|b \geq \sigma_i(\omega_2)\|$, and $\{\omega'\} \subseteq \|b > \sigma_i(\omega_2)\|$. By (L_3), $\mathbb{N}_i\{\omega'\} \supseteq \mathbb{N}_i\|b > \sigma_i(\omega_2)\|$. Therefore, by $\omega_2 \notin \mathbb{N}_i\{\omega'\}$ it follows that $\omega_2 \notin \mathbb{N}_i\|b > \sigma_i(\omega_2)\|$, thus implying $\omega_2 \notin \mathbf{R}_i$, and therefore contradicting $\omega_2 \in \mathcal{B}^*(\omega_1)$.

Inductive Step. Fix an integer $m \geq 1$ and suppose that, for every player $j \in I$ and for every $\omega \in \mathcal{B}^*(\omega_1)$, $\sigma_j(\omega) \in U_j^m$, that is, $\mathcal{B}^*(\omega_1) \subseteq U^m$. We want to show (by contradiction) that, for every player $i \in I$ and for every $\omega \in \mathcal{B}^*(\omega_1)$, $\sigma_i(\omega) \notin W_i^m$. Suppose not. Then there exist a player i and an $\omega_2 \in \mathcal{B}^*(\omega_1)$ such that $\sigma_i(\omega_2) \in W_i^m$, that is, strategy $\sigma_i(\omega_2)$ of player i is U-dominated relative to U_{-i}^m, i.e., for every $X_{-i} \subseteq U_{-i}^m$ and every $s_{-i} \in X_{-i}$ there exists a strategy $s_i \in U_i^m$ such that (1) for all $x_{-i} \in X_{-i}$, $\pi_i(s_i, x_{-i}) \geq \pi_i(\sigma_i(\omega_2), x_{-i})$, and (2) $\pi_i(s_i, s_{-i}) > \pi_i(\sigma_i(\omega_2), s_{-i})$. Set $X_{-i} = \sigma_{-i}(\mathcal{B}_i(\omega_2))$. By the induction hypothesis and the fact that $\mathcal{B}_i(\omega_2) \subseteq \mathcal{B}^*(\omega_2) \subseteq \mathcal{B}^*(\omega_1)$ (the latter inclusion follows from transitivity of $\mathcal{B}^*$), $X_{-i} \subseteq U_{-i}^m$. By our preliminary step, there exists some $\omega' \in \mathcal{B}_i(\omega_2)$ such that $\omega_2 \notin \mathbb{N}_i\{\omega'\}$. Then, set $s_{-i} = \sigma_{-i}(\omega')$. Let $b \in U_i^m$ be the strategy that satisfies conditions (1) and (2) for $(X_{-i}, s_{-i}) = (\sigma_{-i}(\mathcal{B}_i(\omega_2)), \sigma_{-i}(\omega'))$, implying that $\omega_2 \in \mathbb{B}_i\|b \geq \sigma_i(\omega_2)\|$, and $\{\omega'\} \subseteq \|b > \sigma_i(\omega_2)\|$. By (L_3), $\mathbb{N}_i\{\omega'\} \supseteq \mathbb{N}_i\|b > \sigma_i(\omega_2)\|$. Therefore, by $\omega_2 \notin \mathbb{N}_i\{\omega'\}$ it follows that $\omega_2 \notin \mathbb{N}_i\|b > \sigma_i(\omega_2)\|$, thus implying $\omega_2 \notin \mathbf{R}_i$, which contradicts $\omega_2 \in \mathcal{B}^*(\omega_1)$. Thus, for every player $i \in I$ and for every $\omega \in \mathcal{B}^*(\omega_1)$, $\sigma_i(\omega) \in \bigcap_{m=1}^{\infty} U_i^m = U_i^\infty$.

It only remains to show that $\sigma_i(\omega_1) \in U_i^\infty$. Take any $\omega_2 \in \mathcal{B}_i(\omega_1)$. Since $\mathcal{B}_i(\omega_1) \subseteq \mathcal{B}^*(\omega_1)$, $\omega_2 \in \mathcal{B}^*(\omega_1)$. Thus $\sigma_i(\omega_2) \in U_i^\infty$. By (Σ_0), since $\omega_2 \in \mathcal{B}_i(\omega_1)$, $\sigma_i(\omega_2) = \sigma_i(\omega_1)$. Thus $\sigma_i(\omega_1) \in U_i^\infty$.

(B_1) Given a game G construct the following model $M \in \mathcal{M}_1$: $\Omega = U^\infty = U_1^\infty \times \cdots \times U_n^\infty$; for every player i and for every $s \in U^\infty$, $\sigma_i : U^\infty \to S_i$ is defined by $\sigma_i(s) = s_i$ (that is, $\sigma_i(s)$ is the i^{th} coordinate of s). To define $\mathcal{B}_i$ first note that, by Definition of U^∞, every $s_i \in U_i^\infty$ is not U-dominated relative to U_{-i}^∞, that is, there exists an $X_{-i}^{s_i} \subseteq B_{-i}^\infty$ and some $\hat{s}_{-i} \in X_{-i}^{s_i}$ such that, for all $s'_i \in S_i$, either there exists an $x_{-i} \in X_{-i}^{s_i}$ such that:

$$\pi_i(s_i', x_{-i}) < \pi_i(s_i, x_{-i}) \tag{B.1}$$

or

$$\pi_i(s_i', \hat{s}_{-i}) \leq \pi_i(s_i, \hat{s}_{-i}). \tag{B.2}$$

For every $s_i \in U_i^\infty$ fix one such set $X_{-i}^{s_i}$ (there may be several) and define $\mathcal{B}_i(s) = \{s_i\} \times X_{-i}^{s_i}$. Moreover, let $\{(s_i, \hat{s}_{-i})\}$ be the only non-empty non-null event for player i at s, that is, $\mathcal{N}_i(s) = \{\varnothing, \{(s_i, \hat{s}_{-i})\}\}$. Thus, by construction for every $s_i' \in S_i$, either, by (B.1), $s \notin \mathbb{B}_i\|s_i' \geq s_i\|$ or, by (B.2), $\|s_i' > s_i\| \in \mathcal{N}_i(s)$. It follows that, for every $i \in I$ and for every $s \in U^\infty$, $s \in \mathbf{R}_i$ and thus $U^\infty = \mathbf{R} = \mathbb{CBR}$. □

Proof of Corollary 1. Fix an arbitrary $s \in T^\infty$. Then, by Theorem 2, there exists some model in $M \in \mathcal{M}_3$ such that for some state ω (in this model), $\sigma(\omega) = s$ and $\omega \in \mathbb{CB}\mathbf{R}$. Since $M_3 \subseteq M_2$ it follows that $M \in \mathcal{M}_2$, and therefore, by Theorem 1, $s \in B^\infty$, thus proving $T^\infty \subseteq B^\infty$.

Likewise, fix an arbitrary $s' \in B^\infty$. Then, by Theorem 1, there exists some model $M' \in \mathcal{M}_2$ such that for some state ω' (in this model), $\sigma(\omega') = s'$ and $\omega' \in \mathbb{CB}\mathbf{R}$. Since $M_2 \subseteq M_1$ it follows that $M' \in \mathcal{M}_1$, and therefore, by Remark 2, $s' \in U^\infty$, thus proving $B^\infty \subseteq U^\infty$.

Finally, fix an arbitrary $s' \in U^\infty$. Then, by Remark 2, there exists some model $M' \in \mathcal{M}_1$ such that for some state ω' (in this model), $\sigma(\omega') = s'$ and $\omega' \in \mathbb{CB}\mathbf{R}$. Since $M_1 \subseteq M_0$ it follows that $M' \in \mathcal{M}_0$, and therefore, by Theorem 1, $s' \in S^\infty$, thus proving $U^\infty \subseteq S^\infty$. □

Proof of Proposition 1. Fix a strategic-form game and a model $M \in \mathcal{M}_2$. Suppose that $\omega_1 \in \mathbb{CB}\mathbf{R} \cap \mathbb{CB}\mathbf{C}$, i.e., $\mathcal{B}^*(\omega_1) \subseteq \mathbf{R} \cap \mathbf{C}$. We want to show that $\sigma(\omega_1) \in T^\infty$. As before, the proof is by induction.

Initial Step. First we show (by contradiction) that, for every $\omega \in \mathcal{B}^*(\omega_1)$, $\sigma(\omega) \notin I^0$ (see Definition 7). Suppose, that there exists an $\omega_2 \in \mathcal{B}^*(\omega_1)$ such that $\sigma(\omega_2) \in I^0$, that is, $\sigma(\beta)$ is inferior relative to the entire set of strategy profiles S. Then there exists a player i and a strategy $\hat{s}_i \in S_i$ such that

$$\begin{aligned} \pi_i(\hat{s}_i, s_{-i}) &\geq \pi_i(\sigma_i(\omega_2), s_{-i}), \text{ for all } s_{-i} \in S_{-i}, &\text{(B.3)} \\ \pi_i(\hat{s}_i, \sigma_{-i}(\omega_2)) &> \pi_i(\sigma_i(\omega_2), \sigma_{-i}(\omega_2)). &\text{(B.4)} \end{aligned}$$

Hence, for every $\omega \in \mathcal{B}_i(\omega_2)$, $\pi_i(\hat{s}_i, \sigma_{-i}(\omega)) \geq \pi_i(\sigma_i(\omega_2), \sigma_{-i}(\omega))$, that is, $\omega_2 \in \mathbb{B}_i\|\hat{s}_i \geq \sigma_i(\omega_2)\|$. Furthermore, since $\mathcal{B}^*(\omega_1) \subseteq \mathbf{C} \subseteq \mathbf{C}_i$ and $\omega_2 \in \mathcal{B}^*(\omega_1)$, $\omega_2 \in \mathcal{B}_i(\omega_2)$. Since the model has full support, $\{\omega_2\} \notin \mathcal{N}_i(\omega_2)$ and thus, by (B.3), $\|\hat{s}_i > \sigma_i(\omega_2)\| \cap \mathcal{B}_i(\omega_2) \notin \mathcal{N}_i(\omega_2)$ (appealing to (L_2) with $F = \|\hat{s}_i > \sigma_i(\omega_2)\|$ and $E = \{\omega_2\}$), so that, by Definition 2, player i is not rational at state ω_2, contradicting the hypothesis that $\omega_2 \in \mathcal{B}^*(\omega_1)$ and $\omega_1 \in \mathbb{CB}\mathbf{R}$. Thus, for every $\omega \in \mathcal{B}^*(\omega_1)$, $\sigma(\omega) \in T^0 \backslash I^0 = T^1$ (recall that $T^0 = S$).

Inductive Step. Fix an integer $m \geq 1$ and suppose that, for every $\omega \in \mathcal{B}^*(\omega_1)$, $\sigma(\omega) \in T^m$. We want to show that, for every $\omega \in \mathcal{B}^*(\omega_1)$, $\sigma(\omega) \notin I^m$. Suppose, by contradiction, that there exists an $\omega_2 \in \mathcal{B}^*(\omega_1)$ such that $\sigma(\omega_2) \in I^m$, that is, $\sigma(\omega_2)$ is inferior relative to T^m. Then there exist a player i and a strategy $\tilde{s}_i \in S_i$ such that

$$\begin{aligned} \pi_i(\tilde{s}_i, \sigma_{-i}(\omega_2)) &> \pi_i(\sigma_i(\omega_2), \sigma_{-i}(\omega_2)), &\text{(B.5)} \\ \pi_i(\tilde{s}_i, s_{-i}) &\geq \pi_i(\sigma_i(\omega_2), s_{-i}), \text{ for all } s_{-i} \in S_{-i} \text{ such that } (\sigma_i(\omega_2), s_{-i}) \in T^m. &\text{(B.6)} \end{aligned}$$

By the induction hypothesis, for every $\omega \in \mathcal{B}^*(\omega_1)$, $(\sigma_i(\omega), \sigma_{-i}(\omega)) \in T^m$. Thus, since $\mathcal{B}_i(\omega_2) \subseteq \mathcal{B}^*(\omega_2) \subseteq \mathcal{B}^*(\omega_1)$ (the latter inclusion follows from transitivity of $\mathcal{B}^*$), we have that, for every $\omega \in$

$\mathcal{B}_i(\omega_2)$, $(\sigma_i(\omega_2), \sigma_{-i}(\omega)) \in T^m$ (recall that, by (Σ_0), if $\omega \in \mathcal{B}_i(\omega_2)$ then $\sigma_i(\omega) = \sigma_i(\omega_2)$). Since $\mathcal{B}^*(\omega_1) \subseteq \mathbf{C} \subseteq \mathbf{C}_i$ and $\omega_2 \in \mathcal{B}^*(\omega_1)$, $\omega_2 \in \mathcal{B}_i(\omega_2)$. Since the model has full support, $\{\omega_2\} \notin \mathcal{N}_i(\omega_2)$ and thus $\|\hat{s}_i > \sigma_i(\omega_2)\| \cap \mathcal{B}_i(\omega_2) \notin \mathcal{N}_i(\omega_2)$ so that, by (B.5) and Definition 2, player i is not rational at state ω_2, contradicting the hypothesis that $\omega_2 \in \mathcal{B}^*(\omega_1)$ and $\omega_1 \in \mathbb{CB}\mathbf{R}$. Thus, we have shown that, for every $\omega \in \mathcal{B}^*(\omega_1)$, $\sigma(\omega) \in \bigcap_{m=1}^{\infty} T^m = T^\infty$, that is, $\omega_1 \in \mathbb{CB}\mathbf{T}^\infty$. □

Proof of Corollary 2. Fix a strategic-form game with ordinal payoffs and a model $M \in \mathcal{M}_2$. Suppose that $\omega_0 \in \mathbb{CB}\mathbf{R} \cap \mathbb{CB}\mathbf{C} \cap \mathbf{C}_\cup$. Since $\omega_0 \in \mathbf{C}_\cup$, there exists a player $i \in I$ such that $\omega_0 \in \mathbf{C}_i$, that is, $\omega_0 \in \mathcal{B}_i(\omega_0)$. Hence, by definition of $\mathcal{B}^*$, $\omega_0 \in \mathcal{B}^*(\omega_0)$. By Proposition 1, $\omega_0 \in \mathbb{CB}\mathbf{T}^\infty$, that is, for every $\omega \in \mathcal{B}^*(\omega_0)$, $\omega \in \mathbf{T}^\infty$. Hence $\omega_0 \in \mathbf{T}^\infty$. □

References

Apt, K.R. & Zvesper, J.A. (2010). The role of monotonicity in the epistemic analysis of strategic games. *Games* 1, 381–394.

Aumann, R.J. (1999). Interactive epistemology I: Knowledge. *International Journal of Game Theory* 28, 263–300.

Battigalli, P. & Bonanno, G. (1999). Recent results on belief, knowledge and the epistemic foundations of game theory. *Research in Economics* 53, 149–225.

Bernheim, D. (1984). Rationalizable strategic behavior. *Econometrica* 52, 1007–1028.

Bonanno, G. (2008). A syntactic approach to rationality in games with ordinal payoffs. *Logic and the Foundations of Game and Decision Theory (LOFT7)* (Edited by G. Bonanno, W. van der Hoek & M. Wooldridge), vol. 3 of *Texts in Logic and Games*, Amsterdam University Press, 59–86.

Bonanno, G. & Nehring, K. (1998). On Stalnaker's notion of strong rationalizability and Nash equilibrium in perfect information games. *Theory and Decision* 45, 291–295.

Börgers, T. (1993). Pure strategy dominance. *Econometrica* 61, 423–430.

Brandenburger, A. & Dekel, E. (1987). Rationalizability and correlated equilibria. *Econometrica* 55, 1391–1402.

Brandenburger, A. & Keisler, J. (2006). An impossibility theorem on beliefs in games. *Studia Logica* 84, 211–240.

Chateauneuf, A., Eichberger, J. & Grant, S. (2007). Choice under uncertainty with the best and worst in mind: Neo-additive capacities. *Journal of Economic Theory* 137, 538–567.

Chellas, B.F. (1980). *Modal logic: an introduction.* Cambridge University Press, Cambridge.

Chen, Y.C., Long, N.V. & Luo, X. (2007). Iterated strict dominance in general games. *Games and Economic Behavior* 61, 299-351.

de Finetti, B. (1949). La 'logica del plausible' secondo la concezione di Polya. *Atti della XLII Riunione, Societa Italiana per il Progresso delle Scienze*, 227–236.

Dekel, E., Fudenberg, D. & Morris, S. (2007). Interim correlated rationalizability. *Theoretical Economics* 2, 15–40.

Di Tillio, A. (2008). Subjective expected utility in games. *Theoretical Economics* 3, 287–323.

Ely, J. & Peski, M. (2006). Hierarchies of belief and interim rationalizability. *Theoretical Economics* 1, 19–65.

Epstein, L. & Wang, T. (1996). "Beliefs about beliefs" without probabilities. *Econometrica* 64, 1343–1373.

Fagin, R., Halpern, J.Y., Moses, Y. & Vardi, M.Y. (1995). *Reasoning about knowledge.* MIT press.

Fishburn, P. (1986). The axioms of subjective probability. *Statistical Science* 1, 335–358.

Gärdenfors, P. (1975). Qualitative probability as an intentional logic. *Journal of Philosophical Logic* 4, 171–185.

Harsanyi, J. (1967-68). Games with incomplete information played by Bayesian players, I-III. *Management Science* 14, 159–182, 320–334, 486–502.

Hillas, J. & Samet, D. (2014). Weak dominance: a mystery cracked. *Technical Report, Tel Aviv University.*

Hughes, G.E. & Cresswell, M.J. (1968). *An introduction to modal logic.* Methuen, London, UK.

Koopman, B. (1940). The axioms and algebra of intuitive probability. *Annals of Mathematics* 41, 269–292.

Kraft, C., Pratt, J.W. & Seidenberg, A. (1959). Intuitive probability on finite sets. *Annal of Mathematical Statistics* 30, 408–419.

Kripke, S. (1959). A completeness theorem in modal logic. *Journal of Symbolic Logic* 24, 1–14.

Mackenzie, A. (2017). A foundation for probabilistic beliefs with or without atoms. *Working Paper, University of Rochester.*

Pacuit, E. (2017). *Neighborhood semantics for modal logic.* Springer, New York.

Pearce, D.G. (1984). Rationalizable strategic behavior and the problem of perfection. *Econometrica* 52, 1029–1050.

Scott, D. (1964). Measurement structures and linear inequalities. *Journal of Mathematical Psychology* 1, 233–247.

Scott, D. & Suppes, P. (1958). Foundational aspects of theories of measurement. *Journal of Symbolic Logic* 23, 113–128.

Segerberg, K. (1971). Qualitative probability in a modal setting. *Proceedings of the Second Scandinavian Logic Symposium* 63, 341–352.

Stalnaker, R. (1994). On the evaluation of solution concepts of games. *Journal of Economic Theory* 45, 370–391.

Tan, T. & Werlang, S. (1988). The Bayesian foundations of solution concepts of games. *Journal of Economic Theory* 45, 370–391.

Trost, M. (2013). Epistemic characterization of iterated deletion of inferior strategy profiles in preference-based type spaces. *International Journal of Game Theory* 42, 755–776.

van der Hoek, W. (1996). Qualitative modalities. *Journal of Uncertainty, Fuzziness and Knowledge-Based Systems* 4, 45–59.

van Ditmarsch, H., Halpern, J., van der Hoek, W. & Kooi, B. (2015). *Handbook of Epistemic Logic*. College Publications.

Villegas, C. (1967). On qualitative probability. *American Mathematical Monthly* 74, 661–669.